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PULLBACK EXPONENTIAL ATTRACTORS FOR THE THREE DIMENSIONAL NON-AUTONOMOUS PRIMITIVE EQUATIONS OF LARGE SCALE OCEAN AND ATMOSPHERE DYNAMICS*

BO YOU†

Abstract. The main objective of this paper is to study the existence of pullback exponential attractors for the three-dimensional non-autonomous primitive equations of large-scale ocean and atmosphere dynamics. Due to the shortage of the proof of the uniqueness of weak solutions, it is very difficult to define a solution process such that we cannot obtain the existence of pullback exponential attractors by the standard theory of pullback exponential attractor established in [A.N. Carvalho and S. Sonner, Commun. Pure Appl. Anal., 12(6):3047–3071, 2013], [R. Czaja and M. Efendiev, J. Math. Anal. Appl., 381(2):748–765, 2011], [M. Efendiev, S. Zelik, and A. Miranville, Proc. Roy. Soc. Edinb. Sect. A, 135(4):703–730, 2005], [J.A. Langa, A. Miranville, and J. Real, Discrete Contin. Dyn. Syst., 26(4):1329–1357, 2010]. Inspired by the idea of the method of ℓ-trajectories, we will prove the existence of pullback exponential attractors by the abstract results established in [B. You, Math. Meth. Appl. Sci., 44(13):10361–10386, 2021].

Keywords. Pullback exponential attractors; Primitive equations; Aubin-Lions compactness lemma; The method of ℓ-trajectories; Trajectory space.

AMS subject classifications. 35B41; 35Q86; 37C60; 37L25; 37N10.

1. Introduction

In this paper, we mainly consider the existence of pullback exponential attractors for the following three-dimensional non-autonomous primitive equations of large-scale ocean and atmosphere dynamics (see [34])

$$\begin{cases} \frac{\partial v}{\partial t}+(v\cdot\nabla)v+w\frac{\partial v}{\partial z}+\nabla p+\frac{1}{Ro}fv^{\perp}+L_1v=0,\\ \frac{\partial p}{\partial z}+T=0,\\ \nabla\cdot v+\frac{\partial w}{\partial z}=0,\\ \frac{\partial T}{\partial t}+v\cdot\nabla T+w\frac{\partial T}{\partial z}+L_2T=Q \end{cases}\tag{1.1}$$

in the domain

$$\Omega=M\times(-h,0),$$

where $M\subset\mathbb{R}^2$ is a bounded domain with smooth boundary ∂M. The horizontal velocity field $v=(v_1,v_2)$, the three-dimensional velocity field $(v_1,\ v_2,\ w)$, the temperature T and the pressure p are unknown. The vector $v^{\perp}=(-v_2,v_1)$, $f=f_0+\beta y$ is the Coriolis parameter, Ro is the Rossby number which measures the significant influence of the rotation of the earth to the dynamical behavior of the ocean, $Q(x,y,z,t)$ is a given heat source. The viscosity operator L_1 and the heat diffusion operator L_2 are given by

$$L_1=-\frac{1}{Re_1}\Delta-\frac{1}{Re_2}\frac{\partial^2}{\partial z^2},\ L_2=-\frac{1}{Rt_1}\Delta-\frac{1}{Rt_2}\frac{\partial^2}{\partial z^2},$$

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†School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, P.R. China (youb2013@xjtu.edu.cn).

where Re_1, Re_2 are positive constants representing the horizontal and vertical Reynolds numbers, respectively, Rt_1, Rt_2 are positive constants which stand for the horizontal and vertical eddy diffusivity, respectively. For the sake of simplicity, let $\nabla=(\partial_x,\partial_y)$ be the horizontal gradient operator and let $\Delta=\partial_x^2+\partial_y^2$ be the horizontal Laplacian operator.

Let Γ_u, Γ_b and Γ_l be the upper, bottom and the lateral boundaries of Ω, respectively, which are given by

$$\Gamma_u=\{(x,y,z)\in\overline{\Omega}:z=0\},\ \Gamma_b=\{(x,y,z)\in\overline{\Omega}:z=-h\},$$
$$\Gamma_l=\{(x,y,z)\in\overline{\Omega}:(x,y)\in\partial M,-h\le z\le 0\}.$$

Problem (1.1) is subject to the boundary conditions:

$$\begin{cases}\frac{\partial v}{\partial z}|_{\Gamma_u}=0, w|_{\Gamma_u}=0, (\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T)|_{\Gamma_u}=0,\\ \frac{\partial v}{\partial z}|_{\Gamma_b}=0, w|_{\Gamma_b}=0, \frac{\partial T}{\partial z}|_{\Gamma_b}=0,\\ v\cdot\vec{n}|_{\Gamma_l}=0, \frac{\partial v}{\partial \vec{n}}\times\vec{n}|_{\Gamma_l}=0, \frac{\partial T}{\partial \vec{n}}|_{\Gamma_l}=0,\end{cases}\tag{1.2}$$

where $\vec{n}$ is the unit outward normal vector to Γ_l, α is a positive constant related to the turbulent heating on the surface of the ocean.

In addition, we supply system (1.1)-(1.2) with the initial conditions

$$v(x,y,z,\tau)=v_\tau(x,y,z),\ T(x,y,z,\tau)=T_\tau(x,y,z).\tag{1.3}$$

Large-scale dynamics of ocean and atmosphere is governed by the primitive equations which are derived from the Navier-Stokes equations with rotation coupled to thermodynamics and salinity diffusion-transport equations by taking the buoyancy forces and stratification effects into account under the Boussinesq approximation. Moreover, due to the shallowness of the oceans and the atmosphere, i.e., the depth of the fluid layer is very small in comparison to the radius of the earth, the vertical large-scale motion in the oceans and the atmosphere is much smaller than the horizontal one, which in turn leads to modeling the vertical motion by the hydrostatic balance. As a result, one can obtain the system (1.1)-(1.3) which is known as the primitive equations for ocean and atmosphere dynamics (see [34]). We observe that one has to add the diffusion-transport equation of the salinity to the system (1.1)-(1.3) in the case of ocean dynamics, but we omit it here in order to simplify our mathematical presentation. However, we emphasize that our results are equally valid when the salinity effects are taken into account.

In the past several decades, the well-posedness and the long-time behavior of solutions for the primitive equations of the coupled atmosphere-ocean have been extensively studied from the theoretical point of view (see [1, 12, 17–23, 26–28, 31–33, 38]). In particular, in [28], the authors began to study the well-posedness and long-time behavior of solutions for such a system from the mathematical theoretical point of view for the first time, they established the global existence of weak solutions, the existence of (local in time) strong solutions and the finite fractal dimension of its global attractor, but the issue about the uniqueness of weak solutions and the existence of (global in time) strong solutions remains open. Recently, the existence and uniqueness of (global in time) strong solutions for this system have been well-established in [1], but the uniqueness of weak solutions remains unresolved. Since then, many authors started considering the well-posedness and long-time behavior of solutions for the primitive equations or some similar counterparts. In [18], the authors proved the existence of weak solutions as well as trajectory attractors for the moist atmospheric equations in geophysics. The long-time dynamics of the primitive equations of large-scale atmosphere was considered and the existence of a weakly compact global attractor $\mathcal{A}$ attracting all the trajectories was obtained in [20]. In [21], the author proved the existence of a global attractor in V for the primitive equations of large-scale atmosphere and ocean dynamics by using the Aubin-Lions compactness lemma under the assumption $Q \in L^2(\Omega)$. The regularity of the global attractor for the 3D viscous primitive equations has been established in [38]. In [26], the authors have provided the upper bound of the fractal dimension of the global attractor for the primitive equations of atmospheric circulation and have given its physical interpretation. The existence of finite dimensional global attractors for the 3D viscous primitive equations by using the squeezing property was proved in [22, 23]. To the best of our knowledge, there are no results related to the existence of pullback exponential attractors for the three-dimensional non-autonomous primitive equations of large-scale ocean and atmosphere dynamics.

The study of the long-time behavior of infinite dimensional dynamical systems or semigroups generated by autonomous partial differential equations can be usually reduced to the description of global attractor (see [35,37]), which may attract trajectories slowly and be sensitive to small perturbations. The two drawbacks of global attractor obviously lead to essential difficulties in numerical simulations of global attractor and even make the global attractor unobservable in some sense. To overcome these drawbacks, the notion of an exponential attractor was introduced in [13]. Until now, there are main three classical methods of constructing the exponential attractor for autonomous dissipative equations by the squeezing property ([13])/the smoothing property ([14]) of the difference of two solutions or the quasi-stable methods [10]. Moreover, to ensure the finiteness of the fractal dimension of the exponential attractor in these three ways, there is an additional requirement on the Hölder continuity in time of the semigroup, which is, in general, very difficult to prove, in particular, when the solutions lack regularity.

Non-autonomous equations appear in many applications of the natural sciences, so they are of great importance and interest. In recent years, more attention was paid to the processes generated by the non-autonomous differential equations and their long-time behaviors (see [2, 4, 5, 7–9, 15, 36]). The first attempt was to extend the notion of global attractor to the non-autonomous case, leading to the concept of the so-called uniform attractor (see [9]). It is remarkable that the conditions ensuring the existence of the uniform attractor are parallel to those for the autonomous case. However, one disadvantage of the uniform attractor is that it need not to be “invariant”, unlike the global attractor for autonomous systems. Moreover, it is well known that the trajectories may be unbounded for many non-autonomous systems when the time tends to infinity, and there does not exist a uniform attractor for such systems. In order to overcome this drawback, a new counterpart, called pullback attractor, has been introduced for the non-autonomous case. The theory of pullback attractors has been developed for both non-autonomous and random dynamical systems, and it has been also shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems (see [3]).

Similar to the autonomous case, many authors have also proposed the notion of pullback exponential attractor. In particular, the authors in [16] have first extended the way of construction of exponential attractors for discrete semigroups in [14] to non-autonomous problems by using the concept of forwards attractor and developed an explicit algorithm for discrete evolution processes by the smoothing property of the evolution process. Moreover, they also constructed an exponential attractor of the time continuous process generated by non-autonomous reaction-diffusion systems. Later, this construction was modified in the pullback sense, and the algorithm was also extended to time continuous evolution processes in [11, 25] based on the existence of a fixed bounded pullback absorbing set, which leads to the boundedness of the section of exponential pullback attractor in the past, but it may be unbounded in the future. Recently, the authors in [6] proved the existence of pullback exponential attractors for an asymptotically compact process under significantly weak hypothesis that the process lacks the strong regularity property in time, whose sections are not necessarily uniformly bounded in the past. Moreover, they obtained better estimates for the fractal dimension of the sections of pullback attractor based on the existence of a family of time-dependent absorbing sets. In [39], the author developed the abstract framework of pullback exponential attractor based on the method of ℓ-trajectories and applied it to the three-dimensional planetary geostrophic equations.

In this paper, we are mainly concerned with the existence of pullback exponential attractors for the three-dimensional non-autonomous primitive equations of large-scale ocean and atmosphere dynamics. Due to the shortage of the proof of the uniqueness of weak solutions, it is very difficult to define a solution process such that we cannot consider the existence of pullback exponential attractor by the classical theory of pullback exponential attractor established in [6, 11, 16, 25]. In [30], the authors have set up the theoretical framework of a finite dimensional global attractor as well as an exponential attractor for the autonomous evolutionary equations by the method of ℓ-trajectories. Inspired by the idea of the method of ℓ-trajectories, we have generalized the theoretical framework of autonomous case to non-autonomous case in [39]. In this paper, we first construct the pullback exponential attractors in an auxiliary phase space of the trajectories of length ℓ by the smoothing property of the difference of two solution trajectories. By defining the Lipschitz continuous projection operator from the trajectory phase space into the original phase space, we establish the existence of pullback exponential attractors for this system in the original phase space and also provide a method of constructing pullback exponential attractor.

Throughout this paper, let X be a Banach space endowed with the norm $\|\cdot\|_X$, let $\|u\|_p$ be the $L^p(\Omega)$-norm of u, denote by $\mathbb{R}_\tau=[\tau,+\infty)$ and let C be positive constants which may be different from line to line.

2. Preliminaries

2.1. Functional spaces and some lemmas. To study problem (1.1)-(1.3), we first introduce some notations of function space. Define

$$\begin{aligned}\mathcal{V}_1=\Bigg\{&v\in(C^\infty(\bar\Omega))^2:\frac{\partial v}{\partial z}\bigg|_{\Gamma_u}=0,\ \frac{\partial v}{\partial z}\bigg|_{\Gamma_b}=0, v\cdot\vec{n}|_{\Gamma_l}=0,\ \frac{\partial v}{\partial\vec{n}}\times\vec{n}\bigg|_{\Gamma_l}=0,\\ &\int_{-h}^0\nabla\cdot v(x,y,r)\,dr=0\Bigg\},\\ \mathcal{V}_2=\Bigg\{&T\in C^\infty(\bar\Omega):\left(\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T\right)\bigg|_{\Gamma_u}=0,\ \frac{\partial T}{\partial z}\bigg|_{\Gamma_b}=0,\ \frac{\partial T}{\partial\vec{n}}\bigg|_{\Gamma_l}=0\Bigg\}.\end{aligned}$$

Denote the closure of $\mathcal{V}_1$, $\mathcal{V}_2$ by V_1, V_2, respectively, with respect to the following norms defined as follows

$$\begin{aligned}\|v\|&=\left(\frac{1}{Re_1}\int_\Omega|\nabla v(x,y,z)|^2\,dxdydz+\frac{1}{Re_2}\int_\Omega|\partial_z v(x,y,z)|^2\,dxdydz\right)^{\frac12},\\ \|T\|&=\left(\frac{1}{Rt_1}\int_\Omega|\nabla T(x,y,z)|^2\,dxdydz+\frac{1}{Rt_2}\int_\Omega|\partial_z T(x,y,z)|^2\,dxdydz\right.\end{aligned}$$

$$+\alpha\int_M |T(x,y,0)|^2\,dxdy\Big)^{\frac{1}{2}}$$

for any $v\in\mathcal{V}_1$, $T\in\mathcal{V}_2$, and let H_1 be the closure of $\mathcal{V}_1$ with respect to the $(L^2(\Omega))^2$-norm, $V=V_1\times V_2$, $H=H_1\times L^2(\Omega)$. let $\|(v,T)\|_2^2=\|v\|_2^2+\|T\|_2^2$ for any $(v,T)\in H$ and $\|(v,T)\|^2=\|v\|^2+\|T\|^2$ for any $(v,T)\in V$.

Next, we recall some results used to prove the existence of pullback exponential attractor for problem (1.1)-(1.3).

