

VANISHING VISCOSITY LIMIT FOR COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH TRANSVERSE BACKGROUND MAGNETIC FIELD*

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Abstract. We are concerned with the uniform regularity estimates and vanishing viscosity limit of the solution to two-dimensional viscous compressible magnetohydrodynamic (MHD) equations with transverse background magnetic field. When the magnetic field is assumed to be transverse to the boundary and the tangential component of magnetic field satisfies zero Neumann boundary condition, even though the no-slip velocity boundary condition is imposed, the uniform regularity estimates of the solution and its derivatives still can be achieved in suitable conormal Sobolev spaces in the half plane \mathbb{R}_+^2 , and then the vanishing viscosity limit is justified in L^∞ sense based on these uniform regularity estimates and some compactness arguments. At the same time, together with [X. Cui, S. Li, and F. Xie, *Nonlinearity*, 36(1):354–400, 2022], our results show that the transverse background magnetic field can prevent the strong boundary layer from occurring for compressible magnetohydrodynamics whether there is magnetic diffusion or not.

Keywords. Magnetohydrodynamics; Initial-boundary value problem; Inviscid limit; Conormal Sobolev space.

AMS subject classifications. 35M13; 35Q35; 76N10; 76W05.

1. Introduction

In this paper we consider the vanishing viscosity limit of the solution to two-dimensional viscous compressible magnetohydrodynamic (MHD) equations in the half plane $\mathbb{R}_+^2 := \{(x, y) | x \in \mathbb{R}, y \geq 0\}$:

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon \mathbf{v}^\varepsilon) = 0, \\ \partial_t (\rho^\varepsilon \mathbf{v}^\varepsilon) + \operatorname{div}(\rho^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon) - \varepsilon \mu \Delta \mathbf{v}^\varepsilon - \varepsilon (\mu + \lambda) \nabla (\nabla \cdot \mathbf{v}^\varepsilon) + \nabla p^\varepsilon = (\nabla \times \mathbf{B}^\varepsilon) \times \mathbf{B}^\varepsilon, \\ \partial_t \mathbf{B}^\varepsilon - \nabla \times (\mathbf{v}^\varepsilon \times \mathbf{B}^\varepsilon) = \varepsilon \kappa \Delta \mathbf{B}^\varepsilon, \quad \operatorname{div} \mathbf{B}^\varepsilon = 0, \end{cases} \quad (1.1)$$

where ρ^ε is the density, $\mathbf{v}^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ denotes the velocity and $\mathbf{B}^\varepsilon = (b_1^\varepsilon, b_2^\varepsilon)$ stands for the magnetic field. The viscosity coefficients $\varepsilon \mu$, $\varepsilon \lambda$ and the magnetic diffusion coefficient $\varepsilon \kappa$ are assumed to be of the same order in terms of a small parameter ε with $\mu > 0$ and $\mu + \lambda > 0$. The operators $\nabla = (\partial_x, \partial_y)$ and $\Delta = \partial_x^2 + \partial_y^2$. The pressure p^ε is a function of ρ^ε , which takes the following form:

$$p^\varepsilon = (\rho^\varepsilon)^\gamma, \quad \gamma \geq 1, \quad (1.2)$$

where γ is the adiabatic constant. The initial data is given by

$$(\rho^\varepsilon, \mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon)(t, x, y)|_{t=0} = (\rho_0^\varepsilon, \mathbf{v}_0^\varepsilon, \mathbf{B}_0^\varepsilon)(x, y).$$

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The no-slip boundary condition is imposed on the velocity field:

$$\mathbf{v}^\varepsilon|_{y=0} = 0. \quad (1.3)$$

The main goal of this paper is to analyze the effect of transverse background magnetic field on the vanishing viscosity limit process of the solution to (1.1)-(1.3). Consequently, we impose the following boundary conditions on the magnetic field.

$$\partial_y b_1^\varepsilon|_{y=0} = 0, \quad b_2^\varepsilon|_{y=0} = 1. \quad (1.4)$$

As is well-known that the system of magnetohydrodynamic (MHD) equations is an important model in plasma physics, and also attracts many attentions from mathematicians. Extensive works exist for the study of compressible MHD equations [6, 7, 10, 14, 16, 32] and incompressible MHD equations [2–4, 21, 22, 31].

The inviscid limit problem is also an important but challenging problem in both hydrodynamics and applied mathematics, see [1, 5, 8, 15, 28, 29]. Particularly, when the inviscid limit process is considered in a domain with boundaries, it becomes much more challenging due to the possible presence of strong boundary layers [25, 26, 30].

However, when the Navier-slip boundary conditions are imposed for both velocity and magnetic field, the strong boundary layer usually disappears. Thus, under this kind of slip boundary conditions, it is reasonable to justify the inviscid limit of solution to incompressible MHD system directly without studying the boundary layers, see [11, 34, 36]. But, when the velocity is given the no-slip boundary condition, in general the strong boundary layer always occurs. At least, it is the case for the Navier-Stokes equations [12, 17, 24–26, 35]. Consequently, due to the appearance of strong boundary layer, the inviscid limit in L^∞ sense becomes dramatically difficult as the viscosity coefficient goes to zero. And the essential difficulty is due to the uncontrollability of the vorticity of the boundary layer. Recently, although the no-slip velocity boundary condition is imposed, Liu, the third author and Yang not only established the well-posedness of the solution to MHD boundary layer equations but also proved the validity of Prandtl boundary layer expansion in the Sobolev spaces under the condition that the tangential component of magnetic field does not degenerate near the physical boundary initially in [18, 19], where the tangential component of magnetic field plays a key role in the stability of boundary layers and vanishing viscosity limit process. Thereafter, under the no-slip boundary condition on velocity, Wang and the third author established the inviscid limit result for two-dimensional compressible viscoelastic equations in the half plane in [33]. Similar conclusion was proved for the incompressible [20] and compressible [9] non-resistive magnetohydrodynamics equations. These two results reveal a different phenomenon that both non-degeneracy deformation tensor and transverse background magnetic field can prevent the strong boundary layer formation. It is noted that the magnetic diffusion term is included in (1.1) compared with [9].

The main task of this paper is to prove that the solutions to viscous MHD equations (1.1)-(1.4) converge to the solutions to the following ideal MHD equations as the small parameter ε goes to zero.

$$\begin{cases} \partial_t \rho^0 + \nabla \cdot (\rho^0 \mathbf{v}^0) = 0, \\ \partial_t (\rho^0 \mathbf{v}^0) + \operatorname{div}(\rho^0 \mathbf{v}^0 \otimes \mathbf{v}^0) + \nabla p^0 = (\nabla \times \mathbf{B}^0) \times \mathbf{B}^0, \\ \partial_t \mathbf{B}^0 - \nabla \times (\mathbf{v}^0 \times \mathbf{B}^0) = 0, \quad \operatorname{div} \mathbf{B}^0 = 0, \end{cases} \quad (1.5)$$

where $\mathbf{v}^0 = (v_1^0, v_2^0)$ is the velocity and $\mathbf{B}^0 = (b_1^0, b_2^0)$ denotes the magnetic field.

To formulate the main results, we introduce the following conormal derivative operators of functions depending on (t, \mathbf{x}) :

$$\mathcal{Z}_0 = \partial_t, \quad \mathcal{Z}_1 = \partial_x, \quad \mathcal{Z}_2 = \phi(y)\partial_y, \quad \mathcal{Z}^\alpha = \mathcal{Z}_0^{\alpha_0} \mathcal{Z}_1^{\alpha_1} \mathcal{Z}_2^{\alpha_2},$$

where the spatial variables $\mathbf{x} = (x, y)$, the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$. The smooth and bounded function $\phi(y)$ satisfying $\phi(y)|_{y=0} = 0$ and $\phi'(y)|_{y=0} > 0$, typically, we can choose $\phi(y) = \frac{y}{1+y}$.

For any integer $m \in \mathbb{N}$, we denote the conormal Sobolev space

$$H_{co}^m([0, T] \times \mathbb{R}_+^2) = \{f(t, \mathbf{x}) : \mathcal{Z}^\alpha f \in L^2([0, T] \times \mathbb{R}_+^2), \quad |\alpha| \leq m\}.$$

For any $t \geq 0$, we set the norms

$$\|f(t)\|_m^2 = \sum_{|\alpha| \leq m} \|\mathcal{Z}^\alpha f(t, \cdot)\|_{L_x^2 L_y^2}^2,$$

and

$$\|f\|_{H_{co}^m}^2 = \int_0^t \|f(s)\|_m^2 ds.$$

As usual we use the notation

$$W_{co}^{m, \infty}([0, T] \times \mathbb{R}_+^2) = \{f(t, \mathbf{x}) : \mathcal{Z}^\alpha f \in L^\infty([0, T] \times \mathbb{R}_+^2), \quad |\alpha| \leq m\},$$

and

$$\|f\|_{m, \infty} = \sum_{|\alpha| \leq m} \|\mathcal{Z}^\alpha f\|_{L_{t, \mathbf{x}}^\infty}.$$

It is also convenient to introduce the functional setting

$$\Lambda^m(t) = \{(\rho, \mathbf{v}, \mathbf{B}) : \partial_y^i(\rho - 1, \mathbf{v}, \mathbf{B} - \vec{e}_y) \in H_{co}^{m-i}, i = 0, 1\}$$

with $\vec{e}_y = (0, 1)$.

To derive the uniform conormal estimates of the classical solution $(\rho^\varepsilon, \mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon)$ to compressible MHD Equations (1.1)-(1.4), we introduce the following energy functional:

$$\begin{aligned} N_m(t) &= \sum_{|\alpha| \leq m} \sup_{0 \leq s \leq t} \int_{\mathbb{R}_+^2} \left(\rho^\varepsilon(s) |\mathcal{Z}^\alpha \mathbf{v}^\varepsilon(s)|^2 + |\mathcal{Z}^\alpha (\mathbf{B}^\varepsilon - \vec{e}_y)(s)|^2 \right. \\ &\quad \left. + \gamma^{-1} (p^\varepsilon)^{-1} |\mathcal{Z}^\alpha (p^\varepsilon(s) - 1)|^2 \right) d\mathbf{x} + \|\partial_y(\mathbf{v}^\varepsilon, b_1^\varepsilon, p^\varepsilon)\|_{H_{co}^{m-1}}^2 \\ &\quad + \|\partial_y^2 v_2^\varepsilon\|_{H_{co}^{m-2}}^2 + \varepsilon \mu \|\nabla \mathbf{v}^\varepsilon\|_{H_{co}^m}^2 + \varepsilon(\mu + \lambda) \|\nabla \cdot \mathbf{v}^\varepsilon\|_{H_{co}^m}^2 + \varepsilon \kappa \|\nabla \mathbf{B}^\varepsilon\|_{H_{co}^m}^2 \\ &\quad + \varepsilon^2 \mu^2 \kappa \|\partial_y^2 v_1^\varepsilon\|_{H_{co}^{m-1}}^2 + \varepsilon^2 (2\mu + \lambda)^2 \|\partial_y^2 v_2^\varepsilon\|_{H_{co}^{m-1}}^2 + \varepsilon^2 \kappa^2 \mu \|\partial_y^2 b_1^\varepsilon\|_{H_{co}^{m-1}}^2. \end{aligned} \tag{1.6}$$

Now, we are in a position to state the main results of this paper.

THEOREM 1.1 (Uniform regularity estimates and inviscid limit). *Let the integer $m \geq 9$. Suppose the initial data $(\rho_0^\varepsilon, \mathbf{v}_0^\varepsilon, \mathbf{B}_0^\varepsilon)$ satisfies*

$$\sum_{i=0}^1 \|\partial_y^i (p_0^\varepsilon - 1, \mathbf{v}_0^\varepsilon, \mathbf{B}_0^\varepsilon - \vec{e}_y)\|_{m-i}^2 \leq \sigma, \tag{1.7}$$

where $\sigma > 0$ is some sufficiently small constant. Moreover, \mathbf{B}_0^ε satisfies the compatibility condition $\operatorname{div} \mathbf{B}_0^\varepsilon = 0$. Then for the classical solution $U^\varepsilon = (\rho^\varepsilon, \mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon) \in \Lambda^m(T)$ to the initial boundary value problem of viscous compressible MHD Equations (1.1)-(1.4), there exists a time $T > 0$ independent of ε , such that for any $t \in [0, T]$, the following regularity estimate holds:

$$N_m(t) + \gamma^{-1} \varepsilon (2\mu + \lambda) \sum_{\substack{|\alpha|+i \leq m \\ i=1,2}} \int_{\mathbb{R}_+^2} (p^\varepsilon)^{-1}(t) |\mathcal{Z}^\alpha \partial_y^i p^\varepsilon(t)|^2 d\mathbf{x} \leq C\sigma \tag{1.8}$$

where $C > 0$ is some constant, which is independent of ε .

Moreover, there exists a unique solution $U^0 = (\rho^0, \mathbf{v}^0, \mathbf{B}^0) \in \Lambda^m(T)$ to the ideal compressible MHD Equations (1.5), such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|(U^\varepsilon - U^0)(t, \cdot)\|_{L^\infty(\mathbb{R}_+^2)} = 0.$$

REMARK 1.1. It is direct to compute $\partial_t^k (p^\varepsilon - 1, \mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon - \vec{e}_y)|_{t=0}$, $k \in \mathbb{N}$ through the equations (1.1), then the time regularity requirement is converted to the initial data $(p_0^\varepsilon - 1, \mathbf{v}_0^\varepsilon, \mathbf{B}_0^\varepsilon - \vec{e}_y)$ and its spatial derivatives up to order $2k (k \leq m)$. Moreover, the divergence-free condition is guaranteed by the constraint of the initial data $\operatorname{div} \mathbf{B}_0^\varepsilon = 0$.

