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EXISTENCE OF POLYNOMIAL ATTRACTOR FOR A CLASS OF EXTENSIBLE BEAMS WITH NONLOCAL WEAK DAMPING*

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Abstract. In this paper, we put forward the concept of polynomial attractor and study the connection between the polynomial attractors and the estimate of attractive velocity of bounded sets for infinite-dimensional dynamical systems. Then we prove the existence of polynomial attractor for a class of extensible beams with nonlocal weak damping for the case that the nonlinear term f has subcritical growth.

Keywords. Polynomial attractor; attractive velocity; the polynomial decay; extensible beams; nonlocal weak damping.

AMS subject classifications. 35B41; 35B40; 37B55.

1. Introduction

In this paper, we study the existence of polynomial attractors for a class of extensible beams with nonlocal weak damping

$$\begin{cases} u_{tt}+\Delta^2 u - m(\|\nabla u\|^2)\Delta u + k(\|u_t\|)u_t + f(u) = h, \ (x,t)\in\Omega\times\mathbb{R}^+, \\ u\,|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\,|_{\partial\Omega} = 0 \text{ (clamped)} \quad \text{or} \quad u\,|_{\partial\Omega} = \Delta u\,|_{\partial\Omega} = 0 \ \text{(hinged)}, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $u=u(x,t):\Omega\times[0,\infty)\to\mathbb{R}$ is an unknown function and $h\in L^2(\Omega)$. The assumptions on the functions m and f are as follows:

($\mathbf{A_1}$) $m:\mathbb{R}^+\to\mathbb{R}^+$ is a function of class $\mathcal{C}^1$, satisfying

$$m(s)s \ge \frac{1}{2}M(s) - \theta s, \quad \text{where} \quad M(s) = \int_0^s m(\tau)d\tau, \tag{1.2}$$

where $0\le\theta\le\frac{1}{2}\lambda_1^{\frac{1}{2}}$, $\lambda_1>0$ is the first eigenvalue of the bi-harmonic operator Δ^2 with boundary condition (1.1);

($\mathbf{A_2}$) $f\in\mathcal{C}^1(\mathbb{R})$ satisfies the growth condition

$$|f'(s)|\le C(1+|s|^{\varrho}), \tag{1.3}$$

with $1\le\varrho<\infty$ if $n\le4$ and $1\le\varrho<\frac{4}{n-4}$ if $n\ge5$, and the dissipation condition

$$\liminf_{|s|\to\infty} f'(s) > -\lambda_1. \tag{1.4}$$

Set

$$F(s)=\int_0^s f(\tau)d\tau.$$

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Assumption (1.4) yields that

$$\int_\Omega F(u)dx \geq -\frac{\lambda}{2}\|u\|^2 - C, \tag{1.5}$$

$$(f(u),u) \geq \int_\Omega F(u)dx - \frac{\lambda}{2}\|u\|^2 - C, \tag{1.6}$$

for some $\lambda < \lambda_1$ (see [14]).

($\mathbf{A_3}$) θ and λ are chosen so that

$$1 - \frac{\lambda}{\lambda_1} - \frac{2\theta}{\sqrt{\lambda_1}} > 0.$$

It is a significant issue to predict long-time dynamics of nonlinear evolution equations with different kinds of dissipation. The global attractor is a core concept. By definition, once the global attraction exists, it covers all possible permanent regimes of the system. Due to the Hölder-Mañé theorem (see [5, 7]), each compact set with finite fractal dimension is homeomorphic to a compact subset of Euclidean space $\mathbb{R}^n$. Therefore, if the fractal dimension of the global attractor is finite, then the infinite-dimensional dynamical system restricted on the global attractor can be reduced to a finite-dimensional dynamical system (see [9]).

The further expansions of the concept of global attractor are inertial manifold [6] and exponential attractor [4]. In particular, an exponential attractor is a positively invariant, finite fractal dimensional compact set which uniformly exponentially attracts all orbits starting from bounded subsets and contains the global attractor. Actually, to the best of our knowledge, exponential attractors exist indeed for almost all equations with finite-dimensional global attractors.

A necessary prerequisite for a dynamical system to have an exponential attractor is that it possesses a global attractor with finite fractal dimension. On the other hand, many dynamical systems generated by evolution equations have infinite-dimensional global attractors, such as the p-Laplace equations with symmetry (see [26]), some reaction-diffusion equations in unbounded domains (see [20,21]), some hyperbolic equations in unbounded domains (see [19]), and so on. When a dynamical system has an infinite-dimensional global attractor, it has no exponential attractors, but there may still exist positively invariant and exponentially attractive compact sets. Based on this observation, Zhang et. al ([22]) thought that the properties of exponential attractiveness and finite fractal dimension should be discussed separately, and proposed the concept of exponential decay with respect to noncompactness measure for the first time. They have proved that the sufficient and necessary condition for a dissipative dynamical system to have a positively invariant and exponentially attractive compact set $\mathcal{A}^*$ is that the noncompactness measure of each bounded set decays exponentially. They also gave some criteria for exponential decay with respect to noncompactness measure and proved this property for a class of reaction-diffusion equations and a class of wave equations with weak damping via the (C^*) condition.

We notice that for the semilinear wave equation or beam equation with nonlinear damping $g(u_t)$, when $g(0)=0$ and $g'(0)=0$, there is no conclusion as to whether the fractal dimension of the global attractor is finite and whether the noncompactness measure decays exponentially. There are other equations that face the same problem in the degenerate case. And we notice that it is difficult to obtain the exponential decay estimate with respect to noncompactness measure for the degenerate infinite-dimensional dynamical systems, so is it possible to reach the polynomial decay rate?

With this question in mind, we noticed that M. Nakao [10–12] proved that the nonnegative function $\phi(t)$ which satisfies the difference inequality

$$\sup_{s\in[t,t+1]} \phi(s)^{1+\alpha} \le K_0(1+t)^\gamma\big(\phi(t)-\phi(t+1)\big)+g(t)$$

decays to zero at polynomial or logarithmic polynomial rate as $t\to+\infty$. Using such difference inequality, Nakao proved that any solution of the wave equation with nonlinear damping tends to a certain steady-state solution or solution orbit at polynomial rate in [11,12], and that the energy functionals of solutions of the abstract nonlinear evolution equations

$$u''(t)+B(t)u'(t)+A(t)u(t)=f(t) \tag{1.7}$$

and

$$B(t)u'(t)+A(t)u(t)=f(t) \tag{1.8}$$

decay to 0 at polynomial (or logarithmic polynomial) rate in [10,13], where $A(t)$ is the Fréchet derivative of a functional on the Banach space V, $B(t)$ is a bounded operator from the Banach space W to its dual W^* and $V\hookrightarrow W\hookrightarrow H\hookrightarrow W^*\hookrightarrow V^*$. Recently, Silva, Narciso and Vicente [16] applied the difference inequality proposed by Nakao to prove the polynomial decay of the energy functional of solutions for the beam equation with nonlocal energy damping.

Base on the above analysis, in this paper and [25], we put forward the more general concepts of the polynomial decay with respect to noncompactness measure and polynomial attractor (Definition 3.1) (where the polynomial function $\varphi=t^{-\beta}:\mathbb{R}^+\to\mathbb{R}^+,\beta>0$ is decreasing and satisfies $\varphi(t)\to 0$ as $t\to+\infty$) as a generalization of the theory of exponential decay with respect to noncompact measure. And we prove that for every dynamical system which has the property of polynomial decay with respect to noncompactness measure, there exists a positively invariant compact set $\mathcal{A}^*$ that attracts each bounded set B at the rate of $\varphi\big(t-t_*(B)-1\big)$. This means that $\mathcal{A}^*$ is polynomial attractor. Then we also give some criteria for the polynomial decay with respect to noncompactness measure. Indeed, in [25], we establish a quasi-stable inequality concerning the controlling relationship of the distance at time t and the initial distance between any two orbits starting from a positive invariant bounded absorbing set $\mathcal{B}_0$. This quasi-stable inequality is closely related to a difference inequality and contains compact pseudo-metrics. Thus, by the definition of noncompactness measure and the compactness of the pseudo-metrics, from this quasi-stable inequality we can deduce the difference inequality with respect to the noncompactness measure $\alpha(S(t)\mathcal{B}_0)$, which leads to the estimate of the polynomial decay rate of the noncompactness measure. Consequently, the existence of the polynomial attractors and the estimate of their attractive rate are obtained.

In this paper, we apply an abstract theorem on estimating polynomial decay rate of noncompactness measure of bounded sets for infinite-dimensional dynamical systems to a class of extensible beams with nonlocal weak damping for the case that the nonlinear term f has subcritical growth.

The paper is organized as follows. In Section 2, we give some necessary preliminaries. In Section 3, we state the polynomial attractor and the polynomial decay with respect to noncompactness measure. We prove the existence of polynomial attractor for a class of extensible beams with nonlocal weak damping in Section 4.

