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A CLASS OF GLOBAL LARGE SOLUTIONS TO THE OLDROYD-B-TYPE MODEL WITH FRACTIONAL DISSIPATION*

YICHEN DAI†, ZHONG TAN‡, AND JIAHONG WU§

Abstract. The global existence and regularity problem on the Oldroyd-B-type model with fractional dissipation is not well understood for many ranges of fractional powers. This paper examines this open problem from a different perspective. We construct a class of large solutions to the d-dimensional ($d=2,3$) Oldroyd-B-type models with the fractional dissipation $(-\Delta)^\alpha u$ and $(-\Delta)^\beta \tau$ when the fractional powers satisfy $\alpha+\beta\geq 1$. The process presented here actually assesses that an initial data near any function whose Fourier transform lives in a compact set away from the origin always leads to a unique and global solution.

Keywords. Large solutions; Oldroyd-B-type model; fractional dissipation.

AMS subject classifications. 35A05; 35Q35; 76D03.

1. Introduction

The viscoelastic system has attracted numerous attention in recent years and lots of excellent works have been done both in the macroscopic and microscopic regimes. The Oldroyd-B model is a basic macroscopic model which describes the motion of some viscoelastic flows such as the system coupling fluids and polymers. The formulation about viscoelastic flows of Oldroyd-B type was originally introduced by Oldroyd [26]. We refer to [6, 16] for more detailed physical background and the derivation of the Oldroyd-B type model and related models.

This paper examines the global existence to the 2-D and 3-D incompressible Oldroyd-B-type model with fractional dissipation,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu(-\Delta)^\alpha u + \nabla P = \kappa \nabla\cdot\tau, \quad x\in\mathbb{R}^d,\, t>0,\\ \partial_t \tau + u\cdot\nabla\tau + \eta(-\Delta)^\beta\tau + \mu\tau = \gamma\mathcal{D}u + Q(\nabla u,\tau), \quad x\in\mathbb{R}^d,\, t>0,\\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^d,\, t>0,\\ u(x,0)=u_0(x), \quad \tau(x,0)=\tau_0(x), \end{cases} \tag{1.1}$$

where u, P and τ represent the velocity, the pressure and a symmetric tensor which is the non-Newtonian part of the stress tensor, respectively, and $\nu>0$, $\kappa>0$, $\eta>0$, $\mu>0$, $\gamma\geq 0$, $\alpha\geq 0$ and $\beta\geq 0$ are real parameters. The fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx.$$

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†School of Science, Jimei University, Xiamen, 361021, China (yichendai0613@163.com).

‡School of Mathematical Sciences, Xiamen University, Xiamen, Fujian Province 361005, China & Shenzhen Research Institute of Xiamen University, Shenzhen 518057, China (tan85@xmu.edu.cn).

§Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States (jiahong.wu@okstate.edu).

Sometimes we also write $\Lambda=(-\Delta)^{\frac{1}{2}}$.

Here, $\mathcal{D}u$ is the deformation tensor and is the symmetric part of the velocity gradient

$$\mathcal{D}u=\frac{1}{2}\left(\nabla u+(\nabla u)^{\top}\right)$$

and Q is a given bilinear form which can be chosen as

$$Q(\nabla u,\tau)=\Omega\tau-\tau\Omega+b(\mathcal{D}u\tau+\tau\mathcal{D}u),$$

where

$$\Omega(u)=\frac{1}{2}\left(\nabla u-(\nabla u)^{\top}\right)$$

is the skew symmetric part of ∇u and $b\in[-1,1]$ is a constant.

The system (1.1) with the fractional Laplacian operator may be physically relevant. The fractional Laplacian operator can model various anomalous diffusions. Especially, (1.1) allows us to study long-range diffusive interactions. In addition, (1.1) with hyperviscosity can be used in turbulence modeling to control the effective range of the non-local dissipation and to make numerical resolutions more efficient ([13]). Mathematically (1.1) allows us to examine a family of models simultaneously and the study of (1.1) may provide a more complete picture on the behavior of solutions relative to the sizes of the parameters α and β.

There have been extensive studies on the well-posedness of the Oldroyd-B model. Guillopé and Saut in [14, 15] showed that the strong solutions are locally well-posed in Sobolev spaces. The global existence of weak solutions when $b=0$ was proven by Lions and Masmoudi [23]. Chemin and Masmoudi [6] obtained later the local and global well-posedness results in the frame of critical Besov spaces. Elgindi and Liu [11] established the result about the 3D global well-posedness for smooth solutions with small initial data. Zhu [36] proved the global existence of small smooth solutions without damping on the stress tensor. In 2D case, Constantin and Kligl [8] examined the global well-posedness of strong solutions with $\gamma=0$ and $b=-1$. A range of global well-posedness results on (1.1) have been obtained. Ye [31], and Ye and Xu [32] established the global regularity problem of the 2D Oldroyd-B-type model with partial dissipation for $\alpha+\beta=2$. Yuan and Zhang [33] have shown the d-dimensional global regularity of strong solutions if α and β satisfy

$$\alpha\geq\frac{1}{2}+\frac{d}{4},\quad \beta>0,\quad \alpha+\beta\geq 1+\frac{d}{2}.$$

In addition, some interesting results for related Oldroyd-type models of viscoelastic fluids can be found in [2, 5–7, 12, 17, 19, 22].

The issue of whether smooth solutions of the Oldroyd-B-type model (1.1) with large initial data can develop singularity in finite time is still a challenging open problem when α and β are not in the ranges mentioned above. We remark that the local existence result in Sobolev spaces for this system can be obtained by following a standard procedure as in the book of Majda and Bertozzi [24]. The perspective of this paper is different. Our goal here is to offer an effective approach of constructing large solutions of (1.1). A special consequence of our construction assesses that any initial data close to a function whose Fourier transform supported in a suitable domain away from the origin always leads to a unique global solution of (1.1). We now describe the construction in some detail. There are some differences between the 2D and the 3D cases, so we split our consideration into two cases, one for the 3D case and one for the 2D case, for the sake of clarity. We begin with the 3D case.