Before proceeding, let us explain the difficulty and related strategy for the proof of the main theorem. Compared to the previous work [9], the presence of magnetic diffusion term $\varepsilon \kappa \Delta \mathbf{B}^\varepsilon$ in (1.1) will produce more mixed terms of higher-order derivative when we derive the conormal estimates of $\partial_y v_1$ and $\partial_y b_1$, which makes the analysis in this paper different from the arguments in [9]. Moreover, the conormal estimates for $\partial_y v_1$ and $\partial_y b_1$ should be estimated together by using the equations of b_1 and v_1 here. And the mixed terms appearing in the left-hand sides of (4.3) and (4.5) are cancelled by using the boundary condition of $\partial_y b_1|_{y=0} = 0$.

Here, it should be emphasized that the uniform estimates of the first order normal derivative of $\partial_y (\mathbf{v}^\varepsilon, b_1^\varepsilon, p^\varepsilon)$ and only the second order normal derivative of $\partial_y^2 v_2^\varepsilon$ are achieved in (1.8). However, both the first order normal derivative of $\partial_y (\mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon, p^\varepsilon)$ and the second order normal derivative of $\partial_y^2 (\mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon, p^\varepsilon)$ were obtained in [9]. In fact, we believe that the uniform estimates of high order normal derivative of $\partial_y^i (\mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon, p^\varepsilon)$ ($i \geq 3$) still can be derived in [9], where there is no magnetic diffusion term of $\varepsilon \kappa \Delta \mathbf{B}^\varepsilon$ in (1.1), provided that the high order compatibility conditions are assumed there. But, if there exists a diffusion term of $\varepsilon \kappa \Delta \mathbf{B}^\varepsilon$ in (1.1), it seems that it is impossible to derive the uniform estimates of the high order normal derivatives, even for the second order normal derivative of $\partial_y^2 (v_1^\varepsilon, b_1^\varepsilon, p^\varepsilon)$. As a consequence, we can only use the uniform estimates of $\|(\mathbf{v}^\varepsilon, \mathbf{B}^\varepsilon, p^\varepsilon, \partial_y v_2)\|_{L^\infty}$ to close the a priori energy estimates in this paper, which is truly different from the analysis in [9]. In other words, when there is a diffusion term of $\varepsilon \kappa \Delta \mathbf{B}^\varepsilon$ in (1.1), at this moment we only prove that the $O(1)$ boundary layer does not appear when the vanishing viscosity solution to (1.1)-(1.4) is considered.

In addition, the form of boundary condition (1.4) is not essential. Precisely, the value 1 of b_2^ε on the boundary can be replaced by any given function $f(t, x)$, which satisfies $0 < c \leq f(t, x) \leq C$. Then, the results in Theorem 1.1 are still believed to hold true. It is noted that all of these results only depend on the main assumption that the background magnetic field is transverse to the boundary. Finally, in addition to its own significance, this paper can be regarded as a complement of [9]. And both of these results show that the transverse background magnetic field can prevent the strong

boundary layer from forming, whether there is magnetic diffusion term of $\varepsilon\kappa\Delta\mathbf{B}^\varepsilon$ in (1.1) or not.

The paper is organized as follows. In Section 2, we give some elementary lemmas. Section 3 is devoted to the uniform conormal energy estimates of the classical solution to (1.1)-(1.4). We establish the conormal estimates for normal derivatives of the classical solution in Section 4. In Section 5, we prove Theorem 1.1 based on the estimates obtained in Section 3 and Section 4.

In the following parts, we use notation $A \lesssim B$ to represent $A \leq CB$ for some generic constant $C > 0$ independent of ε . And we denote the polynomial functions by $\mathcal{P}(\cdot)$, which may vary from line to line. The commutator is expressed by $[\cdot, \cdot]$.

2. Preliminaries

In this section, we present some elementary lemmas that will be used frequently later. The first one is the Sobolev-Gagliardo-Nirenberg-Moser type inequality for the conormal Sobolev space and its proof can be found in [13].

LEMMA 2.1. *For the functions $f, g \in L^\infty([0, T] \times \mathbb{R}_+^2) \cap H_{co}^m([0, T] \times \mathbb{R}_+^2)$ with $m \in \mathbb{N}$, it holds that for any $\alpha, \beta \in \mathbb{N}^3$ with $|\alpha| + |\beta| = m$,*

$$\int_0^t \|(\mathcal{Z}^\alpha f \mathcal{Z}^\beta g)(s)\|^2 ds \lesssim \|f\|_{L_{t,x}^\infty}^2 \int_0^t \|g(s)\|_m^2 ds + \|g\|_{L_{t,x}^\infty}^2 \int_0^t \|f(s)\|_m^2 ds. \tag{2.1}$$

Next, the anisotropic Sobolev embedding property in the conormal Sobolev space can be found in [27].

LEMMA 2.2. *Suppose that $f(t, \mathbf{x}) \in H_{co}^3([0, t] \times \mathbb{R}_+^2)$ and $\partial_y f \in H_{co}^2([0, t] \times \mathbb{R}_+^2)$, then*

$$\|f\|_{L_{t,x}^\infty}^2 \lesssim \|f(0)\|_2^2 + \|\partial_y f(0)\|_1^2 + \int_0^t (\|f(s)\|_3^2 + \|\partial_y f(s)\|_2^2) ds.$$

To deal with the commutator involving conormal derivatives, we need the following properties of commutators. The proof can be found in [27]. For any integer $m \geq 1$, there exist two families of bounded smooth functions $\{\phi_{k,m}(y)\}_{0 \leq k \leq m-1}$ and $\{\phi^{k,m}(y)\}_{0 \leq k \leq m-1}$ depending only on $\phi(y)$, such that

$$[\mathcal{Z}_2^m, \partial_y] = \sum_{k=0}^{m-1} \phi_{k,m}(y) \mathcal{Z}_y^k \partial_y = \sum_{k=0}^{m-1} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k. \tag{2.2}$$

3. Conormal energy estimates

For simplicity, we omit the superscript ε in the rest of paper without causing confusion. The conormal energy estimates of the classical solution $(\rho, \mathbf{v}, \mathbf{B})$ are considered in this section. We rewrite the system (1.1) as follows.

$$\begin{cases} \partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \\ \rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \varepsilon \mu \Delta \mathbf{v} - \varepsilon (\mu + \lambda) \nabla (\nabla \cdot \mathbf{v}) + \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = \varepsilon \kappa \Delta \mathbf{B}^\varepsilon, \quad \text{div } \mathbf{B} = 0. \end{cases} \tag{3.1}$$

LEMMA 3.1. *Under the assumption in Theorem 1.1, the classical solution $(\rho, \mathbf{v}, \mathbf{B})$ to the viscous compressible MHD Equations (3.1) with the boundary conditions (1.3)-(1.4) satisfies*

$$\sum_{|\alpha| \leq m} \int_{\mathbb{R}_+^2} \left(\rho(t) |\mathcal{Z}^\alpha \mathbf{v}(t)|^2 + |\mathcal{Z}^\alpha (\mathbf{B}(t) - \vec{e}_y)|^2 + \gamma^{-1} p^{-1}(t) |\mathcal{Z}^\alpha (p(t) - 1)|^2 \right) dx$$

$$\begin{aligned}
 & + \varepsilon \mu \int_0^t \|\nabla \mathbf{v}\|_m^2 ds + \varepsilon(\mu + \lambda) \int_0^t \|\nabla \cdot \mathbf{v}\|_m^2 ds + \varepsilon \kappa \int_0^t \|\nabla \mathbf{B}\|_m^2 ds \\
 \lesssim & \sum_{|\alpha| \leq m} \int_{\mathbb{R}_+^2} \left(\rho_0 |\mathcal{Z}^\alpha \mathbf{v}_0|^2 + |\mathcal{Z}^\alpha (\mathbf{B}_0 - \vec{e}_y)|^2 + \gamma^{-1} p_0^{-1} |\mathcal{Z}^\alpha (p_0 - 1)|^2 \right) dx \\
 & + \left[\left(1 + \|(p, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2)\|_{[m/2]+1, \infty}^2 \right)^3 + \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})(0)\|_{[m/2]+2}^2 \right. \\
 & + \left. \int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{[m/2]+2}^2 ds + \varepsilon \|(v_1, \partial_y v_1, \partial_y b_1)\|_{2, \infty}^2 \left(1 + \|(\rho, p^{-1}, b_1, \partial_y v_2)\|_{L_{t,x}^\infty}^2 \right)^2 \right] \\
 & \cdot \sum_{j=0}^1 \int_0^t \|\partial_y^j (\mathbf{v}, \mathbf{B} - \vec{e}_y, p - 1)\|_{m-j}^2 ds.
 \end{aligned}$$

Proof. For any multi-index α satisfying $|\alpha| \leq m$, we apply the conormal derivatives \mathcal{Z}^α to the last two equations in (3.1). By multiplying $(\mathcal{Z}^\alpha \mathbf{v}, \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y))$ on both sides of the resulting equalities and integrating them over $[0, t] \times \mathbb{R}_+^2$, we get

$$\begin{aligned}
 & \frac{1}{2} \int_{\mathbb{R}_+^2} \left(\rho(t) |\mathcal{Z}^\alpha \mathbf{v}(t)|^2 + |\mathcal{Z}^\alpha (\mathbf{B}(t) - \vec{e}_y)|^2 \right) dx - \frac{1}{2} \int_{\mathbb{R}_+^2} \left(\rho_0 |\mathcal{Z}^\alpha \mathbf{v}_0|^2 + |\mathcal{Z}^\alpha (\mathbf{B}_0 - \vec{e}_y)|^2 \right) dx \\
 = & \varepsilon \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \Delta \mathbf{v} \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds + \varepsilon(\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \nabla (\nabla \cdot \mathbf{v}) \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds \\
 & + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds + \varepsilon \kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \Delta \mathbf{B}^\varepsilon \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) dx ds \\
 & - \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [(\nabla \times \mathbf{B}) \times \mathbf{B}] \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds \\
 & + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [\nabla \times (\mathbf{v} \times \mathbf{B})] \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) dx ds, \tag{3.2}
 \end{aligned}$$

where

$$\mathcal{C}_1^\alpha = -[\mathcal{Z}^\alpha, \rho \partial_t] \mathbf{v} = - \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \geq 1}} \mathcal{C}_\alpha^\beta \mathcal{Z}^\beta \rho \mathcal{Z}^\gamma \partial_t \mathbf{v},$$

and

$$\mathcal{C}_2^\alpha = -[\mathcal{Z}^\alpha, \rho \mathbf{v} \cdot \nabla] \mathbf{v} = - \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \geq 1}} \mathcal{C}_\alpha^\beta \mathcal{Z}^\beta (\rho \mathbf{v}) \cdot \mathcal{Z}^\gamma \nabla \mathbf{v} - \rho v_2 \cdot [\mathcal{Z}^\alpha, \partial_y] \mathbf{v}.$$

In what follows, we deal with term by term of (3.2). The first three terms on the right-hand side of (3.2) have the same estimates as [9].

To consider the fourth term on the right-hand side of (3.2), we have

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha \mathbf{v} dx ds \\
 = & \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \geq 1}} \mathcal{C}_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \left(\mathcal{Z}^\beta (\rho v_1) \cdot \mathcal{Z}^\gamma \partial_x \mathbf{v} \cdot \mathcal{Z}^\alpha \mathbf{v} - \rho v_2 [\mathcal{Z}^\alpha, \partial_y] \mathbf{v} \cdot \mathcal{Z}^\alpha \mathbf{v} \right) dx ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_2 \cdot \mathcal{Z}^\alpha v_2 \Big) dx ds + \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \geq 1}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds.
 \end{aligned}
 \tag{3.3}$$

The first term on the right-hand side of (3.3) is estimated by

$$\begin{aligned}
 &\sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \geq 1}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \left(\mathcal{Z}^\beta(\rho v_1) \cdot \mathcal{Z}^\gamma \partial_x \mathbf{v} \cdot \mathcal{Z}^\alpha \mathbf{v} - \rho v_2 [\mathcal{Z}^\alpha, \partial_y] \mathbf{v} \cdot \mathcal{Z}^\alpha \mathbf{v} \right. \\
 &\quad \left. + \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_2 \cdot \mathcal{Z}^\alpha v_2 \right) dx ds \\
 &\lesssim \|(\rho, \mathbf{v}, \partial_y v_2)\|_{1,\infty}^2 \sum_{j=0}^1 \left(\int_0^t \|\partial_y^j(\rho - 1, \mathbf{v})\|_{m-j}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\mathbf{v}\|_m^2 ds \right)^{\frac{1}{2}}.
 \end{aligned}$$

For the second term on the right-hand side of (3.3), we have

$$\begin{aligned}
 &\sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \geq 1}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds \\
 &= \sum_{\substack{\beta+\gamma=\alpha \\ 1 \leq |\beta| \leq |\gamma|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds \\
 &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ 1 \leq |\gamma| < |\beta|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta(\rho v_2) \cdot \mathcal{Z}^\gamma \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds \\
 &\quad + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds.
 \end{aligned}
 \tag{3.4}$$

It is direct to bound the first two terms on the right-hand side of (3.4) by

$$\begin{aligned}
 &\|(\rho, v_2)\|_{[m/2],\infty} \left(\int_0^t \|\partial_y v_1\|_{m-1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|v_1\|_m^2 ds \right)^{\frac{1}{2}} \\
 &\quad + \|(\rho, \phi \partial_y v_1, \phi^{-1} v_2)\|_{[m/2],\infty} \left(\int_0^t \|(\rho - 1, \phi^{-1} v_2)\|_{m-1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|v_1\|_m^2 ds \right)^{\frac{1}{2}} \\
 &\lesssim \left(1 + \|(\rho, \mathbf{v}, \partial_y v_2)\|_{[m/2]+1,\infty}^2 \right) \left(\int_0^t \|(\rho - 1, \partial_y \mathbf{v})\|_{m-1}^2 ds + \int_0^t \|v_1\|_m^2 ds \right).
 \end{aligned}$$