2. Preliminaries

In this section, we give some necessary preliminaries which are required for establishing our results.

2.1. Functions space. Here we consider $H=L^2(\Omega)$ with usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $L^p(\Omega)$ with norm $\|\cdot\|_p$. We also consider the space $V_0=H_0^1(\Omega)$ and

$$V=\mathcal{D}(\mathcal{A}^{\frac{1}{2}})=\begin{cases} H_0^2(\Omega), & \text{for clamped boundary condition,}\\ H^2(\Omega)\cap H_0^1(\Omega), & \text{for hinged boundary condition.}\end{cases}$$

with norm $\|\nabla\cdot\|$ and $\|\Delta\cdot\|$ respectively, where the operator $\mathcal{A}=\Delta^2$.

Let $\lambda_1>0$ be the first eigenvalue of the bi-harmonic operator Δ^2 with boundary condition (1.1), then it holds

$$\|\Delta u\|^2\geq\lambda_1\|u\|^2, \|\Delta u\|^2\geq\lambda_1^{\frac{1}{2}}\|\nabla u\|^2,\ \forall u\in V. \tag{2.1}$$

Finally, we define the space

$$\mathcal{H}=V\times H$$

endowed with norm

$$\|(u,v)\|_{\mathcal{H}}^2=\|\Delta u\|^2+\|v\|^2.$$

Let C denote any positive constant which may be different from line to line and even in the same line. We also denote the different positive constants by $c_i,C_i,i\in\mathbb{N}$ et al.

2.2. Basic concepts and properties. We briefly recall the definition and basics of Kuratowski α-measure of noncompactness . For more details, we refer to [1,3,17].

DEFINITION 2.1 ([1,3]). *Let (X,d) be a metric space and let B be a bounded subset of X. The Kuratowski α-measure of noncompactness is defined by*

$$\alpha(B)=\inf\{\delta>0|B \textit{ has a finite cover of diameter } <\delta\}.$$

LEMMA 2.1 ([1,3]). *Let (X,d) be a complete metric space and α be the Kuratowski measure of noncompactness. Then*

(i) $\alpha(B)=0$ *if and only if B is precompact;*
(ii) $\alpha(A)\leq\alpha(B)$ *whenever $A\subseteq B$;*
(iii) $\alpha(A\cup B)=\max\{\alpha(A),\alpha(B)\}$;
(iv) $\alpha(B)=\alpha(\overline{B})$, *where $\overline{B}$ is the closure of B;*
(v) *if $B_1\supseteq B_2\supseteq B_3\ldots$ are nonempty closed sets in X such that $\alpha(B_n)\to0$ as $n\to\infty$, then $\cap_{n\geq1}B_n$ is nonempty and compact;*
(vi) *if X is a Banach space, then $\alpha(A+B)\leq\alpha(A)+\alpha(B)$.*

LEMMA 2.2 ([17]). *Assume $X\hookrightarrow\hookrightarrow B\hookrightarrow Y$ where X,B,Y are Banach spaces. The following statements hold.*

(i) *Let F be bounded in $L^p(0,T;X)$ where $1\leq p<\infty$, and $\partial F/\partial t=\{\partial f/\partial t: f\in F\}$ be bounded in $L^1(0,T;Y)$, where $\partial/\partial t$ is the weak time derivative. Then F is relatively compact in $L^p(0,T;B)$.*

(ii) *Let F be bounded in $L^\infty(0,T;X)$ and $\partial F/\partial t$ be bounded in $L^r(0,T;Y)$ where $r>1$. Then F is relatively compact in $C(0,T;B)$.*

Next, we will briefly review the definitions and fundamental conclusions of dynamical systems and the global attractor.

DEFINITION 2.2 ([2,15,18]). *A dynamical system is a pair of objects $(X,\{S(t)\}_{t\geq 0})$ consisting of a complete metric space X and a family of continuous mappings $\{S(t)\}_{t\geq 0}$ of X into itself with the semigroup properties:*

(i) $S(0)=I$,

(ii) $S(t+s)=S(t)S(s)$ *for all* $t,s\geq 0$,

where X is called a phase space (or state space) and $\{S(t)\}_{t\geq 0}$ is called an evolution semigroup.

DEFINITION 2.3 ([2, 15, 18]). *Let $\{S(t)\}_{t\geq 0}$ be a semigroup in a complete metric space (X,d). A closed set $\mathcal{B}\subseteq X$ is said to be absorbing for $\{S(t)\}_{t\geq 0}$ iff for any bounded set $B\subseteq X$ there exists $t_0(B)$ (the entering time of B into $\mathcal{B}$) such that $S(t)B\subseteq \mathcal{B}$ for all $t>t_0(B)$. $\{S(t)\}_{t\geq 0}$ is said to be dissipative iff it possesses a bounded absorbing set.*

LEMMA 2.3 ([2]). *Let $\{S(t)\}_{t\geq 0}$ be a semigroup in a complete metric space (X,d). If $\{S(t)\}_{t\geq 0}$ is dissipative, then it possesses a positively invariant bounded absorbing set.*

Moreover, let $\mathcal{B}$ be its bounded absorbing set, then $\mathcal{B}_0=\overline{\bigcup_{t\geq t_{\mathcal{B}}} S(t)\mathcal{B}}$ is a positively invariant bounded absorbing set, where $t_{\mathcal{B}}>0$ is the entering time of $\mathcal{B}$ into itself.

DEFINITION 2.4 ([2,15,18]). *A compact set $\mathcal{A}\subseteq X$ is said to be a global attractor of the dynamical system $(X,\{S(t)\}_{t\geq 0})$ iff*

(i) *$\mathcal{A}\subseteq X$ is an invariant set, i.e., $S(t)\mathcal{A}=\mathcal{A}$ for all $t\geq 0$,*

(ii) *$\mathcal{A}\subseteq X$ is uniformly attracting, i.e., for all bounded sets $B\subseteq X$ we have*

$$\lim_{t\to+\infty} dist(S(t)B,\mathcal{A})=0,$$

where $dist(A,B):=\sup_{x\in A}\mathrm{dist}_X(x,B)$ is the Hausdorff semi-distance.

Ma, Wang and Zhong put forward the concept of ω-limit compact in [8] and proved that ω-limit compactness is a necessary and sufficient condition for a dissipative dynamical system to possess the global attractor. Due to Lemma 2.3, we can weaken the definition of ω-limit compact in [8] to the following form:

DEFINITION 2.5. *The dynamical system $(X,\{S(t)\}_{t\geq 0})$ is said to be ω-limit compact iff for every positively invariant bounded set $B\subseteq X$ we have $\alpha\big(S(t)B\big)\to 0$ as $t\to\infty$, where $\alpha(\cdot)$ is the Kuratowski measure of noncompactness.*

For the above definition, we can still get the same conclusion as in [8]:

THEOREM 2.1. *The dynamical system $(X,\{S(t)\}_{t\geq 0})$ has a global attractor in X if and only if it is both dissipative and ω-limit compact.*

3. The polynomial attractor and the polynomial decay with respect to noncompactness measure

The global attractor gives no information about the attractive rate. And we notice that it is difficult to obtain the exponential decay estimate with respect to noncompactness measure for the degenerate infinite-dimensional dynamical systems. In order to describe the asymptotic behavior of dynamical systems more concretely, we propose the following concept of polynomial attractor in [25]. For the reader's convenience, we restate these theories about polynomial attractor in [25] .

DEFINITION 3.1. *We call a compact* $\mathcal{A}^* \subset X$ *a polynomial attractor for the dynamical system* $(X,\{S(t)\}_{t\geq 0})$*, if* $\mathcal{A}^*$ *is positively invariant with respect to* $S(t)$ *and for every bounded set* $B \subseteq X$ *there exists* $t_B \geq 0$ *such that*

$$\operatorname{dist}(S(t)B,\mathcal{A}^*) \leq C\big((t-t_*(B))\big)^{-\beta},\ \forall t \geq t_*(B)+t_0,$$

for certain positive constants C,β*.*

We emphasize that the finiteness of fractal dimension is not required in the above definition of polynomial attractor. This is because there indeed exist positively invariant compact sets with infinite fractal dimension which can attract all bounded sets at the polynomial speed. We further proposed the following concept of polynomial decay with respect to noncompactness measure as a condition for the existence of polynomial attractor.

