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GLOBAL SMALL SOLUTIONS TO HEAT CONDUCTIVE COMPRESSIBLE NEMATIC LIQUID CRYSTAL SYSTEM: SMALLNESS ON A SCALING INVARIANT QUANTITY*

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Abstract. In this paper, we consider the Cauchy problem to the three dimensional heat conducting compressible nematic liquid crystal system in the presence of vacuum and with vacuum far fields. Global well-posedness of strong solutions is established under the condition that the scaling invariant quantity $(\|\rho_0\|_\infty+1)\big[\|\rho_0\|_3+(\|\rho_0\|_\infty+1)^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)\big]\big[\|\nabla u_0\|_2^2+(\|\rho_0\|_\infty+1)(\|\sqrt{\rho_0}E_0\|_2^2+\|\nabla^2 d_0\|_2^2)\big]$ is sufficiently small with the smallness depending only on the parameters appearing in the system.

Keywords. Heat conducting compressible nematic liquid crystal system; Global well-posedness; Vacuum; Scaling invariant quantity.

AMS subject classifications. 35Q35; 35D35; 76A15; 76N10.

1. Introduction

Liquid crystals are intermediate phases between solids and fluids. The continuum theory of liquid crystals was established by Ericksen [5] and Leslie [25] during the period of 1958 through 1968. The present paper concerns a simplified version of the general Ericksen-Leslie system, which roughly speaking is a coupled system of the compressible Navier-Stokes equations and the harmonic heat flow (see [30,32]). The equations of the heat conducting compressible nematic liquid system read as:

$$\rho_t+\operatorname{div}(\rho u)=0, \tag{1.1}$$

$$\rho(u_t+u\cdot\nabla u)+\nabla P=\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u-\Delta d\cdot\nabla d, \tag{1.2}$$

$$c_v\rho(\theta_t+u\cdot\nabla\theta)+P\operatorname{div}u-\kappa\Delta\theta=\mathcal{Q}(\nabla u)+|\Delta d+|\nabla d|^2 d|^2, \tag{1.3}$$

$$d_t+u\cdot\nabla d=\Delta d+|\nabla d|^2 d, \tag{1.4}$$

where $\rho\in\mathbb{R}_+$ is the density of the fluid, $u\in\mathbb{R}^3$ is the velocity field and $d\in\mathbb{S}^2$ represents macroscopic average of the nematic liquid crystal orientation field, with $\mathbb{S}^2$ denoting the unit spherical surface in $\mathbb{R}^3$. Here, $P=R\rho\theta$ is the pressure with R being a positive constant, λ and μ are constant viscosity coefficients satisfying the physical conditions $\mu>0$ and $2\mu+3\lambda\geq 0$, heat capacity $c_v=\frac{R}{\gamma-1}$ with $\gamma>1$ being the adiabatic constant, $\kappa>0$ is the heat conductive constant coefficient, and

$$\mathcal{Q}(\nabla u)=\frac{\mu}{2}|\nabla u+(\nabla u)^t|^2+\lambda(\operatorname{div}u)^2.$$

The additional assumption that $2\mu>\lambda$ will also be used in this paper.

Mathematical analysis of the nematic liquid crystals have attracted a lot of attention for several decades. For the incompressible case, Lin [30] first introduced and studied a

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simplified Ericksen-Leslie system modeling the incompressible liquid crystal flows. From then on, the solvability and stability of the incompressible liquid crystal flows have been substantially developed. The global existence and partial regularity of weak solutions to the Ginzburg-Landau type approximation system were obtained by Lin and Liu [32,33], while the global existence of weak solutions to the original system was proved by Lin et al. [31], Hong [12], and Hong and Xin [13] for two-dimensional case, and by Lin and Wang [35] for three-dimensional case under some geometric assumptions on the initial director field d_0. Uniqueness of weak solutions in two dimensions was proved by Lin and Wang [34], Li et al. [27], and Wang et al. [40], while the nonuniqueness of weak solutions and finite-time blow up of classic solutions in three dimensions was, respectively, addressed by Gong et al. [10] and Huang et al. [16]. Feireisl et al. [7, 8] considered a non-isothermal Ginzburg-Landau model of nematic liquid crystals and investigated the global existence of weak solutions, while the global existence of weak solutions to the corresponding non-isothermal Ericksen-Leslie system in two dimensions was established by Li and Xin [28]. One can refer to De Anna and Liu [2] for the derivation of the general non-isothermal Ericksen-Leslie system. Huang and Wang [17] established a blow-up criterion for the short-time classical solutions to incompressible liquid crystal flows in dimensions two and three. Hong et al. [14] established the local well-posedness and blow-up criteria of strong solutions to the liquid crystal system with general Oseen-Frank free energy density.

Concerning the compressible case, the model of the liquid crystals becomes more complicated since the density variation affects the mechanical behaviour of the fluid. Global well-posedness of the isentropic compressible nematic liquid crystals in one dimension was proved by Ding et al. [3, 4], while the global existence of weak solutions in multi-dimensions was proved by Jiang et al. [22, 23] under a smallness condition on the third component of initial orientation field. The local existence of unique strong solution to the initial value or initial-boundary value problem was proved in Huang et al. [18,19], where a series of blow-up criterion of strong solutions were established as well. Li et al. [29] obtained the global classical solutions to the Cauchy problem with small initial energy but possibly large oscillations and the initial density may allow vacuum. The long-time behavior of classical solution was considered in [9]. By virtue of the Fourier splitting method, the authors built optimal temporal decay rate of the global solution. For more results on simplified isothermal Ericksen-Leslie system, the readers can refer to [15, 36, 38, 39] and references therein.

Inspired by the introduction of non-isothermal models of incompressible nematic liquid crystals by Feireisl et al. in [7, 8], the compressible non-isothermal nematic liquid crystal flows are now attracting increasing research attention. Fan et al. [6] first investigated the local existence of unique strong solution to the initial boundary value problem. Guo et al. [11] obtained the global existence of smooth solutions for the Cauchy problem provided that the initial datum is close to a steady state and gave the algebraic decay rate of the global solution. A blow up criterion was established in [42] for the strong solutions to the two-dimensional non-isothermal flows in a bounded domain under a geometric condition introduced by Lei et al. in [24]. Recently, Liu and Zhong [37] proved that the global well-posedness of strong solutions with the initial data can have compact support provided that the quantity $\|\rho_0\|_{L^\infty} + \|\nabla d_0\|_{L^3}$ is suitably small with the smallness depending not only on the parameters involved in the system, but also on some high order norm of the initial data.

The purpose of this paper is to investigate the global well-posedness of strong solutions to the Cauchy problem of (1.1)-(1.4) along with the initial condition:

$$(\rho,u,\theta,d)|_{t=0}=(\rho_0,u_0,\theta_0,d_0). \tag{1.5}$$

The initial data allows far field vacuum and the smallness assumption, which depends only on the parameters involved in the system, is imposed exclusively on some quantities that are scaling invariant with respect to the following scaling transform:

$$(\rho_{0\lambda}(x),u_{0\lambda}(x),\theta_{0\lambda}(x),d_{0\lambda}(x))=(\rho_0(\lambda x),\lambda u_0(\lambda x),\lambda^2\theta_0(\lambda x),d_0(\lambda x)),\quad \text{for any} \quad \lambda\neq 0. \tag{1.6}$$

This scaling transform on the initial data is motivated from the natural scaling invariant property of the compressible nematic liquid crystal flow (1.1)-(1.4):

$$\rho_\lambda(x,t)=\rho(\lambda x,\lambda^2 t),\ u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t),\ \theta_\lambda(x,t)=\lambda^2\theta(\lambda x,\lambda^2 t),\ d_\lambda(x,t)=d(\lambda x,\lambda^2 t).$$

More precisely, if (ρ,u,θ,d) is a solution with the initial data $(\rho_0,u_0,\theta_0,d_0)$, then, by straightforward calculations, one can find that $(\rho_\lambda,u_\lambda,\theta_\lambda,d_\lambda)$ is also a solution with the transformed initial data $(\rho_{0\lambda},u_{0\lambda},\theta_{0\lambda},d_{0\lambda})$ for any nonzero λ.

As already explained in [26], imposing smallness assumptions on the scaling invariant quantities is necessary for obtaining globally well-posed system (1.1)-(1.5). In fact, if assuming that $\mathcal{M}$ is a functional such that

$$\mathcal{M}(\rho_{0\lambda},u_{0\lambda},\theta_{0\lambda},d_{0\lambda})=\lambda^\ell(\rho_0,u_0,\theta_0,d_0),\quad \text{for any } \lambda\neq 0 \ \text{ and some constant } \ell\neq 0,$$

and that the global well-posedness holds for any initial data $(\rho_0,u_0,\theta_0,d_0)$ satisfying

$$\mathcal{M}(\rho_0,u_0,\theta_0,d_0)\le\varepsilon_0,$$

then, by suitably choosing the scaling parameter λ, one can show that the global well-posedness holds for arbitrary large initial data. However, such global well-posedness for arbitrary large initial data is far from what we have already known.

Throughout this paper, the following notations are needed. For $1\le p\le\infty$, denote $L^p=L^p(\mathbb{R}^3)$ as the standard L^p Lebesgue spaces with the norm $\|\cdot\|_p$. For $1\le p\le\infty$ and positive integer k, denote by $W^{k,p}=W^{k,p}(\mathbb{R}^3)$ the standard Sobolev spaces, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$ or $\|\cdot\|_{H^k}$ with $H^k=W^{k,2}$. To simplify the expressions, the norm $\sum_{i=1}^K\|f_i\|_X$ or $\left(\sum_{i=1}^K\|f_i\|_X^2\right)^{\frac12}$ are sometimes denoted by $\|(f_1,f_2,\dots,f_K)\|_X$. For $1\le r\le\infty$, $D^{k,r}$ is the homogeneous Sobolev space, which is defined by

$$D^{k,r}=\left\{u\in L^1_{loc}(\mathbb{R}^3))\ \middle|\ \|\nabla^k u\|_r<\infty\right\},\quad D^k=D^{k,2},$$
$$D^1_0=\left\{u\in L^6(\mathbb{R}^3))\ \middle|\ \|\nabla u\|_2<\infty\right\}.$$

For simplicity, let

$$\int f\,dx=\int_{\mathbb{R}^3}f\,dx.$$

Definition 1.1. *Let $T>0$. (ρ,u,θ,d) is called a strong solution to the compressible nematic liquid crystal flow (1.1)-(1.4) in $\mathbb{R}^3\times(0,T)$ with initial condition (1.5), if for some $q\in(3,6]$,*

$$\rho\in C([0,T];H^1\cap W^{1,q}),\quad (u,\theta)\in C([0,T];D^1_0\cap D^2)\cap L^2(0,T;D^{2,q}),$$

$$\nabla d\in C([0,T];H^2)\cap L^2(0,T;H^3),\quad \rho_t\in C([0,T];L^2\cap L^q),\quad (u_t,\theta_t)\in L^2(0,T;D_0^1),$$
$$(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty(0,T;L^2),\quad d_t\in C([0,T];H^1)\cap L^2(0,T;H^2),\quad |d|=1,$$

and (ρ,u,θ,d) *satisfies (1.1)-(1.4) a.e. in* $\mathbb{R}^3\times(0,T)$ *and fulfills the initial condition (1.5).*

THEOREM 1.1. *Assume that the initial data* ρ_0,u_0,θ_0 *and* d_0 *satisfy*

$$\rho_0,\theta_0\ge 0,\quad \rho_0\in H^1\cap W^{1,q},\quad \sqrt{\rho_0}\theta_0\in L^2,$$
$$(u_0,\theta_0)\in D_0^1\cap D^2,\quad \nabla d_0\in H^2,\quad \text{and } |d_0|=1,$$

for some $q\in(3,6]$. *Set* $\overline{\rho}:=\|\rho_0\|_\infty+1$. *In addition, the following compatibility conditions*

$$-\mu\Delta u_0-(\mu+\lambda)\nabla \mathrm{div}u_0+\nabla P_0+\Delta d_0\cdot\nabla d_0=\sqrt{\rho_0}g_1, \tag{1.7}$$
$$\kappa\Delta\theta_0+\mathcal{Q}(\nabla u_0)+|\Delta d_0+|\nabla d_0|^2d_0|^2=\sqrt{\rho_0}g_2, \tag{1.8}$$

hold with $g_1,\ g_2\in L^2$, *where* $P_0=R\rho_0\theta_0$.

Then, there is a positive constant ε_0 *depending only on* R,γ,μ,λ, *and* κ, *such that if*

$$\mathcal{N}_0:=\overline{\rho}\left[\|\rho_0\|_3+\overline{\rho}^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)\right]\left[\|\nabla u_0\|_2^2+\overline{\rho}(\|\sqrt{\rho_0}E_0\|_2^2+\|\nabla^2 d_0\|_2^2)\right]\le\varepsilon_0,$$

where $E_0=\frac{|u_0|^2}{2}+c_v\theta_0$, *the problem (1.1)-(1.5) has a unique global strong solution.*

REMARK 1.1. It is obvious that if there is an initial data $(\rho_0,u_0,\theta_0,d_0)$ satisfying $\mathcal{N}_0\le\varepsilon_0$, then, any $(\rho_{0\lambda},u_{0\lambda},\theta_{0\lambda},d_{0\lambda})$ defined by the scaling transform (1.6) with $\lambda\neq 0$ also satisfies $\mathcal{N}_0\le\varepsilon_0$.

REMARK 1.2. The global well-posedness of strong solutions to the Cauchy problem for compressible non-isothermal nematic liquid crystal flows with vacuum as far field density has recently been proved in [37], which needs the small initial data satisfying

$$\begin{aligned}&\|\rho_0\|_{L^\infty}+\|\nabla d_0\|_{L^3}\\ \le&\epsilon_0=\epsilon_0(\|\rho_0\|_1,\|\sqrt{\rho_0}u_0\|_2,\|\nabla u_0\|_2,\|\sqrt{\rho_0}E_0\|_2,\|\nabla d_0\|_2,\|\nabla^2 d_0\|_2,\mu,\lambda,R,\gamma,\kappa).\end{aligned}$$

Note that the explicit dependence of ϵ_0 on the initial norms was not derived in [37]. Therefore, the scaling invariant quantities may not be expected there.

Note that since the system (1.1)-(1.5) contains the full compressible Navier-Stokes equations as a subsystem, it inherits the difficulties of the full compressible Navier-Stokes equations. A typical difficulty is that the basic energy estimate does not yield the desired dissipation estimates $\int_0^T\|\nabla u\|_2^2dt$. Another difficulty is that the presence of the liquid crystal director field d brings strong coupling term $u\cdot\nabla d$ and nonlinear terms $\nabla d\cdot\Delta d$ and $|\nabla d|^2 d$. In order to overcome these difficulties, we adopt the idea in [26] to get the $L^\infty(0,T;L^3)$ estimate of ρ and introduce the spatial L^2-norm of ∇d and $\nabla^2 d$. This motivates us to put smallness assumptions on $\|\rho_0\|_\infty^2\|\sqrt{\rho_0}u_0\|_2\|\sqrt{\rho_0}|u_0|^2\|_2$ and $\|\nabla d_0\|_2\|\nabla^2 d_0\|_2$, which are both scaling invariant. As a result, by continuity arguments, some necessary lower order time-independent estimates are obtained. Then, we give higher order estimates and eliminate the impact of vacuum by introducing the effective viscous flux and the material derivative.

The paper is organized as follows. In Section 2 and Section 3, we derive some lower order and higher order a priori estimates for the solutions to the Cauchy problem (1.1)-(1.5), respectively. Section 4 is devoted to proving the global well-posedness.

2. Time independent lower order a priori estimates

This section is devoted to the low order time independent a priori estimates to strong solutions to system (1.1)-(1.5), under suitable smallness assumption on the initial data.

We begin with the local existence and uniqueness of strong solutions whose proof can be performed in a similar way as in [1] and [6].