LEMMA 2.1 ([1]). *There exists a positive constant K_1, such that*

$$\frac{1}{K_1}\|T\|^2\le\|T\|^2_{H^1(\Omega)}\le K_1\|T\|^2$$

for any $T\in V_2$. Moreover, we have

$$\|T\|_2^2\le K_2\|T\|^2$$

for any $T\in V_2$, where

$$K_2=\max\{\frac{2h}{\alpha},2Rt_2h^2\}.$$

LEMMA 2.2 ([9]). *Assume that $p_1\in(1,\infty]$, $p_2\in[1,\infty)$. Let X, X_0, X_1 be Banach spaces such that $X_0\subset\subset X\subset X_1$. Then*

$$Y=\{u\in L^{p_1}(0,\ell;X_0):u'\in L^{p_2}(0,\ell;X_1)\}\subset\subset L^{p_1}(0,\ell;X),$$

where ℓ is any fixed positive constant.

DEFINITION 2.1 ([29]). *A process $\{U(t,\tau)\}_{t\ge\tau}$ defined on a Banach space X is said to be τ-continuous, if for every $u_0\in X$ and every $t\in\mathbb{R}$, the X-valued function*

$$\tau\to U(t,\tau)u_0$$

is continuous and bounded on $(-\infty,t]$.

DEFINITION 2.2 ([35,37]). *Let H be a separable real Hilbert space. For any non-empty compact subset $K\subset H$, the fractal dimension $d_F(K)$ of K is defined by*

$$d_F(K)=\limsup_{\epsilon\to0^+}\frac{\log(N_\epsilon(K))}{\log(\frac{1}{\epsilon})},$$

where $N_\epsilon(K)$ denotes the minimum number of open balls in H with radii $\epsilon>0$ that are necessary to cover K.

LEMMA 2.3 ([30]). *Let X, Y be two metric spaces and the function $f:X\to Y$ is α-Hölder continuous on the subset $A\subset X$. Then*

$$d_F(f(A))\le\frac{1}{\alpha}d_F(A).$$

In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

DEFINITION 2.3 ([6, 11, 16]). *Let $\{U(t,s)\}_{t\ge s}$ be an evolution process defined on a metric space X. We call the family $\mathcal{M}=\{\mathcal{M}(t):t\in\mathbb{R}\}$ a pullback exponential attractor for the evolution process $\{U(t,s)\}_{t\ge s}$ in X, if*

(i) *The subsets* $\mathcal{M}(t)\subset X$ *are non-empty and compact in* X *for all* $t\in\mathbb{R}$.

(ii) *The family is positively semi-invariant, that is*

$$U(t,s)\mathcal{M}(s)\subset\mathcal{M}(t),\ \ \forall\, t\geq s.$$

(iii) *The fractal dimension of the section* $\mathcal{M}(t)$ *in* X *is uniformly bounded for all* $t\in\mathbb{R}$.

(iv) *The family* $\{\mathcal{M}(t):t\in\mathbb{R}\}$ *exponentially pullback attracts bounded subsets of* X, *i.e., there exists a positive constant* $\omega>0$, *such that for every bounded subset* $B\subset X$ *and* $t\in\mathbb{R}$,

$$\lim_{s\to+\infty} e^{\omega s} dist(U(t,t-s)B,\mathcal{M}(t))=0.$$

2.2. New formulation. We can reformulate problem (1.1)-(1.3) by integrating the second equation as well as the third equation of (1.1) with respect to z and combining the boundary conditions (1.2) as follows just like in [1]:

$$\begin{cases}\frac{\partial v}{\partial t}+(v\cdot\nabla)v-(\int_{-h}^{z}\nabla\cdot v(x,y,r,t)\,dr)\frac{\partial v}{\partial z}+\nabla p_s(x,y,t)+\frac{1}{Ro}fv^{\perp}+L_1v\\ \quad=\int_0^z\nabla T(x,y,r,t)\,dr,\\ \int_{-h}^{0}\nabla\cdot v(x,y,r,t)\,dr=0,\\ \frac{\partial T}{\partial t}+v\cdot\nabla T-(\int_{-h}^{z}\nabla\cdot v(x,y,r,t)\,dr)\frac{\partial T}{\partial z}+L_2T=Q,\\ \frac{\partial v}{\partial z}|_{\Gamma_u}=0,\frac{\partial v}{\partial z}|_{\Gamma_b}=0,v\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial v}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0,\\ (\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T)|_{\Gamma_u}=0,\frac{\partial T}{\partial z}|_{\Gamma_b}=0,\frac{\partial T}{\partial\vec{n}}|_{\Gamma_l}=0,\\ v(x,y,z,\tau)=v_\tau(x,y,z),\ T(x,y,z,\tau)=T_\tau(x,y,z).\end{cases}\tag{2.1}$$

Define

$$\bar{v}(x,y)=\frac{1}{h}\int_{-h}^{0}v(x,y,r)\,dr$$

and

$$\tilde{v}=v-\bar{v},$$

then it is clear that $\bar{v}$ and $\tilde{v}$ satisfy the following problem:

$$\begin{cases}\frac{\partial\bar{v}}{\partial t}+(\bar{v}\cdot\nabla)\bar{v}+\overline{(\tilde{v}\cdot\nabla)\tilde{v}+(\nabla\cdot\tilde{v})\tilde{v}}+\nabla p_s(x,y,t)-\frac{1}{Re_1}\Delta\bar{v}+\frac{1}{Ro}f\bar{v}^{\perp}\\ \quad=\overline{\int_0^z\nabla T(x,y,r,t)\,dr},\\ \nabla\cdot\bar{v}=0,\bar{v}\cdot\vec{n}|_{\Gamma_l}=0,\ \frac{\partial\bar{v}}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0\end{cases}\tag{2.2}$$

and

$$\begin{cases}\frac{\partial\tilde{v}}{\partial t}+(\tilde{v}\cdot\nabla)\tilde{v}-(\int_{-h}^{z}\nabla\cdot\tilde{v}(x,y,r,t)\,dr)\frac{\partial\tilde{v}}{\partial z}+(\tilde{v}\cdot\nabla)\bar{v}+(\bar{v}\cdot\nabla)\tilde{v}-\int_0^z\nabla T(x,y,r,t)\,dr\\ \quad+\frac{1}{Ro}f\tilde{v}^{\perp}+L_1\tilde{v}-\overline{(\tilde{v}\cdot\nabla)\tilde{v}+(\nabla\cdot\tilde{v})\tilde{v}}+\overline{\int_0^z\nabla T(x,y,r,t)\,dr}=0,\\ \frac{\partial\tilde{v}}{\partial z}|_{\Gamma_u}=0,\frac{\partial\tilde{v}}{\partial z}|_{\Gamma_b}=0,\tilde{v}\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial\tilde{v}}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0.\end{cases}\tag{2.3}$$

3. The existence of a pullback attractor

3.1. The well-posedness. We start with the following general existence and uniqueness of solutions obtained by the standard Faedo-Galerkin methods ([1,21,24,28, 37]). Here, we only state it as follows.

THEOREM 3.1 ([1,21]). *Suppose that* $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$. *Then for any initial data* $(v_\tau,T_\tau)\in H$, *there exists at least one weak solution* $(v(t),T(t))\in \mathcal{C}(\mathbb{R}_\tau;H_w)\cap L^2_{loc}(\mathbb{R}_\tau;V)$ *of problem* (2.1). *Furthermore, if* $(v_\tau,T_\tau)\in V$, *there exists a unique strong solution* $(v(t),T(t))\in C(\mathbb{R}_\tau;V)$ *of problem* (2.1), *which depends continuously on the initial data with respect to the topology of* H *and* V.

By Theorem 3.1, we can define a family of continuous processes $\{U(t,\tau):-\infty<\tau\le t<\infty\}$ on V by

$$U(t,\tau)(v_\tau,T_\tau)=(v(t),T(t)):=(v(t,\tau;(v_\tau,T_\tau)),T(t,\tau;(v_\tau,T_\tau)))$$

for any $t\ge\tau$, where $(v(t),T(t))$ is the strong solution of problem (2.1) with initial data $(v(\tau),T(\tau))=(v_\tau,T_\tau)\in V$. That is, a family of mappings $U(\cdot,\tau):\mathbb{R}_\tau\times V\to V$ satisfies

$$\begin{aligned}&U(\tau,\tau)=id\ (\text{identity}),\\&U(t,\tau)=U(t,r)U(r,\tau)\end{aligned}$$

for all $t\ge r\ge\tau$.

Combining Theorem 3.1 with the similar procedure of the proof of absorbing set in [21], we can easily obtain the following conclusions.

COROLLARY 3.1. *Assume that* $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$. *Then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *associated with problem* (2.1) *is* τ*-continuous.*

COROLLARY 3.2. *Assume that* $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$, $(v^m_\tau,T^m_\tau)\rightharpoonup(v_\tau,T_\tau)$ *in* H, $(v_m(t),T_m(t))$ *is a sequence of weak solutions for problem* (2.1) *such that* $(v_m(\tau),T_m(\tau))=(v^m_\tau,T^m_\tau)$. *For any* $T>\tau$, *if there exists a subsequence converging (∗-) weakly in spaces* $L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\cap H^1(\tau,T;(V\cap H^3(\Omega))')$ *to a certain function* $(v(t),T(t))$. *Then* $(v(t),T(t))$ *is a weak solution of problem* (2.1) *on* $[\tau,T]$ *with* $(v(\tau),T(\tau))=(v_\tau,T_\tau)$.

3.2. The existence of a pullback attractor in X_ℓ. In this subsection, we will consider the existence of a pullback attractor for problem (2.1) by using the method of ℓ-trajectories. From Theorem 3.1, we know that for any $(v_\tau,T_\tau)\in H$, there exists at least one weak solution $(v(t),T(t))\in\mathcal{C}(\mathbb{R}_\tau;H_w)\cap L^2_{loc}(\mathbb{R}_\tau;V)$ of problem (2.1), which implies that many trajectories may start from the same initial data $(v_\tau,T_\tau)\in H$. However, for any $t>\tau$, there exists some $t_0\in(\tau,t)$ such that $(v(t_0),T(t_0))\in V$ and there exists a unique strong solution of problem (2.1) starting from $(v(t_0),T(t_0))$. For the sake of simplicity, denote by $[\chi^\beta(s,\tau;(v_\tau,T_\tau))]_{s\in[\tau,\tau+\ell]}$, for short $\chi^\beta(s,\tau;(v_\tau,T_\tau))$ $(\beta\in\Gamma_{(v_\tau,T_\tau)})$, where $\Gamma_{(v_\tau,T_\tau)}$ is the set of indices marking trajectories starting from (v_τ,T_τ). In the following, we first give the mathematical framework of pullback attractor.

DEFINITION 3.1. *Let* $\ell\in(0,1]$ *be a fixed positive constant. Define*

$$X_\ell=\bigcup_{\tau\in\mathbb{R}}\ \bigcup_{(v_\tau,T_\tau)\in H}\ \bigcup_{\beta\in\Gamma_{(v_\tau,T_\tau)}}\chi^\beta(s,\tau;(v_\tau,T_\tau))$$

equipped with the topology of $L^2(0,\ell;H)$.

Since $\chi^{\beta}(s,\tau;(v_\tau,T_\tau))\in\mathcal{C}([\tau,\tau+\ell];H_w)$ for any $\tau\in\mathbb{R}$, $(v_\tau,T_\tau)\in H$ and $\beta\in\Gamma_{(v_\tau,T_\tau)}$, it makes sense to talk about the point values of each trajectory. However, it is not clear whether X_ℓ is closed in $L^2(0,\ell;H)$ such that X_ℓ in general is not a complete metric space. In what follows, we first give the definition of some operators.

For any $t\in[0,1]$, we define the mapping $e_t:X_\ell\to H$ by

$$e_t(\chi(s,\tau;(v_\tau,T_\tau)))=\chi(t\ell+\tau,\tau;(v_\tau,T_\tau))$$

for any $\chi(s,\tau;(v_\tau,T_\tau))\in X_\ell$.

For any $t\geq\tau$, the operators $L(t,\tau):X_\ell\to X_\ell$ are given by the relation

$$\begin{aligned}&L(t,\tau)\chi(s,\tau;(v_\tau,T_\tau))=(v,T)(t+s-\tau,\tau;(v_\tau,T_\tau))\\ =&U(t+s-\tau,\ell+\tau)e_1(\chi(s,\tau;(v_\tau,T_\tau)))=\chi(t+s-\tau,\tau;(v_\tau,T_\tau)),\ s\in[\tau,\tau+\ell]\end{aligned}$$

for any $\chi(s,\tau;(v_\tau,T_\tau))\in X_\ell$, where $(v,T)(s,\tau;(v_\tau,T_\tau))$ is the unique solution of problem (2.1) on $[\tau,\ell+t]$ such that $(v,T)|_{[\tau,\tau+\ell]}=\chi(s,\tau;(v_\tau,T_\tau))$, we can easily prove the operators $\{L(t,\tau)\}_{t\geq\tau}$ is a process on X_ℓ.

In what follows, let $\mathcal{D}_\ell$ be the family of all nonempty bounded subsets of X_ℓ and let $\mathcal{D}$ be the family of all nonempty bounded subsets of H. In the following, we will perform some a priori estimates of solutions for problem (2.1) to prove the existence of pullback attractors for problem (2.1).

THEOREM 3.2. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

Then there exists a positive constant ρ_1 satisfying for any $B_\ell\in\mathcal{D}_\ell$, there exists a time $\tau_1=\tau_1(B_\ell,t)\leq t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau,T_\tau))\in B_\ell$, we have

$$\|(v(t),T(t))\|_2^2+\int_0^\ell\|(v,T)(t+\zeta)\|_2^2\,d\zeta\leq\rho_1$$

for any $\tau\leq t-\tau_1$.