To construct large solutions for the Oldroyd-B-type model, we define a suitable vector field $\phi_0^\varepsilon \in C_0^\infty(\mathbb{R}^3)$ with its Fourier transform satisfying

$$\widehat{\phi}_0^\varepsilon(\xi) = \left(\varepsilon^{-1}\log\frac{1}{\varepsilon}\right)\chi(\xi), \quad \xi\in\mathbb{R}^3, \tag{1.2}$$

where $0<\varepsilon\le 1$ is a small parameter depending on ν and η and will be specified later. Here χ is a smooth cutoff function,

$$\mathrm{supp}\chi\subset\mathcal{C} \quad \text{and}\ \chi=1 \quad \text{on} \quad \mathcal{C}_1,$$

where $\mathcal{C}$ and $\mathcal{C}_1$ denote the annulus sections,

$$\mathcal{C} := \left\{\xi\in\mathbb{R}^3\,|\,|\xi_i-\xi_j|\le\varepsilon, i,j=1,2,3, 1\le|\xi|^2\le 2\right\},$$
$$\mathcal{C}_1 := \left\{\xi\in\mathbb{R}^3\,|\,|\xi_i-\xi_j|\le\varepsilon/2, i,j=1,2,3, \frac{5}{4}\le|\xi|^2\le\frac{7}{4}\right\}.$$

We remark that the set $\mathcal{C}$ can be realized by restricting $\xi\in\mathbb{R}^3$ to the cylinder of radius $\frac{\varepsilon}{\sqrt{3}}$ centered on the line $x_1=x_2=x_3$. Suppose $\xi\in\mathbb{R}^3$ satisfies, for any $r\in\mathbb{R}$,

$$x_1=x_2=x_3=\frac{r}{3}, \quad \xi_1+\xi_2+\xi_3=r, \tag{1.3}$$

$$(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2\le\frac{1}{3}\varepsilon^2. \tag{1.4}$$

Then it is not difficult to see that $|\xi_i-\xi_j|\le\varepsilon, i,j=1,2,3$. In fact, we have from (1.3) and (1.4)

$$\begin{aligned}&(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2\\ =&\xi_1^2+\xi_2^2+\xi_3^2-2x_1(\xi_1+\xi_2+\xi_3)+3x_1^2\\ =&\xi_1^2+\xi_2^2+\xi_3^2-\frac{r^2}{3}\le\frac{1}{3}\varepsilon^2.\end{aligned}$$

Then,

$$\begin{aligned}&(\xi_1-\xi_2)^2+(\xi_1-\xi_3)^2+(\xi_2-\xi_3)^2\\ =&2(\xi_1^2+\xi_2^2+\xi_3^2)-2(\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3)\\ =&2(\xi_1^2+\xi_2^2+\xi_3^2)+(\xi_1^2+\xi_2^2+\xi_3^2-r^2)\le\varepsilon^2,\end{aligned}$$

which implies $|\xi_i-\xi_j|\le\varepsilon, i,j=1,2,3$.

Our global existence and regularity result for the 3D Oldroyd-B-type model can be stated as follows.

THEOREM 1.1. *Assume ϕ_0^ε is given by (1.2). Define U_0 and B_0 by*

$$U_0=\nabla\times\phi_0^\varepsilon, \qquad B_0=0. \tag{1.5}$$

Let $\nu>0$, $\kappa>0$, $\eta>0$, $\mu>0$, $\gamma=0$, $\alpha>0$, $\beta>0$ and $\alpha+\beta>1$. Let $s>\frac{5}{2}$. Consider the Oldroyd-B-type model in (1.1) with the initial data

$$u_0=U_0+v_0 \quad \text{and} \quad \tau_0=B_0+h_0.$$

If v_0 and h_0 satisfy

$$\left(C_1\nu\eta\|v_0\|_{H^s}^2+\|h_0\|_{H^s}^2+\varepsilon\|\phi_0^\varepsilon\|_{H^{s+2}}^4\right)e^{C_2\left(\|\phi_0^\varepsilon\|_{H^{s+2}}+\|\phi_0^\varepsilon\|_{H^{s+2}}^2\right)}\leq C_3\nu\min\left\{\eta,\nu\right\}$$

for suitable constants $C_1>0$, $C_2>0$ and $C_3>0$, then (1.1) has a unique global solution (u,τ) satisfying

$$u,\tau\in C([0,\infty);H^s(\mathbb{R}^3)),\quad \Lambda^\alpha u,\Lambda^\beta\tau\in L^2(0,\infty;L^2(\mathbb{R}^3)),$$

where $\Lambda=(-\Delta)^{\frac{1}{2}}$.

The initial data (u_0,τ_0) in Theorem 1.1 is not small. In fact,

$$\begin{aligned}
\|u_0\|_{L^2}&\geq\|U_0\|_{L^2}-\|v_0\|_{L^2}\\
&=\left[\int_{\mathbb{R}^3}|\xi|^2|\widehat{\phi_0^\varepsilon}(\xi)|^2\,d\xi\right]^{\frac{1}{2}}-\|v_0\|_{H^s}\\
&\geq\left[\int_{\mathcal{C}_1}|\xi|^2|\widehat{\phi_0^\varepsilon}(\xi)|^2\,d\xi\right]^{\frac{1}{2}}-\|v_0\|_{H^s}\\
&\geq C\left(\varepsilon^{-1}\log\frac{1}{\varepsilon}\right)\varepsilon-\|v_0\|_{H^s}\\
&\geq C\log\frac{1}{\varepsilon}-C_3\nu\max\left\{\eta,\nu\right\},
\end{aligned}\tag{1.6}$$

which can be really large when ε small enough. Similarly any homogeneous $\dot{H}^s$ norm is not small neither. In addition, as we shall see from the proof, Theorem 1.1 remains valid when the annulus in the definition of $\mathcal{C}$ is replaced by any compact set supported away from the origin.

A similar result also holds for the 2D Oldroyd-B-type model in (1.1). There are some minor differences in the construction process. We define a scalar function $\psi_0^\varepsilon\in C_0^\infty(\mathbb{R}^2)$ satisfying

$$\widehat{\psi_0^\varepsilon}(\xi)=\left(\varepsilon^{-\frac{1}{2}}\log\frac{1}{\varepsilon}\right)\chi_1(\xi),\quad \xi\in\mathbb{R}^2,\tag{1.7}$$

where χ_1 is a smooth cutoff function,

$$\mathrm{supp}\chi_1\subset\mathcal{D}\quad \text{and}\,\chi_1=1\quad\text{on}\quad\mathcal{D}_1.$$

Here $\mathcal{D}$ and $\mathcal{D}_1$ denote the annulus sections,

$$\begin{aligned}
\mathcal{D}&:=\left\{\xi\in\mathbb{R}^2\,|\,|\xi_i-\xi_j|\leq\varepsilon,i,j=1,2,1\leq|\xi|^2\leq2\right\},\\
\mathcal{D}_1&:=\left\{\xi\in\mathbb{R}^2\,|\,|\xi_i-\xi_j|\leq\varepsilon/2,i,j=1,2,\frac{5}{4}\leq|\xi|^2\leq\frac{7}{4}\right\}.
\end{aligned}$$

The global existence and regularity result for the 2D Oldroyd-B-type model can be stated as follows.