Next, we consider the third term on the right-hand side of (3.4). Let $\tilde{b}_2 = b_2 - 1$. Then the equation of b_1 can be rewritten as follows.

$$\partial_y v_1 + \varepsilon \kappa \partial_y^2 b_1 = -\varepsilon \kappa \partial_x^2 b_1 + \partial_t b_1 + v_1 \partial_x b_1 - \tilde{b}_2 \partial_y v_1 + v_2 \partial_y b_1 + b_1 \partial_y v_2.
 \tag{3.5}$$

Inserting above equality into the third term on the right-hand side of (3.4), we have

$$\begin{aligned}
 &\int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \partial_y v_1 \cdot \mathcal{Z}^\alpha v_1 dx ds \\
 &= -\varepsilon \kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \partial_y^2 b_1 \cdot \mathcal{Z}^\alpha v_1 dx ds
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \left(-\varepsilon\kappa \partial_x^2 b_1 + \partial_t b_1 + v_1 \partial_x b_1 + b_1 \partial_y v_2 + v_2 \partial_y b_1 - \tilde{b}_2 \partial_y v_1 \right) \cdot \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds.
 \end{aligned}
 \tag{3.6}$$

By (2.2), the first part on the right-hand side of (3.6) can be handled in the following way.

$$\begin{aligned}
 & -\varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \partial_y^2 b_1 \cdot \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds \\
 & = -\varepsilon\kappa \sum_{\beta+\gamma=\alpha} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta \rho \phi^{-1} \mathcal{Z}^\gamma v_2 \cdot \phi \partial_y^2 b_1 \cdot \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds \\
 & = -\varepsilon\kappa \sum_{\beta+\gamma=\alpha} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta \rho \phi^{-1} \mathcal{Z}^\gamma v_2 \cdot \mathcal{Z}_2 \partial_y b_1 \cdot \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds \\
 & \lesssim \varepsilon \|\mathcal{Z}_2 \partial_y b_1\|_{L_{t,x}^\infty} \|(\rho, \partial_y v_2)\|_{L_{t,x}^\infty} \left(\int_0^t \|(\rho-1, \partial_y v_2)\|_m^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t \|v_1\|_m^2 \, ds \right)^{\frac{1}{2}} \\
 & \lesssim \varepsilon \|\partial_y b_1\|_{1,\infty}^2 \|(\rho, \partial_y v_2)\|_{L_{t,x}^\infty}^2 \int_0^t \|(\rho-1, v_1)\|_m^2 \, ds + \frac{\varepsilon\mu^2}{6} \int_0^t \|\partial_y v_2\|_m^2 \, ds.
 \end{aligned}$$

The remaining terms on the right-hand side of (3.6) can be estimated directly.

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(\rho v_2) \cdot \left(-\varepsilon\kappa \partial_x^2 b_1 + \partial_t b_1 + v_1 \partial_x b_1 + b_1 \partial_y v_2 + v_2 \partial_y b_1 - \tilde{b}_2 \partial_y v_1 \right) \cdot \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds \\
 & \lesssim \left(1 + \|(\rho, v_1, \phi \partial_y v_1, v_2, \phi^{-1} v_2, \partial_y v_2, b_1, \phi \partial_y b_1, \phi^{-1} \tilde{b}_2)\|_{2,\infty} \right)^3 \left(\int_0^t \|(\rho-1, v_2)\|_m^2 \, ds \right)^{\frac{1}{2}} \\
 & \quad \cdot \left(\int_0^t \|v_1\|_m^2 \, ds \right)^{\frac{1}{2}} \\
 & \lesssim \left(1 + \|(\rho, v_1, v_2, b_1, \partial_y v_2)\|_{3,\infty} \right)^3 \int_0^t \|(\rho-1, \mathbf{v})\|_m^2 \, ds.
 \end{aligned}$$

To handle the fifth term on the right-hand side of (3.2), by integration by parts, we have

$$\begin{aligned}
 & \varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \Delta \mathbf{B} \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds \\
 & = -\varepsilon\kappa \int_0^t \|\nabla \mathbf{B}\|_m^2 \, ds + \varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} [\mathcal{Z}^\alpha, \nabla \cdot] \nabla \mathbf{B} \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds \\
 & \quad + \varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \nabla \mathbf{B} \cdot [\mathcal{Z}^\alpha, \nabla] (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds.
 \end{aligned}
 \tag{3.7}$$

For the second term on the right-hand side of (3.7), due to (2.2), one has

$$\begin{aligned}
 & \varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} [\mathcal{Z}^\alpha, \nabla \cdot] \nabla \mathbf{B} \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds \\
 & = \varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k \partial_y b_1 \cdot \mathcal{Z}^\alpha b_1 \, d\mathbf{x} ds
 \end{aligned}$$

$$+\varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k \partial_y b_2 \cdot \mathcal{Z}^\alpha (b_2 - 1) \, d\mathbf{x} ds. \tag{3.8}$$

By integration by parts, the first term on the right-hand side of (3.8) is estimated as follows.

$$\begin{aligned} & \varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k \partial_y b_1 \cdot \mathcal{Z}^\alpha b_1 \, d\mathbf{x} ds \\ &= -\varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \partial_y \phi^{k,m}(y) \mathcal{Z}_y^k \partial_y b_1 \cdot \mathcal{Z}^\alpha b_1 \, d\mathbf{x} ds \\ & \quad - \varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \mathcal{Z}_y^k \partial_y b_1 \cdot \partial_y \mathcal{Z}^\alpha b_1 \, d\mathbf{x} ds \\ & \leq \frac{\varepsilon\kappa^2}{6} \int_0^t \|\partial_y b_1\|_m^2 \, ds + \int_0^t (\|b_1\|_m^2 + \|\partial_y b_1\|_{m-1}^2) \, ds. \end{aligned}$$

Since $\nabla \cdot \mathbf{B} = 0$, the second term on the right-hand side of (3.8) satisfies

$$\begin{aligned} & \varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k \partial_y b_2 \cdot \mathcal{Z}^\alpha (b_2 - 1) \, d\mathbf{x} ds \\ &= -\varepsilon\kappa \sum_{k=0}^{m-1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k,m}(y) \partial_y \mathcal{Z}_y^k \partial_x b_1 \cdot \mathcal{Z}^\alpha (b_2 - 1) \, d\mathbf{x} ds \\ & \leq \frac{\varepsilon\kappa^2}{6} \int_0^t \|\partial_y b_1\|_m^2 \, ds + C \int_0^t \|b_2 - 1\|_m^2 \, ds. \end{aligned}$$

For the third term on the right-hand side of (3.7), by (2.2), we have

$$\varepsilon\kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \nabla \mathbf{B} \cdot [\mathcal{Z}^\alpha, \nabla] (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds \leq \frac{\varepsilon\kappa^2}{6} \int_0^t \|\nabla \mathbf{B}\|_m^2 \, ds + C \int_0^t \|\partial_y \mathbf{B}\|_{m-1}^2 \, ds.$$

Then we write the sixth term on the right-hand side of (3.2) as follows

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha \mathbf{v} \, d\mathbf{x} ds \\ &= \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p - 1) \cdot [\nabla \cdot, \mathcal{Z}^\alpha] \mathbf{v} \, d\mathbf{x} ds - \int_0^t \int_{\mathbb{R}_+^2} [\mathcal{Z}^\alpha, \nabla] (p - 1) \cdot \mathcal{Z}^\alpha \mathbf{v} \, d\mathbf{x} ds \\ & \quad + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p - 1) \cdot \mathcal{Z}^\alpha (\nabla \cdot \mathbf{v}) \, d\mathbf{x} ds. \end{aligned} \tag{3.9}$$

The first two terms have the same estimates as [9]. Hence, we only pay attention to the last term of (3.9). We insert the equation of momentum

$$\nabla \cdot \mathbf{v} = -\gamma^{-1} p^{-1} \partial_t p - \gamma^{-1} p^{-1} \mathbf{v} \cdot \nabla p$$

into the third term on the right-hand side of (3.9) to get

$$\int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p - 1) \cdot \mathcal{Z}^\alpha (\nabla \cdot \mathbf{v}) \, d\mathbf{x} ds$$

$$\begin{aligned}
 &= -\gamma^{-1} \sum_{\beta+\gamma=\alpha} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \partial_t \mathcal{Z}^\gamma(p-1) \, dx ds \\
 &\quad - \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathbf{v} \cdot \mathcal{Z}^\alpha \nabla(p-1) \, dx ds \\
 &\quad - \gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma \mathbf{v} \cdot \mathcal{Z}^\iota \nabla(p-1) \, dx ds. \tag{3.10}
 \end{aligned}$$

The first term shares the same estimates as [9]. Then we write the second term on the right-hand side of (3.10) as follows

$$\begin{aligned}
 &-\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathbf{v} \cdot \mathcal{Z}^\alpha \nabla(p-1) \, dx ds \\
 &= -\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathbf{v} \cdot [\mathcal{Z}^\alpha, \nabla](p-1) \, dx ds \\
 &\quad - \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathbf{v} \cdot \nabla \mathcal{Z}^\alpha(p-1) \, dx ds. \tag{3.11}
 \end{aligned}$$

The first term on the right-hand side of the above equation shares the same estimate as [9]. The second term on the right-hand side of (3.11) can be treated as follows.

$$\begin{aligned}
 &\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathbf{v} \cdot \nabla \mathcal{Z}^\alpha(p-1) \, dx ds \\
 &= -\frac{1}{2\gamma} \int_0^t \int_{\mathbb{R}_+^2} \nabla \cdot (p^{-1} \mathbf{v}) |\mathcal{Z}^\alpha(p-1)|^2 \, dx ds \\
 &\lesssim \left(\|\partial_x p^{-1}\|_{L_{t,x}^\infty} \|v_1\|_{L_{t,x}^\infty} + \|\phi \partial_y p^{-1}\|_{L_{t,x}^\infty} \|\phi^{-1} v_2\|_{L_{t,x}^\infty} + \|p^{-1}\|_{L_{t,x}^\infty} \|\nabla \cdot \mathbf{v}\|_{L_{t,x}^\infty} \right) \cdot \int_0^t \|p-1\|_m^2 \, ds \\
 &\lesssim \left(\|p^{-1}\|_{1,\infty} \|v_1\|_{1,\infty} + \|p^{-1}\|_{1,\infty} \|\partial_y v_2\|_{L_{t,x}^\infty} \right) \int_0^t \|p-1\|_m^2 \, ds.
 \end{aligned}$$

Next, we divide the last term on the right-hand side of (3.10) into three parts.

$$\begin{aligned}
 &-\gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma \mathbf{v} \cdot \mathcal{Z}^\iota \nabla(p-1) \, dx ds \\
 &= -\gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma v_1 \cdot \partial_x \mathcal{Z}^\iota(p-1) \, dx ds \\
 &\quad - \gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma v_2 \cdot \mathcal{Z}^\iota \partial_y(p-1) \, dx ds \\
 &\quad - \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \partial_y(p-1) \, dx ds. \tag{3.12}
 \end{aligned}$$

For the first term on the right-hand side of (3.12), we have

$$-\gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma v_1 \cdot \partial_x \mathcal{Z}^\iota(p-1) \, dx ds$$

$$\lesssim \|(v_1, p^{-1}, p)\|_{1, \infty}^2 \left(\int_0^t \|(v_1, p^{-1} - 1, p - 1)\|_m^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|p - 1\|_m^2 ds \right)^{\frac{1}{2}}.$$

As for the second term on the right-hand side of (3.12), it holds

$$\begin{aligned} & -\gamma^{-1} \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \neq |\alpha|}} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot \mathcal{Z}^\beta p^{-1} \mathcal{Z}^\gamma v_2 \cdot \mathcal{Z}^\iota \partial_y(p-1) \, d\mathbf{x} ds \\ & \lesssim \left[\sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_{t, \mathbf{x}}^\infty} \|\phi \mathcal{Z}^\iota \partial_y(p-1)\|_{L_{t, \mathbf{x}}^\infty} \left(\int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \right. \\ & \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma| < |\alpha|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t, \mathbf{x}}^\infty} \|\phi \mathcal{Z}^\iota \partial_y(p-1)\|_{L_{t, \mathbf{x}}^\infty} \left(\int_0^t \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ & \quad \left. + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota| < |\alpha|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t, \mathbf{x}}^\infty} \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_{t, \mathbf{x}}^\infty} \left(\int_0^t \|\phi \mathcal{Z}^\iota \partial_y(p-1)\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \right] \\ & \quad \cdot \left(\int_0^t \|\mathcal{Z}^\alpha(p-1)\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ & \lesssim \|(p, p^{-1}, \partial_y v_2)\|_{[m/2]+1, \infty}^2 \sum_{j=0}^1 \left(\int_0^t \|\partial_y^j(v_2, p^{-1} - 1, p - 1)\|_{m-j}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|p - 1\|_m^2 ds \right)^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

From the second equation in (3.1), we have

$$\partial_y p - \varepsilon(2\mu + \lambda) \partial_y^2 v_2 = -\rho \partial_t v_2 + b_1 \partial_x b_2 - \rho \mathbf{v} \cdot \nabla v_2 - b_1 \partial_y b_1 + \varepsilon \mu \partial_x^2 v_2 + \varepsilon(\mu + \lambda) \partial_y \partial_x v_1. \tag{3.14}$$