LEMMA 2.1 (local well-posedness). Assume that the initial data $(\rho_0\geq 0, u_0, \theta_0\geq 0, d_0)$ satisfies the conditions in Theorem 1.1. Then, for any $\Phi_0>0$ satisfying

$$\|\rho_0\|_{W^{1,q}\cap H^1}+\|(u_0,\theta_0)\|_{D_0^1\cap D^2}+\|\nabla d_0\|_{H^2}+\|(\sqrt{\rho_0}\theta_0,g_1,g_2)\|_2\leq\Phi_0,$$

there exist a positive $T_0>0$, which depends on Φ_0, $R, c_v, \mu, \lambda, \kappa$ and such that system (1.1)-(1.5) admits a unique strong solution in $\mathbb{R}^3\times(0,T_0)$.

By applying Lemma 2.1 inductively, one can extend the local solutions (ρ,u,θ,d) established in Lemma 2.1 uniquely to the maximal time interval $(0,T_{\max})$ of existence. Clearly, in order to show the global existence as stated in Theorem 1.1, it suffices to show that $T_{\max}=\infty$. To this end, we on one hand assume by contradiction that $T_{\max}<\infty$ and on the other hand will show (in the rest of this section and the next section) that some high order norms (as high regularities as the initial data) of the solution are uniformly bounded on the time interval $(0,T_{\max})$. Thanks to the uniform boundedness of the high order norms and by Lemma 2.1, one can further extend the solution beyond $T_{\max}$, which contradicts to the definition of $T_{\max}$, leading to $T_{\max}=\infty$.

The desired estimates are divided into two kinds: the lower order (one derivative lower than those of the initial data) a priori estimates being carried out in the rest of this section and the higher order (as high order derivatives as those of the initial data) a priori estimates being carried out in the next section, Section 3. The low order a priori estimates achieved in this section, see Proposition 2.1, are independent of the length of the time interval, under suitable smallness assumption on the initial data; however, they are insufficient to extend the solution beyond $T_{\max}$ and, thus, the higher order a priori estimates are required. Different from the lower order estimates in this section, the higher order estimates in Section 3 will depend on the length of the time interval, and it may grow if $T_{\max}$ grows; however, this will be enough to show the global well-posedness by using the contradicting arguments.

In the rest of this section, as well as in the next section, we always assume that (ρ,u,θ,d) has already been extended to the maximal existence time interval $(0,T_{\max})$, so that it is a strong solution to system (1.1)-(1.5) in $\mathbb{R}^3\times(0,T)$, for any $T\in(0,T_{\max})$.

The main result of this section is the following proposition, which is a direct corollary of Lemma 2.1 and Lemma 2.9, as below.

PROPOSITION 2.1. *For any $T\in(0,T_{max}]$, define*

$$\mathcal{N}_T:=\sup_{0\leq t<T}\overline{\rho}\big(\|\rho\|_3+\overline{\rho}^2(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)\big)\big(\|\nabla u\|_2^2+\overline{\rho}(\|\sqrt{\rho}E\|_2^2+\|\nabla^2 d\|_2^2)\big)(t), \quad (2.1)$$

let $\mathcal{N}_0$ be as in Theorem 1.1, and denote

$$G=(2\mu+\lambda)\mathrm{div}u-p,\quad \omega=\nabla\times u. \quad (2.2)$$

Then, there is a positive number ε_0 (will be given in Lemma 2.9, as below), such that

$$\mathcal{N}_{T_{max}}\leq\frac{\varepsilon_0}{2}\quad and\quad \sup_{0\leq t<T_{max}}\|\rho\|_\infty\leq 2\overline{\rho},$$

provided $\mathcal{N}_0 \le \varepsilon_0$.

Moreover, the following estimates hold:

$$\sup_{0\le t<T_{max}} \left(\|(\sqrt{\rho}E,\sqrt{\rho}u,\nabla u,d_t,\nabla d,\nabla^2 d)\|_2^2+\|\rho\|_3+\|\rho\|_\infty+\|\nabla d\|_4^4+\|\nabla d\|_3^3\right)\le C,$$

$$\int_0^{T_{max}} \|(\nabla\theta,|u|\nabla u,\sqrt{\rho}u_t,\nabla u,\nabla G,\nabla\omega,d_t,\nabla d_t,\nabla^2 d,\nabla^3 d,|\nabla d||\nabla^2 d|)\|_2^2 dt$$
$$+\int_0^{T_{max}}\|\nabla u\|_6^2 dt+\int_0^{T_{max}}\int \rho^3 P dxdt\le C,$$

where C depends only on $R,c_v,\mu,\lambda,\ \kappa,\bar{\rho},\|\rho_0\|_3,\|\sqrt{\rho_0}u_0\|_2,\|\sqrt{\rho_0}E_0\|_2,\|\nabla u_0\|_2,\|\nabla d_0\|_2,$ $\|\nabla^2 d_0\|_2$, and $\|\nabla d_0\|_3$.

First of all, we need a basic energy inequality.

LEMMA 2.2. *Assume that $2\mu>\lambda$. Then, for any $T\in(0,T_{max})$, it holds that*

$$\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\int_0^T(\|\nabla u\|_2^2+\|d_t\|_2^2+\|\nabla^2 d\|_2^2)dt$$
$$\le C(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)+C\int_0^T\|\rho\|_3^2\|\nabla\theta\|_2^2 dt+C\int_0^T\|\nabla d\|_3^2(\|\nabla u\|_2^2+\|\nabla^2 d\|_2^2)dt,$$

for a positive constant C depending only on μ and λ.

Proof. Multiplying (1.2) by u, integrating it over $\mathbb{R}^3$, and using integration by parts, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_2^2+\mu\|\nabla u\|_2^2+(\mu+\lambda)\|\text{div}u\|_2^2\\
&=-\int\nabla(R\rho\theta)\cdot udx-\int\Delta d\cdot\nabla d\cdot udx\\
&\le R\|\rho\|_3\|\theta\|_6\|\text{div}u\|_2+\|u\|_6\|\nabla d\|_3\|\nabla^2 d\|_2\\
&\le C\|\rho\|_3\|\nabla\theta\|_2\|\text{div}u\|_2+C\|\nabla u\|_2\|\nabla d\|_3\|\nabla^2 d\|_2\\
&\le(\mu+\lambda)\|\text{div}u\|_2^2+C\|\rho\|_3^2\|\nabla\theta\|_2^2+\frac{\mu}{2}\|\nabla u\|_2^2+C\|\nabla d\|_3^2\|\nabla^2 d\|_2^2.
\end{aligned}\tag{2.3}$$

Using (1.4) and the Sobolev inequality, it follows

$$\begin{aligned}
\frac{d}{dt}\|\nabla d\|_2^2+\int(|d_t|^2+|\nabla^2 d|^2)dx&=\int|d_t-\Delta d|^2dx=\int|u\cdot\nabla d-|\nabla d|^2 d|^2dx\\
&\le C(\|u\|_6^2\|\nabla d\|_3^2+\|\nabla d\|_4^4)\le C\|\nabla d\|_3^2(\|\nabla u\|_2^2+\|\nabla^2 d\|_2^2),
\end{aligned}\tag{2.4}$$

where $|\nabla d|^2=-\Delta d\cdot d$ guaranteed by $|d|=1$ was used. Adding (2.3) and (2.4) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\mu\|\nabla u\|_2^2+\|d_t\|_2^2+\|\nabla^2 d\|_2^2\\
&\le C\|\rho\|_3^2\|\nabla\theta\|_2^2+C\|\nabla d\|_3^2(\|\nabla u\|_2^2+\|\nabla^2 d\|_2^2).
\end{aligned}$$

The conclusion follows by integrating the above inequality with respect to t. ∎

Then, we derive the estimate on some necessary derivatives of d.

LEMMA 2.3. *For any $T\in(0,T_{max})$, it holds that*

$$\sup_{0\le t\le T}(\|\nabla^2 d\|_2^2+\|\nabla d\|_4^4)+\int_0^T(\|\nabla d_t\|_2^2+\|\nabla^3 d\|_2^2+\||\nabla d||\nabla^2 d|\|_2^2)dt$$
$$\le C(\|\nabla^2 d_0\|_2^2+\|\nabla d_0\|_4^4)+C\sup_{0\le t\le T}(\|\nabla u\|_2^2\|\nabla d\|_2^2)\int_0^T\|\nabla u\|_2^6 dt$$
$$+C\sup_{0\le t\le T}(\|\nabla d\|_2^3\|\nabla^2 d\|_2^3)\int_0^T\|\nabla^3 d\|_2^2 dt,$$

for an absolute positive constant C.

Proof. Applying the operator ∇ to (1.4) yields

$$\nabla d_t-\nabla\Delta d=-\nabla(u\cdot\nabla d)+|\nabla(\nabla d|^2 d),\tag{2.5}$$

from which one derives

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2 d\|_2^2+\int(|\nabla d_t|^2+|\nabla^3 d|^2)dx\\ =&\int|\nabla d_t-\nabla\Delta d|^2dx\\ =&\int|\nabla(u\cdot\nabla d)-|\nabla(\nabla d|^2 d)|^2dx\\ \le& C\int(|\nabla u|^2|\nabla d|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^6)dx+2\int|\nabla d|^2|\nabla^2 d|^2dx.\end{aligned}\tag{2.6}$$

Multiplying (2.5) by $4|\nabla d|^2\nabla d$, and then integrating it over $\mathbb{R}^3$, one obtains

$$\begin{aligned}&\frac12\frac{d}{dt}\int|\nabla d|^4dx+4\int(|\nabla d|^2|\nabla^2 d|^2+2|\nabla d|^2|\nabla(|\nabla d|)|^2)dx\\ =&4\int|\nabla d|^2\nabla d\big(-\nabla(u\cdot\nabla d)+\nabla(|\nabla d|^2 d)\big)dx\\ \le& C\int\big(|\nabla d|^4|\nabla u|+|\nabla d|^3|\nabla^2 d||u|+|\nabla d|^4|\nabla^2 d|+|\nabla d|^6\big)dx\\ \le&\int|\nabla d|^2|\nabla^2 d|^2dx+C\int(|\nabla u|^2|\nabla d|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^6)dx,\end{aligned}\tag{2.7}$$

where $|\nabla d|^2=-\Delta d\cdot d$ guaranteed by $|d|=1$ was used. Adding (2.6) and (2.7) yields

$$\begin{aligned}&\frac{d}{dt}\int(|\nabla^2 d|^2+|\nabla d|^4)dx+\int(|\nabla d_t|^2+|\nabla^3 d|^2+|\nabla d|^2|\nabla^2 d|^2)\\ \le& C\int(|\nabla u|^2|\nabla d|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^6)dx.\end{aligned}\tag{2.8}$$

It follows from the Gagliardo-Nirenberg and Young inequalities that

$$\begin{aligned}\int|\nabla u|^2|\nabla d|^2dx\le&\|\nabla d\|_\infty^2\|\nabla u\|_2^2\le C\|\nabla d\|_2^{\frac12}\|\nabla^3 d\|_2^{\frac32}\|\nabla u\|_2^2\\ \le&\frac14\|\nabla^3 d\|_2^2+C(\|\nabla d\|_2^2\|\nabla u\|_2^2)\|\nabla u\|_2^6,\end{aligned}\tag{2.9}$$

$$
\begin{aligned}
\int |\nabla d|^6 dx \leq & C\|\nabla d\|_6^2 \||\nabla d|^2\|_6 \||\nabla d|^2\|_2 \leq C\|\nabla d\|_6^2 \|\nabla(|\nabla d|^2)\|_2 \|\nabla d\|_4^2 \\
\leq & \frac{1}{2}\||\nabla d||\nabla^2 d|\|_2^2 + C\|\nabla^2 d\|_2^4 \|\nabla d\|_4^4 \\
\leq & \frac{1}{2}\||\nabla d||\nabla^2 d|\|_2^2 + C\|\nabla^2 d\|_2^4 \|\nabla d\|_3^2 \|\nabla^2 d\|_2^2 \\
\leq & \frac{1}{2}\||\nabla d||\nabla^2 d|\|_2^2 + C\|\nabla d\|_2^2 \|\nabla^3 d\|_2^2 \|\nabla d\|_3^2 \|\nabla^2 d\|_2^2 \\
\leq & \frac{1}{2}\||\nabla d||\nabla^2 d|\|_2^2 + C\|\nabla d\|_2^3 \|\nabla^2 d\|_2^3 \|\nabla^3 d\|_2^2, \qquad (2.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\int |u|^2 |\nabla^2 d|^2 dx \leq & C\|u\|_6^2 \|\nabla^2 d\|_6 \|\nabla^2 d\|_2 \leq C\|\nabla u\|_2^2 \|\nabla d\|_2^{\frac{1}{2}} \|\nabla^3 d\|_2^{\frac{3}{2}} \\
\leq & \frac{1}{4}\|\nabla^3 d\|_2^2 + C(\|\nabla d\|_2^2 \|\nabla u\|_2^2)\|\nabla u\|_2^6. \qquad (2.11)
\end{aligned}
$$

Substituting (2.9)-(2.11) into (2.8) yields

$$
\begin{aligned}
& \frac{d}{dt}\int (|\nabla^2 d|^2 + |\nabla d|^4) dx + \frac{1}{2}\int (|\nabla d_t|^2 + |\nabla^3 d|^2 + |\nabla d|^2 |\nabla^2 d|^2) \\
\leq & C\|\nabla d\|_2^3 \|\nabla^2 d\|_2^3 \|\nabla^3 d\|_2^2 + C(\|\nabla d\|_2^2 \|\nabla u\|_2^2)\|\nabla u\|_2^6,
\end{aligned}
$$

which implies the conclusion by integrating in t. ∎

Then, we will show the following key estimate on the sum of the energy.

LEMMA 2.4. *For any $T \in (0, T_{max})$, it holds that*

$$
\begin{aligned}
& \sup_{0\leq t\leq T} \|\sqrt{\rho} E\|_2^2 + \int_0^T (\|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2) dt \\
\leq & C\|\sqrt{\rho_0} E_0\|_2^2 + C\int_0^T \|\rho\|_\infty \|\rho\|_3^{\frac{1}{2}} \|\sqrt{\rho}\theta\|_2 (\|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2) dt \\
& + C\sup_{0\leq t\leq T} \|\nabla^2 d\|_2^4 \int_0^T \|\nabla u\|_2^2 dt \\
& + C\sup_{0\leq t\leq T} (\|\nabla d\|_2^2 \|\nabla^2 d\|_2^2) \int_0^T \||\nabla d||\nabla^2 d|\|_2^2 dt + C\sup_{0\leq t\leq T} (\|\nabla d\|_2 \|\nabla^2 d\|_2) \int_0^T \|\nabla^3 d\|_2^2 dt,
\end{aligned}
$$

for a positive constant C depending only on R, c_v, μ, λ, and κ, where $E = \frac{|u|^2}{2} + c_v\theta$.