THEOREM 1.2. *Assume ψ_0^ε is given by (1.7). Define $\widetilde{U}_0$ and $\widetilde{B}_0$ by*

$$\widetilde{U}_0=\nabla^\perp\psi_0^\varepsilon:=(\partial_2\psi_0^\varepsilon,-\partial_1\psi_0^\varepsilon),\qquad \widetilde{B}_0=0.\tag{1.8}$$

Let $\nu>0$, $\kappa>0$, $\eta>0$, $\mu>0$, $\gamma=0$, $\alpha>0$, $\beta>0$ and $\alpha+\beta>1$. Let $s>2$. Consider the 2D Olrdroyd-B-type model in (1.1) with the initial data

$$u_0=\widetilde{U}_0+v_0 \quad \textit{and} \quad \tau_0=\widetilde{B}_0+h_0.$$

If v_0 and h_0 satisfy

$$\left(\widetilde{C}_1\nu\eta\|v_0\|_{H^s}^2+\|h_0\|_{H^s}^2+\varepsilon\|\psi_0^\varepsilon\|_{H^{s+2}}^4\right)e^{\widetilde{C}_2\left(\|\psi_0^\varepsilon\|_{H^{s+2}}+\|\psi_0^\varepsilon\|_{H^{s+2}}^2\right)}\le\widetilde{C}_3\nu\min\left\{\eta,\nu\right\}$$

for suitable constants $\widetilde{C}_1>0$, $\widetilde{C}_2>0$ and $\widetilde{C}_3>0$, then (1.1) has a unique global solution (u,τ) satisfying

$$u,\tau\in C([0,\infty);H^s(\mathbb{R}^2)),\quad \Lambda^\alpha u,\Lambda^\beta\tau\in L^2(0,\infty;L^2(\mathbb{R}^2)).$$

Again the initial data (u_0,τ_0) is not small in $H^s(\mathbb{R}^2)$. As in (1.6),

$$\|u_0\|_{L^2},\|\tau_0\|_{L^2}\ge C\log\frac{1}{\varepsilon}-\widetilde{C}_3\nu\max\left\{\eta,\nu\right\}.$$

Theorem 1.2 remains true if the annulus in the definition of $\mathcal{D}$ is changed to any compact set supported away from the origin.

We mention some related results. Lei, Lin and Zhou [18] constructed smooth large solutions to the 3D Navier-Stokes equations with the initial data close to a Beltrami flow. More information on the Beltrami flow can be found in [9] and [24]. Zhou and Zhu [35] obtained a class of large solutions to the 3D damped Euler flow near the Beltrami flow. Families of large solutions have also been obtained for the damped MHD equations and the Hall-MHD equations (see [10,20,21,34]). Our construction presented here is somewhat different and does not involve Beltrami flow for $\alpha,\beta\neq0$.

To prove Theorem 1.1, we seek a solution of the form

$$u=U+v,\quad \tau=B+h,$$

where $U=\nabla\times\phi$ and $B=0$ are the solutions of the corresponding linearized equations

$$\begin{cases}\partial_tU+\nu(-\Delta)^\alpha U=\kappa\nabla\cdot B,\\ \partial_tB+\eta(-\Delta)^\beta B+\mu B=0,\\ \nabla\cdot U=\nabla\cdot B=0,\\ U(x,0)=U_0(x),\quad B(x,0)=0.\end{cases}$$

By the definition of U_0 and B_0 in Theorem 1.1, U can be written as

$$U(t)=e^{-\nu(-\Delta)^\alpha t}U_0=e^{-\nu(-\Delta)^\alpha t}\nabla\times\phi_0^\varepsilon. \tag{1.9}$$

The equations of (v,h) are given by

$$\begin{cases}\partial_tv+u\cdot\nabla v+v\cdot\nabla U+\nu(-\Delta)^\alpha v=-\nabla P+\kappa\nabla\cdot h-U\cdot\nabla U,\\ \partial_th+u\cdot\nabla h+\eta(-\Delta)^\beta h+\mu h=Q(\nabla(v+U),h),\\ \nabla\cdot v=0,\\ v(x,0)=v_0(x),\quad h(x,0)=h_0(x).\end{cases} \tag{1.10}$$

Then it suffices to show that (1.10) has a unique global solution. Since the local well-posedness follows from standard procedure, we focus on the global bound via the bootstrap argument. Details on how to obtain suitable energy inequalities and how the bootstrap argument is applied are given in Section 3. The proof of Theorem 1.2 is similar and is sketched in Section 4. Section 2 provides the definitions of the inhomogeneous Sobolev space, the Besov spaces and related tools.

2. Preparation

This section serves as a preparation. We recall the definitions of the inhomogeneous Sobolev space H^s and of the Besov space $B^s_{2,2}$. The norms of these two spaces are equivalent. More details on these definitions and related basic facts can be found in several books and many papers (see, e.g., [1,3,25,27,29]).

Definition 2.1. *Let $\mathcal{S}$ denote the Schwartz class and $\mathcal{S}'$ its dual, the space of tempered distributions. The inhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ with $s\in\mathbb{R}$ is given by*

$$H^s(\mathbb{R}^d):=\left\{f\in\mathcal{S}'(\mathbb{R}^d): \|f\|_{H^s(\mathbb{R}^d)}<\infty\right\}$$

with the norm

$$\|f\|_{H^s(\mathbb{R}^d)}:=\Big(\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2d\xi\Big)^{\frac{1}{2}}.$$

Definition 2.2. *The inhomogeneous Besov space $B^s_{p,q}$ with $1\le p,q\le\infty$ and $s\in\mathbb{R}$ consists of functions $f\in\mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}}:=\big\|2^{js}\|\Delta_j f\|_{L^p}\big\|_{l^q}<\infty,$$

where Δ_j denotes the standard inhomogeneous dyadic block operator with $j=-1,0,\cdots$.

The norms of these two spaces H^s and $B^s_{2,2}$ are equivalent.

Lemma 2.1. *For any $s\in\mathbb{R}$,*

$$H^s\sim B^s_{2,2}.$$

We will also make use of the following commutator estimates.