DEFINITION 3.2. *The dynamical system* $(X,\{S(t)\}_{t\geq 0})$ *is said to have polynomial decay with respect to noncompactness measure iff it is dissipative and there exists* $t_0 > 0$ *such that*

$$\alpha(S(t)\mathcal{B}_0) \leq Ct^{-\beta},\ \forall t \geq t_0, \tag{3.1}$$

where $\mathcal{B}_0$ *is a positively invariant bounded absorbing set of* $(X,\{S(t)\}_{t\geq 0})$ *and certain positive constants* C,β*.*

LEMMA 3.1. *Assume that the dynamical system* $(X,\{S(t)\}_{t\geq 0})$ *is* φ*-decaying with respect to noncompactness measure, which implies that there exist a positively invariant bounded absorbing set* $\mathcal{B}_0$ *and a positive constant* t_0 *such that*

$$\alpha(S(t)\mathcal{B}_0) \leq Ct^{-\beta},\ \forall t \geq t_0.$$

Then for every bounded subset B *of* X*, we have*

$$\alpha(S(t)B) \leq C\big(t-t_*(B)\big)^{-\beta},\ \forall t \geq t_*(B)+t_0,$$

where $t_*(B)$ *is the entering time of* B *into* $\mathcal{B}_0$*.*

Proof. Since

$$S(t)B \subseteq \mathcal{B}_0,\ \ \forall t \geq t_*(B),$$

we have

$$S(t)B = S(t-t_*(B))S(t_*(B))B \subseteq S(t-t_*(B))\mathcal{B}_0,$$

then

$$\alpha(S(t)B) \leq \alpha(S(t-t_*(B))\mathcal{B}_0) \leq \big(t-t_*(B)\big)^{-\beta},\ \ \forall t \geq t_*(B)+t_0.$$

∎

THEOREM 3.1. *Assume that the dynamical system* $(X,\{S(t)\}_{t\geq 0})$ *has polynomial decay with respect to noncompactness measure, which implies that there exist a positively invariant bounded absorbing set* $\mathcal{B}_0$ *and a positive constant* t_0 *such that*

$$\alpha(S(t)\mathcal{B}_0) \leq Ct^{-\beta},\ \forall t \geq t_0.$$

Then there exists a positively invariant compact set $\mathcal{A}^*$ such that for every bounded set $B \subseteq X$ we have

$$\operatorname{dist}(S(t)B, \mathcal{A}^*) \le C\big((t - t_*(B) - 1)\big)^{-\beta}, \ \forall t \ge t_*(B) + t_0 + 1, \tag{3.2}$$

where $t_*(B)$ is the entering time of B into $\mathcal{B}_0$. That is to say, $\mathcal{A}^*$ is the polynomial attractor for $(X, \{S(t)\}_{t\ge 0})$.

LEMMA 3.2. *Suppose that $w(t)$ is a nonnegative function on $\mathbb{R}^+$ satisfying*

$$\max\{w^{1+\alpha}(t), w^{1+\alpha}(t+T)\} \le h(t)[w(t) - w(t+T)], \ \forall t \ge t_0, \tag{3.3}$$

where α, T, t_0 are positive constants, $h(t)$ is a positive monotone function. Then $w(t)$ satisfies the following estimate:

$$w(t) \le \Big\{ \inf_{s\in[t_0, t_0+T]} w^{-\alpha}(s) + \frac{\alpha}{T}\int_{t_0+T}^{t-T} \frac{ds}{h(s)} \Big\}^{-\frac{1}{\alpha}}, \ \forall t \ge t_0 + 2T.$$

In particular, when $h(t) = K_0$, we have

$$w(t) \le \Big\{ \inf_{s\in[t_0, t_0+T]} w^{-\alpha}(s) + \frac{\alpha}{TK_0}(t - t_0 - 2T) \Big\}^{-\frac{1}{\alpha}}, \ \forall t \ge t_0 + 2T. \tag{3.4}$$

Proof.

$$\begin{aligned}
\omega^{-\alpha}(t+T) - \omega^{-\alpha}(t) &= \int_0^1 \frac{d}{d\theta}\Big\{ \big[\theta\omega(t+T) + (1-\theta)\omega(t)\big]^{-\alpha} \Big\} d\theta \\
&= \alpha \int_0^1 \big[\theta\omega(t+T) + (1-\theta)\omega(t)\big]^{-\alpha-1} d\theta \big[\omega(t) - \omega(t+T)\big] \\
&\ge \alpha\big(\max\{\omega(t), \omega(t+T)\}\big)^{-\alpha-1}\big[\omega(t) - \omega(t+T)\big].
\end{aligned} \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\omega^{-\alpha}(t+T) - \omega^{-\alpha}(t) \ge \frac{\alpha}{h(t)}, \ \forall t \ge t_0.$$

Thus

$$\omega^{-\alpha}(t) \ge \omega^{-\alpha}(t - nT) + \frac{\alpha}{T}\sum_{i=1}^{n} \frac{T}{h(t - iT)}, \ \forall t \ge t_0 + 2T, \tag{3.6}$$

where $n \equiv [\frac{t-t_0}{T}]$ is the integral part of $\frac{t-t_0}{T}$.

If $h(t)$ is non-increasing, then by (3.6),

$$\begin{aligned}
\omega^{-\alpha}(t) &\ge \omega^{-\alpha}(t - nT) + \frac{\alpha}{h(t-nT)} + \frac{\alpha}{T}\sum_{i=1}^{n-1}\int_{t-(i+1)T}^{t-iT} \frac{ds}{h(t - iT)} \\
&\ge \omega^{-\alpha}(t - nT) + \frac{\alpha}{h(t-nT)} + \frac{\alpha}{T}\int_{t-nT}^{t-T} \frac{ds}{h(s)} \\
&\ge \omega^{-\alpha}(t - nT) + \frac{\alpha}{T}\int_{t_0+T}^{t-T} \frac{ds}{h(s)} \\
&\ge \inf_{s\in[t_0, t_0+T]} \omega^{-\alpha}(s) + \frac{\alpha}{T}\int_{t_0+T}^{t-T} \frac{ds}{h(s)}
\end{aligned} \tag{3.7}$$

holds for all $t \geq t_0 + 2T$.

If $h(t)$ is non-decreasing, then by (3.6)

$$
\begin{aligned}
\omega^{-\alpha}(t) \geq & \omega^{-\alpha}(t-nT) + \frac{\alpha}{T}\sum_{i=1}^{n}\int_{t-iT}^{t-(i-1)T}\frac{ds}{h(t-iT)} \\
\geq & \omega^{-\alpha}(t-nT) + \frac{\alpha}{T}\int_{t-nT}^{t}\frac{ds}{h(s)} \\
\geq & \omega^{-\alpha}(t-nT) + \frac{\alpha}{T}\int_{t_0+T}^{t-T}\frac{ds}{h(s)} \\
\geq & \inf_{s\in[t_0,t_0+T]}\omega^{-\alpha}(s) + \frac{\alpha}{T}\int_{t_0+T}^{t-T}\frac{ds}{h(s)}
\end{aligned} \tag{3.8}
$$

holds for all $t \geq t_0 + 2T$.

Combining (3.7) and (3.8), we conclude that if $h(t)$ is monotone, then

$$
\omega(t) \leq \left\{ \inf_{s\in[t_0,t_0+T]}\omega^{-\alpha}(s) + \frac{\alpha}{T}\int_{t_0+T}^{t-T}\frac{ds}{h(s)} \right\}^{-\frac{1}{\alpha}}, \ \forall t \geq t_0 + 2T. \tag{3.9}
$$

The estimate (3.4) follows immediately from (3.9). □

Remark 3.1. Lemma 3.2 estimates the decay rate of nonnegative function $w(t)$ satisfying the difference inequality (3.3) for the case that $h(t)$ is a general positive monotone function. It is a generalization of Theorem 1 in [10] by M. Nakao, which established decay estimate from the difference inequality

$$
\sup_{s\in[t,t+1]}\phi(s)^{1+\alpha} \leq K_0(1+t)^{\gamma}\big(\phi(t)-\phi(t+1)\big) + g(t).
$$

For the exponential decay (namely $Ce^{-\beta t}$) with respect to noncompactness measure, the case of has been discussed in [22, 23], whereas the case of polynomial decay (namely $Ct^{-\beta}$) has not been discussed before. Next, we give a theorem on the polynomial decay estimate with respect to noncompactness measure in this section.

Lemma 3.3. *Let $\{S(t)\}_{t\geq 0}$ be a dissipative dynamical system on a complete metric space (X,d) and $\mathcal{B}_0$ be a positively invariant bounded absorbing set. Assume that there exist positive constants T, δ_0, a continuous non-decreasing function $q:\mathbb{R}^+ \to \mathbb{R}^+$, a function $g:(\mathbb{R}^+)^m \to \mathbb{R}^+$ and pseudometrics ϱ_T^i $(i=1,2,\dots,m)$ on $\mathcal{B}_0$ such that*

- (i) *$q(0)=0$; $q(s)<s$, $s>0$;*
- (ii) *g is non-decreasing with respect to each variable, $g(0,\dots,0)=0$ and g is continuous at $(0,\dots,0)$;*
- (iii) *$\varrho_T^i (i=1,2,\dots,m)$ is precompact on $\mathcal{B}_0$, i.e., any sequence $\{x_n\}\subseteq\mathcal{B}_0$ has a subsequence $\{x_{n_k}\}$ which is Cauchy with respect to ϱ_T^i;*
- (iv) *the inequality*

$$
\begin{aligned}
&\big(d(S(T)y_1, S(T)y_2)\big)^2 \\
\leq & q\Big(\big(d(y_1,y_2)\big)^2 + g\big(\varrho_T^1(y_1,y_2), \varrho_T^2(y_1,y_2), \dots, \varrho_T^m(y_1,y_2)\big)\Big)
\end{aligned} \tag{3.10}
$$

holds for all $y_1, y_2 \in \mathcal{B}_0$ satisfying $\varrho_T^i(y_1,y_2) \leq \delta_0 (i=1,2,\dots,m)$.