Proof. From the specific kinetic energy E, multiplying (1.2) by u and adding the resultant to (1.3), one has

$$
\rho(E_t + u\cdot\nabla E) + \mathrm{div}(uP) - \kappa\Delta\theta = \mathrm{div}(\mathcal{S}\cdot u) - \Delta d\cdot\nabla d\cdot u + |\Delta d + |\nabla d|^2 d|^2, \qquad (2.12)
$$

where $\mathcal{S} = \mu(\nabla u + (\nabla u)^t) + \lambda \mathrm{div} u I$. Then, multiplying (2.12) with E and integrating it over $\mathbb{R}^3$ yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \kappa c_v\|\nabla\theta\|_2^2 \leq & \frac{\kappa c_v}{2}\|\nabla\theta\|_2^2 + C\||u||\nabla u|\|_2^2 + C\int \rho^2\theta^2|u|^2 dx \\
& - \int (\Delta d\cdot\nabla d\cdot u) E dx + \int |\Delta d + |\nabla d|^2 d|^2 E dx
\end{aligned}
$$

and, thus,

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}E\|_2^2+\kappa c_v\|\nabla\theta\|_2^2\\ \le& C\||u||\nabla u|\|_2^2+C\int\rho^2\theta^2|u|^2dx-2\int(\Delta d\cdot\nabla d\cdot u)Edx+2\int|\Delta d+|\nabla d|^2d|^2Edx. \quad (2.13)\end{aligned}$$

One can rewrite the right-hand side of (1.2) in divergence form since

$$-\Delta d\cdot\nabla d=-\mathrm{div}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right),$$

where $\nabla d\odot\nabla d\triangleq(d_{x_i}\cdot d_{x_j})_{3\times3}$ and $\mathbb{I}_3$ denotes the identity matrix of order 3. By the Sobolev inequality and integration by parts, one deduces

$$\begin{aligned}&-\int(\Delta d\cdot\nabla d\cdot u)Edx=-\int\mathrm{div}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\cdot uEdx\\ =&\int\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right):\nabla(uE)dx\\ \le& C\|\nabla d\|_6^2\|\nabla u\|_2\|E\|_6+\frac{\kappa c_v}{16}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\int|\nabla d|^4|u|^2dx\\ \le&\frac{\kappa c_v}{8}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\|\nabla^2 d\|_2^4\|\nabla u\|_2^2+C\||u|^2\|_6\|\nabla d\|_2\|\nabla d\|_6\||\nabla d|^2\|_6\\ \le&\frac{\kappa c_v}{8}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\|\nabla^2 d\|_2^4\|\nabla u\|_2^2+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2,\end{aligned}$$

and

$$\begin{aligned}&\int|\Delta d+|\nabla d|^2d|^2E\le2\int|\Delta d|^2Edx+2\int|\nabla d|^4Edx\\ \le&2\int|\nabla d||\nabla^3 d|Edx+2\int|\nabla d||\nabla^2 d||\nabla E|dx+\int|\nabla d|^4|u|^2dx+2c_v\int|\nabla d|^4\theta dx\\ \le& C\|\nabla d\|_3\|\nabla^3 d\|_2\|E\|_6+C\|\nabla d\|_3\|\nabla^2 d\|_6\|\nabla E\|_2\\ &+C(\||u|^2\|_6+c_v\|\theta\|_6)\|\nabla d\|_2\|\nabla d\|_6\||\nabla d|^2\|_6\\ \le& C(\||u||\nabla u|\|_2+c_v\|\nabla\theta\|_2)(\|\nabla d\|_3\|\nabla^3 d\|_2+\|\nabla d\|_2\|\nabla^2 d\|_2\||\nabla d||\nabla^2 d|\|_2)\\ \le&\frac{\kappa c_v}{8}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\|\nabla d\|_3^2\|\nabla^3 d\|_2^2+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2.\end{aligned}$$

Putting the above two inequalities into (2.13) leads to

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}E\|_2^2+\frac{\kappa c_v}{2}\|\nabla\theta\|_2^2\le C\||u||\nabla u|\|_2^2+C\int\rho^2\theta^2|u|^2dx\\ &\quad+C\|\nabla^2 d\|_2^4\|\nabla u\|_2^2+C\|\nabla d\|_3^2\|\nabla^3 d\|_2^2+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2. \quad (2.14)\end{aligned}$$

To control the term $\||u||\nabla u|\|_2^2$ in (2.14), we need to multiply (1.2) with $|u|^2u$ to obtain that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2-\int(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u)\cdot|u|^2udx\\ =&-\int P\mathrm{div}(|u|^2u)dx-\int\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\mathrm{div}(|u|^2u)dx\end{aligned}$$

$$
\begin{aligned}
&\le C\int(\rho\theta+|\nabla d|^2)|\nabla u||u|^2dx\\
&\le\frac{1}{2}\left(\mu-\frac{\lambda}{2}\right)\||u||\nabla u|\|_2^2+C\int(\rho^2\theta^2|u|^2+|\nabla d|^4|u|^2)dx\\
&\le\frac{1}{2}\left(\mu-\frac{\lambda}{2}\right)\||u||\nabla u|\|_2^2+C\int\rho^2\theta^2|u|^2dx+C\||u|^2\|_6\|\nabla d\|_2\|\nabla d\|_6\||\nabla d|^2\|_6\\
&\le\left(\mu-\frac{\lambda}{2}\right)\||u||\nabla u|\|_2^2+C\int\rho^2\theta^2|u|^2dx+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2. \qquad (2.15)
\end{aligned}
$$

By direct computation, one has

$$
-\int(\mu\Delta u+(\mu+\lambda)\nabla\text{div}u)\cdot|u|^2udx\ge(2\mu-\lambda)\||u||\nabla u|\|_2^2.
$$

Hence, it follows from the above inequality and (2.15) that

$$
\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2+2(2\mu-\lambda)\||u||\nabla u|\|_2^2\le C\int\rho^2\theta^2|u|^2dx+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2. \qquad (2.16)
$$

Now, multiplying (2.16) by $N>0$, which is a sufficiently large number and depending only on R,c_v,μ,λ, and κ, then adding the resultant to (2.14), one gets

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}E\|_2^2+N\|\sqrt{\rho}|u|^2\|_2^2)+\frac{\kappa c_v}{2}\|\nabla\theta\|_2^2+(2\mu-\lambda)N\||u||\nabla u|\|_2^2\\
&\le C\|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\||u||\nabla u|\|_2+C\|\nabla^2 d\|_2^4\|\nabla u\|_2^2+C\|\nabla d\|_2\|\nabla^2 d\|_2\|\nabla^3 d\|_2^2\\
&\quad+C\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\||\nabla d||\nabla^2 d|\|_2^2, \qquad (2.17)
\end{aligned}
$$

where we have used the fact

$$
\int\rho^2\theta^2|u|^2dx\le C\|\sqrt{\rho}\theta\|_2\|\theta\|_6\||u|^2\|_6\|\rho\|_9^{\frac{3}{2}}\le C\|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\||u||\nabla u|\|_2. \qquad (2.18)
$$

The proof can be completed by integrating (2.17) over $[0,t]$. □

Then, we will get the crucial estimate on the time independent $L^\infty(0,T;L^3)$- norm of ρ.

LEMMA 2.5. *For any $T\in(0,T_{max})$, it holds that*

$$
\begin{aligned}
\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3Pdxdt\le C\|\rho_0\|_3^3+C\sup_{0\le t\le T}(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3)\\
+C\int_0^T(\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2)dt+C\int_0^T(\|\rho\|_\infty\|\rho\|_3^2\|\nabla^2 d\|_2^2)dt,
\end{aligned}
$$

for a positive constant C depending only on R,c_v,μ,λ, and κ.

Proof. If we apply the operator $\Delta^{-1}\text{div}$ to (1.2), it holds that

$$
\begin{aligned}
&\Delta^{-1}\text{div}(\rho u)_t+\Delta^{-1}\text{divdiv}(\rho u\otimes u)-(2\mu+\lambda)\text{div}u+P\\
&=-\Delta^{-1}\text{divdiv}\big(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\big). \qquad (2.19)
\end{aligned}
$$

Taking advantage of (1.1), we have

$$(\rho^3)_t+\mathrm{div}(u\rho^3)+2\mathrm{div}u\rho^3=0. \tag{2.20}$$

Then, multiplying (2.19) by ρ^3 and using above equality, we obtain

$$\begin{aligned}&\frac{2\mu+\lambda}{2}\big((\rho^3)_t+\mathrm{div}(u\rho^3)\big)+\rho^3P+\rho^3\Delta^{-1}\mathrm{div}(\rho u)_t+\rho^3\Delta^{-1}\mathrm{divdiv}(\rho u\otimes u)\\&=-\rho^3\Delta^{-1}\mathrm{divdiv}\big(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\big).\end{aligned} \tag{2.21}$$

Using (2.20), it follows

$$\begin{aligned}&\int\rho^3\Delta^{-1}\mathrm{div}(\rho u)_t dx\\&=\frac{d}{dt}\int\rho^3\Delta^{-1}\mathrm{div}(\rho u)dx+\int[\mathrm{div}(\rho^3 u)+2\mathrm{div}u\rho^3]\Delta^{-1}\mathrm{div}(\rho u)dx\\&=\int[2\mathrm{div}u\rho^3\Delta^{-1}\mathrm{div}(\rho u)-\rho^3 u\cdot\nabla\Delta^{-1}\mathrm{div}(\rho u)]dx+\frac{d}{dt}\int\rho^3\Delta^{-1}\mathrm{div}(\rho u)dx.\end{aligned}$$

Thanks to this, it follows from integrating (2.21) over $\mathbb{R}^3$ that

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{2\mu+\lambda}{2}+\Delta^{-1}\mathrm{div}(\rho u)\right)\rho^3dx+\int\rho^3Pdx\\&=\int\left[\rho^3\big(u\cdot\nabla\Delta^{-1}\mathrm{div}(\rho u)-\Delta^{-1}\mathrm{divdiv}(\rho u\otimes u)\big)-2\mathrm{div}u\rho^3\Delta^{-1}\mathrm{div}(\rho u)\right]dx\\&\quad-\int\rho^3\Delta^{-1}\mathrm{divdiv}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)dx.\end{aligned}$$

The conclusion in this lemma then follows from the same estimates as in Proposition 2.4 in [26] and the following bound for the last term in above equality

$$\begin{aligned}&\int\rho^3\left|\Delta^{-1}\mathrm{divdiv}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\right|dx\\&\le C\|\rho\|_\infty\|\rho\|_3^2\left\|\Delta^{-1}\mathrm{divdiv}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\right\|_3\\&\le C\|\rho\|_\infty\|\rho\|_3^2\|\nabla d\|_6^2\le C\|\rho\|_\infty\|\rho\|_3^2\|\nabla^2 d\|_2^2,\end{aligned}$$

where the elliptic estimates were applied. □

In order to obtain bound of $\|\rho\|_{L^\infty(0,T;L^\infty)}$, we need to introduce the effective viscous flux G and the curl of velocity ω, and establish the following estimates for them.

LEMMA 2.6. *Assume that*

$$\sup_{0\le t\le T}\|\rho\|_\infty\le 4\bar\rho.$$

Then, for any $T\in(0,T_{max})$, it holds that

$$\sup_{0\le t\le T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt$$

$$
\begin{aligned}
&\le C\|\nabla u_0\|_2^2+C\overline{\rho}\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_2^2+C\overline{\rho}^3\int_0^T\|\nabla u\|_2^4(\|\nabla u\|_2^2+\overline{\rho}\|\sqrt{\rho}\theta\|_2^2)dt\\
&\quad +C\int_0^T(\overline{\rho}+\overline{\rho}^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt\\
&\quad +C\|\nabla d_0\|_4^4+C\|\nabla d\|_4^4+\varepsilon_1\overline{\rho}\int_0^T\|\nabla\theta\|_2^2dt\\
&\quad +C\overline{\rho}\sup_{0\le t\le T}(\|\nabla d\|_2\|\nabla^2 d\|_2)\int_0^T(\|\nabla d_t\|_2^2+\|\nabla^3 d\|_2^2)dt\\
&\quad +C\overline{\rho}\sup_{0\le t\le T}(\|\nabla d\|_2^2\|\nabla^2 d\|_2^2)\int_0^T\|\nabla(|\nabla d|^2)\|_2^2dt,
\end{aligned}
$$

where G and ω are given by (2.2), and the constant $C>0$ depending only on R,c_v,μ,λ, and κ.

Proof. Multiplying (1.2) by u_t, it follows from integration by parts that

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\mu\|\nabla u\|_2^2+(\mu+\lambda)\|\mathrm{div}u\|_2^2)-\int P\mathrm{div}u_t dx+\|\sqrt{\rho}u_t\|_2^2\\
&=-\int\rho(u\cdot\nabla)u\cdot u_t dx-\int\Delta d\cdot\nabla d\cdot u_t dx. \qquad (2.22)
\end{aligned}
$$

By the definition of effective viscous flux G, it is easy to see $\mathrm{div}u=\frac{G+P}{2\mu+\lambda}$, which implies

$$
\begin{aligned}
-\int P\mathrm{div}u_t dx&=-\frac{d}{dt}\int P\mathrm{div}u dx+\int P_t\mathrm{div}u dx\\
&=-\frac{d}{dt}\int P\mathrm{div}u dx+\frac{1}{2(2\mu+\lambda)}\frac{d}{dt}\|P\|_2^2+\frac{1}{2\mu+\lambda}\int P_t G dx. \qquad (2.23)
\end{aligned}
$$

On the other hand, it follows from (1.3) that

$$
P_t=(\gamma-1)\big(\mathcal{Q}(\nabla u)-P\mathrm{div}u+\kappa\Delta\theta+|\Delta d+|\nabla d|^2d|^2\big)-\mathrm{div}(uP),
$$

which leads to

$$
\int P_t G dx=\int\left[(\gamma-1)\big(\mathcal{Q}(\nabla u)-P\mathrm{div}u+|\Delta d+|\nabla d|^2d|^2\big)G+\big(uP-\kappa(\gamma-1)\nabla\theta\big)\cdot\nabla G\right]dx. \qquad (2.24)
$$

Due to $\|\nabla u\|_2^2=\|\omega\|_2^2+\|\mathrm{div}u\|_2^2$ and combining with (2.22)-(2.24), one can deduce that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\|\sqrt{\rho}u_t\|_2^2\\
&=-\int\rho(u\cdot\nabla)u\cdot u_t dx-\int\Delta d\cdot\nabla d\cdot u_t dx+\frac{1}{2\mu+\lambda}\int\big(\kappa(\gamma-1)\nabla\theta-uP\big)\cdot\nabla G dx\\
&\quad-\frac{\gamma-1}{2\mu+\lambda}\int\big(\mathcal{Q}(\nabla u)-p\mathrm{div}u+|\Delta d+|\nabla d|^2d|^2\big)G dx. \qquad (2.25)
\end{aligned}
$$

In order to bound the right-hand side of (2.25), we need to reformulate (1.2) in the following form with the help of $\Delta u=\nabla\mathrm{div}u-\nabla\times\nabla\times u$:

$$
\rho(u_t+u\cdot\nabla u)=\nabla G-\mu\nabla\times\omega-\Delta d\cdot\nabla d. \qquad (2.26)
$$

Then, multiplying both sides of (2.26) by ∇G, it follows

$$\begin{aligned}\|\nabla G\|_2^2 =& \int \big(\rho(u_t+u\cdot\nabla u)\cdot\nabla G+\Delta d\cdot\nabla d\cdot\nabla G\big)dx\\ \le& \int \left(\frac{|\nabla G|^2}{2}+2\overline{\rho}\rho|u_t|^2\right)dx+\int \big(\rho(u\cdot\nabla)u\cdot\nabla G+\Delta d\cdot\nabla d\cdot\nabla G\big)dx\end{aligned}$$

where $\int \nabla G\cdot\nabla\times\omega dx=0$ and $\|\rho\|_\infty\le 4\overline{\rho}$ were used. This gives that

$$\frac{\|\nabla G\|_2^2}{16\overline{\rho}}\le \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2+\frac{1}{8\overline{\rho}}\int \big(\rho(u\cdot\nabla)u\cdot\nabla G+\Delta d\cdot\nabla d\cdot\nabla G\big)dx. \tag{2.27}$$

Similarly, one has

$$\frac{\mu^2\|\nabla\omega\|_2^2}{16\overline{\rho}}\le \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2+\frac{1}{8\overline{\rho}}\int \big(\rho(u\cdot\nabla)u\cdot\nabla\omega+\Delta d\cdot\nabla d\cdot\nabla\omega\big)dx. \tag{2.28}$$

Putting (2.27) and (2.28) into (2.25), one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\frac{1}{2}\|\sqrt{\rho}u_t\|_2^2+\frac{1}{16\overline{\rho}}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)\\ \le& C\int \rho|u||\nabla u|\left(|u_t|^2+\frac{1}{\overline{\rho}}(|\nabla G|+|\nabla\omega|)\right)dx+C\int(|\nabla\theta|+\rho\theta|u|)|\nabla G|dx\\ &+C\int(|\nabla u|^2+\rho\theta|\nabla u|)|G|dx-\int\Delta d\cdot\nabla d\cdot u_t dx+C\int\big(|\Delta d+|\nabla d|^2d|^2\big)Gdx\\ &+\frac{C}{\overline{\rho}}\int|\Delta d||\nabla d|(|\nabla G|+|\nabla\omega|)dx=:\sum_{i=1}^6 I_i.\end{aligned} \tag{2.29}$$