Lemma 2.2 (Lemma 2.6 in [30]). *Let $s>-1$ and $(p,r)\in[1,+\infty]^2$. Assume u is a divergence-free vector field, namely $\nabla\cdot u=0$. The following inequality holds,*

$$\big\|2^{js}\|[\Delta_j,u\cdot\nabla]v\|_{L^p}\big\|_{l^r_j}\le C\big(\|\nabla u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|\nabla v\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}}\big),$$

where

$$[\Delta_j,u\cdot\nabla]v=\Delta_j(u\cdot\nabla v)-u\cdot\nabla(\Delta_j v).$$

We also recall the paraproducts (see, e.g., [1,4]). Let S_j denote the corresponding inhomogeneous low frequency cutoff operator,

$$S_j=\sum_{k\le j-1}\Delta_k.$$

In terms of the operators Δ_j and S_j, we can write a standard product of two functions as

$$fg=T_f g+T_g f+R(f,g),$$

where

$$T_f g=\sum_j S_{j-1}f\,\Delta_j g,\qquad R(f,g)=\sum_j\sum_{k\ge j-1}\Delta_k f\widetilde{\Delta}_k g$$

with $\widetilde{\Delta}_k=\Delta_{k-1}+\Delta_k+\Delta_{k+1}$.

3. Proof for Theorem 1.1

This section proves Theorem 1.1. As described in the introduction, it suffices to establish that the solutions of (1.10) remain bounded in H^s for all time. This is verified by deriving suitable energy inequalities and applying the bootstrap argument. As a preparation, we first present some bounds on $U(t)$ given by (1.9).

LEMMA 3.1. *Let ϕ_0 and ψ_0 be given by (1.2), U_0 and B_0 by (1.5) and $U(t)$ by (1.9). Then the following estimate holds. For any $s > \frac{5}{2}$,*

$$\begin{aligned} &\|U\|_{H^s} \le Ce^{-C_0 t}\|\phi_0^\varepsilon\|_{H^{s+2}}, \\ &\|\nabla U\|_{H^s} \le Ce^{-C_0 t}\|\phi_0^\varepsilon\|_{H^{s+2}}, \\ &\|f\|_{H^s} \le C\varepsilon e^{-2C_0 t}\|\phi_0^\varepsilon\|_{H^{s+2}}^2, \end{aligned} \tag{3.1}$$

where $f = -U \cdot \nabla U$.

Proof. **(Proof of Lemma 3.1.)** To prove (3.1), we rewrite the terms in f so that each term contains a difference $\partial_i - \partial_j$ with $i,j = 1,2,3$. Clearly, for $\phi = e^{-\nu(-\Delta)^\alpha t}\phi_0^\varepsilon$,

$$U = \nabla \times \phi = (\partial_2\phi_3 - \partial_3\phi_2, \partial_3\phi_1 - \partial_1\phi_3, \partial_1\phi_2 - \partial_2\phi_1).$$

Then, direct calculations show that the first component of $-U \cdot \nabla U$ becomes

$$\begin{aligned} -U\cdot\nabla U^1 = &-\partial_2\phi_3\partial_1\partial_2\phi_3 + \partial_1\phi_3\partial_2\partial_2\phi_3 + \partial_2\phi_3\partial_1\partial_3\phi_2 - \partial_1\phi_3\partial_2\partial_3\phi_2 \\ &+ \partial_3\phi_2\partial_1\partial_2\phi_3 - \partial_1\phi_2\partial_3\partial_2\phi_3 - \partial_3\phi_2\partial_1\partial_3\phi_2 + \partial_1\phi_2\partial_3\partial_3\phi_2 \\ &- \partial_3\phi_1\partial_2\partial_2\phi_3 + \partial_2\phi_1\partial_3\partial_2\phi_3 + \partial_3\phi_1\partial_2\partial_3\phi_2 - \partial_2\phi_1\partial_3\partial_3\phi_2 \\ = &[(\partial_1-\partial_2)\phi_3\partial_1\partial_2\phi_3 + \partial_1\phi_3\partial_2(\partial_2-\partial_1)\phi_3] \\ &+ [(\partial_2-\partial_1)\phi_3\partial_1\partial_3\phi_2 + \partial_1\phi_3\partial_3(\partial_1-\partial_2)\phi_2] \\ &+ [(\partial_3-\partial_1)\phi_2\partial_1\partial_2\phi_3 + \partial_1\phi_2\partial_2(\partial_1-\partial_3)\phi_3] \\ &+ [(\partial_1-\partial_3)\phi_2\partial_1\partial_3\phi_2 + \partial_1\phi_2\partial_3(\partial_3-\partial_1)\phi_2] \\ &+ [(\partial_2-\partial_3)\phi_1\partial_2\partial_2\phi_3 + \partial_2\phi_1\partial_2(\partial_3-\partial_2)\phi_3] \\ &+ [(\partial_3-\partial_2)\phi_1\partial_2\partial_3\phi_2 + \partial_2\phi_1\partial_3(\partial_2-\partial_3)\phi_2]. \end{aligned}$$

Taking the H^s-norm yields,

$$\begin{aligned} \|-U\cdot\nabla U^1\|_{H^s} \le &C\big(\|(\partial_1-\partial_2)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_2-\partial_1)\phi\|_{H^{s+1}} \\ &+ \|(\partial_2-\partial_1)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_1-\partial_2)\phi\|_{H^{s+1}} \\ &+ \|(\partial_3-\partial_1)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_1-\partial_3)\phi\|_{H^{s+1}} \\ &+ \|(\partial_1-\partial_3)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_3-\partial_1)\phi\|_{H^{s+1}} \\ &+ \|(\partial_2-\partial_3)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_3-\partial_2)\phi\|_{H^{s+1}} \\ &+ \|(\partial_3-\partial_2)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_2-\partial_3)\phi\|_{H^{s+1}}\big) \\ \le &C\big(\|(\partial_i-\partial_j)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi\|_{H^{s+1}}\big), \end{aligned}$$

where i,j in the last line are summed over $i,j = 1,2,3$. Similarly,

$$\begin{aligned} \|f\|_{H^s} \le &C\big(\|(\partial_i-\partial_j)\phi\|_{H^s}\|\phi\|_{H^{s+2}} + \|\phi\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi\|_{H^{s+1}}\big) \\ \le &Ce^{-2C_0 t}\big(\|(\partial_i-\partial_j)\phi_0^\varepsilon\|_{H^s}\|\phi_0^\varepsilon\|_{H^{s+2}} + \|\phi_0^\varepsilon\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi_0^\varepsilon\|_{H^{s+1}}\big) \\ \le &C\varepsilon e^{-2C_0 t}\big(\|\phi_0^\varepsilon\|_{H^s}\|\phi_0^\varepsilon\|_{H^{s+2}} + \|\phi_0^\varepsilon\|_{H^{s+1}}\|\phi_0^\varepsilon\|_{H^{s+1}}\big) \end{aligned}$$