Hence, the last part of (3.12) can be rewritten as

$$\begin{aligned} & -\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \partial_y(p-1) \, d\mathbf{x} ds \\ & = -\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \partial_y^2 v_2 \, d\mathbf{x} ds \\ & \quad - \varepsilon(\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \partial_y \partial_x v_1 \, d\mathbf{x} ds \\ & \quad + \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \left(\rho \partial_t v_2 - b_1 \partial_x b_2 + \rho \mathbf{v} \cdot \nabla v_2 - \varepsilon \mu \partial_x^2 v_2 \right) \, d\mathbf{x} ds \\ & \quad + \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot b_1 \partial_y b_1 \, d\mathbf{x} ds. \end{aligned} \tag{3.15}$$

The first term can be estimated in the following way.

$$-\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot \partial_y^2 v_2 \, d\mathbf{x} ds$$

$$\begin{aligned} &\lesssim \varepsilon \|\phi \partial_y^2 v_2\|_{L_{t,x}^\infty} \|p^{-1}\|_{L_{t,x}^\infty} \left(\int_0^t \int_{\mathbb{R}_+^2} \|\mathcal{Z}^\alpha(p-1)\|_{L_x^2 L_y^2}^2 dx ds \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_0^t \int_{\mathbb{R}_+^2} \|\phi^{-1} \mathcal{Z}^\alpha v_2\|_{L_x^2 L_y^2}^2 dx ds \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon \mu^2}{6} \int_0^t \|\partial_y v_2\|_m^2 ds + C\varepsilon \|\partial_y v_2\|_{1,\infty}^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \int_0^t \|p-1\|_m^2 ds. \end{aligned}$$

Similarly, the second term has the bound

$$\frac{\varepsilon \mu^2}{6} \int_0^t \|\partial_y v_2\|_m^2 ds + C\varepsilon \|v_1\|_{2,\infty}^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \int_0^t \|p^{-1}-1\|_m^2 ds$$

and the third term on the right-hand side of (3.15) can be bounded by

$$\|p^{-1}\|_{L_{t,x}^\infty} \left(1 + \|(\rho, \mathbf{v}, \partial_y v_2, \mathbf{B} - \vec{e}_y)\|_{2,\infty} \right)^3 \int_0^t \|(v_2, p-1)\|_m^2 ds.$$

To estimate the last term of (3.15), by the first equation in (3.1), we have

$$\partial_y b_1 = \rho \partial_t v_1 + \partial_x p + b_2 \partial_x b_2 - \tilde{b}_2 \partial_y b_1 + \rho \mathbf{v} \cdot \nabla v_1 - \varepsilon \mu \partial_x^2 v_1 - \varepsilon(\mu + \lambda) \partial_x(\nabla \cdot \mathbf{v}) - \varepsilon \mu \partial_y^2 v_1, \tag{3.16}$$

which yields that

$$\begin{aligned} &\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot b_1 \partial_y b_1 dx ds \\ &= -\varepsilon \mu \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot b_1 \partial_y^2 v_1 dx ds \\ &\quad + \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot b_1 \left(\rho \partial_t v_1 + \partial_x p + b_2 \partial_x b_2 - \phi^{-1} \tilde{b}_2 \phi \partial_y b_1 \right. \\ &\quad \left. + \rho v_1 \partial_x v_1 + \rho \phi^{-1} v_2 \phi \partial_y v_1 - \varepsilon \mu \partial_x^2 v_1 - \varepsilon(\mu + \lambda) \partial_x(\nabla \cdot \mathbf{v}) \right) dx ds. \end{aligned} \tag{3.17}$$

The first term can be handled as follows.

$$\begin{aligned} &-\varepsilon \mu \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha(p-1) \cdot p^{-1} \mathcal{Z}^\alpha v_2 \cdot b_1 \partial_y^2 v_1 dx ds \\ &\lesssim \varepsilon \|p^{-1}\|_{L_{t,x}^\infty} \|b_1\|_{L_{t,x}^\infty} \|\phi \partial_y^2 v_1\|_{L_{t,x}^\infty} \left(\int_0^t \|\mathcal{Z}^\alpha(p-1)\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\phi^{-1} \mathcal{Z}^\alpha v_2\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \frac{\varepsilon \mu^2}{6} \int_0^t \|\partial_y v_2\|_m^2 ds + \varepsilon \|\partial_y v_1\|_{1,\infty}^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \|b_1\|_{L_{t,x}^\infty}^2 \int_0^t \|p-1\|_m^2 ds. \end{aligned}$$

The second term on the right-hand side of (3.17) can be bounded by

$$\|p^{-1}\|_{L_{t,x}^\infty} \|b_1\|_{L_{t,x}^\infty} \left(1 + \|(\rho, v_1, \mathbf{B} - \vec{e}_y, \partial_y v_2)\|_{1,\infty} \right)^3 \int_0^t \|(v_2, p-1)\|_m^2 ds.$$

It remains to estimate the last two terms on the right-hand side of (3.2).

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [(\nabla \times \mathbf{B}) \times \mathbf{B}] \cdot \mathcal{Z}^\alpha \mathbf{v} \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [\nabla \times (\mathbf{v} \times \mathbf{B})] \cdot \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y) \, d\mathbf{x} ds \\
 = & - \sum_{\beta+\gamma=\alpha} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \left(\mathcal{Z}^\beta b_2 \partial_x \mathcal{Z}^\gamma b_2 \mathcal{Z}^\alpha v_1 + \mathcal{Z}^\beta v_1 \partial_x \mathcal{Z}^\gamma b_1 \mathcal{Z}^\alpha b_1 \right. \\
 & - \mathcal{Z}^\beta b_1 \partial_x \mathcal{Z}^\gamma \tilde{b}_2 \mathcal{Z}^\alpha v_2 - \mathcal{Z}^\beta b_2 \partial_x \mathcal{Z}^\gamma v_1 \mathcal{Z}^\alpha \tilde{b}_2 - \mathcal{Z}^\beta v_1 \partial_x \mathcal{Z}^\gamma b_2 \mathcal{Z}^\alpha \tilde{b}_2 \\
 & \left. + \mathcal{Z}^\beta b_1 \partial_x \mathcal{Z}^\gamma v_2 \mathcal{Z}^\alpha \tilde{b}_2 + \mathcal{Z}^\beta v_2 \partial_x \mathcal{Z}^\gamma b_1 \mathcal{Z}^\alpha \tilde{b}_2 \right) d\mathbf{x} ds \\
 & + \sum_{\beta+\gamma=\alpha} C_\alpha^\beta \int_0^t \int_{\mathbb{R}_+^2} \left(\mathcal{Z}^\beta b_2 \mathcal{Z}^\gamma \partial_y b_1 \mathcal{Z}^\alpha v_1 - \mathcal{Z}^\beta b_1 \mathcal{Z}^\gamma \partial_y b_1 \mathcal{Z}^\alpha v_2 + \mathcal{Z}^\beta b_2 \mathcal{Z}^\gamma \partial_y v_1 \mathcal{Z}^\alpha b_1 \right. \\
 & \left. - \mathcal{Z}^\beta b_1 \mathcal{Z}^\gamma \partial_y v_2 \mathcal{Z}^\alpha b_1 - \mathcal{Z}^\beta v_2 \mathcal{Z}^\gamma \partial_y b_1 \mathcal{Z}^\alpha b_1 \right) d\mathbf{x} ds \triangleq \mathbb{I}_1 + \mathbb{I}_2. \tag{3.18}
 \end{aligned}$$

First we consider the case $\gamma = \alpha$. By integration by parts, we write the terms on the right-hand side of (3.18) as follows

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} \partial_x \tilde{b}_2 \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha v_1 \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} \partial_x b_1 \mathcal{Z}^\alpha v_1 \mathcal{Z}^\alpha b_1 \, d\mathbf{x} ds - \int_0^t \int_{\mathbb{R}_+^2} \partial_x b_1 \mathcal{Z}^\alpha v_2 \mathcal{Z}^\alpha \tilde{b}_2 \, d\mathbf{x} ds \\
 & + \frac{1}{2} \int_0^t \int_{\mathbb{R}_+^2} \nabla \cdot \mathbf{v} |\mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y)|^2 \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} \partial_y b_1 \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha v_2 \, d\mathbf{x} ds. \tag{3.19}
 \end{aligned}$$

It is easy to bound the first four terms of (3.19) by

$$\|(v_1, \partial_y v_2, b_1, \tilde{b}_2)\|_{1,\infty} \int_0^t \|(v_1, v_2, b_1, \tilde{b}_2)\|_m^2 \, ds.$$

By (3.16) and the Sobolev embedding inequality, the last term on the right-hand side of (3.19) is solved by

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} \partial_y b_1 \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha v_2 \, d\mathbf{x} ds \\
 = & -\varepsilon \mu \int_0^t \int_{\mathbb{R}_+^2} \partial_y^2 v_1 \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha v_2 \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} \left(\rho \partial_t v_1 + \partial_x p + b_2 \partial_x b_2 - \phi^{-1} \tilde{b}_2 \phi \partial_y b_1 \right. \\
 & \left. + \rho v_1 \partial_x v_1 + \rho \phi^{-1} v_2 \phi \partial_y v_1 - \varepsilon \mu \partial_x^2 v_1 - \varepsilon (\mu + \lambda) \partial_x (\nabla \cdot \mathbf{v}) \right) \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha v_2 \, d\mathbf{x} ds \\
 \lesssim & \varepsilon \|\phi \partial_y^2 v_1\|_{L_{t,x}^\infty} \left(\int_0^t \|\mathcal{Z}^\alpha b_1\|_{L_x^2 L_y^2}^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t \|\phi^{-1} \mathcal{Z}^\alpha v_2\|_{L_x^2 L_y^2}^2 \, ds \right)^{\frac{1}{2}} \\
 & + \left(1 + \|(\rho, v_1, p, b_1, \tilde{b}_2, \partial_y v_2)\|_{2,\infty} \right)^3 \int_0^t \|(v_2, b_1)\|_m^2 \, ds \\
 \lesssim & \frac{\varepsilon \mu^2}{6} \int_0^t \|\partial_y v_2\|_m^2 \, ds + \varepsilon \|\partial_y v_1\|_{1,\infty}^2 \int_0^t \|b_1\|_m^2 \, ds \\
 & + \left(1 + \|(\mathbf{v}, \mathbf{B} - \vec{e}_y, p, \partial_y v_2)\|_{2,\infty} \right)^3 \int_0^t \|(v_2, b_1)\|_m^2 \, ds. \tag{3.20}
 \end{aligned}$$

Next, we consider the case $|\gamma| < |\alpha|$. It is easy to know \mathbb{I}_1 has the bound

$$\|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_{1,\infty} \int_0^t \|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_m^2 \, ds.$$

Special attention is paid to \mathbb{I}_2 . When $1 \leq |\beta| < |\gamma|$, it is bounded by

$$\|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_{[m/2], \infty} \left(\int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{m-1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_m^2 ds \right)^{\frac{1}{2}}.$$

When $|\gamma| \leq |\beta| < |\alpha|$, the Sobolev embedding inequality yields that

$$\begin{aligned} \mathbb{I}_2 &\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma| \leq |\beta| < |\alpha|}} \sup_{0 \leq s \leq t} \|(\mathcal{Z}^\gamma \partial_y \mathbf{v}, \mathcal{Z}^\gamma \partial_y \mathbf{B})(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|(\mathcal{Z}^\beta \mathbf{v}, \mathcal{Z}^\beta (\mathbf{B} - \vec{e}_y))\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_0^t \|(\mathcal{Z}^\alpha \mathbf{v}, \mathcal{Z}^\alpha (\mathbf{B} - \vec{e}_y))\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \left[\|(\partial_y \mathbf{v}(0), \partial_y \mathbf{B}(0))\|_{[m/2]+2} + \left(\int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{[m/2]+2}^2 ds \right)^{\frac{1}{2}} \right] \\ &\quad \cdot \left(\int_0^t \|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_{m-1}^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{m-1}^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_m^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \left(1 + \|(\partial_y \mathbf{v}(0), \partial_y \mathbf{B}(0))\|_{[m/2]+2} + \int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{[m/2]+2}^2 ds \right) \\ &\quad \cdot \left(\int_0^t \|(\partial_y \mathbf{v}, \partial_y \mathbf{B})\|_{m-1}^2 ds + \int_0^t \|(\mathbf{v}, \mathbf{B} - \vec{e}_y)\|_m^2 ds \right). \end{aligned}$$

When $\beta = \alpha$, we write \mathbb{I}_2 as follows

$$\begin{aligned} & - \int_0^t \int_{\mathbb{R}_+^2} \partial_y v_2 \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha b_1 \, dx ds + \int_0^t \int_{\mathbb{R}_+^2} \partial_y b_1 (\mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha v_1 - 2\mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha v_2) \, dx ds \\ & + \int_0^t \int_{\mathbb{R}_+^2} \partial_y v_1 \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha b_1 \, dx ds. \end{aligned} \tag{3.21}$$

The first term of (3.21) can be handled by

$$- \int_0^t \int_{\mathbb{R}_+^2} \partial_y v_2 \mathcal{Z}^\alpha b_1 \mathcal{Z}^\alpha b_1 \, dx ds \lesssim \|\partial_y v_2\|_{L_{t,x}^\infty} \int_0^t \|b_1\|_m^2 ds.$$

The second term has the similar estimates as (3.20). For the last term on the right-hand side of (3.21), by (3.5), we have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^2} \partial_y v_1 \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha b_1 \, dx ds \\ &= -\varepsilon \kappa \int_0^t \int_{\mathbb{R}_+^2} \partial_y^2 b_1 \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha b_1 \, dx ds - \varepsilon \kappa \int_0^t \int_{\mathbb{R}_+^2} \partial_x^2 b_1 \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha b_1 \, dx ds \\ & \quad + \int_0^t \int_{\mathbb{R}_+^2} \left(\partial_t b_1 + v_1 \partial_x b_1 - \tilde{b}_2 \partial_y v_1 + v_2 \partial_y b_1 + b_1 \partial_y v_2 \right) \mathcal{Z}^\alpha b_2 \mathcal{Z}^\alpha b_1 \, dx ds \\ &\lesssim \varepsilon \sup_{0 \leq s \leq t} \|\phi \partial_y^2 b_1(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|\phi^{-1} \mathcal{Z}^\alpha \tilde{b}_2\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\mathcal{Z}^\alpha b_1\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 & + (1 + \|(v_1, b_1, \phi^{-1}v_2, \phi\partial_y b_2, \partial_y v_2)\|_{2,\infty})^2 \int_0^t \|\mathbf{B} - \vec{e}_y\|_m^2 ds \\
 & \lesssim \frac{\varepsilon\kappa^2}{6} \int_0^t \|\nabla \mathbf{B}\|_m^2 ds + \left(\|\partial_y b_1(0)\|_3^4 + \int_0^t \|\partial_y b_1\|_3^4 ds \right) \int_0^t \|b_1\|_m^2 ds \\
 & + \left(1 + \|(v_1, b_1, \tilde{b}_2, \partial_y v_2)\|_{2,\infty} \right)^2 \int_0^t \|\mathbf{B} - \vec{e}_y\|_m^2 ds.
 \end{aligned}$$

Collecting all the above estimates and by Hölder’s inequality, we prove Lemma 3.1. \square

4. Normal derivative estimates

To close the energy estimates in Section 2, it suffices to derive the conormal estimates for the first order normal derivatives of $(\mathbf{v}, \mathbf{B}, p)$ and the second order normal derivatives of v_2 due to Lemma 2.2, which are carried out in the subsequent parts.