Estimates $I_i, i=1,2,\ldots,6$ are given as follows. It follows from the Hölder and Young inequalities that

$$\begin{aligned}I_1\le& C\sqrt{\overline{\rho}}\||u||\nabla u|\|_2\|\sqrt{\rho}u_t\|_2+C\||u||\nabla u|\|_2(\|\nabla G\|_2+\|\nabla\omega\|_2)\\ \le&\frac{1}{12}\|\sqrt{\rho}u_t\|_2^2+\frac{1}{192\overline{\rho}}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)+C\overline{\rho}\||u||\nabla u|\|_2^2,\end{aligned}$$

and

$$\begin{aligned}I_2\le& C\|\nabla\theta\|_2\|\nabla G\|_2+\|\rho\theta u\|_2\|\nabla G\|_2\\ \le& C\|\nabla\theta\|_2\|\nabla G\|_2+C\sqrt{\overline{\rho}}\|\rho\|_3^{\frac14}\|\sqrt{\rho}\theta\|_2^{\frac12}\|\nabla\theta\|_2^{\frac12}\||u||\nabla u|\|_2^{\frac12}\|\nabla G\|_2\\ \le&\frac{1}{192\overline{\rho}}\|\nabla G\|_2^2+C\big(\overline{\rho}^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2+\overline{\rho}\big)(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2),\end{aligned}$$

where (2.18) was used in I_2. For I_3, noticing that

$$\begin{aligned}\|\nabla u\|_6\le& C(\|\omega\|_6+\|\mathrm{div}u\|_6)\le C(\|\omega\|_6+\|G\|_6+\|\rho\theta\|_6)\\ \le& C(\|\nabla\omega\|_2+\|\nabla G\|_2+\overline{\rho}\|\nabla\theta\|_2),\end{aligned} \tag{2.30}$$

it follows from the Hölder, Young and Sobolev inequalities that

$$I_3\le C\|\nabla u\|_2\|\nabla u\|_6\|G\|_3+\|\nabla u\|_2\|\rho\theta\|_6\|G\|_3$$

$$\leq C\|\nabla u\|_2(\|\nabla G\|_2+\|\nabla\omega\|_2+\overline{\rho}\|\nabla\theta\|_2)\|G\|_2^{\frac{1}{2}}\|\nabla G\|_2^{\frac{1}{2}}+C\overline{\rho}\|\nabla u\|_2\|\nabla\theta\|_2\|G\|_2^{\frac{1}{2}}\|\nabla G\|_2^{\frac{1}{2}}$$
$$\leq\frac{1}{192\overline{\rho}}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)+C\overline{\rho}^3\|\nabla u\|_2^4\|G\|_2^2+C\overline{\rho}\|\nabla\theta\|_2^2.$$

Note that

$$I_4=-\int\mathrm{div}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\cdot u_t dx=\int(\nabla d\odot\nabla d):\nabla u_t dx-\frac{1}{2}\int|\nabla d|^2\mathrm{div}u_t dx.$$

Using the Young and Sobolev inequalities, one obtains

$$\begin{aligned}&\int(\nabla d\odot\nabla d):\nabla u_t dx\\ =&\frac{d}{dt}\int(\nabla d\odot\nabla d):\nabla u dx-\int(\nabla d_t\odot\nabla d):\nabla u dx-\int(\nabla d\odot\nabla d_t):\nabla u dx\\ \leq&\frac{d}{dt}\int(\nabla d\odot\nabla d):\nabla u dx+C\|\nabla d\|_3\|\nabla d_t\|_2\|\nabla u\|_6\\ \leq&\frac{d}{dt}\int(\nabla d\odot\nabla d):\nabla u dx+\frac{\varepsilon}{4\overline{\rho}}\|\nabla u\|_6^2+C\overline{\rho}\|\nabla d\|_3^2\|\nabla d_t\|_2^2,\end{aligned}$$

where $\varepsilon>0$ is a sufficiently small constant. Similarly,

$$\int|\nabla d|^2\mathrm{div}u_t dx\leq\frac{d}{dt}\int|\nabla d|^2\mathrm{div}u dx+\frac{\varepsilon}{4\overline{\rho}}\|\nabla u\|_6^2+C\overline{\rho}\|\nabla d\|_3^2\|\nabla d_t\|_2^2.$$

Therefore, we get

$$I_4\leq\frac{d}{dt}\int[(\nabla d\odot\nabla d):\nabla u+|\nabla d|^2\mathrm{div}u]dx+\frac{\varepsilon}{2\overline{\rho}}\|\nabla u\|_6^2+C\overline{\rho}\|\nabla d\|_3^2\|\nabla d_t\|_2^2.$$

Using (2.30) again, we obtain

$$\begin{aligned}I_4\leq&\frac{d}{dt}\int[(\nabla d\odot\nabla d):\nabla u+|\nabla d|^2\mathrm{div}u]dx\\&+\frac{1}{192\overline{\rho}}(\|\nabla\omega\|_2^2+\|\nabla G\|_2^2)+\frac{\varepsilon_1\overline{\rho}}{2}\|\nabla\theta\|_2^2+C\overline{\rho}\|\nabla d\|_3^2\|\nabla d_t\|_2^2,\end{aligned}$$

where $\varepsilon_1=\frac{C\varepsilon}{2}$ is sufficiently small.

Now, let us turn to I_5 and I_6. By virtue of the Hölder, Young and Sobolev inequalities, one deduces

$$\begin{aligned}I_5=&C\int\left(|\Delta d+|\nabla d|^2d|^2\right)G dx\\ \leq&C\int(|\nabla d||\nabla^3 d||G|+|\nabla d||\nabla^2 d||\nabla G|)dx+C\|\nabla d\|_2\|\nabla d\|_6\||\nabla d|^2\|_6\|G\|_6\\ \leq&C\|\nabla d\|_3\|\nabla^3 d\|_2\|\nabla G\|_2+C\|\nabla d\|_2\|\nabla^2 d\|_2\|\nabla(|\nabla d|^2)\|_2\|\nabla G\|_2\\ \leq&\frac{1}{192\overline{\rho}}\|\nabla G\|_2^2+C\overline{\rho}\|\nabla d\|_3^2\|\nabla^3 d\|_2^2+C\overline{\rho}\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\|\nabla(|\nabla d|^2)\|_2^2,\end{aligned}$$

where $|\Delta d+|\nabla d|^2d|^2=|\Delta d|^2+2\Delta d\cdot d|\nabla d|^2+|\nabla d|^4=|\Delta d|^2-|\nabla d|^4$ was used, and

$$I_6\leq\frac{C}{4\overline{\rho}}\|\Delta d\|_6\|\nabla d\|_3(\|\nabla G\|_2+\|\nabla\omega\|_2)\leq\frac{1}{192\overline{\rho}}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)+C\|\nabla d\|_3^2\|\nabla^3 d\|_2^2.$$

Putting all these estimates for $I_i, i=1,2,\ldots,6$ into (2.29) leads to

$$\begin{aligned}
&\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\frac{1}{2}\|\sqrt{\rho}u_t\|_2^2+\frac{1}{16\overline{\rho}}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)\\
\le&2\frac{d}{dt}\int[(\nabla d\odot\nabla d):\nabla u+|\nabla d|^2\mathrm{div}u]dx+C(\overline{\rho}+\overline{\rho}^2\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)\\
&+C\overline{\rho}^3\|\nabla u\|_2^4\|G\|_2^2+\varepsilon_1\overline{\rho}\|\nabla\theta\|_2^2+C\overline{\rho}\|\nabla d\|_3^2(\|\nabla d_t\|_2^2+\|\nabla^3 d\|_2^2)\\
&+C\overline{\rho}\|\nabla d\|_2^2\|\nabla^2 d\|_2^2\|\nabla(|\nabla d|^2)\|_2^2.
\end{aligned}\tag{2.31}$$

Note that

$$\|\nabla u\|_2\le C(\|\omega\|_2+\|G\|_2+\|\rho\theta\|_2)\le C(\|\omega\|_2+\|G\|_2+\sqrt{\overline{\rho}}\|\sqrt{\rho}\theta\|_2)\tag{2.32}$$

and

$$\int[(\nabla d\odot\nabla d):\nabla u+|\nabla d|^2\mathrm{div}u]dx\le C\|\nabla u\|_2\||\nabla d|^2\|_2\le\varepsilon_2\|\nabla u\|_2^2+C\|\nabla d\|_4^4.\tag{2.33}$$

Integrating (2.31) in t, substituting (2.32) and (2.33) into the resultant and choosing ε_2 small enough, one gets the desired result. ∎

LEMMA 2.7. *Assume that*

$$\sup_{0\le t\le T}\|\rho\|_\infty\le 4\overline{\rho}.$$

Then, for any $T\in(0,T_{max})$, it holds that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\rho\|_\infty\le\|\rho_0\|_\infty\\
&\cdot e^{C\overline{\rho}^{\frac{2}{3}}\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}+C\overline{\rho}\int_0^T\|\nabla u\|_2\|(\nabla G,\nabla\omega,\overline{\rho}\nabla\theta)\|_2dt+C\left(\int_0^T\|\nabla^2 d\|_2dt\int_0^T\|\nabla^3 d\|_2dt\right)^{\frac{1}{2}}},
\end{aligned}$$

for a positive constant C depending only on $R,c_v,\mu,\lambda,$ and κ, where G and ω are given by (2.2).

Proof. In view of (2.19), one has

$$\begin{aligned}
&\Delta^{-1}\mathrm{div}(\rho u)_t+u\cdot\nabla\Delta^{-1}\mathrm{div}(\rho u)-(2\mu+\lambda)\mathrm{div}u+P\\
&\quad+\Delta^{-1}\mathrm{div}\mathrm{div}\left(\nabla d\odot\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\\
=&u\cdot\nabla\Delta^{-1}\mathrm{div}(\rho u)-\Delta^{-1}\mathrm{div}\mathrm{div}(\rho u\otimes u)=[u,\mathcal{R}\otimes\mathcal{R}](\rho u),
\end{aligned}$$

where $\mathcal{R}$ is the Riesz transform on $\mathbb{R}^3$. To obtain the estimates of $\|\rho\|_\infty$, we adapt the arguments by [26]. Exactly in the same way as in Proposition 2.6 of [26], one can prove that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\rho\|_\infty\\
\le&\|\rho_0\|_\infty e^{C\overline{\rho}^{\frac{2}{3}}\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}+C\overline{\rho}\int_0^T\|\nabla u\|_2\|(\nabla G,\nabla\omega,\overline{\rho}\nabla\theta)\|_2dt+C\int_0^T\|\nabla d\|_\infty^2dt}.
\end{aligned}\tag{2.34}$$

Thanks to this and noticing that

$$\int_0^T \|\nabla d\|_\infty^2 dt \le C\int_0^T \|\nabla^2 d\|_2\|\nabla^3 d\|_2 dt \le C\left(\int_0^T \|\nabla^2 d\|_2 dt\right)^{\frac12}\left(\int_0^T \|\nabla^3 d\|_2 dt\right)^{\frac12},$$

the conclusion follows. □

Collecting Lemmas 2.2-2.7, we have the following estimates bounded by the initial data.

Lemma 2.8. *Let G and ω be given by (2.2) and $\mathcal{N}_T$ be given by (2.1). Then there is a positive constant η_0 depending only on R, c_v, μ, λ, and κ, such that if*

$$\eta \le \eta_0, \quad \sup_{0\le t\le T}\|\rho\|_\infty \le 4\bar\rho, \quad \text{and } \mathcal{N}_T \le \sqrt{\eta},$$

then it holds that

$$\sup_{0\le t\le T}\|\rho\|_3 + \left(\int_0^T\int \rho^3 P dxdt\right)^{\frac13} \le C\big(\|\rho_0\|_3 + \bar\rho^2(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2)\big), \tag{2.35}$$

$$\begin{aligned}&\bar\rho^2\left(\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2) + \int_0^T \|(\nabla u, d_t, \nabla^2 d)\|_2^2 dt\right)\\ &\le C\big(\|\rho_0\|_3 + \bar\rho^2(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2)\big),\end{aligned} \tag{2.36}$$

$$\begin{aligned}&\sup_{0\le t\le T}\left[\bar\rho(\|\nabla^2 d\|_2^2 + \|\nabla d\|_4^4 + \|\sqrt{\rho}E\|_2^2) + \|\nabla u\|_2^2\right]\\ &\quad + \int_0^T \left(\|(\nabla d_t, \nabla^3 d, |\nabla d||\nabla^2 d|, \nabla\theta, |u||\nabla u|)\|_2^2 + \left\|\left(\sqrt{\rho}u_t, \frac{\nabla G}{\sqrt{\bar\rho}}, \frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2\right)dt\\ &\le C\big(\bar\rho(\|\nabla^2 d_0\|_2^2 + \|\sqrt{\rho_0}E_0\|_2^2) + \|\nabla u_0\|_2^2\big),\end{aligned} \tag{2.37}$$

$$\sup_{0\le t\le T}\|\rho\|_\infty \le \bar\rho e^{C\mathcal{N}_0^{\frac16} + C\mathcal{N}_0^{\frac12}}, \tag{2.38}$$

for any $T\in(0, T_{max})$, where the constant $C>0$ depends only on R, c_v, μ, λ, and κ.