Proof. Taking the inner product of the third equation of problem (2.1) with T in $L^2(\Omega)$ and combining Lemma 2.1 with Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|T(t)\|_2^2+\|T(t)\|^2=&\int_\Omega Q(x,y,z,t)T(x,y,z,t)\,dxdydz\\ \leq&\|Q(t)\|_2\|T(t)\|_2\\ \leq&\frac{1}{2}\|T(t)\|^2+\frac{1}{2}K_2\|Q(t)\|_2^2,\end{aligned}$$

which implies that

$$\frac{d}{dt}\|T(t)\|_2^2+\frac{1}{K_2}\|T(t)\|_2^2\leq K_2\|Q(t)\|_2^2$$

and

$$\frac{d}{dt}\|T(t)\|_2^2+\|T(t)\|^2\leq K_2\|Q(t)\|_2^2. \tag{3.1}$$

It follows from the classical Gronwall inequality that

$$\|T(t)\|_2^2 \le \frac{1}{\ell}\int_0^\ell \|T(\zeta+\tau)\|_2^2 e^{\frac{\zeta+\tau-t}{K_2}}\,d\zeta + \frac{K_2}{\ell}\int_0^\ell \int_{\tau+\zeta}^t e^{\frac{s-t}{K_2}}\|Q(s)\|_2^2\,ds\,d\zeta$$
$$\le \frac{e^{\frac{\ell}{K_2}}}{\ell} e^{\frac{\tau-t}{K_2}} \int_0^\ell \|T(\zeta+\tau)\|_2^2\,d\zeta + K_2(1+K_2)\sup_{r\in\mathbb{R}}\int_{r-1}^r \|Q(s)\|_2^2\,ds. \tag{3.2}$$

Thanks to

$$\frac{d}{ds}(\|T(s)\|_2^2 e^{\frac{s}{K_2}}) \le K_2\|Q(s)\|_2^2 e^{\frac{s}{K_2}} \tag{3.3}$$

for any $\zeta\in(0,\ell)$, integrating inequality (3.3) with respect to s from $\tau+\zeta$ to $t+\zeta$ and integrating the resulting inequality over $(0,\ell)$ with respect to ζ, we obtain

$$\int_0^\ell \|T(t+\zeta)\|_2^2\,d\zeta \le e^{\frac{\tau-t}{K_2}}\int_0^\ell \|T(\tau+\zeta)\|_2^2\,d\zeta + K_2\int_0^\ell\int_{\tau+\zeta}^{t+\zeta} e^{\frac{s-t-\zeta}{K_2}}\|Q(s)\|_2^2\,ds\,d\zeta$$
$$\le e^{\frac{\tau-t}{K_2}}\int_0^\ell \|T(\tau+\zeta)\|_2^2\,d\zeta + K_2(1+K_2)\ell\sup_{r\in\mathbb{R}}\int_{r-1}^r \|Q(s)\|_2^2\,ds. \tag{3.4}$$

Multiplying the first equation of problem (2.1) by v and integrating over Ω, we find

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \|v(t)\|^2 = \int_\Omega \int_0^z \nabla T(x,y,s,t)\,ds\cdot v(x,y,z,t)\,dx\,dy\,dz$$
$$\le h\|T(t)\|_2\|\nabla v(t)\|_2.$$

Let $\lambda = \sup\{\mu < \frac{1}{K_2} : \mu\|v\|_2^2 \le \|v\|^2, \forall v\in V_1\}$, we infer from Young's inequality and Poincáre's inequality that

$$\frac{d}{dt}\|v(t)\|_2^2 + \|v(t)\|^2 \le h^2 Re_1\|T(t)\|_2^2 \tag{3.5}$$

and

$$\frac{d}{dt}\|v(t)\|_2^2 + \lambda\|v(t)\|_2^2 \le h^2 Re_1\|T(t)\|_2^2. \tag{3.6}$$

We infer from the classical Gronwall inequality and inequality (3.2) that

$$\|v(t)\|_2^2 \le \|v(\tau+\zeta)\|_2^2 e^{\lambda(\tau+\zeta-t)} + h^2 Re_1\int_{\tau+\zeta}^t \|T(s)\|_2^2 e^{\lambda(s-t)}\,ds$$
$$\le \|v(\tau+\zeta)\|_2^2 e^{\lambda(\tau+\zeta-t)} + \frac{h^2 Re_1 K_2 e^{\frac{\ell}{K_2}}}{\ell(1-K_2\lambda)} e^{\lambda(\tau-t)}\int_0^\ell \|T(\zeta+\tau)\|_2^2\,d\zeta$$
$$+ \frac{h^2 Re_1}{\lambda} K_2(1+K_2)\sup_{r\in\mathbb{R}}\int_{r-1}^r \|Q(\zeta)\|_2^2\,d\zeta,$$

which implies that

$$\|v(t)\|_2^2 \le \frac{e^{\lambda\ell}}{\ell} e^{\lambda(\tau-t)}\int_0^\ell \|v(\tau+\zeta)\|_2^2\,d\zeta + \frac{h^2 Re_1 K_2 e^{\frac{\ell}{K_2}}}{\ell(1-K_2\lambda)} e^{\lambda(\tau-t)}\int_0^\ell \|T(\zeta+\tau)\|_2^2\,d\zeta$$
$$+ \frac{C}{\lambda} K_2(1+K_2)\sup_{r\in\mathbb{R}}\int_{r-1}^r \|Q(\zeta)\|_2^2\,d\zeta. \tag{3.7}$$

Thanks to

$$\frac{d}{ds}(\|v(s)\|_2^2 e^{\lambda s}) \leq h^2 Re_1 \|T(s)\|_2^2 e^{\lambda s}. \tag{3.8}$$

Integrating inequality (3.8) with respect to s between $\tau+\zeta$ and $t+\zeta$ and integrating the resulting inequality with respect to ζ over $(0,\ell)$, using inequality (3.2), we conclude

$$\begin{aligned}
\int_0^\ell \|v(t+\zeta)\|_2^2 d\zeta \leq & e^{\lambda(\tau-t)} \int_0^\ell \|v(\tau+\zeta)\|_2^2 d\zeta + h^2 Re_1 \int_0^\ell \int_{\tau+\zeta}^{t+\zeta} \|T(s)\|_2^2 e^{\lambda(s-t-\zeta)} ds d\zeta \\
\leq & h^2 Re_1 \int_0^\ell \int_{\tau+\zeta}^{t+\zeta} \left(\frac{e^{\frac{\ell}{K_2}}}{\ell} e^{\frac{\tau-s}{K_2}} \int_0^\ell \|T(\eta+\tau)\|_2^2 d\eta \right) e^{\lambda(s-t-\zeta)} ds d\zeta \\
& + h^2 Re_1 \int_0^\ell \int_{\tau+\zeta}^{t+\zeta} \left(K_2(1+K_2) \sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds \right) e^{\lambda(s-t-\zeta)} ds d\zeta \\
& + e^{\lambda(\tau-t)} \int_0^\ell \|v(\tau+\zeta)\|_2^2 d\zeta \\
\leq & e^{\lambda(\tau-t)} \int_0^\ell \|v(\tau+\zeta)\|_2^2 d\zeta + h^2 Re_1 e^{\frac{\ell}{K_2}} e^{\lambda(\tau-t)} \int_0^\ell \|T(\tau+\zeta)\|_2^2 d\zeta \\
& + \frac{h^2 Re_1 \ell}{\lambda} K_2(1+K_2) \sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds.
\end{aligned} \tag{3.9}$$

Therefore, we deduce from inequalities (3.2), (3.4), (3.7) and (3.9) that there exists a positive constant ρ_1 satisfying for any $B_\ell \in \mathcal{D}_\ell$, there exists a time $\tau_1 = \tau_1(B_\ell, t) \leq t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau,T_\tau)) \in B_\ell$, we have

$$\|(v(t),T(t))\|_2^2 + \int_0^\ell \|(v,T)(t+\zeta)\|_2^2 d\zeta \leq \rho_1$$

for any $\tau \leq t - \tau_1$. ∎

From the proof of Theorem 3.2, we conclude the following result.

COROLLARY 3.3. *Assume that $Q \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds < +\infty.$$

Then there exists a positive constant ρ_1 satisfying for any $B \in \mathcal{D}$, there exists a time $\tau_1' = \tau_1'(B,t) \leq t$ such that for any weak solution of problem (2.1) with any initial data $(v_\tau, T_\tau) \in B$, we have

$$\|(v(t),T(t))\|_2^2 + \int_0^\ell \|(v,T)(t+\zeta)\|_2^2 d\zeta \leq \rho_1$$

for any $\tau \leq t - \tau_1'$.

THEOREM 3.3. *Assume that $Q \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds < +\infty.$$

Then there exists a positive constant ρ_2 satisfying for any $B_\ell \in \mathcal{D}_\ell$, there exists a time $\tau_2 = \tau_2(B_\ell, t) \le t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau, T_\tau)) \in B_\ell$, we have

$$\|v(t)\|_6^2 + \|T(t)\|_6^2 + \int_0^\ell \|(v,T)(t+\zeta)\|^2 d\zeta \le \rho_2$$

for any $\tau \le t - \tau_2$.

Proof. Taking the inner product of the third equation of problem (2.1) with $|T|^4 T$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\|T(t)\|_6^6 + \frac{5}{9}\||T(t)|^3\|^2 &\le \||T(t)|^3\|_{\frac{10}{3}}^{\frac{5}{3}} \|Q(t)\|_2 \\ &\le C\|Q(t)\|_2 \||T(t)|^3\|_2^{\frac{2}{3}} \||T(t)|^3\|.\end{aligned}$$

We infer from Young's inequality and the Sobolev embedding theorem that

$$\frac{d}{dt}\|T(t)\|_6^2 \le C\|Q(t)\|_2^2. \tag{3.10}$$

For any $\zeta \in (0,\ell)$, integrating inequality (3.10) over $(t-\ell+\zeta, t)$ and integrating the resulting inequality with respect to ζ over $(0,\ell)$, we have

$$\begin{aligned}\|T(t)\|_6^2 &\le \frac{1}{\ell}\int_0^\ell \|T(t-\ell+\zeta)\|_6^2 d\zeta + C\frac{1}{\ell}\int_0^\ell \int_{t-\ell+\zeta}^t \|Q(s)\|_2^2 ds\, d\zeta \\ &\le \frac{C}{\ell}\int_{t-\ell}^t \|T(\zeta)\|^2 d\zeta + C\int_{t-1}^t \|Q(s)\|_2^2 ds.\end{aligned} \tag{3.11}$$

Integrating inequality (3.1) from $t-\ell$ to t and combining inequality (3.2), we obtain

$$\begin{aligned}&\|T(t)\|_2^2 + \int_{t-\ell}^t \|T(\zeta)\|^2 d\zeta \\ &\le K_2 \int_{t-\ell}^t \|Q(\zeta)\|_2^2 d\zeta + \|T(t-\ell)\|_2^2 \\ &\le K_2 \int_{t-\ell}^t \|Q(\zeta)\|_2^2 d\zeta + \frac{e^{\frac{2\ell}{K_2}}}{\ell} e^{\frac{\tau-t}{K_2}} \int_0^\ell \|T(\zeta+\tau)\|_2^2 d\zeta + K_2(1+K_2)\sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds \\ &\le \frac{e^{\frac{2\ell}{K_2}}}{\ell} e^{\frac{\tau-t}{K_2}} \int_0^\ell \|T(\zeta+\tau)\|_2^2 d\zeta + K_2(1+K_2)\sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds.\end{aligned} \tag{3.12}$$

We deduce from inequalities (3.11)-(3.12) that

$$\begin{aligned}\|T(t)\|_6^2 \le& \frac{Ce^{\frac{2\ell}{K_2}}}{\ell^2} e^{\frac{\tau-t}{K_2}} \int_0^\ell \|T(\zeta+\tau)\|_2^2 d\zeta + \frac{C}{\ell} K_2(1+K_2)\sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds \\ &+ C\int_{t-1}^t \|Q(s)\|_2^2 ds \\ \le& \frac{Ce^{\frac{2\ell}{K_2}}}{\ell^2} e^{\frac{\tau-t}{K_2}} \int_0^\ell \|T(\zeta+\tau)\|_2^2 d\zeta + \frac{C}{\ell}(1+K_2)^2 \sup_{r\in\mathbb{R}} \int_{r-1}^r \|Q(s)\|_2^2 ds.\end{aligned} \tag{3.13}$$

Multiplying the first equation of problem (2.3) by $|\tilde v|^4\tilde v$ and integrating by parts over Ω, we deduce

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|\tilde v(t)\|_6^6+\frac{1}{Re_1}\||\nabla\tilde v||\tilde v|^2\|_2^2+\frac{1}{Re_2}\||\partial_z\tilde v||\tilde v|^2\|_2^2+\frac{4}{9}\||\tilde v|^3\|^2\\ \le&C\int_\Omega|\bar v||\nabla\tilde v||\tilde v|^5\,dxdydz+C\int_M(\int_{-h}^0|T|\,dz)(\int_{-h}^0|\nabla\tilde v||\tilde v|^4\,dz)\,dxdy\\ &+C\int_M(\int_{-h}^0|\tilde v|^2\,dz)(\int_{-h}^0|\nabla\tilde v||\tilde v|^4\,dz)\,dxdy.\end{aligned}\tag{3.14}$$

It follows from Hölder's inequality and Minkowski inequality that

$$\begin{aligned}&\int_\Omega|\bar v||\nabla\tilde v||\tilde v|^5\,dxdydz\le\int_M|\bar v|(\int_{-h}^0|\nabla\tilde v|^2|\tilde v|^4\,dz)^{\frac12}(\int_{-h}^0|\tilde v|^6\,dz)^{\frac12}\,dxdy\\ \le&(\int_M|\bar v|^4\,dxdy)^{\frac14}(\int_\Omega|\nabla\tilde v|^2|\tilde v|^4\,dxdydz)^{\frac12}(\int_{-h}^0(\int_M|\tilde v|^{12}\,dxdy)^{\frac12}\,dz)^{\frac12}.\end{aligned}\tag{3.15}$$

By virtue of interpolation inequality, we have

$$\begin{aligned}\int_M|\tilde v|^{12}\,dxdy=&\int_M||\tilde v|^3|^4\,dxdy\\ \le&C\int_M|\tilde v|^6\,dxdy\int_M|\nabla|\tilde v|^3|^2\,dxdy,\end{aligned}$$

which entails that

$$(\int_{-h}^0(\int_M|\tilde v|^{12}\,dxdy)^{\frac12}\,dz)^{\frac12}\le C(\int_\Omega|\tilde v|^6\,dxdydz)^{\frac14}(\int_\Omega|\nabla|\tilde v|^3|^2\,dxdydz)^{\frac14}.\tag{3.16}$$

We deduce from inequalities (3.15)-(3.16) that

$$\int_\Omega|\bar v||\nabla\tilde v||\tilde v|^5\,dxdydz\le C\|\tilde v\|_6^{\frac32}\|v\|_2^{\frac12}\|\nabla v\|_2^{\frac12}(\int_\Omega|\nabla|\tilde v|^3|^2\,dxdydz)^{\frac14}(\int_\Omega|\nabla\tilde v|^2|\tilde v|^4\,dxdydz)^{\frac12}.\tag{3.17}$$

Repeating the similar process as the above, we have

$$\int_M(\int_{-h}^0|T|\,dz)(\int_{-h}^0|\nabla\tilde v||\tilde v|^4\,dz)\,dxdy\le C\|T\|_6(\int_\Omega|\nabla\tilde v|^2|\tilde v|^4\,dxdydz)^{\frac12}\|\tilde v\|_6^2\tag{3.18}$$

and

$$\int_M(\int_{-h}^0|\tilde v|^2\,dz)(\int_{-h}^0|\nabla\tilde v||\tilde v|^4\,dz)\,dxdy\le C(\int_\Omega|\nabla\tilde v|^2|\tilde v|^4\,dxdydz)^{\frac12}\|\tilde v\|_6^3\|\tilde v\|_{H^1(\Omega)},\tag{3.19}$$

where we use the inequality $\|\tilde v\|_{L^8(M)}^8\le C\|\tilde v\|_{L^6(M)}^6\|\tilde v\|_{H^1(M)}^2$.