4.1. Conormal estimate of $\partial_y v_2$. We first consider the conormal estimate of $\partial_y v_2$. By the equation of density, one has

$$\partial_y v_2 = -\partial_x v_1 - \gamma^{-1} p^{-1} \partial_t p - \gamma^{-1} p^{-1} \mathbf{v} \cdot \nabla p. \tag{4.1}$$

For any multi-index α satisfying $|\alpha| \leq m - 1$, by applying \mathcal{Z}^α to the above equality and taking the L^2 inner product on both sides of the resulting equality, it follows that

$$\begin{aligned}
 \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y v_2|^2 dx ds & \lesssim \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x v_1|^2 dx ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} \partial_t p)|^2 dx ds \\
 & + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_1 \cdot \partial_x p)|^2 dx ds \\
 & + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_2 \cdot \partial_y p)|^2 dx ds.
 \end{aligned} \tag{4.2}$$

The first three terms on the right-hand side of (4.2) share the same estimates as [9]. On the other hand, by the same trick as (4.6), the last term has the bound.

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_2 \cdot \partial_y p)|^2 dx ds \\
 & \lesssim \|p^{-1}\|_{L_{t,x}^\infty}^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|p - 1\|_m^2 ds \\
 & + \sup_{0 \leq s \leq t} \|\partial_y p(s)\|_{L_x^\infty L_y^2}^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\alpha v_2\|_{L_x^2 L_y^\infty}^2 ds \\
 & + \|p\|_{1,\infty}^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|p^{-1} - 1\|_{m-1}^2 ds.
 \end{aligned}$$

Notice that by the Sobolev embedding inequality, the second term can be estimated as follows.

$$\begin{aligned}
 & \sup_{0 \leq s \leq t} \|\partial_y p(s)\|_{L_x^\infty L_y^2}^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\alpha v_2\|_{L_x^2 L_y^\infty}^2 ds \\
 & \leq C \|p^{-1}\|_{L_{t,x}^\infty}^4 \left(\|\partial_y p(0)\|_2^4 + \int_0^t \|\partial_y p\|_2^4 ds \right) \int_0^t \|v_2\|_{m-1}^2 ds + \frac{1}{2} \int_0^t \|\partial_y v_2\|_{m-1}^2 ds.
 \end{aligned}$$

Thus we obtain the estimate of $\partial_y v_2$.

$$\begin{aligned} \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y v_2|^2 \, d\mathbf{x} ds &\lesssim (1 + \|(v_1, p, p^{-1}, \partial_y v_2)\|_{1, \infty}^2)^2 \int_0^t \|(v_1, p - 1, p^{-1} - 1)\|_m^2 \, ds \\ &\quad + \|p^{-1}\|_{L_{t,x}^\infty}^4 \left(\|\partial_y p(0)\|_2^4 + \int_0^t \|\partial_y p\|_2^4 \, ds \right) \int_0^t \|v_2\|_{m-1}^2 \, ds. \end{aligned}$$

4.2. Conormal estimate of $\partial_y v_1$. Then we consider the conormal estimate of $\partial_y v_1$. For any multi-index α satisfying $|\alpha| \leq m - 1$, by applying $\mu^{\frac{1}{2}} \mathcal{Z}^\alpha$ to (3.5) and taking the L^2 inner product of the resulting equality, we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}_+^2} (\mu |\mathcal{Z}^\alpha \partial_y v_1|^2 + \varepsilon^2 \kappa^2 \mu |\mathcal{Z}^\alpha \partial_y^2 b_1|^2) \, d\mathbf{x} ds + 2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y^2 b_1 \, d\mathbf{x} ds \\ &\lesssim \varepsilon^2 \kappa^2 \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x^2 b_1|^2 \, d\mathbf{x} ds + \mu \int_0^t \int_{\mathbb{R}_+^2} |\partial_t \mathcal{Z}^\alpha b_1|^2 \, d\mathbf{x} ds + \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (v_1 \partial_x b_1)|^2 \, d\mathbf{x} ds \\ &\quad + \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\tilde{b}_2 \partial_y v_1)|^2 \, d\mathbf{x} ds + \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (v_2 \partial_y b_1)|^2 \, d\mathbf{x} ds \\ &\quad + \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (b_1 \partial_y v_2)|^2 \, d\mathbf{x} ds \triangleq \varepsilon^2 \kappa^2 \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x^2 b_1|^2 \, d\mathbf{x} ds + \mathbb{J}. \end{aligned} \tag{4.3}$$

By integration by parts, we handle the second term on the left-hand side of (4.3) as follows.

$$\begin{aligned} &2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y^2 b_1 \, d\mathbf{x} ds \\ &= 2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y v_1 \cdot [\mathcal{Z}^\alpha, \partial_y] \partial_y b_1 \, d\mathbf{x} ds - 2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} [\partial_y, \mathcal{Z}^\alpha] \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 \, d\mathbf{x} ds \\ &\quad - 2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 \, d\mathbf{x} ds, \end{aligned} \tag{4.4}$$

where the boundary condition $\partial_y b_1|_{y=0} = 0$ is used.

For the first term on the right-hand side of (4.4), by (2.2), one has

$$\begin{aligned} &2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y v_1 \cdot [\mathcal{Z}^\alpha, \partial_y] \partial_y b_1 \, d\mathbf{x} ds \\ &= 2\varepsilon \kappa \mu \sum_{k=0}^{m-2} \int_0^t \int_{\mathbb{R}_+^2} \phi_{k, m-1}(y) \mathcal{Z}^\alpha \partial_y v_1 \cdot \mathcal{Z}_y^k \partial_y^2 b_1 \, d\mathbf{x} ds \\ &\leq \frac{\mu}{2} \int_0^t \|\partial_y v_1\|_{m-1}^2 \, ds + C \varepsilon^2 \kappa^2 \mu^2 \int_0^t \|\partial_y^2 b_1\|_{m-2}^2 \, ds. \end{aligned}$$

With the similar line, the second term on the right-hand side of (4.4) has the bound

$$\begin{aligned} &-2\varepsilon \kappa \mu \int_0^t \int_{\mathbb{R}_+^2} [\partial_y, \mathcal{Z}^\alpha] \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 \, d\mathbf{x} ds \\ &= -2\varepsilon \kappa \mu \sum_{k=0}^{m-2} \int_0^t \int_{\mathbb{R}_+^2} \phi^{k, m-1}(y) \mathcal{Z}_y^k \partial_y^2 v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 \, d\mathbf{x} ds \end{aligned}$$

$$\leq \frac{\kappa}{2} \int_0^t \|\partial_y b_1\|_{m-1}^2 ds + C\varepsilon^2 \kappa \mu^2 \int_0^t \|\partial_y^2 v_1\|_{m-2}^2 ds.$$

Next, we turn to consider the terms on the right-hand side of (4.3). For the first term on the right-hand side of (4.3), we have

$$\varepsilon^2 \kappa^2 \mu \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x^2 b_1|^2 dx ds \lesssim \varepsilon^2 \kappa^2 \mu \int_0^t \|\partial_x b_1\|_m^2 ds.$$

With the same calculations in [9], the remaining terms on the right-hand side of (4.3) can be estimated as follows.

$$\mathbb{J} \lesssim (1 + \|(v_1, b_1, \partial_y v_2)\|_{1,\infty}^2) \int_0^t \|(\mathbf{v}, b_1)\|_m^2 ds + \|b_1\|_{1,\infty}^2 \int_0^t \|\partial_y v_2\|_{m-1}^2 ds.$$

Combining above estimates, we obtain by induction that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^2} (\mu |\mathcal{Z}^\alpha \partial_y v_1|^2 + \varepsilon^2 \kappa^2 \mu |\mathcal{Z}^\alpha \partial_y^2 b_1|^2) dx ds - 2\varepsilon \mu \kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 dx ds \\ & \leq \frac{\kappa}{2} \int_0^t \|\partial_y b_1\|_{m-1}^2 ds + C(1 + \|(v_1, b_1, \partial_y v_2)\|_{1,\infty}^2) \int_0^t \|(\mathbf{v}, b_1)\|_m^2 ds \\ & \quad + C\varepsilon^2 \kappa \mu^2 \int_0^t \|\partial_y^2 v_1\|_{m-2}^2 ds + C\|b_1\|_{1,\infty}^2 \int_0^t \|\partial_y v_2\|_{m-1}^2 ds. \end{aligned}$$

4.3. Conormal Estimate of $\partial_y b_1$. To derive the conormal estimate of $\partial_y b_1$, the transverse background magnetic field is also essentially used. For any multi-index α satisfying $|\alpha| \leq m - 1$, applying $\kappa^{\frac{1}{2}} \mathcal{Z}^\alpha$ to (3.16) and taking the L^2 inner product of the resulting equality give that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^2} (\kappa |\mathcal{Z}^\alpha \partial_y b_1|^2 + \varepsilon^2 \mu^2 \kappa |\mathcal{Z}^\alpha \partial_y^2 v_1|^2) dx ds + 2\varepsilon \mu \kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 dx ds \\ & \lesssim \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\rho \partial_t v_1)|^2 dx ds + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x p|^2 dx ds + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (b_2 \partial_x b_2)|^2 dx ds \\ & \quad + \varepsilon^2 \mu^2 \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x^2 v_1|^2 dx ds + \varepsilon^2 (\mu + \lambda)^2 \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x (\nabla \cdot \mathbf{v})|^2 dx ds \\ & \quad + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\phi^{-1} \tilde{b}_2 \phi \partial_y b_1)|^2 dx ds + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\rho v_1 \partial_x v_1)|^2 dx ds \\ & \quad + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\rho v_2 \partial_y v_1)|^2 dx ds \triangleq \mathbb{K} + \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\rho v_2 \partial_y v_1)|^2 dx ds. \end{aligned} \tag{4.5}$$

By the same arguments as in [9], we solve \mathbb{K} by

$$\begin{aligned} \mathbb{K} & \lesssim \left(1 + \|(\rho, v_1, b_1, \tilde{b}_2)\|_{1,\infty}^2\right)^2 \int_0^t \|(v_1, b_1, \tilde{b}_2, p - 1)\|_m^2 ds + \varepsilon^2 \mu^2 \kappa \int_0^t \|\partial_x v_1\|_m^2 ds \\ & \quad + \varepsilon^2 (\mu + \lambda)^2 \int_0^t \|\nabla \cdot \mathbf{v}\|_m^2 ds. \end{aligned}$$

For the last term on the right-hand side of (4.5), by Lemma 2.1, we have

$$\begin{aligned}
 & \kappa \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(\rho v_2 \partial_y v_1)|^2 \, d\mathbf{x} ds \\
 & \lesssim \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(\rho \phi^{-1} v_2 \phi \partial_y v_1)|^2 \, d\mathbf{x} ds \\
 & \lesssim \|\rho \phi^{-1} v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|v_1\|_m^2 \, ds + \|\mathcal{Z} v_1\|_{L_{t,x}^\infty}^2 \int_0^t \|\rho \phi^{-1} v_2\|_{m-1}^2 \, ds \\
 & \lesssim \|\rho\|_{L_{t,x}^\infty}^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|v_1\|_m^2 \, ds + \|v_1\|_{1,\infty}^2 \|\rho\|_{L_{t,x}^\infty}^2 \int_0^t \|\partial_y v_2\|_{m-1}^2 \, ds \\
 & \quad + \|v_1\|_{1,\infty}^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|p-1\|_{m-1}^2 \, ds. \tag{4.6}
 \end{aligned}$$

