COMMUN. MATH. SCI.
Vol. 21, No. 6, pp. 1487–1503

SHARP INTERFACE LIMIT FOR COMPRESSIBLE NAVIER-STOKES/ALLEN-CAHN SYSTEM WITH SHOCK WAVE*

YUNKUN CHEN†, BIN HUANG‡, AND XIAODING SHI§

Abstract. In this paper, the sharp interface limit for the diffusion interface model system of immiscible two-phase flow called compressible Navier-Stokes/Allen-Cahn system is studied in one dimension. The results show that, for the initial perturbations with small energy but possibly large oscillations of shock wave solutions, and the strength of initial phase field is allowed to vary arbitrarily within its physical meaning, then the sharp interface limit of the compressible Navier-Stokes/Allen-Cahn system is the standard two-phase compressible Navier-Stokes equations.

Keywords. Compressible Navier-Stokes equations; Allen-Cahn equation, Shock wave; Stability; Sharp interface limit.

AMS subject classifications. 35Q35; 35B65; 76N10; 35M10; 35B40; 35C20; 76T30.

1. Introduction

Diffusion interface model is an important model to describe immiscible two-phase flow. The advantage of this model is that it is convenient to describe the interface motion between two phases, especially in numerical simulation. This model can capture the motion of the interface by introducing the phase field function, which overcomes the difficulty of interface tracking. However, due to the limitation of computing technology, it is impossible to simulate the diffusion interface with thickness as thin as the actual physical scale in numerical simulation. In fact, in the actual calculation, one often has to choose interfaces that are much thicker than the actual physical scale. Therefore, in order to ensure the accuracy of the simulation and to be able to compare with the data of physical experiments, the sharp interface limit becomes extremely important in immiscible two-phase flow dynamics.

Now we briefly review the establishment of diffusion interface model. Taking any volume element V in the two-phase flow, M_i is assumed to be the mass of the components in the representative material volume V, we define $\chi_i = \frac{\rho_i}{\rho}$ the mass concentration, $\rho_i = \frac{M_i}{V}$ the apparent mass density of the fluid i $(i=1,2)$, $\rho = \rho_1 + \rho_2$ the total density, and $\chi = \chi_1 - \chi_2$ the difference of the two components for the fluid mixture. χ is also known as the phase function or phase field. Obviously, physically speaking, formally, $-1 \le \chi \le 1$, the region with $\chi = -1$ is occupied by one phase field and the region with $\chi = 1$ is occupied by another phase field, and the diffusion interface between the two phases is described below

$$\Gamma_\epsilon(t) \overset{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^n \middle| -1 < \chi(\mathbf{x},t) < 1 \right\}, \tag{1.1}$$

for any time $t \ge 0$, and $\Gamma_\epsilon(t)$ divides the whole domain $\mathbb{R}^n$ into two separated domains $\Omega_\epsilon^-(t)$ and $\Omega_\epsilon^+(t)$ which represents the domains occupied by two phase fields respectively,

*Received: July 24, 2022; Accepted (in revised form): November 09, 2022. Communicated by Feimin Huang.

†School of Mathematics and Computer Science, Anshun University, Anshun 561000, China (cyk 2013@gznu.edu.cn).

‡College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China (abinhuang@gmail.com).

§Corresponding author. College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China (shixd@mail.buct.edu.cn).

more precisely

$$\Omega_\epsilon^-(t) \overset{\text{def}}{=} \{\mathbf{x}\in\mathbb{R}^n \,|\, \chi(\mathbf{x},t)=-1\}, \qquad \Omega_\epsilon^+(t) \overset{\text{def}}{=} \{\mathbf{x}\in\mathbb{R}^n \,|\, \chi(\mathbf{x},t)=1\},$$

and

$$\mathbb{R}^n = \Omega_\epsilon^-(t)\cup\Gamma_\epsilon(t)\cup\Omega_\epsilon^+(t), \quad \forall t\geq 0.$$

With the introduction of the above notation, the Navier-Stokes/Allen-Cahn (called as NSAC) system is proposed by Blesgen [3] and Heida-Málek-Rajagopal [13] to describe the compressible immiscible two-phase flow with diffusion interface, the one-dimensional Cauchy problem model is as follows

$$\begin{cases} \rho_t+(\rho u)_x=0, & x\in\mathbb{R},\ t>0,\\ (\rho u)_t+\big(\rho u^2+p(\rho)\big)_x=\nu(\epsilon)u_{xx}-\frac{1}{2}\eta(\epsilon)\big(\chi_x^2\big)_x, & x\in\mathbb{R},\ t>0,\\ (\rho\chi)_t+(\rho u\chi)_x=-L_d(\epsilon)\mu, & x\in\mathbb{R},\ t>0,\\ \rho\mu=\rho(\chi^3-\chi)-\eta(\epsilon)\chi_{xx}, & x\in\mathbb{R},\ t>0, \end{cases} \tag{1.2}$$

with the initial condition

$$(\rho,u,\chi)(x,0)=(\rho_0,u_0,\chi_0)(x), \quad x\in\mathbb{R}, \tag{1.3}$$

and the asymptotic constraints on initial condition

$$\lim_{x\to\pm\infty}(\rho_0,u_0,\chi_0)(x)=(\rho_\pm,u_\pm,\pm 1), \tag{1.4}$$

where the unknown $\rho(x,t)$ is the total density, $u(x,t)$ the mean velocity, $\chi(x,t)$ the concentration difference of the immiscible two-phase flow, respectively. $\mu(x,t)$ is the chemical potential, and $p=p(\rho)$ the pressure. $\rho_\pm>0, u_\pm$ are the given positive constants. $\epsilon>0$ is the parameter. $\eta(\epsilon)$ represents the gradient energy coefficient related to the interfacial width, $L_d(\epsilon)$ the phenomenological mobility coefficient related to the speed at which the system approaches an equilibrium configuration, and $\nu(\epsilon)$ the viscosity coefficient for the immiscible two-phase flow respectively. In this paper, these parameters satisfy the following relationships:

$$\nu(\epsilon)=\epsilon, \quad L_d(\epsilon)=\frac{1}{\epsilon}, \quad \eta(\epsilon)=\epsilon^2. \tag{1.5}$$