We deduce from inequalities (3.14), (3.17)-(3.19) that

$$\begin{aligned}&\frac{d}{dt}\|\tilde v(t)\|_6^6+\frac{2}{Re_1}\||\nabla\tilde v||\tilde v|^2\|_2^2+\frac{2}{Re_2}\||\partial_z\tilde v||\tilde v|^2\|_2^2+2\||\tilde v|^3\|^2\\ \le&C(\|v\|_2^2\|\nabla v\|_2^2+\|\tilde v\|_{H^1(\Omega)}^2)\|\tilde v\|_6^6+C\|T\|_6^2\|\tilde v\|_6^4,\end{aligned}\tag{3.20}$$

which implies that

$$\begin{aligned}\frac{d}{dt}\|\tilde{v}(t)\|_6^2 \leq & C(\|v\|_2^2\|\nabla v\|_2^2+\|\tilde{v}\|_{H^1(\Omega)}^2)\|\tilde{v}\|_6^2+C\|T\|_6^2\\ \leq & C(\|v\|_2^2\|v\|^2+\|v\|^2)\|\tilde{v}\|_6^2+C\|T\|_6^2.\end{aligned}$$

For any $\zeta\in(0,\ell)$, we deduce from the Gronwall inequality that

$$\begin{aligned}\|\tilde{v}(t)\|_6^2 \leq & C\left(\int_{t+\zeta-\ell}^t\|T(s)\|_6^2\,ds\right)\exp\left\{C\int_{t+\zeta-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\}\\ &+(\|\tilde{v}(t+\zeta-\ell)\|_6^2)\exp\left\{C\int_{t+\zeta-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\}\\ \leq & C\left(\|\tilde{v}(t+\zeta-\ell)\|^2+\int_{t-\ell}^t\|T(s)\|^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\}\\ \leq & C\left(\|v(t+\zeta-\ell)\|^2+\int_{t-\ell}^t\|T(s)\|^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\},\end{aligned}$$

which entails that

$$\begin{aligned}\|\tilde{v}(t)\|_6^2\leq & C\left(\int_{t-\ell}^t\|T(s)\|^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\}\\ &+C\left(\frac{1}{\ell}\int_0^\ell\|v(t+\zeta-\ell)\|^2\,d\zeta\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2\,ds\right\}.\end{aligned}\tag{3.21}$$

Integrating inequality (3.5) from $t-\ell$ to t and combining inequality (3.7), we obtain

$$\begin{aligned}&\|v(t)\|_2^2+\int_{t-\ell}^t\|v(\zeta)\|^2\,d\zeta\\ \leq & h^2Re_1\int_{t-\ell}^t\|T(\zeta)\|_2^2\,d\zeta+\|v(t-\ell)\|_2^2\\ \leq & \frac{e^{2\lambda\ell}}{\ell}e^{(\tau-t)}\int_0^\ell\|v(\tau+\zeta)\|_2^2\,d\zeta+\frac{h^2Re_1K_2e^{\frac{\ell}{K_2}+\lambda\ell}}{\ell(1-K_2\lambda)}e^{\lambda(\tau-t)}\int_0^\ell\|T(\zeta+\tau)\|_2^2\,d\zeta\\ &+h^2Re_1K_2\int_{t-\ell}^t\|T(\zeta)\|^2\,d\zeta+\frac{C}{\lambda}K_2(1+K_2)\sup_{r\in\mathbb{R}}\int_{r-1}^r\|Q(\zeta)\|_2^2\,d\zeta.\end{aligned}\tag{3.22}$$

Taking the inner product of the first equation of problem (2.2) with $-\Delta\bar{v}$ in $L^2(M)$ and using Hölder's inequality, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\bar{v}(t)\|_{L^2(M)}^2+\frac{1}{Re_1}\|\Delta\bar{v}(t)\|_{L^2(M)}^2\\ \leq & C\|\bar{v}\|_{L^4(M)}\|\nabla\bar{v}\|_{L^4(M)}\|\Delta\bar{v}\|_{L^2(M)}+C\||\nabla\tilde{v}||\tilde{v}|\|_2\|\Delta\bar{v}\|_{L^2(M)}\\ &+C\|\bar{v}\|_{L^2(M)}\|\Delta\bar{v}\|_{L^2(M)}+C\|\nabla T\|_2\|\Delta\bar{v}\|_{L^2(M)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla\bar{v}(t)\|_{L^2(M)}^2+\frac{1}{Re_1}\|\Delta\bar{v}(t)\|_{L^2(M)}^2\\ \leq & C\|\bar{v}\|_{L^2(M)}^2\|\nabla\bar{v}\|_{L^2(M)}^4+C\|\nabla T\|_2^2+C\|\nabla v\|_2^2+C\||\nabla\tilde{v}||\tilde{v}|^2\|_2^2\end{aligned}$$

$$\leq C\|v\|_2^2\|v\|^2\|\nabla\bar{v}\|_{L^2(M)}^2+C\|T\|^2+C\|v\|^2+C\||\nabla\tilde{v}||\tilde{v}|^2\|_2^2.$$

For $\zeta\in(0,\ell)$, in view of the Gronwall inequality, we obtain

$$\begin{aligned}
&\|\nabla\bar{v}(t)\|_{L^2(M)}^2\\
\leq& C\left(\int_{t+\zeta-\ell}^t\|T(s)\|^2+\|v(s)\|^2+\||\nabla\tilde{v}(s)||\tilde{v}(s)|^2\|_2^2\,ds\right)\exp\left\{C\int_{t+\zeta-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}\\
&+\|\nabla\bar{v}(t+\zeta-\ell)\|_{L^2(M)}^2\exp\left\{C\int_{t+\zeta-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}\\
\leq& C\left(\|v(t+\zeta-\ell)\|^2\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}\\
&+C\left(\int_{t-\ell}^t\|(v(s),T(s))\|^2+\||\nabla\tilde{v}(s)||\tilde{v}(s)|^2\|_2^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\nabla\bar{v}(t)\|_{L^2(M)}^2\leq C\left(\frac{1}{\ell}\int_0^\ell\|v(t+\zeta-\ell)\|^2\,d\zeta\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}\\
&\quad+C\left(\int_{t-\ell}^t\|(v(s),T(s))\|^2+\||\nabla\tilde{v}(s)||\tilde{v}(s)|^2\|_2^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}\\
\leq& C\left(\int_{t-\ell}^t\|(v(s),T(s))\|^2+\||\nabla\tilde{v}(s)||\tilde{v}(s)|^2\|_2^2\,ds\right)\exp\left\{C\int_{t-\ell}^t\|v(s)\|_2^2\|v(s)\|^2\,ds\right\}. \qquad (3.23)
\end{aligned}$$

Integrating inequality (3.20) from $t-\ell$ to t, we obtain

$$\begin{aligned}
&\|\tilde{v}(t)\|_6^6+\frac{2}{Re_1}\int_{t-\ell}^t\||\nabla\tilde{v}(s)||\tilde{v}(s)|^2\|_2^2\,ds+\frac{2}{Re_2}\||\partial_z\tilde{v}||\tilde{v}|^2\|_2^2+2\||\tilde{v}|^3\|^2\\
\leq&\|\tilde{v}(t-\ell)\|_6^6+C\int_{t-\ell}^t(\|v(s)\|_2^2\|\nabla v(s)\|_2^2+\|\tilde{v}(s)\|_{H^1(\Omega)}^2)\|\tilde{v}(s)\|_6^6+\|T(s)\|_6^2\|\tilde{v}(s)\|_6^4\,ds\\
\leq&\|\tilde{v}(t-\ell)\|_6^6+C\int_{t-\ell}^t(\|v(s)\|_2^2\|v(s)\|^2+\|v(s)\|^2)\|\tilde{v}(s)\|_6^6+\|T(s)\|_6^2\|\tilde{v}(s)\|_6^4\,ds. \qquad (3.24)
\end{aligned}$$

Therefore, we deduce from inequalities (3.2), (3.7), (3.12)-(3.13), (3.21)-(3.24) and the inequality $\|v\|_6\leq Ch^{-\frac{1}{3}}\|v\|_2+Ch^{\frac{1}{6}}\|\nabla\bar{v}\|_2+\|\tilde{v}\|_6$ shown in [21] that there exists a positive constant ρ_2 satisfying for any $B_\ell\in\mathcal{D}_\ell$, there exists a time $\tau_2=\tau_2(B_\ell,t)\leq\tau_1\leq t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau,T_\tau))\in B_\ell$, we have

$$\|v(t)\|_6^2+\|T(t)\|_6^2+\int_0^\ell\|(v,T)(t+\zeta)\|^2\,d\zeta\leq\rho_2 \qquad (3.25)$$

for any $\tau\leq t-\tau_2$. ∎

THEOREM 3.4. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^r\|Q(s)\|_2^2\,ds<+\infty.$$

Then there exists a positive constant ρ_3 satisfying for any $B_\ell\in\mathcal{D}_\ell$, there exists a time $\tau_3=\tau_3(B_\ell,t)\leq t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau,T_\tau))\in B_\ell$, we have

$$\|(v(t),T(t))\|^2+\int_0^\ell\left(\|(v(t+\zeta),T(t+\zeta))\|_{H^2(\Omega)}^2+\|(v_t(t+\zeta),T_t(t+\zeta))\|_2^2\right)d\zeta\leq\rho_3$$

for any $\tau \le t-\tau_3$.

Proof. Denoting $u=v_z$. It is clear that u satisfies the following equation obtained by differentiating the first equation of problem (2.1) with respect to z:

$$\begin{cases} \frac{\partial u}{\partial t}+L_1 u+(v\cdot\nabla)u-(\int_{-h}^{z}\nabla\cdot v(x,y,r,t)dr)\frac{\partial u}{\partial z}+(u\cdot\nabla)v \\ -(\nabla\cdot v)u+\frac{1}{Ro}fu^{\perp}-\nabla T=0, \\ u|_{\Gamma_u}=0, u|_{z=-h}=0, u\cdot\vec{n}|_{\Gamma_l}=0, \frac{\partial u}{\partial \vec{n}}\times\vec{n}|_{\Gamma_l}=0. \end{cases} \tag{3.26}$$

Multiplying the first equation of problem (3.26) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2+\|u(t)\|^2\le\|T\|_2\|\nabla u\|_2+C\|v\|_6\|u\|_2^{\frac{1}{2}}\|u\|^{\frac{3}{2}}.$$

We deduce from Young's inequality that

$$\frac{d}{dt}\|u(t)\|_2^2+\|u(t)\|^2\le C\|v\|_6^4\|u\|_2^2+C\|T\|_2^2. \tag{3.27}$$

For any $\zeta\in(0,\ell)$, it follows from the Gronwall inequality that

$$\begin{aligned}\|u(t)\|_2^2 &\le\left(\|u(t-\ell+\zeta)\|_2^2+C\int_{t-\ell+\zeta}^{t}\|T(s)\|_2^2 ds\right)\exp\left\{C\int_{t-\ell+\zeta}^{t}\|v(s)\|_6^4 ds\right\}\\ &\le\left(\|u(t-\ell+\zeta)\|_2^2+CK_2\int_{t-\ell}^{t}\|T(s)\|^2 ds\right)\exp\left\{C\int_{t-\ell}^{t}\|v(s)\|_6^4 ds\right\}. \end{aligned}\tag{3.28}$$

Integrating inequality (3.28) over $(t-\ell+\zeta,t)$ with respect to ζ over $(0,\ell)$, we have

$$\begin{aligned}\|u(t)\|_2^2 &\le\left(\frac{1}{\ell}\int_0^{\ell}\|u(t-\ell+\zeta)\|_2^2 d\zeta+CK_2\int_{t-\ell}^{t}\|T(s)\|^2 ds\right)\exp\left\{C\int_{t-\ell}^{t}\|v(s)\|_6^4 ds\right\}\\ &\le\left(\frac{1}{\ell}\int_{t-\ell}^{t}\|v(s)\|^2 ds+CK_2\int_{t-\ell}^{t}\|T(s)\|^2 ds\right)\exp\left\{C\int_{t-\ell}^{t}\|v(s)\|_6^4 ds\right\}\\ &\le C\left(\int_{t-\ell}^{t}\|(v(s),T(s))\|^2 ds\right)\exp\left\{C\int_{t-\ell}^{t}\|v(s)\|_6^4 ds\right\}. \end{aligned}\tag{3.29}$$

Moreover, integrating inequality (3.27) over $(t-\ell,t)$, we obtain

$$\|u(t)\|_2^2+\int_{t-\ell}^{t}\|u(s)\|^2 ds\le\|u(t-\ell)\|_2^2+C\int_{t-\ell}^{t}\left(\|v(s)\|_6^4\|u(s)\|_2^2+\|T(s)\|_2^2\right)ds. \tag{3.30}$$

Taking the inner product of the first equation of problem (2.1) with $-\Delta v$ in $L^2(\Omega)$, we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_2^2+\frac{1}{Re_1}\|\Delta v(t)\|_2^2+\frac{1}{Re_2}\|\nabla v_z(t)\|_2^2\\ &\le C\|v_z\|_2^{\frac{1}{2}}\|\nabla v_z\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{3}{2}}+C\|\nabla T\|_2\|\Delta v\|_2+C\|v\|_6\|\nabla v\|_3\|\Delta v\|_2+C\|v\|_2\|\Delta v\|_2.\end{aligned}$$

We infer from Young's inequality that

$$\frac{d}{dt}\|\nabla v(t)\|_2^2+\frac{1}{Re_1}\|\Delta v(t)\|_2^2+\frac{1}{Re_2}\|\nabla v_z(t)\|_2^2$$

$$\leq C(1+\|v\|_6^4+\|v_z\|_2^2\|\nabla v_z\|_2^2)\|\nabla v\|_2^2+C\|\nabla T\|_2^2. \tag{3.31}$$