Thus we conclude that

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} (\kappa |\mathcal{Z}^\alpha \partial_y b_1|^2 + \varepsilon^2 \mu^2 \kappa |\mathcal{Z}^\alpha \partial_y^2 v_1|^2) \, d\mathbf{x} ds + 2\varepsilon \mu \kappa \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_1 \cdot \mathcal{Z}^\alpha \partial_y b_1 \, d\mathbf{x} ds \\
 & \lesssim \left(1 + \|(\rho, v_1, b_1, \tilde{b}_2, \partial_y v_2)\|_{1,\infty}^2\right)^2 \int_0^t \|(\rho, v_1, \mathbf{B} - \vec{e}_y, p-1)\|_m^2 \, ds \\
 & \quad + \|v_1\|_{1,\infty}^2 \int_0^t \|\partial_y v_2\|_{m-1}^2 \, ds + \varepsilon^2 \mu^2 \kappa \int_0^t \|\partial_x v_1\|_m^2 \, ds + \varepsilon^2 (\mu + \lambda)^2 \int_0^t \|\nabla \cdot \mathbf{v}\|_m^2 \, ds.
 \end{aligned}$$

4.4. Conormal estimate of $\partial_y p$. This subsection is devoted to the conormal estimate of $\partial_y p$. For any multi-index α satisfying $|\alpha| \leq m-1$, by applying \mathcal{Z}^α to (3.14) and taking the L^2 inner product on both sides of the resulting equation, we have

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} (|\mathcal{Z}^\alpha \partial_y p|^2 + \varepsilon^2 (2\mu + \lambda)^2 |\mathcal{Z}^\alpha \partial_y^2 v_2|^2) \, d\mathbf{x} ds \\
 & \quad - 2\varepsilon (2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_2 \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & \lesssim \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(\rho \partial_t v_2)|^2 \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(b_1 \partial_x b_2)|^2 \, d\mathbf{x} ds \\
 & \quad + \varepsilon^2 \mu^2 \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_x^2 v_2|^2 \, d\mathbf{x} ds + \varepsilon^2 (\mu + \lambda)^2 \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y \partial_x v_1|^2 \, d\mathbf{x} ds \\
 & \quad + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(\rho \mathbf{v} \cdot \nabla v_2)|^2 \, d\mathbf{x} ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(b_1 \partial_y b_1)|^2 \, d\mathbf{x} ds. \tag{4.7}
 \end{aligned}$$

We first consider the second term on the left-hand side of (4.7). By multiplying the first equation in (3.1) with $\gamma \rho^{\gamma-1}$ and applying ∂_y to the resulting equality, we have

$$\begin{aligned}
 -\partial_y^2 v_2 &= \gamma^{-1} p^{-1} \partial_t \partial_y p + p^{-1} \partial_y p (\partial_x v_1 + \partial_y v_2) + \partial_x \partial_y v_1 + \gamma^{-1} p^{-1} \partial_y v_1 \partial_x p \\
 & \quad + \gamma^{-1} p^{-1} v_1 \partial_x \partial_y p + \gamma^{-1} p^{-1} \partial_y v_2 \partial_y p + \gamma^{-1} p^{-1} v_2 \partial_y^2 p.
 \end{aligned}$$

Substitute it into the second term on the left-hand side of (4.7) to get

$$\begin{aligned}
 & -2\varepsilon(2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha \partial_y^2 v_2 \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 = & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_t \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [p^{-1} \partial_y p (\partial_x v_1 + \partial_y v_2)] \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \partial_x \mathcal{Z}^\alpha \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_y v_1 \partial_x p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} v_1 \partial_x \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_y v_2 \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} v_2 \partial_y^2 p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds. \tag{4.8}
 \end{aligned}$$

We separate the first term on the right-hand side of (4.8) into two parts.

$$\begin{aligned}
 & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_t \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 = & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \geq 1}} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta p^{-1} \partial_t \mathcal{Z}^\gamma \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} \partial_t \mathcal{Z}^\alpha \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds. \tag{4.9}
 \end{aligned}$$

By the Sobolev embedding inequality, the first term can be dealt by

$$\begin{aligned}
 & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| \geq 1}} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta p^{-1} \partial_t \mathcal{Z}^\gamma \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 \lesssim & \varepsilon \sum_{\substack{\beta + \gamma = \alpha \\ |\gamma| \leq |\beta|}} \sup_{0 \leq s \leq t} \|\partial_t \mathcal{Z}^\gamma \partial_y p(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\mathcal{Z}^\alpha \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\
 & + \varepsilon \sum_{\substack{\beta + \gamma = \alpha \\ |\beta| < |\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty} \left(\int_0^t \|\partial_t \mathcal{Z}^\gamma \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\mathcal{Z}^\alpha \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\
 \lesssim & \varepsilon \left[\|\partial_y p(0)\|_{[(m-1)/2]+3} + \left(\int_0^t \|\partial_y p\|_{[(m-1)/2]+3}^2 ds \right)^{\frac{1}{2}} \right] \left(\int_0^t \|p^{-1} - 1\|_{m-1}^2 ds \right)^{\frac{1}{4}} \\
 & \cdot \left(\int_0^t \|\partial_y p^{-1}\|_{m-1}^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|\partial_y p\|_{m-1}^2 ds \right)^{\frac{1}{2}} + \varepsilon \|p^{-1}\|_{[(m-1)/2], \infty} \int_0^t \|\partial_y p\|_{m-2}^2 ds
 \end{aligned}$$

$$\begin{aligned} &\lesssim \varepsilon^2 \left(\|\partial_y p(0)\|_{[(m-1)/2]+3}^2 + \int_0^t \|\partial_y p\|_{[(m-1)/2]+3}^2 ds \right) \cdot \sum_{j=0}^1 \int_0^t \|\partial_y^j (p^{-1} - 1)\|_{m-1}^2 ds \\ &\quad + \varepsilon \|p^{-1}\|_{[(m-1)/2], \infty} \int_0^t \|\partial_y p\|_{m-2}^2 ds + \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 ds. \end{aligned}$$

The second term on the right-hand side of (4.9) is solved by

$$\begin{aligned} &2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} \partial_t \mathcal{Z}^\alpha \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ &= \varepsilon(2\mu + \lambda)\gamma^{-1} \left(\frac{d}{dt} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} |\mathcal{Z}^\alpha \partial_y p|^2 \, d\mathbf{x} ds - \int_0^t \int_{\mathbb{R}_+^2} \partial_t p^{-1} |\mathcal{Z}^\alpha \partial_y p|^2 \, d\mathbf{x} ds \right) \\ &\geq \varepsilon(2\mu + \lambda)\gamma^{-1} \int_{\mathbb{R}_+^2} p^{-1} |\mathcal{Z}^\alpha \partial_y p|^2 \, d\mathbf{x} - \varepsilon(2\mu + \lambda)\gamma^{-1} \int_{\mathbb{R}_+^2} p_0^{-1} |\mathcal{Z}^\alpha \partial_y p_0|^2 \, d\mathbf{x} \\ &\quad - C\varepsilon \|p^{-1}\|_{1, \infty} \int_0^t \|\partial_y p\|_{m-1}^2 ds. \end{aligned}$$

Then we estimate the second term on the right-hand side of (4.8) in the following way.

$$\begin{aligned} &2\varepsilon(2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha [p^{-1} \partial_y p (\partial_x v_1 + \partial_y v_2)] \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ &\lesssim \varepsilon \left[\sum_{\substack{\beta + \gamma + \iota = \alpha \\ |\gamma|, |\iota| \leq |\beta|}} \sup_{0 \leq s \leq t} \|\mathcal{Z}^\gamma \partial_y p(s)\|_{L_x^\infty L_y^2} \|(\partial_x \mathcal{Z}^\iota v_1, \mathcal{Z}^\iota \partial_y v_2)\|_{L_{t, \mathbf{x}}^\infty} \left(\int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \right. \\ &\quad + \sum_{\substack{\beta + \gamma + \iota = \alpha \\ |\beta|, |\iota| \leq |\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t, \mathbf{x}}^\infty} \|(\partial_x \mathcal{Z}^\iota v_1, \mathcal{Z}^\iota \partial_y v_2)\|_{L_{t, \mathbf{x}}^\infty} \left(\int_0^t \|\mathcal{Z}^\gamma \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ &\quad \left. + \sum_{\substack{\beta + \gamma + \iota = \alpha \\ |\beta|, |\gamma| \leq |\iota|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t, \mathbf{x}}^\infty} \sup_{0 \leq s \leq t} \|\mathcal{Z}^\gamma \partial_y p(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|(\partial_x \mathcal{Z}^\iota v_1, \mathcal{Z}^\iota \partial_y v_2)\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \right] \\ &\quad \cdot \left(\int_0^t \|\mathcal{Z}^\alpha \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\ &\lesssim \varepsilon \|(\partial_x v_1, \partial_y v_2, p^{-1})\|_{[(m-1)/2], \infty} \left[\|\partial_y p(0)\|_{[(m-1)/2]+2} + \left(\int_0^t \|\partial_y p\|_{[(m-1)/2]+2}^2 ds \right)^{\frac{1}{2}} \right] \\ &\quad \cdot \left(\int_0^t \|(p^{-1} - 1, \partial_x v_1, \partial_y v_2)\|_{m-1}^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|\partial_y (p^{-1}, \partial_x v_1, \partial_y v_2)\|_{m-1}^2 ds \right)^{\frac{1}{4}} \\ &\quad \cdot \left(\int_0^t \|\partial_y p\|_{m-1}^2 ds \right)^{\frac{1}{2}} + \varepsilon \|p^{-1}\|_{[(m-1)/2], \infty} \|(\partial_x v_1, \partial_y v_2)\|_{[(m-1)/2], \infty} \int_0^t \|\partial_y p\|_{m-1}^2 ds \\ &\lesssim \varepsilon^2 \|(\partial_x v_1, \partial_y v_2, p^{-1})\|_{[(m-1)/2], \infty}^2 \left(\|\partial_y p(0)\|_{[(m-1)/2]+2}^2 + \int_0^t \|\partial_y p\|_{[(m-1)/2]+2}^2 ds \right) \\ &\quad \cdot \left(\int_0^t \|(p^{-1} - 1, \partial_x v_1, \partial_y v_2)\|_{m-1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y (p^{-1} - 1, \partial_x v_1, \partial_y v_2)\|_{m-1}^2 ds \right)^{\frac{1}{2}} \\ &\quad + \varepsilon \|p^{-1}\|_{[(m-1)/2], \infty} \|(\partial_x v_1, \partial_y v_2)\|_{[(m-1)/2], \infty} \int_0^t \|\partial_y p\|_{m-1}^2 ds + \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 ds. \end{aligned}$$

The third term on the right-hand side of (4.8) can be bounded by

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda) \int_0^t \int_{\mathbb{R}_+^2} \partial_x \mathcal{Z}^\alpha \partial_y v_1 \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & \leq 2\varepsilon(2\mu + \lambda) \left(\int_0^t \|\partial_y v_1\|_m^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y p\|_{m-1}^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.1, we handle the fourth term on the right-hand side of (4.8) by

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_y v_1 \partial_x p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & \leq C\varepsilon^2 \|\partial_y v_1\|_{L_{t,x}^\infty}^2 \int_0^t \|p^{-1} \partial_x p\|_{m-1}^2 \, ds + C\varepsilon^2 \|p^{-1} \partial_x p\|_{L_{t,x}^\infty}^2 \int_0^t \|\partial_y v_1\|_{m-1}^2 \, ds + \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 \, ds \\ & \leq C\varepsilon^2 \|\partial_y v_1\|_{L_{t,x}^\infty}^2 \|(p^{-1}, p)\|_{1,\infty}^2 \int_0^t \|(p-1, p^{-1}-1)\|_m^2 \, ds \\ & \quad + C\varepsilon^2 \|(p, p^{-1})\|_{1,\infty}^4 \int_0^t \|\partial_y v_1\|_{m-1}^2 \, ds + \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 \, ds. \end{aligned}$$

For the fifth term on the right-hand side of (4.8), we divide it into two parts.

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} v_1 \partial_x \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & = 2\varepsilon(2\mu + \lambda) \gamma^{-1} \sum_{|\beta| \geq 1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta (\phi^{-1} v_1) \mathcal{Z}^\gamma (p^{-1} \phi \partial_x \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & \quad + 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} v_1 \mathcal{Z}^\alpha \partial_x \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds. \tag{4.10} \end{aligned}$$

The first part can be estimated as the fourth term above, and they have the similar bound.