$$\leq C\varepsilon e^{-2C_0 t}\|\phi_0^\varepsilon\|_{H^{s+2}}^2.$$

This proves (3.1). ∎

We are now ready to prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Let $\theta>0$ be a small parameter to be chosen later. Applying Δ_j to (1.10) and dotting with $\theta\Delta_j v$ and $\Delta_j h$, respectively, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\big(\theta\|\Delta_j v\|_{L^2}^2+\|\Delta_j h\|_{L^2}^2\big)+\nu\theta 2^{2\alpha j}\|\Delta_j v\|_{L^2}^2+\eta 2^{2\beta j}\|\Delta_j h\|_{L^2}^2+\mu\|\Delta_j h\|_{L^2}^2\leq \textstyle\sum_{m=1}^5 I_m, \quad (3.2)$$

where

$$\begin{aligned}
I_1 :=& -\theta\int_{\mathbb{R}^3}[\Delta_j,u\cdot\nabla]v\cdot\Delta_j v\,dx-\int_{\mathbb{R}^3}[\Delta_j,u\cdot\nabla]h\cdot\Delta_j h\,dx,\\
I_2 :=& -\theta\int_{\mathbb{R}^3}\Delta_j(v\cdot\nabla U)\cdot\Delta_j v\,dx,\\
I_3 :=& \kappa\theta\int_{\mathbb{R}^3}(\nabla\cdot\Delta_j h)\cdot\Delta_j v\,dx,\\
I_4 :=& \theta\int_{\mathbb{R}^3}\Delta_j f\cdot\Delta_j v\,dx,\\
I_5 :=& \int_{\mathbb{R}^3}\Delta_j Q(\nabla(v+U),h)\cdot\Delta_j h\,dx.
\end{aligned}$$

The reason that we need the factor θ is to bound the linear term I_3 above suitably. Multiplying (3.2) by 2^{2js} and summing over $j\geq -1$ yield

$$\frac{1}{2}\frac{d}{dt}\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big)+\nu\theta\|v\|_{H^{s+\alpha}}^2+\eta\|h\|_{H^{s+\beta}}^2+\mu\|h\|_{H^s}^2\leq\sum_{j\geq -1}2^{2js}\sum_{m=1}^5 I_m. \quad (3.3)$$

We estimate the terms on the right of (3.3). By Hölder's inequality, Young's inequality and Lemma 2.2,

$$\begin{aligned}
\sum_{j\geq -1}2^{2js}I_1 =& -\theta\sum_{j\geq -1}2^{2js}\int_{\mathbb{R}^3}[\Delta_j,u\cdot\nabla]v\cdot\Delta_j v\,dx-\sum_{j\geq -1}2^{2js}\int_{\mathbb{R}^3}[\Delta_j,u\cdot\nabla]h\cdot\Delta_j h\,dx\\
\leq& \theta\sum_{j\geq -1}2^{2js}\|[\Delta_j,u\cdot\nabla]v\|_{L^2}\|\Delta_j v\|_{L^2}+\sum_{j\geq -1}2^{2js}\|[\Delta_j,u\cdot\nabla]h\|_{L^2}\|\Delta_j h\|_{L^2}\\
=& \theta\sum_{j\geq -1}2^{(s-\alpha)j}\|[\Delta_j,u\cdot\nabla]v\|_{L^2}\cdot 2^{(s+\alpha)j}\|\Delta_j v\|_{L^2}\\
&+\sum_{j\geq -1}2^{(s-\beta)j}\|[\Delta_j,u\cdot\nabla]h\|_{L^2}\cdot 2^{(s+\beta)j}\|\Delta_j h\|_{L^2}\\
\leq& \theta\Big(\sum_{j\geq -1}2^{-2\alpha j}2^{2sj}\|[\Delta_j,u\cdot\nabla]v\|_{L^2}^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1}2^{2(s+\alpha)j}\|\Delta_j v\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{j\geq -1}2^{-2\beta j}2^{2sj}\|[\Delta_j,u\cdot\nabla]h\|_{L^2}^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1}2^{2(s+\beta)j}\|\Delta_j h\|_{L^2}^2\Big)^{\frac{1}{2}}\\
\leq& C\theta\|\nabla u\|_{H^{s-1}}\|v\|_{H^s}\|v\|_{H^{s+\alpha}}+C\|\nabla u\|_{H^{s-1}}\|h\|_{H^s}\|h\|_{H^{s+\beta}}\\
\leq& \frac{\nu}{4}\theta\|v\|_{H^{s+\alpha}}^2+\frac{\eta}{4}\|h\|_{H^{s+\beta}}^2+C\|\nabla u\|_{H^{s-1}}^2\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\nu}{4}\theta\|v\|_{H^{s+\alpha}}^2+\frac{\eta}{4}\|h\|_{H^{s+\beta}}^2+C\big(\|U\|_{H^s}^2+\|v\|_{H^s}^2\big)\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big)\\
&\le \frac{\nu}{4}\theta\|v\|_{H^{s+\alpha}}^2+\frac{\eta}{4}\|h\|_{H^{s+\beta}}^2+C\|U\|_{H^s}^2\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big)\\
&\quad +C_4\theta\|v\|_{H^s}^2\|v\|_{H^{s+\alpha}}^2+C_5\|v\|_{H^s}^2\|h\|_{H^{s+\beta}}^2. \qquad (3.4)
\end{aligned}
$$

By Hölder's inequality, we get

$$
\sum_{j\ge -1}2^{2js}I_2=-\theta\sum_{j\ge -1}2^{2js}\int_{\mathbb{R}^3}\Delta_j(v\cdot\nabla U)\cdot\Delta_j v\,dx\le C\theta\|\nabla U\|_{H^s}\|v\|_{H^s}^2. \qquad (3.5)
$$