Proof. In view of Lemma 2.2 and by choosing $\eta_0 < 1$ small enough, we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2) + \int_0^T (\|\nabla u\|_2^2 + \|d_t\|_2^2 + \|\nabla^2 d\|_2^2)dt\\ &\le C(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2) + C\sup_{0\le t\le T}\|\rho\|_3^2\int_0^T \|\nabla\theta\|_2^2 dt.\end{aligned} \tag{2.39}$$

It follows from Lemma 2.3 and the assumptions that

$$\begin{aligned}&\bar\rho\sup_{0\le t\le T}(\|\nabla^2 d\|_2^2 + \|\nabla d\|_4^4) + \bar\rho\int_0^T (\|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2 + \||\nabla d||\nabla^2 d|\|_2^2)dt\\ &\le C\bar\rho(\|\nabla^2 d_0\|_2^2 + \|\nabla d_0\|_4^4) + C\eta^{\frac12}\int_0^T \|\nabla u\|_2^6 dt + C\eta^{\frac32}\int_0^T \|\nabla^3 d\|_2^2 dt.\end{aligned} \tag{2.40}$$

With the help of (2.39) and since $\overline{\rho}=\|\rho_0\|_\infty+1$, one deduces from the assumption that

$$\begin{aligned}\int_0^T\|\nabla u\|_2^6dt\le&\sup_{0\le t\le T}\|\nabla u\|_2^4\int_0^T\|\nabla u\|_2^2dt\\ \le& C\sup_{0\le t\le T}\|\nabla u\|_2^4\Big(\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\sup_{0\le t\le T}\|\rho\|_3^2\int_0^T\|\nabla\theta\|_2^2dt\Big)\\ \le& C\eta^{\frac12}\sup_{0\le t\le T}\|\nabla u\|_2^2+C\eta\int_0^T\|\nabla\theta\|_2^2dt,\end{aligned}$$

which together with (2.40) and by choosing η_0 small enough, implies that

$$\begin{aligned}&\overline{\rho}\sup_{0\le t\le T}(\|\nabla^2d\|_2^2+\|\nabla d\|_4^4)+\overline{\rho}\int_0^T(\|\nabla d_t\|_2^2+\|\nabla^3d\|_2^2+\||\nabla d||\nabla^2d|\|_2^2)dt\\ &\le C\overline{\rho}(\|\nabla^2d_0\|_2^2+\|\nabla d_0\|_4^4)+C\eta^{\frac12}\sup_{0\le t\le T}\|\nabla u\|_2^2+C\eta\int_0^T\|\nabla\theta\|_2^2dt.\end{aligned}\tag{2.41}$$

Next, applying Lemma 2.4, using the assumptions and (2.39), we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt\\ &\le C\|\sqrt{\rho_0}E_0\|_2^2+C\eta^{\frac14}\int_0^T(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt\\ &\quad+C\eta\int_0^T\|\nabla^3d\|_2^2dt+C\eta^{\frac12}\int_0^T\||\nabla d||\nabla^2d|\|_2^2dt\\ &\quad+C\sup_{0\le t\le T}\|\nabla^2d\|_2^4(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+C\sup_{0\le t\le T}(\|\nabla^2d\|_2^4\|\rho\|_3^2)\int_0^T\|\nabla\theta\|_2^2dt\\ &\le C\|\sqrt{\rho_0}E_0\|_2^2+C\eta^{\frac14}\int_0^T(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt\\ &\quad+C\eta\int_0^T\|\nabla^3d\|_2^2dt+C\eta^{\frac12}\int_0^T\||\nabla d||\nabla^2d|\|_2^2dt+C\eta^{\frac12}\sup_{0\le t\le T}\|\nabla^2d\|_2^2+C\eta\int_0^T\|\nabla\theta\|_2^2dt.\end{aligned}$$

This, combined with the fact η_0 is small enough, implies that

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt\\ &\le C\|\sqrt{\rho_0}E_0\|_2^2+C\eta\textstyle\int_0^T\|\nabla^3d\|_2^2dt+C\eta^{\frac12}\int_0^T\||\nabla d||\nabla^2d|\|_2^2dt+C\eta^{\frac12}\sup_{0\le t\le T}\|\nabla^2d\|_2^2.\end{aligned}\tag{2.42}$$

Then, using the assumptions and Sobolev inequality, it follows from Lemma 2.6 that

$$\begin{aligned}&\sup_{0\le t\le T}\|\nabla u\|_2^2+\int_0^T\|(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\rho}},\frac{\nabla\omega}{\sqrt{\rho}})\|_2^2dt\\ &\le C\|\nabla u_0\|_2^2+C\overline{\rho}\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2+C\overline{\rho}^3\int_0^t\|\nabla u\|_2^4(\|\nabla u\|_2^2+\overline{\rho}\|\sqrt{\rho}E\|_2^2)dt\\ &\quad+C\int_0^T(\overline{\rho}+\overline{\rho}^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2+\||u||\nabla u|\|_2^2)dt+\eta\overline{\rho}\int_0^T\|\nabla\theta\|_2^2dt\\ &\quad+C\|\nabla d_0\|_4^4+C\|\nabla d\|_3^2\|\nabla^2d\|_2^2+C\eta\overline{\rho}\int_0^T(\|\nabla d_t\|_2^2+\|\nabla^3d\|_2^2)dt+C\eta^{\frac12}\overline{\rho}\int_0^T\|\nabla(|\nabla d|^2)\|_2^2dt,\end{aligned}\tag{2.43}$$

where we choose $\varepsilon_1 \leq \eta$ small enough. By (2.39) and (2.42), we get

$$
\begin{aligned}
&\overline{\rho}^3 \int_0^t \|\nabla u\|_2^4 (\|\nabla u\|_2^2 + \overline{\rho}\|\sqrt{\rho}E\|_2^2) dt \\
\leq & C\overline{\rho}^3 \sup_{0\leq t\leq T} (\|\nabla u\|_2^2 + \overline{\rho}\|\sqrt{\rho}E\|_2^2) \sup_{0\leq t\leq T} \|\nabla u\|_2^2 \\
& \cdot \left((\sup_{0\leq t\leq T} (\|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2) + \sup_{0\leq t\leq T} \|\rho\|_3^2 \int_0^T \|\nabla\theta\|_2^2 dt) \right) \\
\leq & C\eta^{\frac{1}{2}} \sup_{0\leq t\leq T} \|\nabla u\|_2^2 + C\eta \int_0^T \|\nabla\theta\|_2^2 dt \qquad (2.44)
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{\rho} \sup_{0\leq t\leq T} \|\sqrt{\rho}E\|_2^2 + \int_0^T (\overline{\rho} + \overline{\rho}^2 \|\rho\|_3^{\frac{1}{2}} \|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2) dt \\
\leq & \overline{\rho} \sup_{0\leq t\leq T} \|\sqrt{\rho}E\|_2^2 + (\overline{\rho} + \overline{\rho}^2 \sup_{0\leq t\leq T} (\|\rho\|_3^{\frac{1}{2}} \|\sqrt{\rho}E\|_2) \int_0^T (\|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2) dt \\
\leq & \overline{\rho} \sup_{0\leq t\leq T} \|\sqrt{\rho}E\|_2^2 + (\overline{\rho} + \overline{\rho}\eta^{\frac{1}{4}}) \int_0^T (\|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2) dt \\
\leq & C\overline{\rho}\|\sqrt{\rho_0}E_0\|_2^2 + C\overline{\rho}\eta \int_0^T \|\nabla^3 d\|_2^2 dt + C\overline{\rho}\eta^{\frac{1}{2}} \int_0^T \||\nabla d||\nabla^2 d|\|_2^2 dt + C\overline{\rho}\eta^{\frac{1}{2}} \sup_{0\leq t\leq T} \|\nabla^2 d\|_2^2. \qquad (2.45)
\end{aligned}
$$

Substituting (2.44) and (2.45) into (2.43) and using η_0 small enough, one obtains

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \|\nabla u\|_2^2 + \int_0^T \|(\sqrt{\rho}u_t, \frac{\nabla G}{\sqrt{\overline{\rho}}}, \frac{\nabla\omega}{\sqrt{\overline{\rho}}})\|_2^2 dt \\
\leq & C(\|\nabla u_0\|_2^2 + \overline{\rho}\|\sqrt{\rho_0}E_0\|_2^2 + \|\nabla d_0\|_4^4) + C\eta\overline{\rho} \int_0^T \|\nabla\theta\|_2^2 dt \\
& + C\overline{\rho}\eta^{\frac{1}{2}} \int_0^T \||\nabla d||\nabla^2 d|\|_2^2 dt + C\overline{\rho}\eta^{\frac{1}{2}} \sup_{0\leq t\leq T} \|\nabla^2 d\|_2^2 \\
& + C\eta\overline{\rho} \int_0^T \|\nabla d_t\|_2^2 dt + C\eta\overline{\rho} \int_0^T \|\nabla^3 d\|_2^2 dt. \qquad (2.46)
\end{aligned}
$$

The combination of (2.41), (2.42) and (2.46) yields that

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \big(\overline{\rho}(\|\nabla^2 d\|_2^2 + \|\nabla d\|_4^4 + \|\sqrt{\rho}E\|_2^2) + \|\nabla u\|_2^2\big) + \int_0^T \left\| \left(\sqrt{\rho}u_t, \frac{\nabla G}{\sqrt{\overline{\rho}}}, \frac{\nabla\omega}{\sqrt{\overline{\rho}}} \right) \right\|_2^2 dt \\
& + \overline{\rho} \int_0^T \big(\|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2 + \||\nabla d||\nabla^2 d|\|_2^2 + \|\nabla\theta\|_2^2 + \||u||\nabla u|\|_2^2\big) dt \\
\leq & C\big(\overline{\rho}(\|\nabla^2 d_0\|_2^2 + \|\sqrt{\rho_0}E_0\|_2^2) + \|\nabla u_0\|_2^2\big) \\
& + C\overline{\rho} \sup_{0\leq t\leq T} \|\nabla d\|_3^2 \|\nabla^2 d\|_2^2 + C\eta^{\frac{1}{2}} \sup_{0\leq t\leq T} \|\nabla u\|_2^2 + C\eta\overline{\rho} \int_0^T \|\nabla\theta\|_2^2 dt \\
& + C\overline{\rho}\eta^{\frac{1}{2}} \int_0^T \||\nabla d||\nabla^2 d|\|_2^2 dt + C\overline{\rho}\eta^{\frac{1}{2}} \sup_{0\leq t\leq T} \|\nabla^2 d\|_2^2 + C\eta\overline{\rho} \int_0^T (\|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2) dt
\end{aligned}
$$

$$
\begin{aligned}
&\le C\big(\bar\rho(\|\nabla^2 d_0\|_2^2+\|\sqrt{\rho_0}E_0\|_2^2)+\|\nabla u_0\|_2^2\big)+C\eta^{\frac12}\sup_{0\le t\le T}\|\nabla u\|_2^2+C\eta\bar\rho\int_0^T\|\nabla\theta\|_2^2dt\\
&\quad+C\bar\rho\eta^{\frac12}\int_0^T\||\nabla d||\nabla^2 d|\|_2^2dt+C\bar\rho(\eta^{\frac12}+\eta)\sup_{0\le t\le T}\|\nabla^2 d\|_2^2\\
&\quad+C\eta\bar\rho\int_0^T(\|\nabla d_t\|_2^2+\|\nabla^3 d\|_2^2)dt,
\end{aligned}
$$

from which, choosing η_0 small enough, one gets (2.37) and

$$
\bar\rho\int_0^T\|\nabla\theta\|_2^2dt\le C\big(\bar\rho(\|\nabla^2 d_0\|_2^2+\|\sqrt{\rho_0}E_0\|_2^2)+\|\nabla u_0\|_2^2\big). \tag{2.47}
$$

Recalling (2.39) and the assumptions, and using (2.47), we have

$$
\begin{aligned}
&\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\int_0^T(\|\nabla u\|_2^2+\|d_t\|_2^2+\|\nabla^2 d\|_2^2)dt\\
&\le C(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)+C\frac{1}{\bar\rho}\sup_{0\le t\le T}\|\rho\|_3^2\big(\bar\rho(\|\nabla^2 d_0\|_2^2+\|\sqrt{\rho_0}E_0\|_2^2)+\|\nabla u_0\|_2^2\big)\\
&\le C(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)+C\eta^{\frac12}\frac{1}{\bar\rho^2}\sup_{0\le t\le T}\|\rho\|_3.
\end{aligned}\tag{2.48}
$$

It follows from Lemma 2.5, (2.48), the Young inequality and the assumptions that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3P\,dxdt\\
&\le C\|\rho_0\|_3^3+C\sup_{0\le t\le T}(\|\rho\|_\infty^{\frac23}\|\sqrt{\rho}u\|_2^{\frac13}\|\sqrt{\rho}E\|_2^{\frac13}\|\rho\|_3^3)\\
&\quad+C\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3^2\int_0^T(\|\nabla u\|_2^2+\|\nabla^2 d\|_2^2)dt\\
&\le C\|\rho_0\|_3^3+C\eta^{\frac{1}{12}}\sup_{0\le t\le T}\|\rho\|_3^3+C\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3^2\left(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2+\eta^{\frac12}\frac{1}{\bar\rho^2}\sup_{0\le t\le T}\|\rho\|_3\right)\\
&\le C\|\rho_0\|_3^3+C\big(\eta^{\frac{1}{12}}+\eta^{\frac12}\big)\sup_{0\le t\le T}\|\rho\|_3^3+C\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)\\
&\le C\|\rho_0\|_3^3+C\big(\eta^{\frac{1}{12}}+\eta^{\frac12}+\frac14\big)\sup_{0\le t\le T}\|\rho\|_3^3+C\bar\rho^6(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)^3,
\end{aligned}
$$

which implies (2.35) by choosing η_0 sufficiently small.

Now, substituting (2.35) into (2.48) yields that

$$
\begin{aligned}
&\bar\rho^2\left(\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\int_0^T(\|\nabla u\|_2^2+\|d_t\|_2^2+\|\nabla^2 d\|_2^2)dt\right)\\
&\le C\bar\rho^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)+C\eta^{\frac12}\sup_{0\le t\le T}\|\rho\|_3\\
&\le C\big(\|\rho_0\|_3+\bar\rho^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)\big),
\end{aligned}
$$

which gives (2.36).

Finally, (2.38) follows immediately from Lemma 2.7, (2.36) and (2.37), and the proof is complete. ∎

Now, we are able to get time-independent estimates on the scaling invariant quantity $\mathcal{N}_T$.

LEMMA 2.9. *Let $\eta_0, \mathcal{N}_T$, and $\mathcal{N}_0$ be as in Lemma 2.8, (2.1), and Theorem 1.1, respectively. Then, there exists a number $\varepsilon_0 \in (0,\eta_0)$ such that if*

$$\sup_{0\le t\le T}\|\rho\|_\infty \le 4\bar\rho,\ \mathcal{N}_T \le \sqrt{\varepsilon_0}\ \text{ and }\ \mathcal{N}_0\le\varepsilon_0,$$

then

$$\sup_{0\le t\le T}\|\rho\|_\infty \le 2\bar\rho\ \text{ and }\ \mathcal{N}_T\le\frac{\sqrt{\varepsilon_0}}{2},$$

where ε_0 depends only on R, c_v, μ, λ, and κ.

Proof. If $\varepsilon_0\le\eta_0$ is sufficiently small, all the conditions in Lemma 2.8 hold. Therefore, we obtain

$$\begin{aligned}\mathcal{N}_T &\le C\bar\rho\big(\|\rho_0\|_3+\bar\rho^2(\|\sqrt{\rho_0}u_0\|_2^2+\|\nabla d_0\|_2^2)\big)\big(\|\nabla u_0\|_2^2+\bar\rho(\|\sqrt{\rho_0}E_0\|_2^2+\|\nabla^2 d_0\|_2^2)\big)\\ &\le C\varepsilon_0\le\frac{\sqrt{\varepsilon_0}}{2}.\end{aligned}$$

At the same time,

$$\sup_{0\le t\le T}\|\rho\|_\infty\le\bar\rho e^{C\mathcal{N}_0^{\frac16}+C\mathcal{N}_0^{\frac12}}\le\bar\rho e^{C\varepsilon_0^{\frac16}+C\varepsilon_0^{\frac12}}\le 2\bar\rho.$$

We complete the proof of the lemma. ∎

3. Time dependent higher order estimates

Recalling that the low order a priori estimates established in Proposition 2.1 are one order derivative lower than the orders of derivatives of the initial data, the a priori estimates obtained in the previous section are insufficient to extend the solution beyond $T_{\max}$ (in case that $T_{\max}<\infty$) through Lemma 2.1. Therefore, besides the a priori estimates obtained before, the higher order a priori estimates are also required to prove our main theorem. The desired higher order estimates are presented in this section. As will be seen in Section 4, our main theorem is proved by contradiction argument, that is we assume by contradiction that $T_{\max}<\infty$, and then show that this is not true based on Lemma 2.1 and the a priori estimates, where $T_{\max}$ as in the previous section is the maximal time of existence of the extended solution (ρ,u,θ,d). Therefore, throughout this section, we always assume that $T_{\max}<\infty$. The following estimate will be proved in this section:

$$\sup_{0\le t<T_{\max}}\big(\|\rho\|_{H^1\cap W^{1,q}}+\|\nabla\theta\|_{H^1}^2+\|(\nabla^2u,\sqrt{\rho}\dot u,\sqrt{\rho}\dot\theta,\nabla^3 d,\nabla d_t)\|_2^2\big)<\infty.$$

Here and in what follows,

$$\dot f:=f_t+u\cdot\nabla f$$

denotes the material derivative of f. These a priori estimates can be established by modifying the methods of [20, 21, 41] for the compressible Navier-Stokes equations and magnetohydrodynamic equations.

Let us begin with the following estimate on $\dot u$.

LEMMA 3.1. *Assume $\mathcal{N}_0 \le \varepsilon_0$. Then, for any $T \in (0,T_{max})$, it holds that*

$$\sup_{0\le t\le T}(\|\nabla\theta\|_2^2+\|\sqrt{\rho}\dot u\|_2^2+\|\nabla d_t\|_2^2)+\int_0^T\|(\sqrt{\rho}\dot\theta,\nabla\dot u,d_{tt},\Delta d_t)\|_2^2dt\le C_{T_{max}},$$

where $C_{T_{max}}$ depends only on $R,c_v,\mu,\lambda,\kappa,\Phi_0$, and T_{max}.