For any $\zeta\in(0,\ell)$, it follows from the Gronwall inequality that

$$\begin{aligned}\|\nabla v(t)\|_2^2\leq&\left(\|\nabla v(t-\ell+\zeta)\|_2^2\right)\exp\left\{C\int_{t-\ell+\zeta}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}\\&+C\left(\int_{t-\ell+\zeta}^t\|\nabla T(s)\|_2^2ds\right)\exp\left\{C\int_{t-\ell+\zeta}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}\\\leq&\left(\|\nabla v(t-\ell+\zeta)\|_2^2\right)\exp\left\{C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}\\&+C\left(\int_{t-\ell}^t\|\nabla T(s)\|_2^2ds\right)\exp\left\{C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\},\end{aligned}$$

which entails that

$$\begin{aligned}\|\nabla v(t)\|_2^2\leq&\left(\frac{1}{\ell}\int_0^\ell\|\nabla v(t-\ell+\zeta)\|_2^2d\zeta\right)\exp\left\{C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}\\&+C\left(\int_{t-\ell}^t\|\nabla T(s)\|_2^2ds\right)\exp\left\{C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}\\\leq&C\left(\int_{t-\ell}^t\|(v(s),T(s))\|^2ds\right)\exp\left\{C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\,ds\right\}.\end{aligned}\tag{3.32}$$

Integrating inequality (3.31) over $(t-\ell,t)$, we obtain

$$\begin{aligned}&\|\nabla v(t)\|_2^2+\frac{1}{Re_1}\int_{t-\ell}^t\|\Delta v(s)\|_2^2ds+\frac{1}{Re_2}\int_{t-\ell}^t\|\nabla v_z(s)\|_2^2ds\\\leq&\|\nabla v(t-\ell)\|_2^2+C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\|\nabla v(s)\|_2^2ds+C\int_{t-\ell}^t\|\nabla T(s)\|_2^2ds\\\leq&\|\nabla v(t-\ell)\|_2^2+C\int_{t-\ell}^t(1+\|v(s)\|_6^4+\|v_z(s)\|_2^2\|\nabla v_z(s)\|_2^2)\|\nabla v(s)\|_2^2ds+C\int_{t-\ell}^t\|T(s)\|^2ds.\end{aligned}\tag{3.33}$$

Multiplying the third equation of problem (2.1) by L_2T and integrating over Ω, we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|T(t)\|^2+\|L_2T(t)\|_2^2\\\leq&C\|v\|_6\|\nabla T\|_3\|L_2T\|_2+\|Q(t)\|_2\|L_2T\|_2+C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|\partial_zT\|_2^{\frac{1}{2}}\|\nabla T_z\|_2^{\frac{1}{2}}\|L_2T\|_2,\end{aligned}$$

which entails

$$\frac{d}{dt}\|T(t)\|^2+\|L_2T(t)\|_2^2\leq C(\|v\|_6^4+\|\nabla v\|_2^2\|\Delta v\|_2^2)\|T\|^2+C\|Q(t)\|_2^2. \tag{3.34}$$

For any $\zeta\in(0,\ell)$, we infer from the Gronwall inequality that

$$\begin{aligned}\|T(t)\|^2\leq&C\left(\int_{t-\ell+\zeta}^t\|Q(s)\|_2^2ds\right)\exp\left\{C\int_{t-\ell+\zeta}^t(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\\&+\left(\|T(t-\ell+\zeta)\|^2\right)\exp\left\{C\int_{t-\ell+\zeta}^t(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\\\leq&C\left(\int_{t-1}^t\|Q(s)\|_2^2ds\right)\exp\left\{C\int_{t-\ell}^t(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\end{aligned}$$

$$+\left(\|T(t-\ell+\zeta)\|^2\right)\exp\left\{C\int_{t-\ell}^{t}(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\},$$

which entails that

$$\begin{aligned}\|T(t)\|^2\leq & C\left(\int_{t-1}^{t}\|Q(s)\|_2^2\,ds\right)\exp\left\{C\int_{t-\ell}^{t}(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\\ &+\left(\frac{1}{\ell}\int_0^{\ell}\|T(t-\ell+\zeta)\|^2\,d\zeta\right)\exp\left\{C\int_{t-\ell}^{t}(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\\ \leq & C\left(\int_{t-1}^{t}\|Q(s)\|_2^2\,ds\right)\exp\left\{C\int_{t-\ell}^{t}(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}\\ &+\left(\frac{1}{\ell}\int_{t-\ell}^{t}\|T(\zeta)\|^2\,d\zeta\right)\exp\left\{C\int_{t-\ell}^{t}(\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\,ds\right\}. \end{aligned}\tag{3.35}$$

Integrating inequality (3.34) over $(t-\ell,t)$, we conclude

$$\begin{aligned}&\|T(t)\|^2+\int_{t-\ell}^{t}\|L_2T(s)\|_2^2\,ds\\ \leq&\|T(t-\ell)\|^2+C\int_{t-\ell}^{t}\left((\|v(s)\|_6^4+\|\nabla v(s)\|_2^2\|\Delta v(s)\|_2^2)\|T(s)\|^2+\|Q(s)\|_2^2\right)ds.\end{aligned}\tag{3.36}$$

Thanks to

$$\|T_t\|_2\leq\|Q(t)\|_2+C\|v\|_6\|\nabla T\|_3+\|L_2T\|_2+C\|\nabla v\|_2^{\frac12}\|\Delta v\|_2^{\frac12}\|\partial_zT\|_2^{\frac12}\|\nabla T_z\|_2^{\frac12}\tag{3.37}$$

and

$$\begin{aligned}\|v_t\|_2\leq & C\|v\|_6\|\nabla v\|_3+C\|\nabla v\|_2^{\frac12}\|\Delta v\|_2^{\frac12}\|v_z\|_2^{\frac12}\|\nabla v_z\|_2^{\frac12}\\ &+C\|v\|_2+C\|\nabla T\|_2+\|L_1v\|_2.\end{aligned}\tag{3.38}$$

We derive from inequalities (3.37)-(3.38) that

$$\|(v_t,T_t)(t)\|_2^2\leq C\|Q(t)\|_2^2+C(1+\|(v,T)(t)\|^2)(1+\|v(t)\|_6^4+\|(v,T)(t)\|_{H^2(\Omega)}^2).\tag{3.39}$$

Therefore, we deduce from inequalities (3.25), (3.29)-(3.30), (3.32)-(3.33), (3.35)-(3.36) and (3.39) that there exists a positive constant ρ_3 satisfying for any $B_\ell\in\mathcal{D}_\ell$, there exists a time $\tau_3=\tau_3(B_\ell,t)\leq\tau_2\leq t$ such that for any weak solution of problem (2.1) with short trajectory $\chi(s,\tau;(v_\tau,T_\tau))\in B_\ell$, we have

$$\|(v(t),T(t))\|^2+\int_{t-\ell}^{t}\left(\|(v(\zeta),T(\zeta))\|_{H^2(\Omega)}^2+\|(v_t(\zeta),T_t(\zeta))\|_2^2\right)d\zeta\leq\rho_3$$

for any $\tau\leq t-\tau_3$. □

Corollary 3.4. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

Then there exists a positive constant ρ_3 satisfying for any $B\in\mathcal{D}$, there exists a time $\tau_2'=\tau_2'(B,t)\leq t$ such that for any weak solution of problem (2.1) with initial data $(v_\tau,T_\tau)\in B$, we have

$$\|(v(t),T(t))\|^2+\int_0^{\ell}\left(\|(v(t+\zeta),T(t+\zeta))\|_{H^2(\Omega)}^2+\|(v_t(t+\zeta),T_t(t+\zeta))\|_2^2\right)d\zeta\leq\rho_3$$

for any $\tau \le t - \tau_2'$.

Let

$$B_0 = \left\{(v,T)\in H : \|(v,T)\|^2 \le \rho_3\right\},$$

then $B_0\in\mathcal{D}$, we infer from Corollary 3.4 that for any $t\in\mathbb{R}$, there exists a time $\tau_0 = \tau_0(B_0,t)\ge 0$ such that for any initial data $(v_\tau,T_\tau)\in B_0$ and any $\tau\le t-\tau_0$, we have

$$U(t,\tau)B_0\subset B_0.$$

For any $t\in\mathbb{R}$, define

$$\begin{aligned} B_1(t) &= \bigcup_{\tau\in[t-\tau_0,t]}\{U(t,\tau)(v_\tau,T_\tau):(v_\tau,T_\tau)\in B_0\},\\ B_2(t) &= \overline{B_1(t)}^H \end{aligned}$$

and

$$B_0^\ell(t) = \{\chi\in X_\ell : e_0(\chi)\in B_2(t)\}.$$

From the proof of some a priori estimates in the Section 3 of [1] and Corollary 3.4, we deduce for any $\tau\le t$,

$$U(t,\tau)B_1(\tau)\subset B_1(t)$$

and there exists a positive constant $\varrho(\tau_0)$ depending on τ_0 and ρ_3 such that $B_1(t)\subset \{(v,T)\in H : \|(v,T)\|\le\varrho(\tau_0)\}\in\mathcal{D}$ for any $t\in\mathbb{R}$. Moreover, we have the following conclusion.

Proposition 3.1. *Assume that for any* $t\in\mathbb{R}$, $B_1(t)\in\mathcal{D}$ *defined above. Then*

$$B_2(t) = \overline{B_1(t)}^H \in\mathcal{D}$$

for any $t\in\mathbb{R}$ *and*

$$U(t,\tau)B_2(\tau)\subset B_2(t)$$

for any $t\ge\tau$.

Proof. For any $\tau\in\mathbb{R}$, from the definition of $B_2(\tau)$, we infer that for any $(v_\tau,T_\tau)\in B_2(\tau)$, there exists a sequence $\{(v_{n,\tau},T_{n,\tau})\}_{n=1}^\infty\subset B_1(\tau)$ such that

$$(v_{n,\tau},T_{n,\tau})\to(v_\tau,T_\tau)\ in\, H,\ as\, n\to\infty.$$

It follows from Theorem 3.1 that for any fixed $\tau\in\mathbb{R}$, $\{(v_{n,\tau},T_{n,\tau})\}_{n=1}^\infty$ is uniformly bounded in V. Therefore, we deduce from the reflexivity of V and the compactness of $V\subset H$ that there exist some $(v_1,T_1)\in V$ and a subsequence $\{(v_{n_j,\tau},T_{n_j,\tau})\}_{j=1}^\infty$ of $\{(v_{n,\tau},T_{n,\tau})\}_{n=1}^\infty$ such that

$$\begin{aligned} &(v_{n_j,\tau},T_{n_j,\tau})\rightharpoonup(v_1,T_1)\ in\, V,\ as\, j\to\infty,\\ &(v_{n_j,\tau},T_{n_j,\tau})\to(v_1,T_1)\ in\, H,\ as\, j\to\infty, \end{aligned}$$

which entails that

$$(v_1,T_1)=(v_\tau,T_\tau).$$

From

$$\|(v_\tau,T_\tau)\|\le\liminf_{j\to+\infty}\|(v_{n_j,\tau},T_{n_j,\tau})\|\le\varrho(\tau_0),$$

we conclude that $B_2(t)=\overline{B_1(t)}^H\in\mathcal{D}$ for any $t\in\mathbb{R}$.

For any fixed $\tau\in\mathbb{R}$ and any fixed $t>\tau$, we conclude from the definition of $B_2(\tau)$, that for any $(v_\tau,T_\tau)\in B_2(\tau)$, there exists a sequence $\{(v_{n,\tau},T_{n,\tau})\}_{n=1}^{\infty}\subset B_1(\tau)$ such that

$$(v_{n,\tau},T_{n,\tau})\to(v_\tau,T_\tau)\ in\, H,\ as\, n\to\infty.$$

Since $(v_{n,\tau},T_{n,\tau})$ $(n\ge1)$ and (v_τ,T_τ) are bounded in V for any fixed $\tau\in\mathbb{R}$, we conclude from Theorem 3.1 that

$$U(t,\tau)(v_{n,\tau},T_{n,\tau})\to U(t,\tau)(v_\tau,T_\tau)\ in\, H,\ as\, n\to\infty.$$

From $U(t,\tau)(v_{n,\tau},T_{n,\tau})\in B_1(t)$, we obtain $U(t,\tau)(v_\tau,T_\tau)\in B_2(t)$. Therefore, we deduce that

$$U(t,\tau)B_2(\tau)\subset B_2(t)$$

for any $t\ge\tau$. ∎

Let

$$Y=\left\{\chi\in X_\ell:\chi\in L^2(0,\ell;V),\chi_t\in L^1(0,\ell;(H^2(\Omega)\cap V_2)')\right\}$$

equipped with the following norm

$$\|\chi\|_Y=\left\{\int_0^\ell\|\chi(r)\|^2\,dr+\left(\int_0^\ell\|\chi_t(r)\|_{(H^2(\Omega)\cap V_2)'}\,dr\right)^2\right\}^{\frac{1}{2}}.$$

Define $\hat{B}_1^\ell=\{B_1^\ell(t):t\in\mathbb{R}\}$, where

$$B_1^\ell(t)=\left\{\chi\in X_\ell:\|e_0(\chi)\|^2+\|\chi\|_Y^2\le\rho_3\right\}.$$

From Proposition 3.1 and Theorem 3.4, we know that $L(t,\tau)B_0^\ell(\tau)\subset B_0^\ell(t)$ for any $t\ge\tau$ as well as $L(t,\tau)B_0^\ell(\tau)\subset B_1^\ell(t)$ for any $\tau\le t-\tau_3$.

LEMMA 3.1. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^r\|Q(s)\|_2^2\,ds<+\infty.$$

Then $\overline{L(t,\tau)B_0^\ell(\tau)}^{L^2(0,\ell;H)}\subset B_0^\ell(t)$ for any $t\ge\tau$.

Proof. Thanks to $L(t,\tau)B_0^\ell(\tau)\subset B_0^\ell(t)$ for any $t\ge\tau$, it is enough to prove that for any $t\in\mathbb{R}$,

$$\overline{B_0^\ell(t)}^{L^2(0,\ell;H)}\subset B_0^\ell(t).$$

For any fixed $\tau\in\mathbb{R}$ and any $\chi_0\in\overline{B_0^\ell(\tau)}^{L^2(0,\ell;H)}$, there exists a sequence of short trajectories $\chi_n\in B_0^\ell(\tau)$ such that $\chi_n\to\chi_0$ in $L^2(0,\ell;H)$. Since $e_0(\chi_n)\in B_2(\tau)$ for any $n\in\mathbb{N}$, there exists a subsequence $\{e_0(\chi_{n_j})\}_{j=1}^\infty$ of $\{e_0(\chi_n)\}_{n=1}^\infty$ and $(v_\tau,T_\tau)\in V$ such that $e_0(\chi_{n_j})\rightharpoonup(v_\tau,T_\tau)$ in V, which implies that $e_0(\chi_{n_j})\to(v_\tau,T_\tau)$ in H. From the proof of the existence of weak solutions for problem (2.1), we deduce that for any $S>\tau$, there exists a subsequence converging (∗-) weakly in spaces $L^\infty(\tau,S;H)\cap L^2(\tau,S;V)\cap H^1(\tau,S;(V\cap H^3(\Omega))'))$ to a certain function $(v(t),T(t))$ with $(v(\tau),T(\tau))=(v_\tau,T_\tau)$. Therefore, we obtain $\chi_0\in X_\ell$ from Corollary 3.2. It remains to show that $e_0(\chi_0)\in B_2(\tau)$. From the closedness of $B_2(\tau)$, we deduce that $e_0(\chi_0)=(v_\tau,T_\tau)\in B_2(\tau)$. Therefore, we obtain $\chi_0\in B_0^\ell(\tau)$. ∎

LEMMA 3.2. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^r\|Q(s)\|_2^2\,ds<+\infty.$$

Then for any $\tau\in\mathbb{R}$, the mapping $L(t,\tau):X_\ell\to X_\ell$ is locally Lipschitz continuous on $B_0^\ell(\tau)$ for all $t\ge\tau$.