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \sum_{|\beta| \geq 1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta (\phi^{-1} v_1) \mathcal{Z}^\gamma (p^{-1} \phi \partial_x \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & \leq C\varepsilon^2 \|\partial_y v_1\|_{1,\infty}^2 \|(p^{-1}, p)\|_{1,\infty}^2 \int_0^t \|(p-1, p^{-1}-1)\|_{m-1}^2 \, ds \\ & \quad + C\varepsilon^2 \|(p, p^{-1})\|_{1,\infty}^4 \int_0^t \|\partial_y v_1\|_{m-1}^2 \, ds + \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 \, ds. \end{aligned}$$

As for the second part of (4.10), by integration by parts, we have

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} v_1 \mathcal{Z}^\alpha \partial_x \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\ & = -2\varepsilon(2\mu + \lambda) \gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \partial_x (p^{-1} v_1) (\mathcal{Z}^\alpha \partial_y p)^2 \, d\mathbf{x} ds \\ & \lesssim \varepsilon \|(p^{-1}, v_1)\|_{1,\infty}^2 \int_0^t \|\partial_y p\|_{m-1}^2 \, ds. \end{aligned}$$

For the sixth term on the right-hand side of (4.8), we have

$$\begin{aligned}
 & 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} \partial_y v_2 \partial_y p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & \lesssim \varepsilon \left[\sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_{t,\mathbf{x}}^\infty} \sup_{0 \leq s \leq t} \|\mathcal{Z}^\iota \partial_y p(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \right. \\
 & \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,\mathbf{x}}^\infty} \sup_{0 \leq s \leq t} \|\mathcal{Z}^\iota \partial_y p(s)\|_{L_x^\infty L_y^2} \left(\int_0^t \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_x^2 L_y^\infty}^2 ds \right)^{\frac{1}{2}} \\
 & \quad \left. + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota| < |\alpha|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,\mathbf{x}}^\infty} \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_{t,\mathbf{x}}^\infty} \left(\int_0^t \|\mathcal{Z}^\iota \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \right] \\
 & \quad \cdot \left(\int_0^t \|\mathcal{Z}^\alpha \partial_y p\|_{L_x^2 L_y^2}^2 ds \right)^{\frac{1}{2}} \\
 & \leq \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 ds + \frac{\varepsilon^2}{8} \int_0^t \|\partial_y^2 v_2\|_{m-1}^2 ds + C\varepsilon^2 \|\partial_y v_2\|_{[(m-1)/2], \infty}^2 \\
 & \quad \cdot \left(\|\partial_y p(0)\|_{[(m-1)/2]+2}^2 + \int_0^t \|\partial_y p\|_{[(m-1)/2]+2}^2 ds \right) \sum_{j=0}^1 \int_0^t \|\partial_y^j (p^{-1} - 1)\|_{m-1}^2 ds \\
 & \quad + \varepsilon^2 \|p\|_{[(m-1)/2], \infty}^4 \left[\|\partial_y p(0)\|_{[(m-1)/2]+2}^4 + \left(\int_0^t \|\partial_y p\|_{[(m-1)/2]+2}^2 ds \right)^2 \right] \\
 & \quad \cdot \int_0^t \|\partial_y v_2\|_{m-1}^2 ds.
 \end{aligned}$$

Next we consider the last term on the right-hand side of (4.8) and separate it into three terms.

$$\begin{aligned}
 & 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\alpha (p^{-1} v_2 \partial_y^2 p) \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & = 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} v_2 \partial_y \mathcal{Z}^\alpha \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & \quad + 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} v_2 [\mathcal{Z}^\alpha, \partial_y] \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & \quad + 2\varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{|\beta| \geq 1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta (p^{-1} v_2) \mathcal{Z}^\gamma \partial_y^2 p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds. \tag{4.11}
 \end{aligned}$$

The first one can be estimated by integration by parts.

$$\begin{aligned}
 & 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1} v_2 \partial_y \mathcal{Z}^\alpha \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x} ds \\
 & \lesssim \varepsilon \left(\|\phi \partial_y p^{-1} \phi^{-1} v_2\|_{L_{t,\mathbf{x}}^\infty} + \|p^{-1} \partial_y v_2\|_{L_{t,\mathbf{x}}^\infty} \right) \int_0^t \|\partial_y p\|_{m-1}^2 ds \\
 & \lesssim \varepsilon \|p^{-1}\|_{1, \infty} \|\partial_y v_2\|_{L_{t,\mathbf{x}}^\infty} \int_0^t \|\partial_y p\|_{m-1}^2 ds.
 \end{aligned}$$

Then we estimate the second part of (4.11). By Lemma 2.2, one has

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda)\gamma^{-1} \int_0^t \int_{\mathbb{R}_+^2} p^{-1}v_2[\mathcal{Z}^\alpha, \partial_y]\partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x}ds \\ & \lesssim \varepsilon \sum_{k=0}^{m-2} \int_0^t \int_{\mathbb{R}_+^2} p^{-1}v_2\phi^{k,m}(y)\partial_y \mathcal{Z}_y^k \partial_y p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x}ds \\ & \lesssim \varepsilon \|p^{-1}\|_{L_{t,x}^\infty} \|\partial_y v_2\|_{L_{t,x}^\infty} \int_0^t \|\partial_y p\|_{m-1}^2 \, ds. \end{aligned}$$

By using the Hardy trick, we find the last part of (4.11) can be estimated as the sixth term of (4.8).

$$\begin{aligned} & 2\varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{|\beta| \geq 1} \int_0^t \int_{\mathbb{R}_+^2} \mathcal{Z}^\beta (p^{-1}v_2) \mathcal{Z}^\gamma \partial_y^2 p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x}ds \\ & \lesssim \varepsilon \sum_{|\beta| \geq 1} \int_0^t \int_{\mathbb{R}_+^2} \phi^{-1} \mathcal{Z}^\beta (p^{-1}v_2) \phi \mathcal{Z}^\gamma \partial_y^2 p \cdot \mathcal{Z}^\alpha \partial_y p \, d\mathbf{x}ds \\ & \leq \frac{1}{8} \int_0^t \|\partial_y p\|_{m-1}^2 \, ds + \frac{\varepsilon^2}{8} \int_0^t \|\partial_y^2 v_2\|_{m-1}^2 \, ds + C\varepsilon^2 \|p^{-1}\|_{L_{t,x}^\infty}^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|\partial_y p\|_{m-1}^2 \, ds \\ & \quad + C\varepsilon^2 \|\partial_y v_2\|_{L_{t,x}^\infty}^2 \left(\|\partial_y p(0)\|_2^2 + \int_0^t \|\partial_y p\|_2^2 \, ds \right) \left(\int_0^t \|p^{-1} - 1\|_{m-1}^2 \, ds \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\int_0^t \|\partial_y p\|_{m-1}^2 \, ds \right)^{\frac{1}{2}} \\ & \quad + C\varepsilon^4 \|p^{-1}\|_{L_{t,x}^\infty}^4 \left(\|\partial_y p(0)\|_2^4 + \int_0^t \|\partial_y p\|_2^4 \, ds \right) \int_0^t \|\partial_y v_2\|_{m-1}^2 \, ds. \end{aligned}$$

Finally, for the terms on the right-hand side of (4.7), we can get the following bound directly.

$$\begin{aligned} & \left(1 + \|(\rho, \mathbf{v}, b_1, \tilde{b}_2, \partial_y v_2)\|_{1,\infty}^2 \right)^2 \int_0^t \|(\rho, \mathbf{v}, b_1, \tilde{b}_2)\|_m^2 \, ds + \varepsilon^2 \mu^2 \int_0^t \|\partial_x v_2\|_m^2 \, ds \\ & \quad + \varepsilon^2 (\mu + \lambda)^2 \int_0^t \|\partial_y v_1\|_m^2 \, ds + \|b_1\|_{L_{t,x}^\infty}^2 \int_0^t \|\partial_y b_1\|_{m-1}^2 \, ds \\ & \quad + \left(\|\partial_y b_1(0)\|_2^2 + \int_0^t \|\partial_y b_1\|_2^2 \, ds \right) \left(\int_0^t \|b_1\|_{m-1}^2 \, ds + \int_0^t \|\partial_y b_1\|_{m-1}^2 \, ds \right). \end{aligned}$$

Summing up all the estimates together, we conclude by induction that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^2} (|\mathcal{Z}^\alpha \partial_y p|^2 + \varepsilon^2 (2\mu + \lambda)^2 |\mathcal{Z}^\alpha \partial_y^2 v_2|^2) \, d\mathbf{x}ds + \varepsilon(2\mu + \lambda)\gamma^{-1} \int_{\mathbb{R}_+^2} p^{-1} |\mathcal{Z}^\alpha \partial_y p|^2 \, ds \\ & \lesssim \varepsilon(2\mu + \lambda)\gamma^{-1} \int_{\mathbb{R}_+^2} p_0^{-1} |\mathcal{Z}^\alpha \partial_y p_0|^2 \, ds + \varepsilon^2 \int_0^t \|\partial_x v_2\|_m^2 \, ds + \varepsilon^2 \int_0^t \|\partial_y v_1\|_m^2 \, ds \\ & \quad + \left(1 + \|(p, p^{-1}, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2)\|_{[(m-1)/2], \infty}^2 + \varepsilon^2 \|\partial_y v_1\|_{L_{t,x}^\infty}^2 \right)^2 \\ & \quad \cdot \int_0^t \|(p-1, \rho-1, \mathbf{v}, \mathbf{B} - \vec{e}_y)\|_m^2 \, ds + \|b_1\|_{L_{t,x}^\infty}^2 \int_0^t \|\partial_y b_1\|_{m-1}^2 \, ds \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon^2 \left(1 + \|(p^{-1}, v_1, \partial_y v_2, p)\|_{[(m-1)/2]+1, \infty}^2 \right)^2 \int_0^t \|(\partial_y p, \partial_y v_1)\|_{m-1}^2 ds \\
 & + \varepsilon^2 \left(1 + \|(p, p^{-1}, \partial_x v_1, \partial_y v_2)\|_{[(m-1)/2], \infty} \right)^2 \left(1 + \|(\partial_y p, \partial_y b_1)(0)\|_{[(m-1)/2]+3}^2 \right. \\
 & \left. + \int_0^t \|(\partial_y p, \partial_y b_1)\|_{[(m-1)/2]+3}^2 ds \right)^2 \cdot \sum_{j=0}^1 \int_0^t \|\partial_y^j (\partial_x v_1, v_2, b_1, p^{-1} - 1, p - 1)\|_{m-1}^2 ds.
 \end{aligned}$$

4.5. Conormal Estimate of $\partial_y^2 v_2$. The conormal estimates of $\partial_y^2 v_2$ are derived in this part to control $\|\partial_y v_2\|_{L_{t,x}^\infty}$. Similar to (4.2), we have

$$\begin{aligned}
 \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y^2 v_2|^2 dx ds & \lesssim \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y \partial_x v_1|^2 dx ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y (p^{-1} \partial_t p)|^2 dx ds \\
 & \quad + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y (p^{-1} \mathbf{v} \cdot \nabla p)|^2 dx ds. \tag{4.12}
 \end{aligned}$$

For the first and the second terms on the right-hand side of (4.12), one has

$$\int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y \partial_x v_1|^2 dx ds \lesssim \int_0^t \|\partial_y v_1\|_{m-1}^2 ds,$$

and

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y (p^{-1} \partial_t p)|^2 dx ds \\
 & \lesssim \sup_{0 \leq s \leq t} \|(\partial_y p^{-1}, \partial_t \partial_y p)(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|(\mathcal{Z}^\alpha \partial_t p, \mathcal{Z}^\alpha p^{-1})\|_{L_x^2 L_y^\infty}^2 ds \\
 & \quad + \|(\partial_t p, p^{-1})\|_{L_{t,x}^\infty}^2 \int_0^t \|(\mathcal{Z}^\alpha \partial_y p^{-1}, \mathcal{Z}^\alpha \partial_t \partial_y p)\|_{L_x^2 L_y^2}^2 ds \\
 & \lesssim \left(\|(\partial_y p^{-1}, \partial_y p)(0)\|_{[m/2]+2}^2 + \int_0^t \|(\partial_y p^{-1}, \partial_y p)\|_{[m/2]+2}^2 ds \right) \left(\int_0^t \|\partial_y (p, p^{-1})\|_{m-1}^2 ds \right)^{\frac{1}{2}} \\
 & \quad \cdot \left(\int_0^t \|(p - 1, p^{-1} - 1)\|_{m-1}^2 ds \right)^{\frac{1}{2}} + \|(p, p^{-1})\|_{1,\infty}^2 \int_0^t \|(\partial_y p, \partial_y p^{-1})\|_{m-1}^2 ds.
 \end{aligned}$$

Then we write the third term on the right-hand side of (4.12) as

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha \partial_y (p^{-1} \mathbf{v} \cdot \nabla p)|^2 dx ds \\
 & \lesssim \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\partial_y p^{-1} v_1 \partial_x p)|^2 dx ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\partial_y p^{-1} v_2 \partial_y p)|^2 dx ds \\
 & \quad + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} \partial_y \mathbf{v} \cdot \nabla p)|^2 dx ds + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_1 \partial_x \partial_y p)|^2 dx ds \\
 & \quad + \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_2 \partial_y^2 p)|^2 dx ds. \tag{4.13}
 \end{aligned}$$

The first term on the right-hand side of (4.13) is dealt by

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\partial_y p^{-1} v_1 \partial_x p)|^2 dx ds \\
 \lesssim & \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \|\mathcal{Z}^\gamma v_1\|_{L_{t,x}^\infty}^2 \|\mathcal{Z}^\iota \partial_x p\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\beta \partial_y p^{-1}\|_{L_x^2 L_y^2}^2 ds \\
 & + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma|}} \|\phi \mathcal{Z}^\beta \partial_y p^{-1}\|_{L_{t,x}^\infty}^2 \|\mathcal{Z}^\iota \partial_x p\|_{L_{t,x}^\infty}^2 \int_0^t \|\phi^{-1} \mathcal{Z}^\gamma v_1\|_{L_x^2 L_y^2}^2 ds \\
 & + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota|}} \sup_{0 \leq s \leq t} \|\mathcal{Z}^\beta \partial_y p^{-1}(s)\|_{L_x^\infty L_y^2}^2 \|\mathcal{Z}^\gamma v_1\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\iota \partial_x p\|_{L_x^2 L_y^\infty}^2 ds \\
 \lesssim & \| (v_1, p^{-1}) \|_{[m/2], \infty}^2 \|p\|_{[m/2], \infty}^2 \int_0^t \|(\partial_y v_1, \partial_y p^{-1})\|_{m-2}^2 ds \\
 & + \|v_1\|_{[m/2], \infty}^2 \left(\|\partial_y p^{-1}(0)\|_{[m/2]+2}^2 + \int_0^t \|\partial_y p^{-1}\|_{[m/2]+2}^2 ds \right) \\
 & \cdot \left(\int_0^t \|p-1\|_{m-1}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y p\|_{m-1}^2 ds \right)^{\frac{1}{2}}.
 \end{aligned}$$