Then

$$\alpha(S(t)\mathcal{B}_0) \to 0 \ \textit{as}\ t \to +\infty,$$

and $\{S(t)\}_{t\geq 0}$ *is* ω*-limit compact.*

Proof. For each $B \subseteq \mathcal{B}_0$ and any $\epsilon > 0$, by Definition 2.1, there exist sets $F_1, F_2, \ldots, F_n$ such that

$$B \subseteq \cup_{j=1}^{n} F_n,\ \operatorname{diam} F_j < \alpha(B) + \epsilon. \tag{3.11}$$

It follows from assumption (ii) that there exists $\delta > 0$ such that $g(x_1, x_2, \ldots, x_m) < \epsilon$ whenever $x_i \in [0, \delta]$ $(i = 1, 2, \ldots, m)$. By the precompactness of $\varrho_T^i (i = 1, 2, \ldots, m)$, there exists a finite set $\mathcal{N}^i = \{x_j^i : j = 1, 2, \ldots, k_i\} \subseteq B$ such that for every $y \in B$ there is $x_j^i \in \mathcal{N}^i$ with the property $\varrho_T^i(y, x_j^i) \leq \frac{1}{2}\min\{\delta, \delta_0\}$, i.e.,

$$B \subseteq \cup_{j=1}^{k_i} C_j^i,\ C_j^i = \big\{y \in B:\ \varrho_T^i(y, x_j^i) \leq \frac{1}{2}\min\{\delta, \delta_0\}\big\},\ i = 1, \ldots, m. \tag{3.12}$$

Consequently, we have

$$B \subseteq \cup_{j_1, j_2, \ldots, j_m, j} (C_{j_1}^1 \cap C_{j_2}^2 \cap \ldots \cap C_{j_m}^m \cap F_j)$$

and

$$S(T)B \subseteq \cup_{j_1, j_2, \ldots, j_m, j} \big(S(T)(C_{j_1}^1 \cap C_{j_2}^2 \cap \ldots \cap C_{j_m}^m \cap F_j)\big).$$

By (3.11) and (3.12), for any $y_1, y_2 \in C_{j_1}^1 \cap C_{j_2}^2 \cap \ldots \cap C_{j_m}^m \cap F_j$, we have

$$d\big(y_1, y_2\big) \leq \operatorname{diam} F_j < \alpha(B) + \epsilon \tag{3.13}$$

and

$$\varrho_T^i\big(y_1, y_2\big) \leq \min\{\delta, \delta_0\}, i = 1, 2, \ldots, m. \tag{3.14}$$

Inequality (3.14) implies

$$g\Big(\varrho_T^1\big(y_1, y_2\big), \ldots, \varrho_T^m\big(y_1, y_2\big)\Big) < \epsilon. \tag{3.15}$$

We deduce from (3.10), (3.13), (3.14) and (3.15) that

$$\big(d(S(T)y_1, S(T)y_2)\big)^2 \leq q\Big(\big(\alpha(B) + \epsilon\big)^2 + \epsilon\Big) \tag{3.16}$$

for any $y_1, y_2 \in C_{j_1}^1 \cap C_{j_2}^2 \cap \ldots \cap C_{j_m}^m \cap F_j$. Therefore according to the definition of non-compactness measure α, we obtain

$$\big(\alpha(S(T)B)\big)^2 \leq q\Big(\big(\alpha(B) + \epsilon\big)^2 + \epsilon\Big). \tag{3.17}$$

Since q is continuous and non-decreasing, combining (3.17) and the arbitrariness of ϵ gives

$$\big(\alpha(S(T)B)\big)^2 \leq q\Big(\big(\alpha(B)\big)^2\Big). \tag{3.18}$$

We infer from (3.18) that

$$\left(\alpha(S(kT)\mathcal{B}_0)\right)^2 \le q\Big(\left(\alpha(S((k-1)T)\mathcal{B}_0)\right)^2\Big),\ k=1,2,\ldots. \tag{3.19}$$

Since $q(s)\le s$, the sequence $\left\{\Big(\alpha\big(S(kT)\mathcal{B}_0\big)\Big)^2\right\}_{k=1}^{+\infty}$ is non-increasing and thus there exists $\alpha_0=\lim_{k\to+\infty}\Big(\alpha\big(S(kT)\mathcal{B}_0\big)\Big)^2$. By the continuity of q, (3.19) implies $\alpha_0\le q(\alpha_0)$, which yields $\alpha_0=0$ by assumption (ii). Consequently, we obtain

$$\alpha(S(t)\mathcal{B}_0)\to 0 \text{ as } t\to+\infty. \tag{3.20}$$

For any bounded set $D\subseteq X$, there exists $t_D>0$ such that $S(t_D)D\subseteq\mathcal{B}_0$, which implies $S(t+t_D)D\subseteq S(t)\mathcal{B}_0$. Thus, we have $\alpha\big(S(t+t_D)D\big)\le\alpha\big(S(t)\mathcal{B}_0\big)$, which, together with (3.20), gives

$$\alpha(S(t)D)\to 0 \text{ as } t\to+\infty. \tag{3.21}$$

Therefore, $\{S(t)\}_{t\ge0}$ is ω-limit compact. □

THEOREM 3.2 (The Polynomial Decay Theorem). *Let $\{S(t)\}_{t\ge0}$ be a dissipative dynamical system on a complete metric space (X,d) and $\mathcal{B}_0$ be a positively invariant bounded absorbing set. Assume that there exist positive constants C,T,δ_0, $\beta\in(0,1)$, functions $g_l:(\mathbb{R}^+)^m\to\mathbb{R}^+$ $(l=1,2)$ and pseudometrics ϱ_T^i $(i=1,2,\ldots,m)$ on $\mathcal{B}_0$ such that*

(i) *g_l is non-decreasing with respect to each variable, $g_l(0,\ldots,0)=0$ and g_l is continuous at $(0,\ldots,0)$;*

(ii) *$\varrho_T^i(i=1,2,\ldots,m)$ is precompact on $\mathcal{B}_0$, i.e., any sequence $\{x_n\}\subseteq\mathcal{B}_0$ has a subsequence $\{x_{n_k}\}$ which is Cauchy with respect to ϱ_T^i;*

(iii) *the inequalities*

$$\begin{aligned}&(d(S(T)y_1,S(T)y_2))^2\\ &\le(d(y_1,y_2))^2+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\ldots,\varrho_T^m(y_1,y_2)\Big)\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}(d(S(T)y_1,S(T)y_2))^2\le C\Big[&(d(y_1,y_2))^2-(d(S(T)y_1,S(T)y_2))^2\\ &+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\ldots,\varrho_T^m(y_1,y_2)\Big)\Big]^\beta\\ &+g_2\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\ldots,\varrho_T^m(y_1,y_2)\Big)\end{aligned} \tag{3.23}$$

hold for all $y_1,y_2\in\mathcal{B}_0$ satisfying $\varrho_T^i(y_1,y_2)\le\delta_0(i=1,2,\ldots,m)$.
Then there exists $t_0>0$ such that for each bounded $B\subseteq X$ the estimate

$$\alpha(S(t)B)\le\left\{(\alpha(\mathcal{B}_0))^{\frac{2(\beta-1)}{\beta}}+\frac{1-\beta}{T\beta(1+2C)^{\frac{1}{\beta}}}\big(t-t_*(B)-t_0-2T\big)\right\}^{\frac{\beta}{2(\beta-1)}} \tag{3.24}$$

holds for all $t\ge t_0+2T+t_*(B)$, where $t_*(B)$ satisfies

$$S(t)B\subseteq\mathcal{B}_0,\ \forall t\ge t_*(B).$$

Thus, $(X,\{S(t)\}_{t\ge 0})$ possesses a polynomial attractor $\mathcal{A}^*$ (see Definition 3.1) such that for every bounded set $B\subseteq X$,

$$\operatorname{dist}(S(t)B,\mathcal{A}^*)\le\Big\{(\alpha(\mathcal{B}_0))^{\frac{2(\beta-1)}{\beta}}+\frac{1-\beta}{T\beta(1+2C)^{\frac{1}{\beta}}}(t-t_0-2T-t_*(B)-1)\Big\}^{\frac{\beta}{2(\beta-1)}} \tag{3.25}$$

holds for all $t\ge t_0+2T+t_*(B)+1$.