Proof. Assume that $\tau\in\mathbb{R}$ and χ^1, $\chi^2\in B_0^\ell(\tau)$. For any $t>\tau+\ell$, let $(v_1(t),T_1(t))=L(t,\tau)\chi^1$, $(v_2(t),T_2(t))=L(t,\tau)\chi^2$ and let $v=v_1-v_2$, $T=T_1-T_2$, $p=p_1-p_2$. From the proof of Theorem 2 in [1], we conclude

$$\frac{d}{dt}\|(v(t),T(t))\|_2^2+\|(v(t),T(t))\|^2\le\mathbb{L}(t)\|(v(t),T(t))\|_2^2, \tag{3.40}$$

where

$$\mathbb{L}(t)=C(\|(v_2,T_2)\|^4+\|\partial_zv_2\|^2\|\nabla\partial_zv_2\|_2^2+\|\partial_zT_2\|^2\|\nabla\partial_zT_2\|_2^2).$$

Let $s\in(0,\ell)$ and integrating inequality (3.40) from $\tau+s$ to $t+s$, we obtain

$$\|(v(t+s),T(t+s))\|_2^2\le\int_{\tau+s}^{t+s}\mathbb{L}(r)\|(v(r),T(r))\|_2^2\,dr+\|(v(\tau+s),T(\tau+s))\|_2^2. \tag{3.41}$$

From the classical Gronwall inequality, we deduce

$$\begin{aligned}\|(v(t+s),T(t+s))\|_2^2&\le\|(v(\tau+s),T(\tau+s))\|_2^2\exp\Big(\int_{\tau+s}^{t+s}\mathbb{L}(r)\,dr\Big)\\&\le\mathcal{M}_\ell(t,\tau)\|(v(\tau+s),T(\tau+s))\|_2^2,\end{aligned} \tag{3.42}$$

where

$$\mathcal{M}_\ell(t,\tau)=\mathcal{M}_\ell(|t+\ell-\tau|)=\exp\Big(\int_\tau^{t+\ell}\mathbb{L}(r)\,dr\Big) \tag{3.43}$$

is a finite number depending on $\varrho(\tau_0)$ and $|t+\ell-\tau|$ for any fixed $t>\tau+\ell$ from the proof of a priori estimates in the Section 3 of [1], since $e_0(\chi^1)$, $e_0(\chi^2)\in B_2(\tau)$ are uniformly bounded in V for any χ^1, $\chi^2\in B_0^\ell(\tau)$.

Integrating inequality (3.42) with respect to s for 0 to ℓ, we obtain

$$\int_0^\ell\|(v(t+s),T(t+s))\|_2^2\,ds\le\mathcal{M}_\ell(t,\tau)\int_0^\ell\|(v(\tau+s),T(\tau+s))\|_2^2\,ds, \tag{3.44}$$

which implies that for any $\tau\in\mathbb{R}$, the mapping $L(t,\tau):X_\ell\to X_\ell$ is locally Lipschitz continuous on $B_0^\ell(\tau)$ for all $t\geq\tau$. ◻

We can immediately obtain the existence of a pullback attractor in X_ℓ stated as follows.

THEOREM 3.5. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

Then the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (2.1) possesses a pullback attractor $\hat{\mathcal{A}}_\ell=\{\mathcal{A}^\ell(t):t\in\mathbb{R}\}$ in X_ℓ and $e_1(\mathcal{A}_\ell(t))$ is included in $B_2(t+\ell)$ for any $t\in\mathbb{R}$, where

$$e_1(\mathcal{A}_\ell(t))=\{e_1(\chi):\chi\in\mathcal{A}_\ell(t)\}$$

for any $t\in\mathbb{R}$.

4. The existence of pullback exponential attractors

In this section, we construct the pullback exponential attractors of problem (2.1) by combining the method of ℓ-trajectories with the smoothing property of the process $\{L(t,\tau)\}_{t\geq\tau}$.

4.1. The existence of pullback exponential attractors. In this subsection, we prove the smoothness property of the process $\{L(t,\tau)\}_{t\geq\tau}$ to construct the pullback exponential attractors of problem (2.1).

THEOREM 4.1. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

For any fixed $\tau\in\mathbb{R}$, let χ^1 and χ^2 be two short trajectories belonging to $B_0^\ell(\tau)$. Then there exists a positive constant κ independent of t such that for arbitrary $t\geq\tau+\ell$, we have

$$\|L(t,\tau)\chi^1-L(t,\tau)\chi^2\|_Y^2\leq\kappa\mathcal{M}_\ell(t,\tau)\int_0^\ell\|\chi^1(\tau+r)-\chi^2(\tau+r)\|_2^2\,dr,$$

where $\mathcal{M}_\ell(t,\tau)$ is given in (3.43).

Proof. Suppose that $\tau\in\mathbb{R}$ and any $\chi^1,\ \chi^2\in B_0^\ell(\tau)$, for any $t>\tau+\ell$, let $(v_1(t),T_1(t))=L(t,\tau)\chi^1$, $(v_2(t),T_2(t))=L(t,\tau)\chi^2$ and let $v=v_1-v_2$, $T=T_1-T_2$. From inequality (3.40), we conclude

$$\frac{d}{dt}\|(v(t),T(t))\|_2^2+\|(v(t),T(t))\|^2\leq\mathbb{L}(t)\|(v(t),T(t))\|_2^2.\tag{4.1}$$

For any $t\geq\tau+\ell$, integrating inequality (4.1) from $t-s$ to $t+\ell$ with $s\in[0,\frac{\ell}{2}]$, we conclude

$$\begin{aligned}&\|(v(t+\ell),T(t+\ell))\|_2^2+\int_{t-s}^{t+\ell}\|(v(\zeta),T(\zeta))\|^2\,d\zeta\\&\leq\int_{t-s}^{t+\ell}\mathbb{L}(\zeta)\|(v(\zeta),T(\zeta))\|_2^2\,d\zeta+\|(v(t-s),T(t-s))\|_2^2.\end{aligned}$$

It follows from the classical Gronwall inequality that

$$\|(v(t+\ell),T(t+\ell))\|_2^2+\int_{t-s}^{t+\ell}\|(v(\zeta),T(\zeta))\|^2\,d\zeta\le\exp(\int_{t-s}^{t+\ell}\mathbb{L}(\zeta)\,d\zeta)\|(v(t-s),T(t-s))\|_2^2. \quad (4.2)$$

For any $t\ge\tau+\ell$ and any $s\in[0,\frac{\ell}{2}]$, integrating inequality (4.1) from $\tau+s$ to $t-s$, we obtain

$$\|(v(t-s),T(t-s))\|_2^2\le\int_{\tau+s}^{t-s}\mathbb{L}(r)\|(v(r),T(r))\|_2^2\,dr+\|(v(\tau+s),T(\tau+s))\|_2^2.$$

We deduce from the classical Gronwall inequality that

$$\begin{aligned}\|(v(t-s),T(t-s))\|_2^2&\le\|(v(\tau+s),T(\tau+s))\|_2^2\exp(\int_{\tau+s}^{t-s}\mathbb{L}(r)\,dr)\\&\le\|(v(\tau+s),T(\tau+s))\|_2^2\exp(\int_{\tau}^{t-s}\mathbb{L}(r)\,dr).\end{aligned} \quad (4.3)$$

Combining inequality (4.2) with inequality (4.3), we obtain

$$\begin{aligned}\int_0^{\ell}\|(v(t+\zeta),T(t+\zeta))\|^2\,d\zeta&\le\exp(\int_{\tau}^{t+\ell}\mathbb{L}(\zeta)\,d\zeta)\|(v(\tau+s),T(\tau+s))\|_2^2\\&=\mathcal{M}_\ell(t,\tau)\|(v(\tau+s),T(\tau+s))\|_2^2.\end{aligned}$$

Integrating the above inequality over $(0,\frac{\ell}{2})$ with respect to s, we obtain

$$\int_0^{\ell}\|(v(t+\zeta),T(t+\zeta))\|^2\,d\zeta\le\frac{2\mathcal{M}_\ell(t,\tau)}{\ell}\int_0^{\frac{\ell}{2}}\|(v(\tau+s),T(\tau+s))\|_2^2\,ds. \quad (4.4)$$

Since $\mathcal{M}_\ell(t,\tau)$ is bounded for any fixed $t\in[\tau+\ell,S]$, we obtain

$$\int_0^{\ell}\|(v(t+\zeta),T(t+\zeta))\|^2\,d\zeta\le\frac{2\mathcal{M}_\ell(t,\tau)}{\ell}\int_0^{\ell}\|(v(\tau+s),T(\tau+s))\|_2^2\,ds. \quad (4.5)$$

Thanks to

$$\begin{aligned}\|v_t\|_{((H^2(\Omega))^2\cap V_1)'}\le&\|v\|+C\|v_1\|_3\|v\|_6+C\|\nabla v_1\|_2\|v\|_3\\&+C\|v_2\|_3\|v\|_6+C\|\nabla v\|_2\|v_2\|_3+C\|T\|_2+C\|v\|_2\end{aligned} \quad (4.6)$$

and

$$\begin{aligned}\|T_t\|_{(H^2(\Omega)\cap V_2)'}\le&\|T\|+C\|v_1\|_3\|T\|_6\\&+C\|\nabla v_1\|_2\|T\|_3+C\|T_2\|_3\|v\|_6+C\|\nabla v\|_2\|T_2\|_3,\end{aligned} \quad (4.7)$$

we infer from Theorem 3.3, (4.5)-(4.7) that

$$\left(\int_0^{\ell}\|(v_t(t+r),T_t(t+r))\|_{((H^2(\Omega))^3\cap V)'}\,dr\right)^2\le\kappa_2\mathcal{M}_\ell(t,\tau)\int_0^{\ell}\|(v(\tau+s),T(\tau+s))\|_2^2\,ds. \quad (4.8)$$

□

THEOREM 4.2. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

Then for any $\theta\in(0,\frac{1}{2})$, there exists a pullback exponential attractor $\mathcal{M}_\ell=\mathcal{M}_\ell^\theta=\{\mathcal{M}_\ell(t): t\in\mathbb{R}\}$ for the process $\{L(t,\tau)\}_{t\geq\tau}$ generated by problem (2.1), the sections $\mathcal{M}_\ell(t)$ are compact subsets of $X=L^2(0,\ell;V)$ and their fractal dimension in $L^2(0,\ell;V)$ can be estimated by

$$dim_F(\mathcal{M}_\ell(t))\leq\log_{\frac{1}{2\theta}}(N_\theta^X(B_Y(0;1))),$$

where $B_X(x_0;R)$ denotes a R-ball in X centered at x_0 and $N_\theta^X(B_Y(0;1))$ denotes the smallest number of balls in X of radius θ necessary to cover the unit ball in Y.

Proof. From Theorem 3.4, we know that there exists a $\mathcal{D}_\ell$-pullback absorbing set $\hat{B}_0^\ell=\{B_0^\ell(t):t\in\mathbb{R}\}$ in X satisfying $L(t,\tau)B_0^\ell(\tau)\subset B_0^\ell(t)$ for any $t\geq\tau$ and $\mathcal{A}_\ell(\tau)\subset B_0^\ell(\tau)$ for any $\tau\in\mathbb{R}$. Therefore, we infer from Theorem 4.1 that there exists some time $t_1>0$ such that the mapping $L(kt_1,(k-1)t_1):B_0^\ell((k-1)t_1)\to B_0^\ell(kt_1)$ enjoys the smoothness property

$$\|L(kt_1,(k-1)t_1)\chi^1-L(kt_1,(k-1)t_1)\chi^2\|_Y\leq K\|\chi^1-\chi^2\|_X \tag{4.9}$$

for any χ^1, $\chi^2\in B_0^\ell((k-1)t_1)$ and

$$L(kt_1,(k-1)t_1)B_1^\ell((k-1)t_1)\subset B_1^\ell(kt_1) \tag{4.10}$$

where $K^2=\kappa\mathcal{M}_\ell(t_1)=\kappa\exp(\int_{(k-1)t_1}^{kt_1}\mathbb{L}(r)\,dr)$ is a fixed positive constant.

For any natural number $k\in\mathbb{Z}$, $B_0^\ell(kt_1)$ is uniformly bounded in X, which implies that there exists a positive constant R and $\chi_k\in B_0^\ell(kt_1)$ such that $B_0^\ell(kt_1)\subset B_X(\chi_k;R)$ for all $k\in\mathbb{Z}$, denote by $W^0(k)=\{\chi_k\}$. Moreover, for any $\theta\in(0,\frac{1}{2})$, we can choose some elements $\eta_1,\eta_2,\cdots,\eta_N\in Y$ such that

$$B_Y(0;1)\subset\bigcup_{j=1}^{N}B_X(\eta_j;\frac{\theta}{K}),$$

where $N=N^X_{\frac{\theta}{K}}(B_Y(0;1))$.