Proof. Applying $\dot u_j\big(\partial_t+\mathrm{div}(u\cdot)\big)$ to $(1.2)^j$ and integrating over $\mathbb{R}^3$, it follows

$$\begin{aligned}\frac12\frac{d}{dt}\|\sqrt{\rho}\dot u\|_2^2=&-\int\dot u_j\big(\partial_jP_t+\mathrm{div}(u\partial_jP)\big)dx+\mu\int\dot u_j\big(\partial_t\Delta u_j+\mathrm{div}(u\Delta u_j)\big)dx\\&+(\mu+\lambda)\int\dot u_j\big(\partial_j\mathrm{div}u_t+\mathrm{div}(u\partial_j\mathrm{div}u)\big)dx\\&-\int\partial_i(M_{i,j}(d))_t\cdot\dot u_jdx-\int\partial_k\big(u_k\partial_i(M_{i,j}(d))\big)\dot u_jdx=:\sum_{i=1}^5J_i,\end{aligned}\tag{3.1}$$

where $M_{i,j}(d)=\partial_id\cdot\partial_jd-\frac12|\nabla d|^2\delta_{i,j}$. It follows from the Hölder, Young and Sobolev inequalities that

$$\begin{aligned}J_1=&-\int\dot u_j\big[\partial_jP_t+\partial_j\mathrm{div}(uP)-\mathrm{div}(\partial_juP)\big]dx\\=&\int\mathrm{div}\dot u(P_t+\mathrm{div}(uP))dx-\int\nabla\dot u_j\cdot\partial_juPdx\\=&R\int\mathrm{div}\dot u\rho\dot\theta dx-R\int\nabla\dot u_j\cdot\partial_ju\rho\theta dx\\\le&\frac{\mu}{8}\|\nabla\dot u\|_2^2+C\|\rho\dot\theta\|_2^2+C\int\rho^2\theta^2|\nabla u|^2dx\\\le&\frac{\mu}{8}\|\nabla\dot u\|_2^2+C\|\rho\dot\theta\|_2^2+C\|\rho\theta\|_2^{\frac12}\|\theta\|_6^{\frac32}\|\nabla u\|_4^2\\\le&\frac{\mu}{8}\|\nabla\dot u\|_2^2+C\big(1+\|\rho\dot\theta\|_2^2+\|\nabla\theta\|_2^4+\|\nabla u\|_4^4\big),\end{aligned}$$

where Proposition 2.1 was used. By virtue of integration by parts, we compute

$$\begin{aligned}J_2=&-\mu\int\big(\partial_i\dot u_j(\partial_iu_j)_t+\Delta u_ju\cdot\nabla\dot u_j\big)dx\\=&-\mu\int\big(|\nabla\dot u|^2-\partial_i\dot u_ju_k\partial_k\partial_iu_j-\partial_i\dot u_j\partial_iu_k\partial_ku_j+\Delta u_ju\cdot\nabla\dot u_j\big)dx\\=&-\mu\int\big(|\nabla\dot u|^2+\partial_i\dot u_j\partial_iu_j\mathrm{div}u-\partial_i\dot u_j\partial_iu_k\partial_ku_j-\partial_iu_j\partial_iu_k\partial_k\dot u_j\big)dx\\\le&-\frac{7\mu}{8}\|\nabla\dot u\|_2^2+C\|\nabla u\|_4^4.\end{aligned}$$

In the same way, one gets

$$J_3\le-\frac{7(\mu+\lambda)}{8}\|\mathrm{div}\dot u\|_2^2+C\|\nabla u\|_4^4.$$

For J_4 and J_5, by integration by parts and the Hölder, Young and Sobolev inequalities, one has for $\eta_1\in(0,1]$

$$J_4\le\int|\nabla d||\nabla d_t||\nabla\dot u|dx\le C\|\nabla d\|_6\|\nabla d_t\|_3\|\nabla\dot u\|_2\le C\|\nabla^2d\|_2\|\nabla d_t\|_2^{\frac12}\|\nabla^2d_t\|_2^{\frac12}\|\nabla\dot u\|_2$$

$$\le \varepsilon\|\nabla\dot{u}\|_2^2+\eta_1\|\nabla^2 d_t\|_2^2+C(\varepsilon,\eta_1)\|\nabla d_t\|_2^2,$$

where $\sup\limits_{0\le t\le T}\|\nabla^2 d\|_2\le C$ guaranteed by Proposition 2.1 was used, and

$$J_5\le\int|u||\nabla d||\nabla^2 d||\nabla\dot{u}|dx\le C\|u\|_6\|\nabla d\|_6\|\nabla^2 d\|_6\|\nabla\dot{u}\|_2\le\varepsilon\|\nabla\dot{u}\|_2^2+C(\varepsilon)\|\nabla^3 d\|_2^2,$$

where Proposition 2.1 was used.

Substituting $J_i, i=1,2,\dots,5$ into (3.1), one obtains after choosing ε suitably small that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}\dot{u}\|_2^2+\mu\|\nabla\dot{u}\|_2^2\\ &\le C\eta_1\|\nabla^2 d_t\|_2^2+C\big(1+\|\rho\dot{\theta}\|_2^2+\|\nabla\theta\|_2^4+\|\nabla u\|_4^4+\|\nabla^3 d\|_2^2\big)+C(\eta_1)\|\nabla d_t\|_2^2.\end{aligned}\tag{3.2}$$

Next, multiplying (1.3) by $\dot{\theta}$ and integrating the resultant over $\mathbb{R}^3$ yield

$$\begin{aligned}&\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_2^2+c_v\|\sqrt{\rho}\dot{\theta}\|_2^2\\ &=-\kappa\int\nabla\theta\cdot\nabla(u\cdot\nabla\theta)dx+\lambda\int|\mathrm{div}u|^2\dot{\theta}dx+\frac{\mu}{2}\int|\nabla u+(\nabla u)^t|^2\dot{\theta}dx\\ &\quad-R\int\rho\theta\mathrm{div}u\dot{\theta}dx+\int|\Delta d+|\nabla d|^2 d|^2\dot{\theta}dx:=\sum_{i=1}^5 K_i.\end{aligned}\tag{3.3}$$

It follows from elliptic estimates, Proposition 2.1, and Gagliardo-Nirenberg and Young inequalities that

$$\begin{aligned}\|\nabla\theta\|_{H^1}^2&\le C+C\|\sqrt{\rho}\dot{\theta}\|_2^2+C\|\nabla\theta\|_2^2+C\int\rho^2\theta^2|\nabla u|^2dx+C\|\nabla u\|_4^4+C\|\Delta d+|\nabla d|^2 d\|_4^4\\ &\le C+C\|\sqrt{\rho}\dot{\theta}\|_2^2+C\|\nabla\theta\|_2^2+C\|\nabla u\|_2^2\|\theta\|_\infty^2+C\|\nabla u\|_4^4+C\|\Delta d+|\nabla d|^2 d\|_4^4\\ &\le\frac{1}{2}\|\nabla\theta\|_{H^1}^2+C\big(1+\|\sqrt{\rho}\dot{\theta}\|_2^2+\|\nabla\theta\|_2^2+\|\nabla u\|_4^4+\|\Delta d+|\nabla d|^2 d\|_4^4\big),\end{aligned}$$

which implies

$$\|\nabla\theta\|_{H^1}^2\le C\big(1+\|\sqrt{\rho}\dot{\theta}\|_2^2+\|\nabla\theta\|_2^2+\|\nabla u\|_4^4+\|\Delta d+|\nabla d|^2 d\|_4^4\big).\tag{3.4}$$

Moveover, by (1.4), the Hölder, Young and Sobolev inequalities and Proposition 2.1, one can get by the elliptic estimates that

$$\begin{aligned}\|\nabla^3 d\|_2&\le C\|\nabla d_t\|_2+\|\nabla u\cdot\nabla d\|_2+C\|u\cdot\nabla^2 d\|_2+C\||\nabla d|^3\|_2+C\||\nabla d|\nabla^2 d\|_2\\ &\le C\|\nabla d_t\|_2+C\|\nabla u\|_4\|\nabla d\|_4+C\|u\|_6\|\nabla^2 d\|_3+C\|\nabla d\|_6^3+C\|\nabla d\|_6\|\nabla^2 d\|_3\\ &\le C\|\nabla d_t\|_2+C\|\nabla u\|_4+C\|\nabla^2 d\|_2^{\frac12}\|\nabla^3 d\|_2^{\frac12}+C\\ &\le\frac12\|\nabla^3 d\|_2+C\big(1+\|\nabla d_t\|_2+\|\nabla u\|_4\big),\end{aligned}$$

which gives

$$\|\nabla d\|_{H^2}\le C\big(1+\|\nabla d_t\|_2+\|\nabla u\|_4\big).\tag{3.5}$$

According to (3.5), one gets by the Sobolev embedding inequality and Proposition 2.1 that

$$\|\Delta d+|\nabla d|^2 d\|_4^4 \le C(\|\Delta d\|_{H^1}^4+\|\nabla d\|_\infty^4\|\nabla d\|_4^4)\le C\|\nabla d\|_{H^2}^4\le C\big(1+\|\nabla d_t\|_2^4+\|\nabla u\|_4^4\big). \tag{3.6}$$

Thus, (3.4), (3.6), the Sobolev and Young inequalities yield

$$\begin{aligned} K_1 =& -\kappa\int \nabla\theta\cdot\nabla(u\cdot\nabla\theta)dx \le C\int |\nabla\theta|\big(|u||\nabla^2\theta|+|\nabla u||\nabla\theta|\big)dx\\ \le& C\big(\|\nabla\theta\|_3\|u\|_6\|\nabla^2\theta\|_2+\|\nabla u\|_2\|\nabla\theta\|_6\|\nabla\theta\|_3\big)\\ \le& C\|\nabla u\|_2\|\nabla\theta\|_2^{\frac12}\|\nabla^2\theta\|_2^{\frac32}\le \varepsilon\|\nabla\theta\|_2^2+C(\varepsilon)\|\nabla\theta\|_2^2\\ \le& C\varepsilon\|\sqrt{\rho}\dot\theta\|_2^2+C(\varepsilon)\big(1+\|\nabla\theta\|_2^2+\|\nabla u\|_4^4+\|\nabla d_t\|_2^4\big). \end{aligned}$$

By integration by parts, it follows from the Hölder, Young and Sobolev inequalities and Proposition 2.1 that

$$\begin{aligned} K_2 =& \lambda\int (\mathrm{div}u)^2\theta_t dx+\lambda\int(\mathrm{div}u)^2 u\cdot\nabla\theta dx\\ =& \lambda\left(\int(\mathrm{div}u)^2\theta dx\right)_t-2\lambda\int\theta\mathrm{div}u\mathrm{div}(\dot u-u\cdot\nabla u)dx+\lambda\int(\mathrm{div}u)^2u\cdot\nabla\theta dx\\ =& \lambda\left(\int(\mathrm{div}u)^2\theta dx\right)_t-2\lambda\int\theta\mathrm{div}u\mathrm{div}\dot u dx+2\lambda\int\theta\mathrm{div}u\partial_iu_j\partial_ju_idx+\lambda\int u\cdot\nabla\big(\theta(\mathrm{div}u)^2\big)dx\\ \le& \lambda\left(\int(\mathrm{div}u)^2\theta dx\right)_t+C\|\theta\|_6\|\nabla u\|_2^{\frac13}\|\nabla u\|_4^{\frac23}(\|\nabla\dot u\|_2+\|\nabla u\|_4^2)\\ \le& \lambda\left(\int(\mathrm{div}u)^2\theta dx\right)_t+\eta_1\|\nabla\dot u\|_2^2+C(\eta_1)\big(1+\|\nabla u\|_4^4+\|\nabla\theta\|_2^4\big). \end{aligned}$$

Similarly, one has

$$K_3\le\frac{\mu}{2}\left(\int|\nabla u+(\nabla u)^t|^2\theta dx\right)_t+\eta_1\|\nabla\dot u\|_2^2+C(\eta_1)\big(1+\|\nabla u\|_4^4+\|\nabla\theta\|_2^4\big).$$

Using Proposition 2.1 again, we get

$$K_4\le C\|\sqrt{\rho}\dot\theta\|_2\|\sqrt{\rho}\theta\|_2^{\frac14}\|\theta\|_6^{\frac34}\|\nabla u\|_4\le\varepsilon\|\sqrt{\rho}\dot\theta\|_2^2+C(\varepsilon)\big(1+\|\nabla\theta\|_2^4+\|\nabla u\|_4^4\big).$$

At last, for K_5, noticing that $|\Delta d+|\nabla d|^2d|^2=|\Delta d|^2-|\nabla d|^4$ and $\Delta d\cdot d=-|\nabla d|^2$ guaranteed by $|d|=1$, it follows from the Hölder, Young and Sobolev inequalities that

$$\begin{aligned} K_5 =& \int|\Delta d+|\nabla d|^2d|^2\theta_tdx+\int|\Delta d+|\nabla d|^2d|^2u\cdot\nabla\theta dx\\ =& \left(\int|\Delta d+|\nabla d|^2d|^2\theta dx\right)_t-2\int\big(\Delta d\cdot\Delta d_t-4|\nabla d|^2\nabla d:\nabla d_t\big)\theta dx\\ &+\int|\Delta d-(\Delta d\cdot d)d|^2u\cdot\nabla\theta dx\\ =& \left(\int|\Delta d+|\nabla d|^2d|^2\theta dx\right)_t-2\int\big(\Delta d\cdot\Delta d_t+4\Delta d\cdot d\nabla d:\nabla d_t\big)\theta dx\\ &+\int|\Delta d-(\Delta d\cdot d)d|^2u\cdot\nabla\theta dx \end{aligned}$$

$$
\begin{aligned}
&\le \left(\int |\Delta d+|\nabla d|^2 d|^2\theta dx\right)_t + C\|\theta\|_6\|\Delta d_t\|_2\|\Delta d\|_3 + C\|\Delta d\|_2\|\nabla d\|_6\|\nabla d_t\|_6\|\theta\|_6\\
&\quad + C\|\nabla\theta\|_2\|u\|_6\|\nabla^2 d\|_6^2\\
&\le \left(\int |\Delta d+|\nabla d|^2 d|^2\theta dx\right)_t + C\|\nabla\theta\|_2\|\Delta d_t\|_2\|\nabla^2 d\|_2^{\frac12}\|\nabla^2 d\|_2^{\frac12} + C\|\Delta d_t\|_2\|\nabla\theta\|_2\\
&\quad + C\|\nabla\theta\|_2\|\nabla u\|_2\|\nabla^3 d\|_2^2\\
&\le \left(\int |\Delta d+|\nabla d|^2 d|^2\theta dx\right)_t + \eta_1\|\Delta d_t\|_2^2 + C(\eta_1)\big(1+\|\nabla\theta\|_2^2\big)(1+\|\nabla^3 d\|_2^2\big).
\end{aligned}
$$

Now, substituting the estimates for $K_i, i=1,2,\ldots,5$ into (3.3) and then choosing ε small enough, we deduce that

$$
\begin{aligned}
\frac{d}{dt}\int \Phi dx + c_v\|\sqrt{\rho}\dot\theta\|_2^2 \le{}& C(1+\|\nabla\theta\|_2^2)(1+\|\nabla^3 d\|_2^2+\|\nabla\theta\|_2^2) + C\eta_1\|\nabla\dot u\|_2^2 + C\eta_1\|\Delta d_t\|_2^2\\
&+ C\|\nabla u\|_4^4 + C\|\nabla d_t\|_2^4 + C,
\end{aligned}\tag{3.7}
$$

where

$$
\Phi := \kappa|\nabla\theta|^2 - 2\theta\big(\lambda(\mathrm{div} u)^2 + \frac{\mu}{2}|\nabla u+(\nabla u)^t|^2 + |\Delta d+|\nabla d|^2 d|^2\big). \tag{3.8}
$$