Proof. For all $y_1,y_2\in\mathcal{B}_0$ satisfying $\varrho_T^i(y_1,y_2)\le\delta_0(i=1,2,\dots,m)$, it follows from (3.23) that

$$\begin{aligned}\big(d(S(T)y_1,S(T)y_2)\big)^{\frac{2}{\beta}}\le&(2C)^{\frac{1}{\beta}}\Big[\big(d(y_1,y_2)\big)^2-\big(d(S(T)y_1,S(T)y_2)\big)^2\\&+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\Big]\\&+2^{\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big),\end{aligned}$$

which yields

$$\begin{aligned}&(2C)^{-\frac{1}{\beta}}\big(d(S(T)y_1,S(T)y_2)\big)^{\frac{2}{\beta}}+\big(d(S(T)y_1,S(T)y_2)\big)^2\\\le&\big(d(y_1,y_2)\big)^2+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\\&+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big).\end{aligned} \tag{3.26}$$

We rewrite (3.26) as

$$\begin{aligned}&w\Big(\big(d(S(T)y_1,S(T)y_2)\big)^2\Big)\\\le&\big(d(y_1,y_2)\big)^2+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\\&+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\end{aligned} \tag{3.27}$$

with $w(s)=(2C)^{-\frac{1}{\beta}}s^{\frac{1}{\beta}}+s,\ s\ge 0$. We denote by w^{-1} the inverse function of w on $\mathbb{R}^+$. Since w^{-1} is increasing, (3.27) implies that

$$\begin{aligned}&\big(d(S(T)y_1,S(T)y_2)\big)^2\\\le&w^{-1}\Big(\big(d(y_1,y_2)\big)^2+g_1\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\\&+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\Big(\varrho_T^1(y_1,y_2),\varrho_T^2(y_1,y_2),\dots,\varrho_T^m(y_1,y_2)\Big)\Big).\end{aligned} \tag{3.28}$$

Moreover, it is easy to check that $w^{-1}(0)=0$ and $w^{-1}(s)<0,\ s>0$. Thus, by Lemma 3.3 we deduce from inequality (3.28) that

$$\alpha(S(t)\mathcal{B}_0)\to 0 \text{ as } t\to+\infty. \tag{3.29}$$

Consequently, there exists $t_0>0$ such that

$$\alpha(S(t)\mathcal{B}_0)<1,\ \forall t\ge t_0. \tag{3.30}$$

For each fixed $t\geq t_0$ and each $\epsilon>0$, by Definition 2.1, there exist sets $F_1,F_2,\dots,F_n$ such that

$$S(t)\mathcal{B}_0\subseteq\cup_{j=1}^{n}F_n,\ \operatorname{diam}F_j<\alpha(S(t)\mathcal{B}_0)+\epsilon. \tag{3.31}$$

It follows from assumption (i) that there exists $\delta>0$ such that $g_l(x_1,x_2,\dots,x_m)<\epsilon$ ($l=1,2$) whenever $x_i\in[0,\delta]$ ($i=1,2,\dots,m$). By the precompactness of $\varrho_T^i(i=1,2,\dots,m)$, there exists a finite set $\mathcal{N}^i=\{x_j^i:j=1,2,\dots,k_i\}\subseteq\mathcal{B}_0$ such that for every $y\in\mathcal{B}_0$ there is $x_j^i\in\mathcal{N}^i$ with the property $\varrho_T^i(S(t)y,S(t)x_j^i)\leq\frac{1}{2}\min\{\delta,\delta_0\}$, i.e.,

$$S(t)\mathcal{B}_0\subseteq\cup_{j=1}^{k_i}C_j^i,\ C_j^i=\big\{S(t)y:y\in\mathcal{B}_0,\ \varrho_T^i(S(t)y,S(t)x_j^i)\leq\frac{1}{2}\min\{\delta,\delta_0\}\big\},\ i=1,\dots,m. \tag{3.32}$$

Consequently, we have

$$S(t)\mathcal{B}_0\subseteq\cup_{j_1,j_2,\dots,j_m,j}(C_{j_1}^1\cap C_{j_2}^2\cap\ldots\cap C_{j_m}^m\cap F_j)$$

and

$$S(t+T)\mathcal{B}_0\subseteq\cup_{j_1,j_2,\dots,j_m,j}\big(S(T)(C_{j_1}^1\cap C_{j_2}^2\cap\ldots\cap C_{j_m}^m\cap F_j)\big).$$

By (3.31) and (3.32), for any $S(t)y_1,S(t)y_2\in C_{j_1}^1\cap C_{j_2}^2\cap\ldots\cap C_{j_m}^m\cap F_j$, we have

$$d\big(S(t)y_1,S(t)y_2\big)\leq\operatorname{diam}F_j<\alpha(S(t)\mathcal{B}_0)+\epsilon \tag{3.33}$$

and

$$\varrho_T^i\big(S(t)y_1,S(t)y_2\big)\leq\min\{\delta,\delta_0\}(i=1,2,\dots,m). \tag{3.34}$$

Inequality (3.34) implies

$$g_l\Big(\varrho_T^1\big(S(t)y_1,S(t)y_2\big),\dots,\varrho_T^m\big(S(t)y_1,S(t)y_2\big)\Big)<\epsilon,\ l=1,2. \tag{3.35}$$

We deduce from (3.22), (3.23) and (3.34) that

$$\begin{aligned}\big(d(S(T+t)y_1,S(T+t)y_2)\big)^{\frac{2}{\beta}}\leq(2C)^{\frac{1}{\beta}}\big[&\big(d(S(t)y_1,S(t)y_2)\big)^2-\big(d(S(T+t)y_1,S(T+t)y_2)\big)^2\\&+g_1\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big)\big]\\&+2^{\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big),\end{aligned}$$

which yields

$$\begin{aligned}&(2C)^{-\frac{1}{\beta}}\big(d(S(T+t)y_1,S(T+t)y_2)\big)^{\frac{2}{\beta}}+\big(d(S(T+t)y_1,S(T+t)y_2)\big)^2\\&\leq\big(d(S(t)y_1,S(t)y_2)\big)^2+g_1\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big)\\&\quad+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big),\end{aligned} \tag{3.36}$$

i.e.,

$$\begin{aligned}&w\Big(\big(d(S(T+t)y_1,S(T+t)y_2\big)^2\Big)\\&\leq\big(d(S(t)y_1,S(t)y_2)\big)^2+g_1\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big)\end{aligned}$$

$$+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big). \tag{3.37}$$

Since $w(s)$ is increasing on $\mathbb{R}^+$, (3.37) implies

$$\begin{aligned}&\big(d(S(T+t)y_1,S(T+t)y_2)^2\\ \le & w^{-1}\Big(\big(d(S(t)y_1,S(t)y_2)\big)^2+g_1\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big)\\ &+C^{-\frac{1}{\beta}}g_2^{\frac{1}{\beta}}\big(\varrho_T^1(S(t)y_1,S(t)y_2),\dots,\varrho_T^m(S(t)y_1,S(t)y_2)\big)\Big).\end{aligned} \tag{3.38}$$

We derive from (3.33), (3.35), (3.38) and the monotonically increasing property of w^{-1} that

$$\big(d(S(T+t)y_1,S(T+t)y_2)^2\le w^{-1}\Big(\big(\alpha(S(t)\mathcal{B}_0)+\epsilon\big)^2+\epsilon+C^{-\frac{1}{\beta}}\epsilon^{\frac{1}{\beta}}\Big).$$

As a consequence,

$$\big(\alpha(S(t+T)\mathcal{B}_0)\big)^2\le w^{-1}\Big(\big(\alpha(S(t)\mathcal{B}_0)+\epsilon\big)^2+\epsilon+C^{-\frac{1}{\beta}}\epsilon^{\frac{1}{\beta}}\Big).$$

Hence by the arbitrariness of ϵ, we have

$$w\Big(\big(\alpha(S(t+T)\mathcal{B}_0)\big)^2\Big)\le\big(\alpha(S(t)\mathcal{B}_0)\big)^2. \tag{3.39}$$

Inequality (3.39) is equivalent to

$$\big(\alpha(S(t+T)\mathcal{B}_0)\big)^2\le 2C\Big[\big(\alpha(S(t)\mathcal{B}_0)\big)^2-\big(\alpha(S(t+T)\mathcal{B}_0)\big)^2\Big]^\beta. \tag{3.40}$$