We infer from inequality (4.9) that for any $\chi\in B_X(\chi_{k-1};R)$,

$$\|L(kt_1,(k-1)t_1)\chi-L(kt_1,(k-1)t_1)\chi_{k-1}\|_Y\leq K\|\chi-\chi_{k-1}\|_X\leq KR,$$

i.e.,

$$L(kt_1,(k-1)t_1)B_X(\chi_{k-1};R)\subset B_Y(L(kt_1,(k-1)t_1)\chi_{k-1};KR),$$

which implies that for any $\chi\in B_Y(L(kt_1,(k-1)t_1)\chi_{k-1};KR)$, we have

$$\frac{\chi-L(kt_1,(k-1)t_1)\chi_{k-1}}{KR}\in B_Y(0;1)\subset\bigcup_{j=1}^{N}B_X(\eta_j;\frac{\theta}{K}),$$

and

$$B_Y(L(kt_1,(k-1)t_1)\chi_{k-1};KR)\subset\bigcup_{j=1}^{N}B_X(L(kt_1,(k-1)t_1)\chi_{k-1}+KR\eta_j;R\theta),$$

which yields that there exist z_1, z_2, $\cdots$, $z_N\in L(kt_1,(k-1)t_1)B_0^\ell((k-1)t_1)$ and y_1, y_2, $\cdots$, $y_N\in B_0^\ell((k-1)t_1)$ such that

$$\begin{aligned}L(kt_1,(k-1)t_1)B_0^\ell((k-1)t_1)=&L(kt_1,(k-1)t_1)\left(B_X(\chi_{k-1};R)\cap B_0^\ell((k-1)t_1)\right)\\ \subset&\bigcup_{j=1}^{N}B_X(z_j;2\theta R)\end{aligned}$$

and

$$L(kt_1,(k-1)t_1)y_j=z_j$$

for any $j=1,2,\cdots,N$. Denoting the new set of centers by $W^1(k)$, it follows

$$L(kt_1,(k-1)t_1)B_0^\ell((k-1)t_1)\subset\bigcup_{\chi\in W^1(k)}B_X(\chi;2\theta R)$$

with $W^1(k)\in L(kt_1,(k-1)t_1)B_0^\ell((k-1)t_1)$ and $\sharp W^1(k)\le N$.

In what follows, we assume that the sets $W^m(k)\subset L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1)\subset B_0^\ell(kt_1)$ are already constructed for all $m\le n$, which satisfies

$$L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1)\subset\bigcup_{\chi\in W^m(k)}B_X(\chi;(2\theta)^mR)$$

and

$$\sharp W^m(k)\le N^m.$$

In order to construct a covering of

$$\begin{aligned}&L(kt_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1)\\ =&L(kt_1,(k-1)t_1)L((k-1)t_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1)\\ \subset&\bigcup_{\chi\in W^n(k-1)}L(kt_1,(k-1)t_1)B_X(\chi;(2\theta)^nR)\\ \subset&\bigcup_{\chi\in W^n(k-1)}B_Y(L(kt_1,(k-1)t_1)\chi;(2\theta)^nKR),\end{aligned}$$

let $\chi\in W^n(k-1)$, we proceed as before and use the covering of the unit ball $B_Y(0;1)$ by $\frac{\theta}{K}$-balls in X to conclude

$$B_Y(L(kt_1,(k-1)t_1)\chi;(2\theta)^nKR)\subset\bigcup_{j=1}^{N}B_X(L(kt_1,(k-1)t_1)\chi+(2\theta)^nKR\eta_j;(2\theta)^nR\theta),$$

which entails that

$$B_Y(L(kt_1,(k-1)t_1)\chi;(2\theta)^nKR)\subset\bigcup_{j=1}^{N}B_X(L(kt_1,(k-1)t_1)y_j^\chi;(2\theta)^{n+1}R)$$

for some $y_1^\chi, \cdots, y_N^\chi \in L((k-1)t_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1)$. Constructing in the same way such a covering by balls with radius $(2\theta)^{n+1}R$ in X for every $\chi \in W^n(k-1)$, we obtain a covering of the set $L(kt_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1)$ and denote the new set of centres by $W^{n+1}(k)$, which yields $\sharp W^{n+1}(k) \le N \sharp W^n(k-1) \le N^{n+1}$ and $W^{n+1}(k) \subset L(kt_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1)$ as well as

$$L(kt_1,(k-n-1)t_1)B_0^\ell((k-n-1)t_1) \subset \bigcup_{\chi \in W^{n+1}(k)} B_X(\chi;(2\theta)^{n+1}R).$$

In order to obtain the existence of the pullback exponential attractor, for any $k \in \mathbb{Z}$ and any $n \in \mathbb{N}$, we define

$$\begin{aligned}
E^0(k) &= W^0(k) = \{\chi_k\},\\
E^1(k) &= L(kt_1,(k-1)t_1)E^0(k-1) \cup W^1(k),\\
&\vdots\\
E^n(k) &= L(kt_1,(k-1)t_1)E^{n-1}(k-1) \cup W^n(k) = \bigcup_{j=0}^{n} L(kt_1,(k-j)t_1)W^{n-j}(k-j).
\end{aligned}$$

From the fact that $L(t,\tau)B_0^\ell(\tau) \subset B_0^\ell(t)$ for any $t \ge \tau$, we conclude for any $k \in \mathbb{Z}$,

$$L(kt_1,(k-n)t_1)B_0^\ell((k-n)t_1) \subset L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1)$$

for any $n,m \in \mathbb{N}$ with $n \ge m$. Moreover, for any $k \in \mathbb{Z}$, the family of sets $E^n(k)(n \in \mathbb{N})$, satisfies the following properties

(i) $L(kt_1,(k-1)t_1)E^n(k-1) \subset E^{n+1}(k)$, $E^0(k) = W^0(k) \subset B_0^\ell(kt_1)$, $E^n(k) \subset L(kt_1,(k-n)t_1)B_0^\ell((k-n)t_1) \subset B_0^\ell(kt_1)$,

(ii) $\sharp E^n(k) \le \sum_{i=0}^n N^i \le (n+1)N^n$,

(iii) $L(kt_1,(k-n)t_1)B_0^\ell((k-n)t_1) \subset \bigcup_{\chi \in W^n(k)} B_X(\chi;(2\theta)^n R) \subset \bigcup_{\chi \in E^n(k)} B_X(\chi;(2\theta)^n R)$.

For any $k \in \mathbb{Z}$, define

$$\tilde{\mathcal{M}}_\ell(k) = \bigcup_{n=0}^{\infty} E^n(k).$$

In what follows, we will prove that for any $k \in \mathbb{Z}$, the set $\tilde{\mathcal{M}}_\ell(k)$ is pre-compact, finite-dimensional and positively semi-invariant with respect to the process $\{L(mt_1,nt_1): m \ge n\}$.

First of all, for any $k \in \mathbb{Z}$ and any $m \in \mathbb{N}$, it follows from the property (i) that

$$\begin{aligned}
L((m+k)t_1,kt_1)\tilde{\mathcal{M}}_\ell(k) &= \bigcup_{n=0}^{\infty} L((m+k)t_1,kt_1)E^n(k)\\
&\subset \bigcup_{n=0}^{\infty} E^{n+m}(m+k) = \bigcup_{n=m}^{\infty} E^n(m+k)\\
&\subset \bigcup_{n=0}^{\infty} E^n(m+k) = \tilde{\mathcal{M}}_\ell(m+k).
\end{aligned}$$

Furthermore, for any $k\in\mathbb{Z}$, since $E^n(k)\subset L(kt_1,(k-n)t_1)B_0^\ell((k-n)t_1)\subset L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1)$ for any $n\geq m$, we deduce

$$\begin{aligned}\tilde{\mathcal{M}}_\ell(k)=\bigcup_{n=0}^{\infty}E^n(k)&\subset\bigcup_{n=0}^{m}E^n(k)\cup\bigcup_{n=m+1}^{\infty}E^n(k)\\&\subset\bigcup_{n=0}^{m}E^n(k)\cup L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1).\end{aligned}$$

We infer from properties (ii) and (iii) that for any $k\in\mathbb{Z}$,

$$\sharp(\bigcup_{n=0}^{m}E^n(k))=\sum_{n=0}^{m}\sharp E^n(k)\leq(m+1)\sharp E^m(k)\leq(m+1)^2N^m$$

and

$$L(kt_1,(k-m)t_1)B_0^\ell((k-m)t_1)\subset\bigcup_{\chi\in W^m(k)}B_X(\chi;(2\theta)^mR).$$

For any $\epsilon>0$, there exists some positive integer m sufficiently large such that

$$(2\theta)^mR\leq\epsilon<(2\theta)^{m-1}R.$$

Therefore, for any $k\in\mathbb{Z}$, we can estimate the number of ϵ-balls needed to cover $\tilde{\mathcal{M}}_\ell(k)$ as follows

$$N_\epsilon^X(\tilde{\mathcal{M}}_\ell(k))\leq\sharp(\bigcup_{n=0}^{m}E^n(k))+\sharp W^m(k)\leq(m+1)^2N^m+N^m\leq2(m+1)^2N^m,$$

which implies that for any $k\in\mathbb{Z}$, there exists a finite number of ϵ-net to cover $\tilde{\mathcal{M}}_\ell(k)$. Therefore, $\tilde{\mathcal{M}}_\ell(k)$ is a pre-compact subset of $B_0^\ell(k)$ in X for any $k\in\mathbb{Z}$.

For any fixed $k\in\mathbb{Z}$, we conclude the fractal dimension of the set $\tilde{\mathcal{M}}_\ell(k)$,

$$dim_F(\tilde{\mathcal{M}}_\ell(k))=\limsup_{\epsilon\to0^+}\frac{\ln(N_\epsilon^X(\tilde{\mathcal{M}}_\ell(k)))}{-\ln\epsilon}\leq\log_{\frac{1}{2\theta}}(N)=\log_{\frac{1}{2\theta}}(N_\theta^X(B_Y(0;1))).$$

In what follows, we will prove that for any $k\in\mathbb{Z}$, the set $\tilde{\mathcal{M}}_\ell(k)$ exponentially attracts all bounded subsets of X. For any bounded subset B^ℓ of X_ℓ, we infer from Theorem 3.4 that for any $k\in\mathbb{Z}$, there exists some $t_2=t_2(B^\ell)>0$ such that $L(kt_1,kt_1-\tau)B^\ell\subset B_0^\ell(kt_1)$ for any $kt_1-\tau\geq t_2$, which implies that there exists some natural number $n_0\in\mathbb{N}$ with $n_0t_1\geq t_2$ such that $L(kt_1,(k-n)t_1)B^\ell\subset B_0^\ell(kt_1)$ for any $n\geq n_0$. Therefore, if $n\geq n_0+1$, we obtain

$$\begin{aligned}&dist_X(L(kt_1,(k-n)t_1)B^\ell,\tilde{\mathcal{M}}_\ell(k))\\\leq&dist_X(L(kt_1,(k-n+n_0)t_1)L((k-n+n_0)t_1,(k-n)t_1)B^\ell,\bigcup_{n=0}^{\infty}E^n(k))\\\leq&dist_X(L(kt_1,(k-n+n_0)t_1)B_0^\ell((k-n+n_0)t_1),\bigcup_{n=0}^{\infty}E^n(k))\\\leq&dist_X(L(kt_1,(k-n+n_0)t_1)B_0^\ell((k-n+n_0)t_1),E^{n-n_0}(k))\end{aligned}$$

$$
\begin{aligned}
&\leq (2\theta)^{n-n_0} R \\
&= (2\theta)^{-n_0} R e^{-ln(\frac{1}{2\theta})n}.
\end{aligned} \tag{4.11}
$$

To obtain the existence of a pullback exponential attractor for the continuous time process $\{L(t,\tau)\}_{t\geq\tau}$, we define

$$\tilde{\mathcal{E}}_\ell(t) := L(t,kt_1)\tilde{\mathcal{M}}_\ell(k),\ for t\in[kt_1,(k+1)t_1).$$

From Lemma 3.1, we know that $\hat{B}_0^\ell$ is a family of closed subsets of X. Let $\mathcal{E}_\ell(t)$ be the closure of $\tilde{\mathcal{E}}_\ell(t)$ in X for any $t\in\mathbb{R}$.

Due to the Lipschitz-continuity of the process, the sets $\tilde{\mathcal{E}}_\ell(t)$ are compact in X. Moreover, we deduce from Lemma 2.3 that the same (uniform) bound on the fractal dimension of the sections $\mathcal{E}_\ell(t)$,

$$
\begin{aligned}
&dim_F(\mathcal{E}_\ell(t)) = dim_F(\tilde{\mathcal{E}}_\ell(t)) = dim_F(L(t,kt_1)\tilde{\mathcal{M}}_\ell(k)) \\
&\leq dim_F(\tilde{\mathcal{M}}_\ell(k)) \leq \log_{\frac{1}{2\theta}}(N_\theta^X(B_Y(0;1))),\ for t\in[kt_1,(k+1)t_1).
\end{aligned}
$$

In the following, we will prove that the sets $\{\mathcal{E}_\ell(t): t\in\mathbb{R}\}$ are positively semi-invariant.

Let $t,s\in\mathbb{R}$ and $t\geq s$, then $s=kt_1+s_1'$ and $t=lt_1+t_1'$ for some $k,l\in\mathbb{Z}$, $k\leq l$ and $s_1',t_1'\in[0,t_1)$. If $l\geq k+1$, we obtain

$$
\begin{aligned}
&L(t,s)\tilde{\mathcal{E}}_\ell(s) = L(lt_1+t_1',kt_1+s_1')\tilde{\mathcal{E}}_\ell(kt_1+s_1') = L(lt_1+t_1',kt_1+s_1')L(kt_1+s_1',kt_1)\tilde{\mathcal{M}}_\ell(k) \\
&= L(lt_1+t_1',lt_1)L(lt_1,kt_1)\tilde{\mathcal{M}}_\ell(k) \subset L(lt_1+t_1',lt_1)\tilde{\mathcal{M}}_\ell(l) = \tilde{\mathcal{E}}_\ell(lt_1+t_1') = \tilde{\mathcal{E}}_\ell(t),
\end{aligned}
$$

where we used the semi-invariance of the family $\{\tilde{\mathcal{M}}_\ell(k): k\in\mathbb{Z}\}$ under the action of the process $\{L(mt_1,nt_1): m\geq n\}$. On the other hand, if $l=k$, then $s=kt_1+s_1'$ and $t=kt_1+t_1'$ for some s_1', $t_1'\in[0,t_1)$ with $t_1'\geq s_1'$ and

$$
\begin{aligned}
&L(t,s)\tilde{\mathcal{E}}_\ell(s) = L(kt_1+t_1',kt_1+s_1')\tilde{\mathcal{E}}_\ell(kt_1+s_1') \\
&= L(kt_1+t_1',kt_1+s_1')L(kt_1+s_1',kt_1)\tilde{\mathcal{M}}_\ell(k) \\
&= L(kt_1+t_1',kt_1)\tilde{\mathcal{M}}_\ell(k) = \tilde{\mathcal{E}}_\ell(lt_1+t_1') = \tilde{\mathcal{E}}_\ell(t).
\end{aligned}
$$

By the continuity of the process, we obtain the semi-invariance of the family $\{\mathcal{E}_\ell(t): t\in\mathbb{R}\}$.