On the other hand, applying ∂_t to (1.4), we have

$$
d_{tt} - \Delta d_t = (-u\cdot\nabla d + |\nabla d|^2 d)_t.
$$

It follows from integration by parts, the Sobolev and Young inequalities and Proposition 2.1 that

$$
\begin{aligned}
\frac{d}{dt}\|\nabla d_t\|_2^2 + \|(d_{tt},\Delta d_t)\|_2^2 &= \int |d_{tt}-\Delta d_t|^2 dx\\
&= \int |(-u\cdot\nabla d)_t + (|\nabla d|^2 d)_t|^2 dx\\
&\le \int \big(|u_t|^2|\nabla d|^2 + |u|^2|\nabla d_t|^2 + |\nabla d|^4|d_t|^2 + |\nabla d|^2|\nabla d_t|^2\big)dx\\
&\le \int \big(|\dot u|^2|\nabla d|^2 + |u|^2|\nabla u|^2|\nabla d|^2\big)dx + C\|\nabla d\|_6^4\|d_t\|_6^2\\
&\quad + C(\|u\|_6^2+\|\nabla d\|_6^2)\|\nabla d_t\|_2\|\nabla d_t\|_6\\
&\le \frac12\|\Delta d_t\|_2^2 + C\|\dot u\|_6^2\|\nabla d\|_3^2 + C\|u\|_6^2\|\nabla u\|_6^2\|\nabla d\|_6^2 + C\|\nabla d_t\|_2^2\\
&\le \frac12\|\Delta d_t\|_2^2 + C\|\nabla\dot u\|_2^2 + C\|\nabla u\|_6^2 + C\|\nabla d_t\|_2^2,
\end{aligned}
$$

which yields

$$
\frac{d}{dt}\|\nabla d_t\|_2^2 + \|d_{tt}\|_2^2 + \frac12\|\Delta d_t\|_2^2 \le C\|\nabla\dot u\|_2^2 + C\|\nabla u\|_6^2 + C\|\nabla d_t\|_2^2. \tag{3.9}
$$

Thus, multiplying (3.2) and (3.9) by $\eta_1^{\frac14}$ and $\eta_1^{\frac12}$, respectively, then adding the result with (3.7) and choosing η_1 suitably small, we finally obtain

$$
2\frac{d}{dt}\int \big(\Phi + \eta_1^{\frac12}|\nabla d_t|^2 + \eta_1^{\frac14}\rho|\dot u|^2\big)dx + c_v\|\sqrt{\rho}\dot\theta\|_2^2 + \eta_1^{\frac12}\|(d_{tt},\Delta d_t)\|_2^2 + \mu\eta_1^{\frac14}\|\nabla\dot u\|_2^2
$$

$$\le C\big(1+\|\nabla\theta\|_2^2\big)\big(1+\|\nabla^3 d\|_2^2+\|\nabla\theta\|_2^2\big)+C\big(\|\nabla u\|_4^4+\|\nabla d_t\|_2^4+\|\nabla d_t\|_2^4+\|\nabla u\|_6^2\big), \quad (3.10)$$

where Φ is given by (3.8).

Next, one needs to show the estimate of $\|\nabla u\|_6$ in order to bound $\|\nabla u\|_4$. To this end, decompose $u=v+w$, where v satisfies

$$\mu\Delta v+(\mu+\lambda)\nabla \mathrm{div} v=\nabla P. \quad (3.11)$$

According to Lemma 2.3 in [20], there exists a unique $v(\cdot,t)\in D_0^1\cap D^{2,2}\cap D^{2,q}$ satisfying (3.11) and the following L^p, $p\in[2,6]$ and L^∞ estimates for $t\in[0,T]$:

$$\|\nabla v\|_p\le C\|\rho\theta\|_p, \quad (3.12)$$

and

$$\|\nabla v\|_\infty\le C\big(1+\log(e+\|\nabla(\rho\theta)\|_q)\|\rho\theta\|_\infty+\|\rho\theta\|_2\big), \quad q\in(3,6]. \quad (3.13)$$

While, w satisfies

$$\mu\Delta w+(\mu+\lambda)\nabla \mathrm{div} w=\rho\dot u+\Delta d\cdot\nabla d. \quad (3.14)$$

By the elliptic estimates, it holds that

$$\|\nabla w\|_6+\|\nabla^2 w\|_2\le C\|\rho\dot u\|_2+C\|\Delta d\cdot\nabla d\|_2, \quad (3.15)$$

and

$$\|\nabla^2 w\|_6\le C\|\rho\dot u\|_6+C\|\Delta d\cdot\nabla d\|_6. \quad (3.16)$$

Hence, by the Hölder and Sobolev inequalities, it follows from Proposition 2.1 that

$$\begin{aligned}\|\nabla u\|_6\le& C\|\rho\theta\|_6+C\|\rho\dot u\|_2+C\|\Delta d\cdot\nabla d\|_2\\ \le& C\|\nabla\theta\|_2+C\|\rho\dot u\|_2+C\|\nabla d\|_6\|\nabla^2 d\|_2^{\frac12}\|\nabla^2 d\|_6^{\frac12}\\ \le& C\|\nabla\theta\|_2+C\|\rho\dot u\|_2+C\|\nabla^3 d\|_2^{\frac12}. \end{aligned}\quad (3.17)$$

Thus, by the Young and Sobolev inequalities, one has

$$\|\nabla u\|_4^4\le C\|\nabla u\|_2\|\nabla u\|_6^3\le C\|\nabla u\|_2^4+C\|\nabla u\|_6^4\le C\big(1+\|\nabla\theta\|_2^4+\|\sqrt{\rho}\dot u\|_2^4+\|\nabla^3 d\|_2^2\big). \quad (3.18)$$

Substituting (3.18) into (3.10), we obtain

$$\begin{aligned}&2\frac{d}{dt}\int\big(\Phi+\eta_1^{\frac12}|\nabla d_t|^2+\eta_1^{\frac14}\rho|\dot u|^2\big)dx+c_v\|\sqrt{\rho}\dot\theta\|_2^2+\eta_1^{\frac12}\|(d_{tt},\Delta d_t)\|_2^2+\mu\eta_1^{\frac14}\|\nabla\dot u\|_2^2\\ &\le C\big(1+\|\nabla\theta\|_2^2\big)\big(1+\|\nabla^3 d\|_2^2+\|\nabla\theta\|_2^2\big)+C\big(\|\sqrt{\rho}\dot u\|_2^4+\|\nabla d_t\|_2^4+\|\nabla d_t\|_2^2+\|\nabla u\|_6^2\big),\end{aligned} \quad (3.19)$$

where Φ is given by (3.8).

Now, we want to show the lower bound of Φ. By the elliptic estimates, it follows from (1.4), the Hölder, Sobolev and Young inequalities that

$$\begin{aligned}&\|\nabla^3 d\|_2\le C\big(\|\nabla d_t\|_2+\|\nabla u\cdot\nabla d\|_2+\|u\cdot\nabla^2 d\|_2+\||\nabla d|^2\nabla d\|_2+\||\nabla d|\nabla^2 d\|_2\big)\\ &\le C\big(\|\nabla d_t\|_2+\|\nabla u\|_4\|\nabla d\|_4+\|u\|_6\|\nabla^2 d\|_2^{\frac12}\|\nabla^3 d\|_2^{\frac12}+\|\nabla d\|_6^3+\|\nabla d\|_6\|\nabla^2 d\|_2^{\frac12}\|\nabla^3 d\|_2^{\frac12}\big)\\ &\le\frac14\|\nabla^3 d\|_2+C\big(1+\|\nabla d_t\|_2+\|\nabla u\|_4\big),\end{aligned}$$

where we have used Proposition 2.1. By (3.17) and the Cauchy inequality, it holds that

$$\|\nabla u\|_6 \le C\big(1+\|\nabla\theta\|_2+\|\rho\dot{u}\|_2\big)+\frac{1}{4}\|\nabla^3 d\|_2.$$

Combining the above two inequalities, together with (3.18), lead to

$$\|\nabla u\|_6+\|\nabla^3 d\|_2 \le C\big(1+\|\nabla\theta\|_2+\|\rho\dot{u}\|_2+\|\nabla d_t\|_2\big). \tag{3.20}$$

Thus, from the definition of Φ, (3.20), the Young and Sobolev inequalities, one deduces by Proposition 2.1 that

$$\begin{aligned}
&2\int\Big(\Phi+\eta_1^{\frac14}\rho|\dot{u}|^2+\eta_1^{\frac12}|\nabla d_t|^2\Big)\,dx\\
\ge{}&2\kappa\|\nabla\theta\|_2^2-C\|\theta\|_6\|\nabla u\|_2^{\frac32}\|\nabla u\|_6^{\frac12}-C\|\theta\|_6\|\Delta d\|_2^{\frac32}\|\Delta d\|_6^{\frac12}-C\|\theta\|_6\|\nabla d\|_{\frac{24}{5}}^4\\
&\quad+2\int\Big(\eta_1^{\frac14}\rho|\dot{u}|^2+\eta_1^{\frac12}|\nabla d_t|^2\Big)\,dx\\
\ge{}&\frac32\kappa\|\nabla\theta\|_2^2-C\big(1+\|\nabla u\|_6+\|\nabla^3 d\|_2\big)+2\int\Big(\eta_1^{\frac14}\rho|\dot{u}|^2+\eta_1^{\frac12}|\nabla d_t|^2\Big)\,dx\\
\ge{}&\frac32\kappa\|\nabla\theta\|_2^2-C\big(\|\nabla\theta\|_2+\|\sqrt{\rho}\dot{u}\|_2+\|\nabla d_t\|_2\big)+2\int\Big(\eta_1^{\frac14}\rho|\dot{u}|^2+\eta_1^{\frac12}|\nabla d_t|^2\Big)\,dx\\
\ge{}&\kappa\|\nabla\theta\|_2^2-C(\eta_1)+\int\Big(\eta_1^{\frac14}\rho|\dot{u}|^2+\eta_1^{\frac12}|\nabla d_t|^2\Big)\,dx.
\end{aligned}\tag{3.21}$$

Finally, integrating (3.19) over $[0,t]$, and then using (3.21) and Grönwall's inequality, the conclusion follows. ∎

As a straightforward consequence of Lemma 3.1, and using (3.20), we have the following corollary:

Corollary 3.1. *Assume $\mathcal{N}_0\le\varepsilon_0$. Then, for any $T\in(0,T_{max})$, it holds that*

$$\sup_{0\le t\le T}\big(\|\nabla u\|_6+\|\nabla^3 d\|_2+\||\nabla d||\nabla^2 d|\|_2\big)\le C_{T_{max}},$$

where $C_{T_{max}}$ depends only on R, c_v, μ, λ, κ, T_{max} and the initial data.

Then, we focus on the bound of $\dot{\theta}$.

Lemma 3.2. *Assume $\mathcal{N}_0\le\varepsilon_0$. Then, for any $T\in(0,T_{max})$, it holds that*

$$\sup_{0\le t\le T}\big(\|\sqrt{\rho}\dot{\theta}\|_2^2+\|\nabla^2\theta\|_2^2\big)+\int_0^T\|\nabla\dot{\theta}\|_2^2dt\le C_{T_{max}},$$

where $C_{T_{max}}$ depends only on $R,c_v,\mu,\lambda,\kappa,\Phi_0$, and T_{max}.

Proof. Recalling (3.4), by the Sobolev inequality, Proposition 2.1 and Lemma 3.1, in order to get this result, it remains to bound the term $\sup_{0\le t\le T}\|\sqrt{\rho}\dot{\theta}\|_2^2$. Applying the operator $\partial_t+\mathrm{div}(u\cdot)$ to (1.3), by tedious computations developed in the Appendix, it follows

$$\begin{aligned}
&c_v\rho(\dot{\theta}_t+u\cdot\nabla\dot{\theta})\\
={}&\kappa\Delta\dot{\theta}+\kappa\big(\mathrm{div}u\Delta\theta-\partial_i(\partial_i u\cdot\nabla\theta)-\partial_i u\cdot\nabla\partial_i\theta\big)+\Big(\lambda(\mathrm{div}u)^2+\frac{\mu}{2}|\nabla u+(\nabla u)^t|^2\Big)\mathrm{div}u
\end{aligned}$$

$$
\begin{aligned}
&+R\rho\theta\partial_k u_l\partial_l u_k - R\rho\dot{\theta}\mathrm{div}u - R\rho\theta\mathrm{div}\dot{u} + 2\lambda(\mathrm{div}\dot{u} - \partial_k u_l\partial_l u_k)\mathrm{div}u\\
&+\mu(\partial_i u_j + \partial_j u_i)(\partial_i\dot{u}_j + \partial_j\dot{u}_i - \partial_i u_k\partial_k u_j - \partial_j u_k\partial_k u_i)\\
&+\partial_t(|\Delta d + |\nabla d|^2 d|^2) + \mathrm{div}(|\Delta d + |\nabla d|^2 d|^2 u). \qquad (3.22)
\end{aligned}
$$

Recalling that $|\Delta d + |\nabla d|^2 d|^2 = |\Delta d|^2 - |\nabla d|^4$ and $\Delta d + |\nabla d|^2 d = \Delta d - (d\cdot\Delta d)d$ guaranteed by $|d| = 1$, one has $\partial_t|\Delta d + |\nabla d|^2 d|^2 = 2\Delta d\cdot\Delta d_t - 4|\nabla d|^2\nabla d : \nabla d_t$. Thanks to this, multiplying (3.22) by $\dot{\theta}$, using integration by parts, Proposition 2.1, Lemma 3.1, and Corollary 3.1, we have

$$
\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\|\sqrt{\rho}\dot{\theta}\|_2^2 + \kappa\|\nabla\dot{\theta}\|_2^2\\
\le& C\int|\nabla u|(|\nabla^2\theta||\dot{\theta}| + |\nabla\theta||\nabla\dot{\theta}|)dx + \int|\nabla u|^2|\dot{\theta}|(|\nabla u| + \theta)dx\\
&+C\int\rho|\dot{\theta}|^2|\nabla u|dx + C\int\rho\theta|\nabla\dot{u}||\dot{\theta}|dx + C\int|\nabla u||\nabla\dot{u}||\dot{\theta}|dx\\
&+C\int(|\Delta d||\Delta d_t||\dot{\theta}| + |\nabla d|^3|\nabla d_t||\dot{\theta}| + |\Delta d - (d\cdot\Delta d)d|^2|u||\nabla\dot{\theta}|)dx\\
\le& C\|\nabla u\|_3\big(\|\nabla^2\theta\|_2\|\dot{\theta}\|_6 + \|\nabla\theta\|_6\|\nabla\dot{\theta}\|_2\big) + C\|\nabla u\|_3^2\|\dot{\theta}\|_6(\|\nabla u\|_6 + \|\theta\|_6)\\
&+C\|\nabla u\|_3\|\rho\dot{\theta}\|_2\|\dot{\theta}\|_6 + C\|\sqrt{\rho}\theta\|_2^{\frac12}\|\theta\|_6^{\frac12}\|\nabla\dot{u}\|_2\|\dot{\theta}\|_6 + C\|\nabla u\|_3\|\nabla\dot{u}\|_2\|\dot{\theta}\|_6\\
&+C\|\Delta d\|_3\|\Delta d_t\|_2\|\dot{\theta}\|_6 + C\|\nabla d\|_{\frac92}^3\|\nabla d_t\|_6\|\dot{\theta}\|_6 + C\|\Delta d\|_6^2\|u\|_6\|\nabla\dot{\theta}\|_2\\
\le&\frac{\kappa}{2}\|\nabla\dot{\theta}\|_2^2 + C\big(1 + \|\nabla^2\theta\|_2^2 + \|\sqrt{\rho}\dot{\theta}\|_2^2 + \|\nabla\dot{u}\|_2^2 + \|\Delta d_t\|_2^2\big).
\end{aligned}
$$

Thanks to (1.8), Lemma 3.1 and Corollary 3.1, applying Grönwall's inequality, we arrive at

$$
\sup_{0\le t\le T}\|\sqrt{\rho}\dot{\theta}\|_2^2 + \int_0^T\|\nabla\dot{\theta}\|_2^2 dt \le C_{T_{\max}},
$$

which completes the proof. ◻

Finally, a higher regularity on ρ is obtained.