Since $\alpha(S(t+T)\mathcal{B}_0)\le\alpha(S(t)\mathcal{B}_0)<1$ holds for all $t\ge t_0$, we have

$$(\alpha(S(t)\mathcal{B}_0))^2-(\alpha(S(t+T)\mathcal{B}_0))^2\le\big[(\alpha(S(t)\mathcal{B}_0))^2-(\alpha(S(t+T)\mathcal{B}_0))^2\big]^\beta. \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\begin{aligned}(\alpha(S(t)\mathcal{B}_0))^2=&(\alpha(S(t+T)\mathcal{B}_0))^2+(\alpha(S(t)\mathcal{B}_0))^2-(\alpha(S(t+T)\mathcal{B}_0))^2\\ \le&(1+2C)\big[(\alpha(S(t)\mathcal{B}_0))^2-(\alpha(S(t+T)\mathcal{B}_0))^2\big]^\beta,\end{aligned}$$

i.e.,

$$(\alpha(S(t)\mathcal{B}_0))^{\frac{2}{\beta}}\le(1+2C)^{\frac{1}{\beta}}\big[(\alpha(S(t)\mathcal{B}_0))^2-(\alpha(S(t+T)\mathcal{B}_0))^2\big] \tag{3.42}$$

holds for all $t\ge t_0$. Since $\mathcal{B}_0$ is positively invariant, $\alpha(S(t)\mathcal{B}_0)$ is non-increasing with respect to t. Therefore it follows from (3.42) that (3.3) holds with $w(t)=(\alpha(S(t)\mathcal{B}_0))^2$, $1+\alpha=\frac{1}{\beta}$ and $h(t)=(1+2C)^{\frac{1}{\beta}}$. Consequently, by Lemma 3.2, we have

$$\alpha(S(t)\mathcal{B}_0)\le\Big\{(\alpha(\mathcal{B}_0))^{\frac{2(\beta-1)}{\beta}}+\frac{1-\beta}{T\beta(1+2C)^{\frac{1}{\beta}}}\big(t-t_0-2T\big)\Big\}^{\frac{\beta}{2(\beta-1)}},\ \forall t\ge t_0+2T,$$

which, together with lemma 3.1 , gives (3.24).

By Theorem 3.1, $(X,\{S(t)\}_{t\ge0})$ possesses a polynomial attractor $\mathcal{A}^*$ such that for every bounded set $B\subseteq X$,

$$\operatorname{dist}(S(t)B,\mathcal{A}^*)\le\Big\{(\alpha(\mathcal{B}_0))^{\frac{2(\beta-1)}{\beta}}+\frac{1-\beta}{T\beta(1+2C)^{\frac{1}{\beta}}}\big(t-t_0-2T-t_*(B)-1\big)\Big\}^{\frac{\beta}{2(\beta-1)}} \tag{3.43}$$

holds for all $t\ge t_0+2T+t_*(B)+1$. □

4. The main result

Recently, we have proved in [24] the following result.

LEMMA 4.1. *Let $T>0$ be arbitrary. Under the assumptions $(A_1)-(A_3)$, for every $(u_0,u_1)\in\mathcal{H}=V\times H$ the initial boundary value problem (1.1) has a unique weak solution $u\in\mathcal{C}([0,T];\mathcal{H})$, which implies that the family of evolution operators $S_t:\mathcal{H}\to\mathcal{H}$ defined by*

$$S_t(u_0;u_1)=(u(t);u_t(t)),t\geq 0, \tag{4.1}$$

where (u,u_t) solves (1.1) with the initial data $(u_0;u_1)$, defines a nonlinear C_0-semigroup, generates a dynamical system $(\mathcal{H},S_t)$ in the phase space $\mathcal{H}=V\times H$.

Furthermore, the semigroup $\{S_t\}_{t\geq 0}$ is dissipative and asymptotically smooth, which imply the existence of a positively invariant bounded absorbing set $\mathcal{B}_0$ as well as the global attractor $\mathcal{A}$.

LEMMA 4.2. *Let $u,v\in H$, H is a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$. Then there exists some constant C_γ which depends on γ such that*

$$\left(\|u\|_H^{\gamma-2}u-\|v\|_H^{\gamma-2}v,u-v\right)\geq\begin{cases}C_\gamma\|u-v\|_H^\gamma,\ if\ \gamma\geq 2,\\ C_\gamma\frac{\|u-v\|_H^2}{(\|u\|_H+\|v\|_H)^{2-\gamma}},\ if\ 1\leq\gamma\leq 2.\end{cases} \tag{4.2}$$

Now we will apply Theorem 3.2 to give the estimate of polynomial dissipativity of noncompactness measure of bounded subsets for problem (1.1).

THEOREM 4.1. *Under conditions (A_1), (A_2) and (A_3), the dynamical system $(\mathcal{H},S_t)$ generated by problem (1.1) has the property of polynomial dissipativity of noncompactness measure of bounded subsets. More precisely, there exists $t_0>0$ such that for any bounded $B\subseteq V\times H$ we have*

$$\alpha(S(t)B)\leq\left\{(\alpha(\mathcal{B}_0))^{-p}+\frac{pC_p}{2^{p+2}}\left(t-t_0-t_*(B)\right)\right\}^{-\frac{1}{p}},\ \forall t\geq t_0+T+t_*(B),$$

where $t_*(B)$ satisfies

$$S(t)B\subseteq\mathcal{B}_0,\ \forall t\geq t_*(B).$$

Proof. Let $w(t)$ and $v(t)$ be two weak solutions to the problem (1.1) corresponding to two different initial data in the invariant set $\mathcal{B}_0$:

$$(w(t),w_t(t))\equiv S_ty_0,\quad (v(t),v_t(t))\equiv S_ty_1,\qquad y_0,y_1\in\mathcal{B}_0. \tag{4.3}$$

By Lemma 4.1, we know $\mathcal{B}_0$ is positively invariant, then

$$\begin{cases}\|(w(t),w_t(t))\|_{\mathcal{H}}\leq C,\\ \|(v(t),v_t(t))\|_{\mathcal{H}}\leq C,\end{cases}\quad \forall t>0,y_1,y_2\in\mathcal{B}_0. \tag{4.4}$$

Note that $z(t)=w(t)-v(t)$ satisfies the following equality

$$\begin{aligned}&z_{tt}+\Delta^2z-m(\|\nabla w\|^2)\Delta z-(m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v\\&+\|w_t\|^pw_t-\|v_t\|^pv_t+f(w)-f(v)=0.\end{aligned} \tag{4.5}$$

Multiplying (4.5) by $z_t(t)$ and integrating over Ω, we obtain

$$\begin{aligned}&(z_{tt},z_t)+(\Delta^2z,z_t)-(m(\|\nabla w\|^2)\Delta z,z_t)+(\|w_t\|^pw_t-\|v_t\|^pv_t,z_t)\\&=((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,z_t)-(f(w)-f(v),z_t).\end{aligned} \tag{4.6}$$

In view of

$$m(\|\nabla w\|^2)(\Delta z, z_t) = -\frac{1}{2}\frac{d}{dt}m(\|\nabla w\|^2)\|\nabla z\|^2 - m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t),$$

we rewrite (4.6) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|z_t(t)\|^2 + \|\Delta z(t)\|^2 + m(\|\nabla w\|^2)\|\nabla z\|^2\Big) + (\|w_t\|^p w_t - \|v_t\|^p v_t, z_t)\\
=&-m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t) + ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t)\\
&-(f(w) - f(v), z_t),
\end{aligned} \tag{4.7}$$

from which, by integrating over $[t,T]$, we obtain

$$\begin{aligned}
&E_m(T) + \int_t^T (\|w_t\|^p w_t - \|v_t\|^p v_t, z_t)d\tau\\
=&E_m(t) - \int_t^T m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t)d\tau\\
&+\int_t^T ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t)d\tau - \int_t^T (f(w) - f(v), z_t)d\tau,
\end{aligned} \tag{4.8}$$

where

$$E_m(t) = \frac{1}{2}(\|z_t(t)\|^2 + \|\Delta z(t)\|^2 + m(\|\nabla w\|^2)\|\nabla z(t)\|^2). \tag{4.9}$$

Moreover, integrating (4.8) from 0 to T gives,

$$\begin{aligned}
TE_m(T) =& \int_0^T E_m(t)dt - \int_0^T \int_t^T (\|w_t\|^p w_t - \|v_t\|^p v_t, z_t)d\tau\\
&-\int_0^T dt \int_t^T m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t)d\tau\\
&+\int_0^T dt \int_t^T ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t)d\tau\\
&-\int_0^T dt \int_t^T (f(w) - f(v), z_t)d\tau.
\end{aligned} \tag{4.10}$$

Multiplying (4.5) by $z(t)$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}(z_t, z) - \|z_t\|^2 + \|\Delta z\|^2 + m(\|\nabla w\|^2)\|\nabla z\|^2 + (\|w_t\|^p w_t - \|v_t\|^p v_t, z)\\
=&((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z) - (f(w) - f(v), z),
\end{aligned} \tag{4.11}$$

which implies

$$\begin{aligned}
\int_0^T E_m(t)dt =& \int_0^T \|z_t\|^2 dt + \frac{1}{2}\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, z)dt + \frac{1}{2}(z_t, z)|_0^T\\
=&\frac{1}{2}\int_0^T ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z)dt\\
&-\frac{1}{2}\int_0^T (f(w) - f(v), z)dt.
\end{aligned} \tag{4.12}$$

Combining (4.10) with (4.12), one gets that

$$\begin{aligned}TE_m(T)=&\int_0^T\|z_t\|^2dt+\frac{1}{2}(z_t,z)|_0^T+\frac{1}{2}\int_0^T(\|w_t\|^pw_t-\|v_t\|^pv_t,z)dt\\&-\int_0^T\int_t^T(\|w_t\|^pw_t-\|v_t\|^pv_t,z_t)d\tau\\&+\frac{1}{2}\int_0^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,z)dt\\&-\int_0^Tdt\int_t^Tm'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w,w_t)d\tau\\&+\int_0^Tdt\int_t^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,z_t)d\tau\\&-\frac{1}{2}\int_0^T(f(w)-f(v),z)dt-\int_0^Tdt\int_t^T(f(w)-f(v),z_t)d\tau.\end{aligned}\tag{4.13}$$

Now, we will deal with each term in (4.13) one by one.