Finally, the set $\mathcal{E}_\ell(t)$ exponentially pullback attracts all bounded subsets of X at time $t\in\mathbb{R}$. This follows immediately from the exponential pullback attracting property of the sets $\{\tilde{\mathcal{M}}_\ell(k): k\in\mathbb{Z}\}$ and the Lipschitz-continuity property of the process $\{L(t,\tau): t\geq\tau\}$. Therefore, the family $\mathcal{E}_\ell = \{\mathcal{E}_\ell(t): t\in\mathbb{R}\}$ is a pullback exponential attractor for the process $\{L(t,\tau)\}_{t\geq\tau}$ in X. ∎

4.2. The existence of a pullback exponential attractor in H. In this subsection, we prove the existence of a pullback exponential attractor in H of problem (2.1).

THEOREM 4.3. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds < +\infty.$$

Then for any fixed $\tau\in\mathbb{R}$, the mapping $e_1: B_0^\ell(\tau)\to B_2(\tau+\ell) = e_1(B_0^\ell(\tau))$ is Lipschitz continuous. That is, for any two short trajectories χ^1, $\chi^2\in B_0^\ell(\tau)$, there exists a positive *constant η dependent on ℓ such that*

$$\|e_1(\chi^1)-e_1(\chi^2)\|_2^2 \le \eta \int_0^\ell \|\chi^1(r)-\chi^2(r)\|_2^2\,dr.$$

Proof. Assume that $\tau\in\mathbb{R}$ and $\chi^1,\ \chi^2\in B_0^\ell(\tau)$. For any $t>\tau+\ell$, let $(v_1(t),T_1(t))=L(t,\tau)\chi^1$, $(v_2(t),T_2(t))=L(t,\tau)\chi^2$ and let $v=v_1-v_2$, $T=T_1-T_2$. From inequality (3.40), we conclude

$$\frac{d}{dt}\|(v(t),T(t))\|_2^2+\|(v(t),T(t))\|^2 \le \mathbb{L}(t)\|(v(t),T(t))\|_2^2.$$

For any $\tau\in\mathbb{R}$ and any $\zeta\in(0,\ell)$, we infer from the classical Gronwall inequality that

$$\begin{aligned}\|(v(\tau+\ell),T(\tau+\ell))\|_2^2 &\le \|(v(\tau+\zeta),T(\tau+\zeta))\|_2^2 \exp\Big(\int_{\tau+\zeta}^{\tau+\ell}\mathbb{L}(r)\,dr\Big)\\ &\le \|(v(\tau+\zeta),T(\tau+\zeta))\|_2^2 \exp\Big(\int_{\tau}^{\tau+\ell}\mathbb{L}(r)\,dr\Big).\end{aligned}\tag{4.12}$$

Integrating inequality (4.12) over $(0,\ell)$, we obtain

$$\|(v(\tau+\ell),T(\tau+\ell))\|_2^2 \le \frac{1}{\ell}\exp\Big(\int_{\tau}^{\tau+\ell}\mathbb{L}(r)\,dr\Big)\int_0^\ell \|(v(\tau+\zeta),T(\tau+\zeta))\|_2^2\,d\zeta.$$

Thanks to (3.43), we know that

$$\mathcal{M}_\ell(\tau,\tau)=\exp\Big(\int_{\tau}^{\tau+\ell}\mathbb{L}(r)\,dr\Big)<+\infty,$$

which implies that the mapping $e_1:\mathcal{A}_\ell\to\mathcal{A}$ is Lipschitz continuous. □

THEOREM 4.4. *Assume that $Q\in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}}\int_{r-1}^{r}\|Q(s)\|_2^2\,ds<+\infty.$$

Then for any $\theta\in(0,\frac{1}{2})$, there exists a pullback exponential attractor $\mathcal{E}=\mathcal{E}^\theta=\{\mathcal{E}(t):t\in\mathbb{R}\}=\{e_1(\mathcal{E}_\ell(t-\ell)):t\in\mathbb{R}\}$ for the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1).

Proof. From Lemma 2.3, Theorem 4.2 and Theorem 4.3, we know that for any $t\in\mathbb{R}$, $\mathcal{E}(t)=e_1(\mathcal{E}_\ell(t-\ell))$ is compact and its fractal dimension is uniformly finite. As a result of $L(t-\ell,s-\ell)\mathcal{E}_\ell(s-\ell)\subset\mathcal{E}_\ell(t-\ell)$ for any $t\ge s$, we have

$$U(t,s)\mathcal{E}(s)=U(t,s)e_1(\mathcal{E}_\ell(s-\ell))=e_1(L(t-\ell,s-\ell)\mathcal{E}_\ell(s-\ell))\subset e_1(\mathcal{E}_\ell(t-\ell))=\mathcal{E}(t)$$

for any $t\ge s$. From the definition of $B_2(t)$ and $B_0^\ell(t)$, we deduce that for any $t\in\mathbb{R}$ and any bounded subset B of H, there exists some time $\bar{t}=\bar{t}(B)$ such that

$$U(t,t-\tau)B\subset B_2(t)=e_0(B_0^\ell(t))$$

for any $\tau\ge\bar{t}$, which implies that there exists some natural number n_0 with $n_0t_1\ge\bar{t}$ such that $L(t,t-nt_1)B^\ell\subset B_0^\ell(t)$ for any $n\ge n_0$. Therefore, for any $s\ge(n_0+1)t_1$, there exists some $k_0 \in \mathbb{N}$ and $s_1 \in [0,t_1)$ such that $s = k_0 t_1 + s_1$, we conclude from Theorem 4.3 and inequality (4.11) that

$$\begin{aligned} dist_X(U(t,t-s)B,\mathcal{E}(t)) &= dist_X(U(t,t-s+\bar{t})U(t-s+\bar{t},t-s)B,\mathcal{E}(t)) \\ &\le dist_X(U(t,t-s+\bar{t})B_2(t-s+\bar{t}),\mathcal{E}(t)) \\ &= dist_X(U(t,t-s+\bar{t})e_1(B_0^\ell(t-s+\bar{t}-\ell)),e_1(\mathcal{E}_\ell(t-\ell))) \\ &= dist_X(e_1(L(t-\ell,t-s+\bar{t}-\ell)B_0^\ell(t-s+\bar{t}-\ell)),e_1(\mathcal{E}_\ell(t-\ell))) \\ &\le \eta\, dist_X(L(t-\ell,t-s+\bar{t}-\ell)B_0^\ell(t-s+\bar{t}-\ell),\mathcal{E}_\ell(t-\ell)), \end{aligned}$$

which implies that the family $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ exponentially attracts all bounded subsets of H uniformly. Therefore, the family $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ is a pullback exponential attractor for the process $\{U(t,\tau)\}_{t\ge\tau}$ in H. □

COROLLARY 4.1. *Assume that $Q \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies*

$$\sup_{r\in\mathbb{R}} \int_{r-1}^{r} \|Q(s)\|_2^2 ds < +\infty.$$

Then the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) possesses a pullback attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} = \{e_1(A_\ell(t-\ell)) : t \in \mathbb{R}\}$, where $A_\ell(t-\ell)$ is the section of pullback attractor $\hat{\mathcal{A}}_\ell = \{A_\ell(t) : t \in \mathbb{R}\}$ in X_ℓ for the process $\{L(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) obtained in Theorem 3.5.

REMARK 4.1. Assume that $Q \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies

$$\sup_{r\in\mathbb{R}} \int_{r-1}^{r} \|Q(s)\|_2^2 ds < +\infty.$$

Then each member $\mathcal{E}(t)$ of the pullback exponential attractor $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ for the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) in H contains the section $A(t)$ of the pullback attractor established in Corollary 4.1.

REMARK 4.2. Assume that $Q \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies

$$\sup_{r\in\mathbb{R}} \int_{r-1}^{r} \|Q(s)\|_2^2 ds < +\infty.$$

If the Hölder continuity in time of the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) in H can be obtained, the exponential attractor $\mathcal{M} = \{M(t) : t \in \mathbb{R}\}$ for the time continuous process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (2.1) can be constructed in the usual way:

$$\mathcal{M}_\ell(t) = \bigcup_{s\in[0,t_1]} L(t,s)\mathcal{E}_\ell(s)$$

and

$$M(t) = e_1(\mathcal{M}_\ell(t-\ell)).$$

REFERENCES

[1] C.S. Cao and E.S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large-scale ocean and atmosphere dynamics*, Ann. Math., 166(1):245–267, 2007. 1, 2.1, 2.2, 3.1, 3.1, 3.2, 3.2, 3.2

[2] T. Caraballo and P.E. Kloeden, *Non-autonomous attractor for integro-differential evolution equations*, Discrete Contin. Dyn. Syst. Ser. S, 2(1):17–36, 2009. 1

[3] T. Caraballo, J.A. Langa, and J.C. Robinson, *Upper semicontinuity of attractors for small random perturbations of dynamical systems*, Commun. Partial Differ. Equ., 23(9-10):1557–1581, 1998. 1

[4] T. Caraballo, G. Łukasiewicz, and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal., 64(3):484–498, 2006. 1

[5] A.N. Carvalho, J.A. Langa, and J.C. Robinson, *Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems*, Springer, New York, 2013. 1

[6] A.N. Carvalho and S. Sonner, *Pullback exponential attractors for evolution processes in Banach spaces: theoretical results*, Commun. Pure Appl. Anal., 12(6):3047–3071, 2013. 1, 2.3

[7] D.N. Cheban, *Global Attractors of Nonautonomous Dissipative Dynamical Systems*, World Scientific, 2004. 1

[8] D.N. Cheban, P.E. Kloeden, and B. Schmalfuß, *The relationship between pullback, forward and global attractors of nonautonomous dynamical systems*, Nonlinear Dyn. Syst. Theory, 2(2):125–144, 2002. 1

[9] V.V. Chepyzhov and M.I. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc., Providence, RI, 2002. 1, 2.2

[10] I. Chueshov, *Dynamics of Quasi-Stable Dissipative Systems*, Springer, Cham., 2015. 1

[11] R. Czaja and M. Efendiev, *Pullback exponential attractors for nonautonomous equations. Part I: Semilinear parabolic problems*, J. Math. Anal. Appl., 381(2):748–765, 2011. 1, 2.3

[12] A. Debussche, N. Glatt-Holtz, R. Temam, and M. Ziane, *Global existence and regularity for the 3D stochastic primitive equations of the ocean and atmosphere with multiplicative white noise*, Nonlinearity, 25(7):2093–2118, 2012. 1

[13] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Res. Appl. Math. 37, Masson, Paris, 1994. 1

[14] M. Efendiev, A. Miranville, and S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in* $\mathbb{R}^3$, C.R. Acad. Sci. Paris Sér. I Math., 330(8):713–718, 2000. 1

[15] M. Efendiev, A. Miranville, and S. Zelik, *Infinite dimensional exponential attractors for a non-autonomous reaction-diffusion system*, Math. Nachr., 248-249(1):72–96, 2003. 1

[16] M. Efendiev, S. Zelik, and A. Miranville, *Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems*, Proc. Roy. Soc. Edinb. Sect. A, 135(4):703–730, 2005. 1, 2.3

[17] L.C. Evans and R. Gastler, *Some results for the primitive equations with physical boundary conditions*, Z. Angew. Math. Phys., 64(6):1729–1744, 2013. 1

[18] B.L. Guo and D.W. Huang, *Existence of weak solutions and trajectory attractors for the moist atmospheric equations in geophysics*, J. Math. Phys., 47(8):083508, 2006. 1

[19] B.L. Guo and D.W. Huang, *3D stochastic primitive equations of the large-scale ocean: global well-posedness and attractors*, Commun. Math. Phys., 286(2):697–723, 2009. 1

[20] B.L. Guo and D.W. Huang, *Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere*, J. Differ. Equ., 251(3):457–491, 2011. 1

[21] N. Ju, *The global attractor for the solutions to the 3D viscous primitive equations*, Discrete Contin. Dyn. Syst., 17(1):159–179, 2007. 1, 3.1, 3.1, 3.2

[22] N. Ju, *The finite dimensional global attractor for the 3D viscous primitive equations*, Discrete Contin. Dyn. Syst., 36(12):7001–7020, 2016. 1

[23] N. Ju and R. Temam, *Finite dimensions of the global attractor for 3D primitive equations with viscosity*, J. Nonlinear Sci., 25(1):131–155, 2015. 1

[24] G.M. Kobelkov, *Existence of a solution "in the large" for ocean dynamics equations*, J. Math. Fluid Mech., 9(4):588–610, 2007. 3.1

[25] J.A. Langa, A. Miranville, and J. Real, *Pullback exponential attractors*, Discrete Contin. Dyn. Syst., 26(4):1329–1357, 2010. 1

[26] J.L. Lions, O.P. Manley, R. Temam, and S. Wang, *Physical interpretation of the attractor dimension for the primitive equations of atmospheric circulation*, J. Atmos. Sci., 54(9):1137–1143, 1997. 1

[27] J.L. Lions, R. Temam, and S. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity, 5(2):237–288, 1992. 1

[28] J.L. Lions, R. Temam, and S. Wang, *On the equations of the large-scale ocean*, Nonlinearity,

5(5):1007–1053, 1992. 1, 3.1

[29] G. Łukaszewicz and J.C. Robinson, *Invariant measures for non-autonomous dissipative dynamical systems*, Discrete Contin. Dyn. Syst., 34(10):4211–4222, 2014. 2.1

[30] J. Málek and D. Pražák, *Large time behavior via the method of ℓ-trajectories*, J. Differ. Equ., 181(2):243–279, 2002. 1, 2.3

[31] T. Tachim Medjo, *On the uniqueness of z-weak solutions of the three-dimensional primitive equations of the ocean*, Nonlinear Anal. Real World Appl., 11(3):1413–1421, 2010. 1

[32] T. Tachim Medjo, *Non-autonomous 3D primitive equations with oscillating external force and its global attractor*, Discrete Contin. Dyn. Syst., 32(1):265–291, 2012. 1

[33] T. Tachim Medjo, *Averaging of a 3D primitive equations with oscillating external forces*, Appl. Anal., 92(5):869–900, 2013. 1

[34] J. Pedlosky, *Geophysical Fluid Dynamics*, New York, Springer, 1987. 1, 1

[35] J.C. Robinson, *Infinite-dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001. 1, 2.2

[36] B. Schmalfuß, *Attractors for the non-autonomous dynamical systems*, in B. Fiedler (ed.), Proc. Int. Conf. Differ. Equ., World Sci. Publ., 99:684–689, 2000. 1

[37] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Appl. Math. Sci., Springer-Verlag, New York, 68, 1997. 1, 2.2, 3.1

[38] B. You and F. Li, *Global attractor of the three-dimensional primitive equations of large-scale ocean and atmosphere dynamics*, Z. Angew. Math. Phys., 69(5):114, 2018. 1

[39] B. You, *Pullback exponential attractors for some non-autonomous dissipative dynamical systems*, Math. Meth. Appl. Sci., 44(13):10361–10386, 2021. 1