I_3 can be bounded by

$$
\begin{aligned}
\sum_{j\ge -1}2^{2js}I_3&=\kappa\theta\sum_{j\ge -1}2^{2js}\int_{\mathbb{R}^3}(\nabla\cdot\Delta_j h)\cdot\Delta_j v\,dx\\
&\le C\kappa\theta\sum_{j\ge -1}2^{2sj}\cdot 2^j\|\Delta_j h\|_{L^2}\cdot\|\Delta_j v\|_{L^2}\\
&=C\kappa\theta\sum_{j\ge -1}2^{(1-\alpha-\beta)j}\cdot 2^{(s+\beta)j}\|\Delta_j h\|_{L^2}\cdot 2^{(s+\alpha)j}\|\Delta_j v\|_{L^2}\\
&\le \frac{\eta}{4}\|h\|_{H^{s+\beta}}^2+C_6\theta^2\|v\|_{H^{s+\alpha}}^2. \qquad (3.6)
\end{aligned}
$$

The term with I_4 is bounded by

$$
\begin{aligned}
\sum_{j\ge -1}2^{2js}I_4&=\theta\sum_{j\ge -1}2^{2js}\int_{\mathbb{R}^3}\Delta_j f\cdot\Delta_j v\,dx\\
&\le \theta\sum_{j\ge -1}2^{(s-\alpha)j}\|\Delta_j f\|_{L^2}\cdot 2^{(s+\alpha)j}\|\Delta_j v\|_{L^2}\\
&\le C\theta\|f\|_{H^s}\|v\|_{H^{s+\alpha}}\\
&\le \frac{\nu}{4}\theta\|v\|_{H^{s+\alpha}}^2+C\|f\|_{H^s}^2. \qquad (3.7)
\end{aligned}
$$

It remains to bound I_5. By the notion of paraproducts and Bernstein's inequality,

$$
\begin{aligned}
\sum_{j\ge -1}2^{2js}I_5&=\sum_{j\ge -1}2^{2js}\int_{\mathbb{R}^3}\Delta_j Q(\nabla(v+U),h)\cdot\Delta_j h\,dx\\
&\le C\|\nabla U\|_{H^s}\|h\|_{H^s}^2+\sum_{j\ge -1}2^{2js}\|\Delta_j h\|_{L^2}\Big(\sum_{|j-k|\le 2}\|S_{k-1}h\|_{L^\infty}\|\Delta_k\nabla v\|_{L^2}\\
&\quad +\sum_{|j-k|\le 2}\|\Delta_k h\|_{L^2}\|S_{k-1}\nabla v\|_{L^\infty}+\sum_{k\ge j-1}\|\Delta_k h\|_{L^2}\|\widetilde{\Delta_k}\nabla v\|_{L^\infty}\Big)\\
&\le C\|\nabla U\|_{H^s}\|h\|_{H^s}^2+\sum_{j\ge -1}2^{2js}\|\Delta_j h\|_{L^2}\Big(\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m h\|_{L^2}\cdot 2^j\|\Delta_j v\|_{L^2}\\
&\quad +\sum_{m\le j-1}\|\Delta_j h\|_{L^2}\cdot 2^{(1+\frac{d}{2})m}\|\Delta_m v\|_{L^2}+\sum_{k\ge j-1}2^{(1+\frac{d}{2})k}\|\Delta_k h\|_{L^2}\|\widetilde{\Delta_k}v\|_{L^2}\Big)\\
&:=C\|\nabla U\|_{H^s}\|h\|_{H^s}^2+I_5^{(1)}+I_5^{(2)}+I_5^{(3)}. \qquad (3.8)
\end{aligned}
$$

The term with $I_5^{(1)}$ is bounded by using Hölder's inequality, Young's convolution inequality with $\frac{1}{\infty}+1=\frac{1}{2}+\frac{1}{2}$ and the assumption of $\alpha+\beta>1$ and $s>\frac{5}{2}$,

$$
I_5^{(1)}=\sum_{j\ge -1}2^{2js}\|\Delta_j h\|_{L^2}\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m h\|_{L^2}\cdot 2^j\|\Delta_j v\|_{L^2}
$$

$$
\begin{aligned}
&= \sum_{j\geq -1} 2^{(s+\beta)j}\|\Delta_j h\|_{L^2} \sum_{m\leq j-1} 2^{(1-\alpha-\beta)j+(\frac{d}{2}-s)m}2^{sm}\|\Delta_m h\|_{L^2}\cdot 2^{(s+\alpha)j}\|\Delta_j v\|_{L^2}\\
&= \sum_{j\geq -1} 2^{(s+\beta)j}\|\Delta_j h\|_{L^2}\\
&\quad \times \sum_{m\leq j-1} 2^{(1-\alpha-\beta)(j-m)+(\frac{d}{2}-s+1-\alpha-\beta)m}2^{sm}\|\Delta_m h\|_{L^2}\cdot 2^{(s+\alpha)j}\|\Delta_j v\|_{L^2}\\
&\leq \Big(\sum_{j\geq -1} 2^{2(s+\beta)j}\|\Delta_j h\|_{L^2}^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1} 2^{2(s+\alpha)j}\|\Delta_j v\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\quad \times \Big\|\sum_{m\leq j-1} 2^{(1-\alpha-\beta)(j-m)}2^{sm}\|\Delta_m h\|_{L^2}\Big\|_{l^\infty}\\
&\leq C\|h\|_{H^{s+\beta}}\|v\|_{H^{s+\alpha}}\cdot\Big(\sum_{j\geq -1} 2^{2(1-\alpha-\beta)j}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1} 2^{2sj}\|\Delta_j h\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\leq C\|h\|_{H^{s+\beta}}\|h\|_{H^s}\|v\|_{H^{s+\alpha}}\\
&\leq \frac{\eta}{8}\|h\|_{H^{s+\beta}}^2+\frac{C_7}{3}\|h\|_{H^s}^2\|v\|_{H^{s+\alpha}}^2. \qquad (3.9)
\end{aligned}
$$