According to Lemma 4.1 and $m\in\mathcal{C}^1(\mathbb{R}^+)$, by using estimate(4.4), the mean value theory, and embedding $V\hookrightarrow V_0$, we have

$$m(\|\nabla w\|^2)\le C,\tag{4.14}$$
$$|m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w,w_t)|\le C\|\nabla z\|^2,\tag{4.15}$$
$$|m(\|\nabla w\|^2)-m(\|\nabla v\|^2)|\le C\|\nabla z\|,\tag{4.16}$$
$$|(m(\|\nabla w\|^2-m(\|\nabla v\|^2)))(\Delta v,z)|\le C\|\nabla z\|^2,\tag{4.17}$$
$$|(m(\|\nabla w\|^2)-m(\|\nabla v\|^2))(\Delta v,z_t)|\le C\|\nabla z\|\|z_t\|.\tag{4.18}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\int_0^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,z)dt-\int_0^Tdt\int_t^Tm'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w,w_t)d\tau\\&\quad+\int_0^Tdt\int_t^T((m(\|\nabla w\|^2)-m(\|\nabla v\|^2))\Delta v,z_t)d\tau\\&\le CT\|\nabla z\|.\end{aligned}\tag{4.19}$$

By Lemma 4.1 and the restriction (1.3) on the growth of f in (A_2) along with Sobolev's embedding theorems, for $n\ge5$, let $r=\frac{n}{(n-4)\varrho}$ and $\bar r=\frac{n}{n-(n-4)\varrho}$ are Hölder's conjugate exponents and for $n\le4$, taking r large enough, then we have

$$\begin{aligned}\|f(w)-f(v)\|^2=\int_\Omega|f(w)-f(v)|^2dx=&\int_\Omega|f'(v+\theta_1(w-v))(w-v)|^2dx\\&\le C\int_\Omega(1+|v+\theta_1(w-v)|^\varrho)^2|w-v|^2dx\\&\le C\int_\Omega(1+|w|^{2\varrho}+|v|^{2\varrho})|w-v|^2dx\\&\le C[\int_\Omega(1+|w|^{2\varrho}+|v|^{2\varrho})^rdx]^{\frac{1}{r}}(\int_\Omega|w-v|^{2\bar r}dx)^{\frac{1}{\bar r}}\\&\le C\|w-v\|^2_{2\bar r}\end{aligned}\tag{4.20}$$

where $0<\theta_1<1$. Therefore,

$$-\frac{1}{2}\int_0^T (f(w)-f(v),z)dt \le \frac{T}{2}\sup_{t\in[0,T]}\|f(w)-f(v)\|\sup_{t\in[0,T]}\|z(t)\| \le CT\sup_{t\in[0,T]}\|z(t)\|, \tag{4.21}$$

and

$$-\int_0^T dt\int_t^T (f(w)-f(v),z_t)d\tau \le T^2\sup_{t\in[0,T]}\|f(w)-f(v)\|\sup_{t\in[0,T]}\|z_t(t)\| \le CT^2\sup_{t\in[0,T]}\|f(w)-f(v)\| \le CT^2\|w-v\|_{2\bar{r}}. \tag{4.22}$$

According to Lemma 4.2, one gets that

$$(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)\ge C_p\|w_t-v_t\|^{p+2}, \tag{4.23}$$

taking $g(s)=C_p^{\frac{-2}{p+2}}s^{\frac{2}{p+2}}, s>0$, which is a strictly increasing, concave function, and $g\in\mathcal{C}(\mathbb{R}^+)$ with the property $g(0)=0$ such that

$$\|w_t-v_t\|^2=g(C_p\|w_t-v_t\|^{p+2}) \le g\big((\|u+v\|^p(u+v)-\|u\|^p u,v)\big) =C_p^{\frac{-2}{p+2}}(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)^{\frac{2}{p+2}}, \tag{4.24}$$

which, together with Jensen's inequality, yields that

$$\int_0^T\|z_t\|^2dt\le C_p^{\frac{-2}{p+2}}\int_0^T(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)^{\frac{2}{p+2}}dt \le C_p^{\frac{-2}{p+2}}T\left(\frac{1}{T}\int_0^T(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)dt\right)^{\frac{2}{p+2}} =C_p^{\frac{-2}{p+2}}T^{\frac{p}{p+2}}\left(\int_0^T(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)dt\right)^{\frac{2}{p+2}}. \tag{4.25}$$

From energy relation (4.8) with $t=0$, (4.19), (4.22) and (4.20) with compact embedding theorem, there exists a suitably small constant δ such that

$$\begin{aligned}\int_0^T(\|w_t\|^p w_t-\|v_t\|^p v_t,z_t)dt&=E_m(0)-E_m(T)-\int_0^T m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w,w_t)dt\\&+\int_0^T((m(\|\nabla w\|^2)+m(\|\nabla v\|^2))\Delta v,z_t)dt-\int_0^T(f(w)-f(v),z_t)dt\\&\le E_m(0)-E_m(T)+C\left(\int_0^T\|\nabla z\|^2dt+\int_0^T\|\nabla z\|dt+\int_0^T\|f(w)-f(v)\|dt\right)\\&\le E_m(0)-E_m(T)+TC\sup_{t\in[0,T]}\|w-v\|_{2\bar{r}}\\&\le E_m(0)-E_m(T)+TC\sup_{t\in[0,T]}\|\mathcal{A}^{\frac{1}{2}-\delta}z\|.\end{aligned} \tag{4.26}$$

Then it follows from (4.25) and (4.26) that

$$\int_0^T \|z_t\|^2 dt \leq C_p^{\frac{-2}{p+2}} T^{\frac{p}{p+2}} \left(E_m(0) - E_m(T) + TC \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\delta} z\| \right)^{\frac{2}{p+2}}.$$

Furthermore, in view of Cauchy's inequality along with Sobolev's embedding theorems, there exists a small constant $0<\eta<\frac{1}{2}$ such that

$$\begin{aligned}\int_0^T (\|w_t\|^p w_t - \|v_t\|^p v_t, z) dt &\leq \int_0^T \|z\| \left(\int_\Omega (\|w_t\|^p w_t - \|v_t\|^p v_t)^2 dx \right)^{\frac{1}{2}} dt \\ &\leq C \int_0^T \|z\| \left(\|w_t\|^{2p} \|w_t\|^2 + \|v_t\|^{2p} \|v_t\|^2 \right)^{\frac{1}{2}} dt \\ &\leq TC \sup_{t\in[0,T]} \|z\| \leq TC \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\eta} z(t)\|, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned}\frac{1}{2}(z_t, z)|_0^T &\leq \frac{1}{2}\big(\|z_t(T)\|\|z(T)\| + \|z_t(0)\|\|z(0)\|\big) \\ &\leq C\big(\|z(T)\| + \|z(0)\|\big) \\ &\leq C \sup_{t\in[0,T]} \|z\| \\ &\leq C \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\eta} z(t)\|. \end{aligned} \tag{4.28}$$

Therefore, combining with (4.13) and taking $\tilde{\eta} = \min\{\delta, \eta\}$, one gets that

$$\begin{aligned} E_m(T) \leq & C_T \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\| \\ & + C_p^{\frac{-2}{p+2}} T^{\frac{p}{p+2}} \left(E_m(0) - E_m(T) + TC \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\| \right)^{\frac{2}{p+2}}. \end{aligned} \tag{4.29}$$

Since

$$E_z(t) = \frac{1}{2}(\|z_t(t)\|^2 + \|\Delta z(t)\|^2) = \|S_T y_1 - S_T y_2\|_{\mathcal{H}}^2 \leq E_m(t),$$

applying interpolation theorem and (4.9) it follows that

$$\begin{aligned} m(\|\nabla w\|^2)\|\nabla z(t)\|^2) &\leq C\|z\|^{1-\theta_1} \|\Delta z(t)\|^{\theta_1} \\ &\leq \varepsilon \|\Delta z(t)\|^2 + C_\varepsilon \|z\|^2 \\ &\leq \varepsilon \|\Delta z(t)\|^2 + C_\varepsilon \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\|, \end{aligned}$$

for some constant $C>0$ and $\theta_1 = \frac{1}{2}$. Then by the definition of $E_m(t)$, we can rewrite (4.29) as

$$\begin{aligned} E_z(T) \leq & C_T \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\| \\ & + [C_p T(1+\varepsilon)]^{\frac{-2}{p+2}} \left(E_z(0) - E_z(T) + TC_\varepsilon \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\| \right)^{\frac{2}{p+2}}. \end{aligned} \tag{4.30}$$