By the Sobolev embedding inequality, we estimate the second term on the right-hand side of (4.13) by

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (\partial_y p^{-1} v_2 \partial_y p)|^2 dx ds \\
 \lesssim & \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_{t,x}^\infty}^2 \|\phi \mathcal{Z}^\iota \partial_y p\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\beta \partial_y p^{-1}\|_{L_x^2 L_y^2}^2 ds \\
 & + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma|}} \|\phi \mathcal{Z}^\beta \partial_y p^{-1}\|_{L_{t,x}^\infty}^2 \sup_{0 \leq s \leq t} \|\mathcal{Z}^\iota \partial_y p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_x^2 L_y^\infty}^2 ds \\
 & + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota|}} \|\phi \mathcal{Z}^\beta \partial_y p^{-1}\|_{L_{t,x}^\infty}^2 \|\phi^{-1} \mathcal{Z}^\gamma v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\iota \partial_y p\|_{L_x^2 L_y^2}^2 ds \\
 \leq & C \|(p^{-1}, \partial_y v_2)\|_{[m/2], \infty}^2 \|(p, \partial_y v_2)\|_{[m/2], \infty}^2 \int_0^t \|(\partial_y p^{-1}, \partial_y p)\|_{m-2}^2 ds \\
 & + \|p^{-1}\|_{[m/2], \infty}^4 \left(\|\partial_y p(0)\|_{[m/2]+1}^2 + \int_0^t \|\partial_y p\|_{[m/2]+1}^2 ds \right)^2 \int_0^t \|\partial_y v_2\|_{m-2}^2 ds \\
 & + \frac{1}{6} \int_0^t \|\partial_y^2 v_2\|_{m-2}^2 ds.
 \end{aligned}$$

The third term on the right-hand side of (4.13) is handled by

$$\int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} \partial_y \mathbf{v} \cdot \nabla p)|^2 dx ds$$

$$\begin{aligned}
 &\lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|,|\iota|\leq|\beta|}} \sup_{0\leq s\leq t} \|\mathcal{Z}^\gamma \partial_y v_1(s)\|_{L_x^\infty L_y^2}^2 \|\mathcal{Z}^\iota \partial_x p\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|,|\iota|\leq|\beta|}} \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_{t,x}^\infty}^2 \sup_{0\leq s\leq t} \|\mathcal{Z}^\iota \partial_y p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \|\mathcal{Z}^\iota \partial_x p\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\gamma \partial_y v_1\|_{L_x^2 L_y^2}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \sup_{0\leq s\leq t} \|\mathcal{Z}^\iota \partial_y p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\gamma|\leq|\iota|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \sup_{0\leq s\leq t} \|\mathcal{Z}^\gamma \partial_y v_1(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\iota \partial_x p\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\gamma|\leq|\iota|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \|\mathcal{Z}^\gamma \partial_y v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\iota \partial_y p\|_{L_x^2 L_y^2}^2 ds \\
 &\lesssim \|(p^{-1}, p, \partial_y v_2)\|_{[m/2],\infty}^2 \left(\|(\partial_y v_1, \partial_y p)(0)\|_{[m/2]+1}^2 + \int_0^t \|(\partial_y v_1, \partial_y p)\|_{[m/2]+1}^2 ds \right) \\
 &\quad \cdot \left(\int_0^t \|(p^{-1} - 1, \partial_y v_2, \partial_x p)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|(\partial_y p^{-1}, \partial_y^2 v_2, \partial_x \partial_y p)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \\
 &\quad + \|p^{-1}\|_{[m/2],\infty}^2 \|(p, \partial_y v_2)\|_{[m/2],\infty}^2 \int_0^t \|(\partial_y v_1, \partial_y p)\|_{m-2}^2 ds.
 \end{aligned}$$

For the fourth term on the right-hand side of (4.13), we have

$$\begin{aligned}
 &\int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha (p^{-1} v_1 \partial_x \partial_y p)|^2 dx ds \\
 &\lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|,|\iota|\leq|\beta|}} \|\mathcal{Z}^\gamma v_1\|_{L_{t,x}^\infty}^2 \sup_{0\leq s\leq t} \|\mathcal{Z}^\iota \partial_x \partial_y p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\beta p^{-1}\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\iota|\leq|\gamma|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \sup_{0\leq s\leq t} \|\mathcal{Z}^\iota \partial_x \partial_y p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\gamma v_1\|_{L_x^2 L_y^\infty}^2 ds \\
 &\quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|,|\gamma|\leq|\iota|}} \|\mathcal{Z}^\beta p^{-1}\|_{L_{t,x}^\infty}^2 \|\mathcal{Z}^\gamma v_1\|_{L_{t,x}^\infty}^2 \int_0^t \|\mathcal{Z}^\iota \partial_x \partial_y p\|_{L_x^2 L_y^2}^2 ds \\
 &\lesssim \|(v_1, p^{-1})\|_{[m/2],\infty}^2 \left(\|\partial_y p(0)\|_{[m/2]+2}^2 + \int_0^t \|\partial_y p\|_{[m/2]+2}^2 ds \right) \\
 &\quad \cdot \left(\int_0^t \|(p^{-1} - 1, v_1)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|(\partial_y p^{-1}, \partial_y v_1)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \\
 &\quad + \|p^{-1}\|_{[m/2],\infty}^2 \|v_1\|_{[m/2],\infty}^2 \int_0^t \|\partial_y p\|_{m-1}^2 ds.
 \end{aligned}$$

The last term on the right-hand side of (4.13) is dealt by

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}_+^2} |\mathcal{Z}^\alpha(p^{-1}v_2\partial_y^2 p)|^2 dx ds \\
 & \lesssim \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\gamma|, |\iota| \leq |\beta|}} \|\phi^{-1}\mathcal{Z}^\gamma v_2\|_{L_{t,x}^\infty}^2 \sup_{0 \leq s \leq t} \|\phi\mathcal{Z}^\iota \partial_y^2 p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\mathcal{Z}^\beta p\|_{L_x^2 L_y^\infty}^2 ds \\
 & \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\iota| \leq |\gamma|}} \|\mathcal{Z}^\beta p\|_{L_{t,x}^\infty}^2 \sup_{0 \leq s \leq t} \|\phi\mathcal{Z}^\iota \partial_y^2 p(s)\|_{L_x^\infty L_y^2}^2 \int_0^t \|\phi^{-1}\mathcal{Z}^\gamma v_2\|_{L_x^2 L_y^\infty}^2 ds \\
 & \quad + \sum_{\substack{\beta+\gamma+\iota=\alpha \\ |\beta|, |\gamma| \leq |\iota|}} \|\mathcal{Z}^\beta p\|_{L_{t,x}^\infty}^2 \|\phi^{-1}\mathcal{Z}^\gamma v_2\|_{L_{t,x}^\infty}^2 \int_0^t \|\phi\mathcal{Z}^\iota \partial_y^2 p\|_{L_x^2 L_y^2}^2 ds \\
 & \lesssim \|(p, \partial_y v_2)\|_{[m/2], \infty}^2 \left(\|\partial_y p(0)\|_{[m/2]+2}^2 + \int_0^t \|\partial_y p\|_{[m/2]+2}^2 ds \right) \\
 & \quad \cdot \left(\int_0^t \|(p-1, \partial_y v_2)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|(\partial_y p, \partial_y^2 v_2)\|_{m-2}^2 ds \right)^{\frac{1}{2}} \\
 & \quad + \|p\|_{[m/2], \infty}^2 \|\partial_y v_2\|_{[m/2], \infty}^2 \int_0^t \|\partial_y p\|_{m-1}^2 ds.
 \end{aligned}$$

Thus, we conclude by Hölder’s inequality that

$$\begin{aligned}
 & \int_0^t \|\partial_y^2 v_2\|_{m-2}^2 ds \lesssim \left(1 + \|(\mathbf{v}, p, p^{-1}, \partial_y v_2)\|_{[m/2], \infty}^2\right)^2 \int_0^t \|(\partial_y v_1, \partial_y p, \partial_y p^{-1})\|_{m-1}^2 ds \\
 & \quad + \left(1 + \|(\mathbf{v}, p, p^{-1}, \partial_y v_2)\|_{[m/2], \infty}^2\right)^2 \left(1 + \|(\partial_y v_1, \partial_y p, \partial_y p^{-1})(0)\|_{[m/2]+2}^2\right. \\
 & \quad \left. + \int_0^t \|(\partial_y v_1, \partial_y p, \partial_y p^{-1})\|_{[m/2]+2}^2 ds\right)^2 \cdot \sum_{j=0}^1 \int_0^t \|\partial_y^j(\mathbf{v}, p-1, p^{-1}-1)\|_{m-1}^2 ds.
 \end{aligned}$$

5. Proof of Theorem 1.1

Now we prove Theorem 1.1. According to the estimates in Section 3 and Section 4, one has

$$\begin{aligned}
 & N_m(t) + \varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{\substack{|\alpha|+i \leq m \\ i=1,2}} \int_{\mathbb{R}_+^2} p^{-1}(t) |\mathcal{Z}^\alpha \partial_y^i p(t)|^2 dx \\
 & \lesssim N_m(0) + \varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{\substack{|\alpha|+i \leq m \\ i=1,2}} \int_{\mathbb{R}_+^2} p_0^{-1} |\mathcal{Z}^\alpha \partial_y^i p_0|^2 dx \\
 & \quad + \left\{ \left(1 + \|(p, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2)\|_{[m/2]+1, \infty}^2\right)^3 + \left(1 + \|(p, \mathbf{v}, \partial_y v_2)\|_{[m/2], \infty}^2\right)^2 \right. \\
 & \quad \cdot \left(1 + \|(\partial_y \mathbf{v}, \partial_y p, \partial_y \mathbf{B})(0)\|_{[(m-1)/2]+3}^2 + \int_0^t \|(\partial_y \mathbf{v}, \partial_y p, \partial_y \mathbf{B})\|_{[(m-1)/2]+3}^2 ds\right)^2 \\
 & \quad \left. + \varepsilon\|(v_1, \partial_y v_1, \partial_y b_1)\|_{2, \infty}^2 \left(1 + \|(\rho, p^{-1}, b_1, \partial_y v_2)\|_{L_{t,x}^\infty}^2\right) \right\} \\
 & \quad \cdot \sum_{j=0}^1 \int_0^t \|\partial_y^j(\mathbf{v}, \mathbf{B} - \vec{e}_y, p-1)\|_{m-j}^2 ds. \tag{5.1}
 \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} & \| (p, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2) \|_{[m/2]+1, \infty}^2 \\ & \lesssim \| (p-1, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2)(0) \|_{[m/2]+3}^2 + \| \partial_y (p, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2) \|_{[m/2]+3}^2 \\ & \quad + \int_0^t \left(\| (p-1, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2) \|_{[m/2]+4}^2 + \| \partial_y (p, \mathbf{v}, \mathbf{B} - \vec{e}_y, \partial_y v_2) \|_{[m/2]+3}^2 \right) ds \\ & \lesssim \mathcal{P}(N_m(0)) + t\mathcal{P}(N_m(t)). \end{aligned}$$

On the other hand, by Sobolev embedding, we also have

$$\begin{aligned} \varepsilon \| (\partial_y v_1, \partial_y b_1) \|_{2, \infty}^2 & \lesssim \varepsilon \sum_{|\alpha| \leq 2} \sup_{0 \leq s \leq t} \| \mathcal{Z}^\alpha (\partial_y v_1, \partial_y b_1)(s) \|_{L_x^\infty L_y^2} \| \mathcal{Z}^\alpha (\partial_y^2 v_1, \partial_y^2 b_1)(s) \|_{L_x^\infty L_y^2} \\ & \lesssim \varepsilon \left[\| (\partial_y v_1, \partial_y b_1)(0) \|_4 + \left(\int_0^t \| (\partial_y v_1, \partial_y b_1) \|_4^2 ds \right)^{\frac{1}{2}} \right] \\ & \quad \cdot \left[\| (\partial_y^2 v_1, \partial_y^2 b_1)(0) \|_4 + \left(\int_0^t \| (\partial_y^2 v_1, \partial_y^2 b_1) \|_4^2 ds \right)^{\frac{1}{2}} \right] \\ & \lesssim \| (\partial_y v_1, \partial_y b_1)(0) \|_4^2 + \varepsilon^2 \| (\partial_y^2 v_1, \partial_y^2 b_1)(0) \|_4^2 + \int_0^t \| (\partial_y v_1, \partial_y b_1) \|_4^2 ds \\ & \quad + \varepsilon^2 \int_0^t \| (\partial_y^2 v_1, \partial_y^2 b_1) \|_4^2 ds. \end{aligned}$$

Thus, for any $m \geq 9$, by inserting the above inequalities into (5.1), we obtain

$$\begin{aligned} & N_m(t) + \varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{\substack{|\alpha|+i \leq m \\ i=1,2}} \int_{\mathbb{R}_+^2} p^{-1}(t) |\mathcal{Z}^\alpha \partial_y^i p(t)|^2 d\mathbf{x} \\ & \lesssim \mathcal{P}(N_m(0)) + [t + \varepsilon(2\mu + \lambda)]\mathcal{P}(N_m(t)). \end{aligned}$$

Let the time t and ε be suitably small, then we achieve that

$$N_m(t) + \varepsilon(2\mu + \lambda)\gamma^{-1} \sum_{\substack{|\alpha|+i \leq m \\ i=1,2}} \int_{\mathbb{R}_+^2} p^{-1}(t) |\mathcal{Z}^\alpha \partial_y^i p(t)|^2 d\mathbf{x} \lesssim \mathcal{P}(N_m(0)).$$

Based on the above uniform conormal estimates achieved, the inviscid limit in Theorem 1.1 can be verified as in [23].

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