Similarly for $I_5^{(2)}$, we have

$$
\begin{aligned}
I_5^{(2)} &= \sum_{j\geq -1} 2^{2js}\|\Delta_j h\|_{L^2} \sum_{m\leq j-1} \|\Delta_j h\|_{L^2}\cdot 2^{(1+\frac{d}{2})m}\|\Delta_m v\|_{L^2}\\
&= \sum_{j\geq -1} 2^{(s+\beta)j}\|\Delta_j h\|_{L^2} \sum_{m\leq j-1} 2^{-\beta j}2^{(1+\frac{d}{2}-s-\alpha)m}2^{(s+\alpha)m}\|\Delta_m v\|_{L^2}\cdot 2^{sj}\|\Delta_j h\|_{L^2}\\
&= \sum_{j\geq -1} 2^{(s+\beta)j}\|\Delta_j h\|_{L^2}\\
&\quad \times \sum_{m\leq j-1} 2^{-\beta(j-m)}2^{(1+\frac{d}{2}-s-\alpha-\beta)m}2^{(s+\alpha)m}\|\Delta_m v\|_{L^2}\cdot 2^{sj}\|\Delta_j h\|_{L^2}\\
&\leq \sum_{j\geq -1} 2^{(s+\beta)j}\|\Delta_j h\|_{L^2} \sum_{m\leq j-1} 2^{-\beta(j-m)}2^{(s+\alpha)m}\|\Delta_m v\|_{L^2}\cdot 2^{sj}\|\Delta_j h\|_{L^2}\\
&\leq \Big(\sum_{j\geq -1} 2^{2(s+\beta)j}\|\Delta_j h\|_{L^2}^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1} 2^{2sj}\|\Delta_j h\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\quad \times \Big\|\sum_{m\leq j-1} 2^{-\beta(j-m)}2^{(s+\alpha)m}\|\Delta_m v\|_{L^2}\Big\|_{l^\infty}\\
&\leq C\|h\|_{H^{s+\beta}}\|h\|_{H^s}\cdot\Big(\sum_{j\geq -1} 2^{-2\beta j}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{j\geq -1} 2^{2(s+\alpha)j}\|\Delta_j v\|_{L^2}^2\Big)^{\frac{1}{2}}\\
&\leq C\|h\|_{H^{s+\beta}}\|h\|_{H^s}\|v\|_{H^{s+\alpha}}\\
&\leq \frac{\eta}{8}\|h\|_{H^{s+\beta}}^2+\frac{C_7}{3}\|h\|_{H^s}^2\|v\|_{H^{s+\alpha}}^2. \qquad (3.10)
\end{aligned}
$$

The term involving $I_5^{(3)}$ is bounded by

$$
\begin{aligned}
I_5^{(3)} &= \sum_{j\geq -1} 2^{2js}\|\Delta_j h\|_{L^2} \sum_{k\geq j-1} 2^{(1+\frac{d}{2})k}\|\Delta_k h\|_{L^2}\|\widetilde{\Delta_k} v\|_{L^2}\\
&= \sum_{j\geq -1} 2^{sj}\|\Delta_j h\|_{L^2} \sum_{k\geq j-1} 2^{sj+(1+\frac{d}{2}-2s-\alpha-\beta)k}2^{(s+\alpha)k}\|\widetilde{\Delta_k} v\|_{L^2}\cdot 2^{(s+\beta)k}\|\Delta_k h\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j\ge -1} 2^{sj}\|\Delta_j h\|_{L^2} \\
&\quad\times \sum_{k\ge j-1} 2^{(1+\frac{d}{2}-s-\alpha-\beta)j+(1+\frac{d}{2}-2s-\alpha-\beta)(k-j)} 2^{(s+\alpha)k}\|\widetilde{\Delta}_k v\|_{L^2}\cdot 2^{(s+\beta)k}\|\Delta_k h\|_{L^2} \\
&\le \sum_{j\ge -1} 2^{sj}\|\Delta_j h\|_{L^2} \sum_{k\ge j-1} 2^{-(j-k)} 2^{(s+\alpha)k}\|\widetilde{\Delta_k} v\|_{L^2}\cdot 2^{(s+\beta)k}\|\Delta_k h\|_{L^2} \\
&\le \Big(\sum_{j\ge -1} 2^{2sj}\|\Delta_j h\|_{L^2}^2\Big)^{\frac12}\cdot \Big\|\sum_{k\ge j-1} 2^{-(j-k)} 2^{(s+\alpha)k}\|\widetilde{\Delta_k} v\|_{L^2}\cdot 2^{(s+\beta)k}\|\Delta_k h\|_{L^2}\Big\|_{l^2} \\
&\le \Big(\sum_{j\ge -1} 2^{2sj}\|\Delta_j h\|_{L^2}^2\Big)^{\frac12}\cdot \Big(\sum_{j\ge -1} 2^{-j}\Big)^{\frac12}\cdot \Big\|2^{(s+\alpha)j}\|\widetilde{\Delta_j} v\|_{L^2}\cdot 2^{(s+\beta)j}\|\Delta_j h\|_{L^2}\Big\|_{l^1} \\
&\le \Big(\sum_{j\ge -1} 2^{2sj}\|\Delta_j h\|_{L^2}^2\Big)^{\frac12}\cdot \Big(\sum_{j\ge -1} 2^{2(s+\beta)j}\|\Delta_j h\|_{L^2}^2\Big)^{\frac12}\cdot \Big(\sum_{j\ge -1} 2^{2(s+\alpha)j}\|\widetilde{\Delta_j} v\|_{L^2}^2\Big)^{\frac12} \\
&\le C\|h\|_{H^{s+\beta}}\|h\|_{H^s}\|v\|_{H^{s+\alpha}} \\
&\le \frac{\eta}{8}\|h\|_{H^{s+\beta}}^2 + \frac{C_7}{3}\|h\|_{H^s}^2\|v\|_{H^{s+\alpha}}^2, \qquad (3.11)
\end{aligned}
$$

where we have used Young's convolution inequality.

Combining these estimates from (3.3) to (3.11) and using the Lemma 3.1, we have

$$
\begin{aligned}
&\frac{d}{dt}(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2)+\Big(\nu\theta - C_4\theta\|v\|_{H^s}^2 - C_6\theta^2 - C_7\|h\|_{H^s}^2\Big)\|v\|_{H^{s+\alpha}}^2 \\
&\quad + \Big(\frac{\eta}{4} - C_5\|v\|_{H^s}^2\Big)\|h\|_{H^{s+\beta}}^2 + 2\mu\|h\|_{H^s}^2 \\
&\le C\|f\|_{H^s}^2 + C\big(\|\nabla U\|_{H^s}+\|U\|_{H^s}^2\big)\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big) \\
&\le C\varepsilon e^{-4C_0 t}\|\phi_0^\varepsilon\|_{H^{s+2}}^4 + Ce^{-C_0 t}\big(\|\phi_0^\varepsilon\|_{H^{s+2}}+\|\phi_0^\varepsilon\|_{H^{s+2}}^2\big)\big(\theta\|v\|_{H^s}^2+\|h\|_{H^s}^2\big). \qquad (3.12)
\end{aligned}
$$