Since $z(t)$ is uniformly bounded in $V = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ with $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow\hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\tilde{\eta}})$ $\hookrightarrow\hookrightarrow H = L^2(\Omega)$, by interpolation we have that

$$\|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z\| \le \|z(t)\|^{\theta_2}_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \cdot \|z(t)\|^{1-\theta_2} \quad \le C_R \|z(t)\|^{1-\theta_2}, \quad \theta_2 \in (0,1).$$

Therefore,

$$\begin{aligned} E_z(T) \le & C_T \sup_{t\in[0,T]} \|\mathcal{A}^{\frac{1}{2}-\tilde{\eta}} z(t)\| \\ & + [C_p T(1+\varepsilon)]^{\frac{-2}{p+2}} \left(E_z(0) - E_z(T) + TC_{\mathcal{B},\varrho} \sup_{t\in[0,T]} \|z(t)\|^{\kappa} \right)^{\frac{2}{p+2}}, \end{aligned} \tag{4.31}$$

for some $\kappa \in (0,1]$. Note

$$\rho_T(y_1, y_2) = \sup_{t\in[0,T]} \|z(t)\|^{\kappa},$$

then ρ_T is precompact on the set $\mathcal{B}_0$. In fact, for every bounded set F of $\mathcal{C}([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})) \cap \mathcal{C}^1([0,T];L^2(\Omega))$, that is to say, there exists a constant C such that

$$\|u(t)\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|u_t(t)\| \le C, \quad \forall u(t) \in F(t) = \{u(t) : u \in F\}.$$

Since $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow\hookrightarrow L^2(\Omega)$, we infer that

$$F(t) \text{ is relatively compact in } L^2(\Omega), \quad \forall\, 0 < t < T.$$

On the other hand, $\forall \varepsilon > 0, u \in F$, we have

$$\begin{aligned} \|u(t) - u(t_1)\| = \| \int_{t_1}^{t} u_t(s) ds \| & \le \int_{t_1}^{t} \|u_t(s)\| ds \\ & \le (t-t_1)^{\frac{1}{2}} (\int_{t_1}^{t} \|u_t(s)\| ds)^{\frac{1}{2}} \\ & \le C(t-t_1)^{\frac{1}{2}} \\ & \le C\varepsilon, \end{aligned}$$

$\forall 0 \le t \le t_1 \le T$ satisfying $|t - t_1| \le \varepsilon^2$ i.e. F is uniformly equicontinuous. By the Ascoli Theorem in [17], we obtain the compactness of embedding

$$\mathcal{C}([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})) \cap \mathcal{C}^1([0,T];L^2(\Omega)) \subset \mathcal{C}([0,T];L^2(\Omega)).$$

Therefore, the pseudometric $\rho_T = C_{\mathcal{B},T} \sup_{t\in[0,T]} \|w(t) - v(t)\|^{\kappa}$ is precompact set $\mathcal{B}_0$.

Thus by Theorem 3.2, we deduce from (4.31) that there exists $t_0 > 0$ such that for any bounded $B \subseteq X$,

$$\alpha(S(t)B) \le \left\{ (\alpha(\mathcal{B}_0))^{-p} + \frac{p}{2\left(T^{\frac{2}{p+2}} + 2^{\frac{p}{p+2}+1}[C_p(1+\varepsilon)]^{\frac{-2}{p+2}}\right)^{\frac{p+2}{2}}} \left(t - t_0 - 2T - t_*(B)\right) \right\}^{-\frac{1}{p}} \tag{4.32}$$

holds for all $t \ge t_0 + 2T + t_*(B)$, where $t_*(B)$ satisfies

$$S(t)B \subseteq \mathcal{B}_0, \ \forall t \ge t_*(B).$$

Since $\left\{(\alpha(\mathcal{B}_0))^{-p}+\frac{p}{2\left(T^{\frac{2}{p+2}}+2^{\frac{p}{p+2}+1}[C_p(1+\varepsilon)]^{\frac{-2}{p+2}}\right)^{\frac{p+2}{2}}}\big(t-t_0-2T-t_*(B)\big)\right\}^{-\frac{1}{p}}$ is continuous and increasing with respect to T, where T is an arbitrary positive constant, by the arbitrariness of ε and taking $T\to 0$ in (4.32) we have

$$\alpha(S(t)B)\leq\left\{(\alpha(\mathcal{B}_0))^{-p}+\frac{pkC_p}{2^{p+2}}\big(t-t_0-t_*(B)\big)\right\}^{-\frac{1}{p}},\ \forall\ \ t>t_0+t_*(B).$$

□

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REFERENCES

[1] R.R. Akhmerov, M.I. Kamenskiĭ, A.S. Potapov, A.E. Rodkina, and B.N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Birkhäuser Verlag, Basel, 1992. 2.2, 2.1, 2.1

[2] I. Chueshov and I. Lasiecka, *Long-time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Memoirs of AMS, 195(912), 2008. 2.2, 2.3, 2.3, 2.4

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985. 2.2, 2.1, 2.1

[4] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Masson, Paris, 182, 1994. 1

[5] C. Foias and E. Olson, *Finite fractal dimension and Hölder-Lipshitz parametrization*, Indiana Univ. Math. J., 45:603–616, 1996. 1

[6] C. Foias, G.R. Sell, and R. Temam, *Inertial manifolds for nonlinear evolution equations*, J. Differ. Equ., 73:309–353, 1988. 1

[7] B.R. Hunt and V.Y. Kaloshin, *Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces*, Nonlinearity, 12:1263–1275, 1999. 1

[8] Q. Ma, S. Wang, and C. Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications*, Indiana Univ. Math. J., 51:1541–1559, 2002. 2.2, 2.2

[9] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, in C.M. Dafermos and M. Pokorny (eds.), Handbook of Differential Equations: Evolutionary Equations, Elsevier/North-Holland, Amsterdam, 4:103–200, 2008. 1

[10] M. Nakao, *A difference inequality and its application to nonlinear evolution equations*, J. Math. Soc. Japan, 30:747–762, 1978. 1, 1, 3.1

[11] M. Nakao, *Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state*, Mem. Fac. Sci. Kyushu Univ. Ser. A, 30:257–265, 1976. 1

[12] M. Nakao, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term*, J. Math. Anal. Appl., 58(2):336–343, 1977. 1

[13] M. Nakao, *Decay of solutions of some nonlinear evolution equations*, J. Math. Anal. Appl., 60:542–549, 1977. 1

[14] V. Pata and S. Zelik, *Smooth attractors for strongly damped wave equations*, Nonlinearity, 19:1495–1506, 2006. 1

[15] J.C. Robinson, *Infinite-dimensional Dynamical Systems*, Cambridge University Press, Cambridge, 2001. 2.2, 2.3, 2.4

[16] M.A.J. Silva, V. Narciso, and A. Vicente, *On a beam model related to flight structures with nonlocal energy damping*, Discrete Contin. Dyn. Syst. Ser. B, 24:3281–3298, 2019. 1

[17] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146:65–96, 1986. 2.2, 2.2, 4

[18] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Springer-Verlag, New York, 1997. 2.2, 2.3, 2.4

[19] S. Zelik, *The attractor for a nonlinear hyperbolic equation in the unbounded domain*, Discrete Contin. Dyn. Syst., 7:593–641, 2001. 1

[20] S. Zelik, *The attractor for a nonlinear reaction-diffusion system in the unbounded domain and Kolmogorov's* ϵ*-entropy*, Math. Nachr., 232:129–179, 2001. 1

[21] S. Zelik, *Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity*, Commun. Pure Appl. Math., 56:584–637, 2003. 1

[22] J. Zhang, P. Kloeden, M. Yang, and C. Zhong, *Global exponential $\kappa-$dissipative semigroups and exponential attraction*, Discrete Contin. Dyn. Syst., 37:3487–3502, 2017. 1, 3

[23] J. Zhang, Y. Wang, and C. Zhong, *Robustness of exponentially κ-dissipative dynamical systems with perturbations*, Discrete Contin. Dyn. Syst., Ser. B, 22:3875–3890, 2017. 3

[24] C. Zhao, C. Zhao, and C. Zhong, *The global attractor for a class of extensible beams with nonlocal weak damping*, Discrete Contin. Dyn. Syst. Ser. B, 25:935–955, 2020. 4

[25] C. Zhao, C. Zhong, and C. Zhao, *Estimate of the attractive velocity of attractors for some dynamical systems (in Chinese)*, Sci. Sin. Math., 52:881–900, 2022. 1, 3

[26] C. Zhong and W. Niu, *On the Z_2 index of the global attractor for a class of p-Laplacian equations*, Nonlinear Anal., 73:3698–3704, 2010. 1