LEMMA 3.3. *Assume $\mathcal{N}_0 \le \varepsilon_0$. Then, for any $T\in(0,T_{max})$, it holds that*

$$
\sup_{0\le t\le T}(\|\rho\|_{H^1\cap W^{1,q}} + \|\nabla^2 u\|_2) \le C_{T_{max}},
$$

where $C_{T_{max}}$ depends only on $R, c_v, \mu, \lambda, \kappa, \Phi_0$, and T_{max}.

Proof. By (3.13), Lemma 3.1 and Lemma 3.2, it follows

$$
\|\nabla v\|_\infty \le C_{T_{\max}}\log(e + \|\nabla\rho\|_q), \quad q\in(3,6]. \qquad (3.23)
$$

Meanwhile, it follows from (3.15), Lemma 3.1 and Corollary 3.1, that

$$
\|\nabla w\|_6 + \|\nabla^2 w\|_2 \le C\|\sqrt{\rho}\dot{u}\|_2 + C\|\nabla^2 d\|_2^{\frac32}\|\nabla^3 d\|_2^{\frac12} \le C_{T_{\max}}. \qquad (3.24)
$$

Due to (3.16), Proposition 2.1, and Corollary 3.1, one deduces by the Sobolev inequality that

$$
\begin{aligned}
\|\nabla^2 w\|_6 \le& C\big(\|\nabla\dot{u}\|_2 + \|\nabla^3 d\cdot\nabla d\|_2 + \||\nabla^2 d|^2\|_2\big)\\
\le& C\big(\|\nabla\dot{u}\|_2 + \|\nabla d\|_{H^2}^2\big) \le C\|\nabla\dot{u}\|_2 + C_{T_{\max}}.
\end{aligned} \qquad (3.25)
$$

Hence, by the Sobolev inequality, (3.24) and (3.25) give us

$$\|\nabla w\|_\infty \le C\|\nabla \dot{u}\|_2 + C_{T_{\max}},$$

which combined with (3.23) implies

$$\|\nabla u\|_\infty \le C_{T_{\max}} \log(e+\|\nabla\rho\|_q) + C\|\nabla \dot{u}\|_2, \quad q\in(3,6]. \tag{3.26}$$

Applying the elliptic estimates to (1.2), one has for $2\le p\le q$

$$\begin{aligned}\|\nabla^2 u\|_p &\le C(\|\rho\dot{u}\|_p + \|\Delta d\cdot\nabla d\|_p + \|\nabla P\|_p)\\ &\le C(\|\rho\dot{u}\|_p + \|\Delta d\|_p\|\nabla d\|_\infty + \|\rho\nabla\theta\|_p + \|\nabla\rho\theta\|_p)\\ &\le C(\|\rho\dot{u}\|_p + \|\nabla d\|_{H^2}^2 + \|\nabla\theta\|_{H^1} + \|\nabla\rho\|_p\|\theta\|_\infty)\\ &\le C(1+\|\rho\dot{u}\|_p + \|\nabla\rho\|_p)\\ &\le C_{T_{\max}}(1+\|\nabla\dot{u}\|_2 + \|\nabla\rho\|_p),\end{aligned} \tag{3.27}$$

where Proposition 2.1 and Lemma 3.2 were used. On the other hand, some straightforward calculations show that, for $2\le p\le q$

$$\frac{d}{dt}\|\nabla\rho\|_p \le C(1+\|\nabla u\|_\infty)\|\nabla\rho\|_p + C\|\nabla^2 u\|_p, \tag{3.28}$$

which together with (3.26) and (3.27) yields

$$\frac{d}{dt}\|\nabla\rho\|_p \le C_{T_{\max}}\big(1+\log(e+\|\nabla\rho\|_q)+\|\nabla\dot{u}\|_2\big)\|\nabla\rho\|_p + C_{T_{\max}}(1+\|\nabla\dot{u}\|_2+\|\nabla\rho\|_p).$$

Set

$$f(t) = e + \|\nabla\rho\|_q \quad \text{and} \quad g(t) = 1+\|\nabla\dot{u}\|_2,$$

then

$$\frac{d}{dt}f(t) \le C_{T_{\max}} g(t) f(t)\log f(t).$$

By solving the above ordinary differential inequality and using Lemma 3.1, one gets

$$\sup_{0\le t\le T}\|\nabla\rho\|_q \le C_{T_{\max}}. \tag{3.29}$$

Combining (3.29) with (3.26) yields

$$\int_0^T \|\nabla u\|_\infty^2 dt \le C_{T_{\max}}. \tag{3.30}$$

Choosing $p=2$ in (3.28), it follows from Lemma 3.1, (3.27), (3.30) and Grönwall's inequality that

$$\sup_{0\le t\le T}\|\nabla\rho\|_2 \le C_{T_{\max}}.$$

This together with (3.27) and Lemma 3.1 implies

$$\sup_{0\le t\le T}\|\nabla^2 u\|_2 \le C(\|\rho\dot{u}\|_2 + \|\nabla\rho\|_2) + C_{T_{\max}} \le C_{T_{\max}}.$$

Thus this lemma is proved. ∎

As a direct consequence of Lemmas 3.1-3.3, and Corollary 3.1, we have the following corollary.

COROLLARY 3.2. *Assume that $\mathcal{N}_0 \leq \varepsilon_0$. Then, for any $T \in (0, T_{max})$, it holds that*

$$\sup_{0 \leq t \leq T} (\|\nabla\theta\|_{H^1}^2 + \|(\nabla^2 u, \sqrt{\rho}\dot{u}, \sqrt{\rho}\dot{\theta}, \nabla^3 d, \nabla d_t)\|_2^2 + \|\rho\|_{H^1 \cap W^{1,q}}) \leq C_{T_{max}},$$

$$\int_0^T \|(\nabla\dot{u}, d_{tt}, \nabla^2 d_t, \nabla\dot{\theta})\|_2^2 dt \leq C_{T_{max}},$$

for a positive constant $C_{T_{max}}$ depending only on $R, c_v, \mu, \lambda, \kappa, \Phi_0$, and T_{max}.

4. Proof of Theorem 1.1

Proof. Let (ρ, u, θ, d) be the unique local solution guaranteed by Lemma 2.1. By applying the local well-posedness, i.e. Lemma 2.1, inductively, one can extend the (ρ, u, θ, d) uniquely to the maximal time $T_{\max}$ of existence. We claim that $T_{\max} = \infty$ and thus the conclusion holds. Assume by contradiction that $T_{\max} < \infty$. Let ε_0 be as in Proposition 2.1 and assume that $\mathcal{N}_0 \leq \varepsilon_0$. Then, it follows from Proposition 2.1 and Corollary 3.2 that for any $T \in (0, T_{\max})$

$$\begin{aligned}&\sup_{0 \leq t \leq T} \Big(\|\rho\|_{W^{1,q} \cap H^1}(t) + \|(u, \theta)\|_{D_0^1 \cap D^2}(t) + \|\nabla d\|_{H^2}(t) + \|(\sqrt{\rho}\theta, \sqrt{\rho}\dot{u}, \sqrt{\rho}\dot{\theta})\|_2(t) \Big) \\ &\leq C_{T_{\max}},\end{aligned} \tag{4.1}$$

where $C_{T_{\max}}$ is a positive constant depending on $T_{\max}$ and remains uniformly bounded for any $T < T_{\max}$.

Let δ be a sufficiently small positive number to be determined later and denote

$$(\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{d}_0) := (\rho, u, \theta, d)|_{t = T_{\max} - \delta}.$$

By the regularities of (ρ, u, θ, d) and (4.1), it is clear that

$$\begin{aligned}&\tilde{\rho}_0, \tilde{\theta}_0 \geq 0, \quad \tilde{\rho}_0 \in H^1 \cap W^{1,q}, \quad \sqrt{\tilde{\rho}_0}\tilde{\theta}_0 \in L^2, \\ &(\tilde{u}_0, \tilde{\theta}_0) \in D_0^1 \cap D^2, \quad \nabla\tilde{d}_0 \in H^2, \quad \text{and } |\tilde{d}_0| = 1,\end{aligned} \tag{4.2}$$

and

$$\|\tilde{\rho}_0\|_{W^{1,q} \cap H^1} + \|(\tilde{u}_0, \tilde{\theta}_0)\|_{D_0^1 \cap D^2} + \|\nabla\tilde{d}_0\|_{H^2} + \|\sqrt{\tilde{\rho}_0}\tilde{\theta}_0\|_2 \leq C_{T_{\max}}. \tag{4.3}$$

Since system (1.1)–(1.4) is satisfied a.e. in $\mathbb{R}^3 \times (0, T_{\max})$ and recalling (4.1), one can choose δ such that

$$\begin{aligned}&-\mu\Delta\tilde{u}_0 - (\mu + \lambda)\nabla \mathrm{div}\tilde{u}_0 + \nabla\tilde{P}_0 - \Delta\tilde{d}_0 \cdot \nabla\tilde{d}_0 = \sqrt{\tilde{\rho}_0}\tilde{g}_1, \\ &\kappa\Delta\tilde{\theta}_0 + \mathcal{Q}(\nabla\tilde{u}_0) + |\Delta\tilde{d}_0 + |\nabla\tilde{d}_0|^2\tilde{d}_0|^2 = \sqrt{\tilde{\rho}_0}\tilde{g}_2,\end{aligned} \tag{4.4}$$

with

$$\|\tilde{g}_1\|_2 + \|\tilde{g}_2\|_2 \leq C_{T_{\max}}, \tag{4.5}$$

where

$$\tilde{g}_1 := (\sqrt{\rho}\dot{u})|_{t = T_{\max} - \delta}, \quad \tilde{g}_2 := (c_v\sqrt{\rho}\dot{\theta} + R\sqrt{\rho}\theta \mathrm{div} u)|_{t = T_{\max} - \delta}.$$

With the aid of (4.2)–(4.5), viewing $T_{\max}-\delta$ as the new initial time, and applying Lemma 2.1, there is a positive number T_* depending only on $R, c_v, \mu, \lambda, \kappa$, and $C_{T_{\max}}$, but independent of δ, such that the solution (ρ, u, θ, d) can be extended uniquely from time $T_{\max}-\delta$ to another time $T_{\max}-\delta+T_*$. By choosing δ sufficiently small, it holds that $T_{\max}-\delta+T_*>T_{\max}$. In other words, one can extend the solution beyond $T_{\max}$, if $T_{\max}$ is a finite number, which contradicts to the definition of $T_{\max}$. Therefore, it must have $T_{\max}=\infty$, proving the theorem. ◻

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Appendix. Calculations on (3.22). The details about the calculations on (3.22) are given as follows. Applying the operator $\partial_t+\operatorname{div}(u\cdot)$ to (1.3), from the definition of material derivative, one has

$$\begin{aligned}
&c_v[\partial_t(\rho\dot\theta)+\operatorname{div}(u\rho\dot\theta)]+\partial_t(P\operatorname{div}u)+\operatorname{div}(uP\operatorname{div}u)-\kappa[\partial_t\Delta\theta+\operatorname{div}(u\Delta\theta)]\\
=&\frac{\mu}{2}\left[\partial_t\left(|\partial_j u_i+\partial_i u_j|^2\right)+\operatorname{div}(u|\partial_j u_i+\partial_i u_j|^2)\right]+\lambda[\partial_t(\operatorname{div}u)^2+\operatorname{div}(u(\operatorname{div}u)^2)]\\
&+\partial_t(|\Delta d+|\nabla d|^2 d|^2)+\operatorname{div}(|\Delta d+|\nabla d|^2 d|^2 u).
\end{aligned}\tag{A.1}$$

By (1.1) and some straightforward calculations, it follows that

$$c_v[\partial_t(\rho\dot\theta)+\operatorname{div}(u\rho\dot\theta)]=c_v\rho(\dot\theta_t+u\cdot\nabla\dot\theta),\tag{A.2}$$

$$\begin{aligned}
\partial_t(P\operatorname{div}u)+\operatorname{div}(uP\operatorname{div}u)&=R\big[\partial_t\rho\theta\operatorname{div}u+\rho\partial_t\theta\operatorname{div}u+\rho\theta\operatorname{div}u_t+\rho\theta(\operatorname{div}u)^2\\
&\qquad+u\cdot\nabla\rho\theta\operatorname{div}u+\rho u\cdot\nabla\theta\operatorname{div}u+\rho\theta u\cdot\nabla(\operatorname{div}u)\big]\\
&=R\big[\rho\dot\theta\operatorname{div}u+\rho\theta\operatorname{div}u_t+\rho\theta u\cdot\nabla(\operatorname{div}u)\big]\\
&=R\big[\rho\dot\theta\operatorname{div}u+\rho\theta\operatorname{div}\dot u-\rho\theta\operatorname{div}(u\cdot\nabla u)+\rho\theta u\cdot\nabla(\operatorname{div}u)\big]\\
&=R(\rho\dot\theta\operatorname{div}u+\rho\theta\operatorname{div}\dot u)-R\rho\theta\partial_k u_l\partial_l u_k,
\end{aligned}\tag{A.3}$$

$$\begin{aligned}
-\kappa[\partial_t\Delta\theta+\operatorname{div}(u\Delta\theta)]&=-\kappa[\Delta\dot\theta-\Delta(u\cdot\nabla\theta)+\operatorname{div}(u\Delta\theta)]\\
&=-\kappa[\Delta\dot\theta-\Delta(u\cdot\nabla\theta)+\operatorname{div}u\Delta\theta+u\cdot\nabla(\Delta\theta)]\\
&=-\kappa\Delta\dot\theta-\kappa\big[\operatorname{div}u\Delta\theta-\partial_i(\partial_i u\cdot\nabla\theta)-\partial_i u\cdot\nabla\partial_i\theta\big].
\end{aligned}\tag{A.4}$$

Similarly, for the terms on the right-hand side of (A.1), we also have

$$\begin{aligned}
&\frac{\mu}{2}\left[\partial_t\left(|\partial_j u_i+\partial_i u_j|^2\right)+\operatorname{div}(u|\partial_j u_i+\partial_i u_j|^2\right]\\
=&\mu(\partial_j u_i+\partial_i u_j)(\partial_t\partial_j u_i+\partial_t\partial_i u_j)+\frac{\mu}{2}|\nabla u+(\nabla u)^t|^2\operatorname{div}u\\
&+\mu u\cdot(\partial_j u_i+\partial_i u_j)\nabla(\partial_j u_i+\partial_i u_j)\\
=&\frac{\mu}{2}|\nabla u+(\nabla u)^t|^2\operatorname{div}u+\mu(\partial_i u_j+\partial_j u_i)(\partial_i\dot u_j+\partial_j\dot u_i-\partial_i u_k\partial_k u_j-\partial_j u_k\partial_k u_i),
\end{aligned}\tag{A.5}$$

and

$$\lambda[\partial_t(\operatorname{div}u)^2+\operatorname{div}(u(\operatorname{div}u)^2)]$$

$$=2\lambda(\partial_t \text{div}u)\text{div}u + 2\lambda(u \cdot \nabla \text{div}u)\text{div}u + \lambda(\text{div}u)^3$$
$$=2\lambda \text{div}\dot{u}\text{div}u - 2\lambda \text{div}(u \cdot \nabla u)\text{div}u + 2\lambda(u \cdot \nabla \text{div}u)\text{div}u + \lambda(\text{div}u)^3$$
$$=\lambda(\text{div}u)^3 + 2\lambda(\text{div}\dot{u} - \partial_k u_l \partial_l u_k)\text{div}u. \quad (A.6)$$

Thus, (3.22) follows by combining (A.1)-(A.6) together.

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