We now choose θ satisfying

$$\theta = \frac{\nu}{2C_6},$$

for which

$$\nu\theta - C_6\theta^2 = \frac12 \nu\theta.$$

We apply the bootstrap argument to (3.12) to establish that $\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2$ remains uniformly bounded if $\theta\|v_0\|_{H^s}^2+\|h_0\|_{H^s}^2$ is taken to be sufficiently small. The bootstrap argument starts with an ansatz that $\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2$ is bounded, say

$$\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2 \le \sigma$$

and shows that $\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2$ actually admits a smaller bound, say

$$\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2 \le \frac{\sigma}{2}$$

when $\theta\|v_0\|_{H^s}^2+\|h_0\|_{H^s}^2$ is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [28, p.21]). To apply the bootstrap argument to (3.12), we assume that

$$\theta\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2 \le \sigma := \min\Big\{\frac{\eta}{4C_5}, \frac{\nu}{4C_4}\,\frac{\nu}{4C_7}\Big\}\theta. \qquad (3.13)$$

Clearly, when (3.13) is fulfilled, we have

$$\nu\theta - C_4\theta\|v\|_{H^s}^2 - C_6\theta^2 - C_7\|h\|_{H^s}^2 \geq 0 \quad \text{and} \quad \frac{\eta}{4} - C_5\|v\|_{H^s}^2 \geq 0.$$

It then follows from (3.12) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\theta\|v\|_{H^s}^2 + \|h\|_{H^s}^2\right)\\ \leq & C\varepsilon e^{-4C_0t}\|\phi_0^\varepsilon\|_{H^{s+2}}^4 + Ce^{-C_0t}\left(\|\phi_0^\varepsilon\|_{H^{s+2}} + \|\phi_0^\varepsilon\|_{H^{s+2}}^2\right)\left(\theta\|v\|_{H^s}^2 + \|h\|_{H^s}^2\right).\end{aligned}$$

Integrating in time and using the assumption in Theorem 1.1 yield

$$\begin{aligned}&\theta\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2\\ \leq & C\Big(\theta\|v_0\|_{H^s}^2 + \|h_0\|_{H^s}^2 + \varepsilon\|\phi_0^\varepsilon\|_{H^{s+2}}^4\Big)e^{C\left(\|\phi_0^\varepsilon\|_{H^{s+2}} + \|\phi_0^\varepsilon\|_{H^{s+2}}^2\right)} \leq C_8\delta.\end{aligned}$$

Choosing

$$\delta = \min\left\{\frac{\eta}{8C_5C_8}, \frac{\nu}{8C_4C_8}\ \frac{\nu}{8C_7C_8}\right\}\theta,$$

then we have

$$\theta\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2 \leq \frac{\sigma}{2}.$$

This proves Theorem 1.1. ∎

4. Proof of Theorem 1.2

This section proves Theorem 1.2. Since the proof shares many similarities with that for Theorem 1.1, we shall just provide the details for those parts that differ.

As a preparation of the proof, the following lemma provides upper bounds for $\widetilde{U}$ and f in the 2D case.

LEMMA 4.1. *Let ψ_0^ε be given by (1.7), and $\widetilde{U}_0$ and $\widetilde{B}_0$ by (1.8). Let $\widetilde{U}(t)$ and $\widetilde{B}(t)$ be given by*

$$\widetilde{U}(t) = e^{-\nu(-\Delta)^\alpha t}\widetilde{U}_0, \qquad \widetilde{B}(t) = 0.$$

Let $\widetilde{f}$ be given by

$$\widetilde{f} = -\widetilde{U}\cdot\nabla\widetilde{U}.$$

Then the following estimate holds. For any $s > 1$,

$$\begin{aligned}&\|\widetilde{U}\|_{H^s} \leq Ce^{-C_0t}\|\psi_0^\varepsilon\|_{H^{s+2}},\\ &\|\nabla\widetilde{U}\|_{H^s} \leq Ce^{-C_0t}\|\psi_0^\varepsilon\|_{H^{s+2}},\\ &\|\widetilde{f}\|_{H^s} \leq C\varepsilon e^{-2C_0t}\|\psi_0^\varepsilon\|_{H^{s+2}}^2.\end{aligned} \tag{4.1}$$

Proof. **(Proof of Theorem 1.2.)** The first part of the estimates can be similarly proven as in the proof of Lemma 3.1. To prove (4.1), we write

$$\psi = e^{-\nu(-\Delta)^\alpha t}\psi_0^\varepsilon, \quad \widetilde{U} = \nabla\times\psi$$

and rewrite the first component of $\widetilde{f}$ as

$$\begin{aligned}\widetilde{f}^1 &:= -\widetilde{U}\cdot\nabla\widetilde{U}^1\\ &= \partial_1\psi\partial_2\partial_2\psi - \partial_2\psi\partial_1\partial_2\psi\\ &= (\partial_1-\partial_2)\psi\partial_2\partial_2\psi + \partial_2\psi\partial_2(\partial_2-\partial_1)\psi.\end{aligned}$$

By Hölder's inequality and Sobolev embedding, for $s>1$,

$$\begin{aligned}\|\widetilde{f}^1\|_{H^s} &\le C\big(\|(\partial_1-\partial_2)\psi\|_{H^s}\|\psi\|_{H^{s+2}} + \|\psi\|_{H^{s+1}}\|(\partial_2-\partial_1)\psi\|_{H^{s+1}}\\ &\le Ce^{-2C_0t}\big(\|(\partial_1-\partial_2)\psi_0^\varepsilon\|_{H^s}\|\psi_0^\varepsilon\|_{H^{s+2}} + \|\psi_0^\varepsilon\|_{H^{s+1}}\|(\partial_2-\partial_1)\psi_0^\varepsilon\|_{H^{s+1}}\big)\\ &\le C\varepsilon e^{-2C_0t}\big(\|\psi_0^\varepsilon\|_{H^s}\|\psi_0^\varepsilon\|_{H^{s+2}} + \|\psi_0^\varepsilon\|_{H^{s+1}}\|\psi_0^\varepsilon\|_{H^{s+1}}\big)\\ &\le C\varepsilon e^{-2C_0t}\|\psi_0^\varepsilon\|_{H^{s+2}}^2.\end{aligned}$$

The second component of $\widetilde{f}$ admits the same bound. Therefore,

$$\|\widetilde{f}\|_{H^s} \le \|\widetilde{f}^1\|_{H^s} + \|\widetilde{f}^2\|_{H^s} \le C\varepsilon e^{-2C_0t}\|\psi_0^\varepsilon\|_{H^{s+2}}^2.$$

This completes the proof of Lemma 4.1. ☐

The proof of Theorem 1.2 is close to that for Theorem 1.1 and we omit the details.

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