Moreover, we assume that initial phase field χ_0 satisfies the following physical assumption

$$-1\leq\chi_0\leq 1. \tag{1.6}$$

Remark 1.1. The physical meaning of hypothesis (1.5) is that, the diffusion coefficient of the phase field decreases with the increase of viscosity or the thickness of the interface for the immiscible two-phase flow. What we notice from mathematical model (1.2), the interface between different fluids is a thin layer determined by the phase field χ, i.e. $\Gamma_\epsilon(t)$. This thin layer can essentially be thought of as being caused by a chemical potential (μ) imbalance. Following the conclusions in Heida-Malek-Rajagopal [13] and Lowengrub-Truskinovsky [16], the generalized chemical potential μ is defined by

$$\rho\mu \overset{\text{def}}{=} \rho\frac{\partial f}{\partial\chi}-\mathrm{div}\big(\rho\frac{\partial f}{\partial\nabla\chi}\big), \tag{1.7}$$

where f is the phase-phase interfacial free energy density, and satisfies

$$f(\rho,\chi,\nabla\chi) \overset{\text{def}}{=} \frac{1}{4}(1-\chi^2)^2 + \frac{\eta(\epsilon)}{2\rho}|\nabla\chi|^2, \tag{1.8}$$

substituting (1.8) into (1.7), $(1.2)_4$ is achieved.

For the convenience of analyzing density and velocity, the Lagrange coordinates are introduced. Without losing generality, we still use (x,t) to represent this coordinate system

$$t \Rightarrow t, \quad x \Rightarrow \int_{(0,0)}^{(x,t)} \rho dx - \rho u dt. \tag{1.9}$$

Letting $v = \frac{1}{\rho}$, by using (1.5), the system (1.2)–(1.4) can be rewritten as follows:

$$\begin{cases} v_t - u_x = 0, & x\in\mathbb{R}, t>0, \\ u_t + p_x(v) = \epsilon\left(\frac{u_x}{v}\right)_x - \frac{\epsilon^2}{2}\left(\frac{\chi_x^2}{v^2}\right)_x, & x\in\mathbb{R}, t>0, \\ \chi_t = -\frac{v}{\epsilon}\mu, & x\in\mathbb{R}, t>0, \\ \mu = (\chi^3 - \chi) - \epsilon^2\left(\frac{\chi_x}{v}\right)_x, & x\in\mathbb{R}, t>0, \\ (v,u,\chi)(x,0) = (v_0,u_0,\chi_0)(x), & x\in\mathbb{R}, \end{cases} \tag{1.10}$$

with

$$\lim_{x\to\pm\infty}(v_0,u_0,\chi_0)(x) = (v_\pm, u_\pm, \pm 1), \quad -1\le\chi_0\le 1, \tag{1.11}$$

here $v_0 = \frac{1}{\rho_0}$, $v_\pm = \frac{1}{\rho_\pm}$. Formally, as the parameter $\epsilon \to 0^+$, the interface thickness of two-phase flow tends to zero, and the system (1.10) tends to the following standard two-phase compressible inviscid Navier-Stokes equations:

$$\begin{cases} v_t - u_x = 0, & \text{in } \Omega^\pm(t), \\ u_t + p_x(v) = 0, & \text{in } \Omega^\pm(t), \\ \chi = \pm 1, & \text{in } \Omega^\pm(t), \end{cases} \tag{1.12}$$

where for fixed $t \ge 0$,

$$\Omega^-(t) \overset{\text{def}}{=} \{x\in\mathbb{R} \,|\, \chi(x,t) = -1\}, \qquad \Omega^+(t) \overset{\text{def}}{=} \{x\in\mathbb{R} \,|\, \chi(x,t) = 1\},$$

and

$$\Gamma(t) \overset{\text{def}}{=} \mathbb{R}\backslash\{\Omega^-(t)\cup\Omega^+(t)\}, \qquad \text{meas}\Gamma(t) = 0.$$

The diffuse-interface model (1.2) for compressible immiscible two-phase flow has been studied extensively, both theoretically and numerically. In all of these works, the sharp interface limit to the diffusion interface model of compressible immiscible two-phase flow was an open and challenging problem. Even for the smooth solution, there is almost no rigorous analysis work.

We briefly review the recent analytical and computational work on compressible Navier-Stokes/Allen-Cahn system. The global existence of finite energy weak solutions in 3-D is established for the adiabatic exponent of pressure $\gamma > 6$ by Feireisl-Petzeltová-Rocca-Schimperna [12], this result was subsequently generalized to $\gamma > 2$ by Chen-Wen-Zhu [10]. The existence and uniqueness of strong solutions are obtained by Kotschote [15], Ding-Li-Lou [11], Chen-Wang-Xu [9], Chen-Guo [8], Chen-He-Huang-Shi [4]-[5], etc. The generalized Navier boundary condition and the relaxation boundary condition are established and discussed by Chen-He-Huang-Shi [6]. The large-time behavior of strong solutions to the Cauchy problem with small perturbations for the perturbation near a particular initial phase field in 3D was discussed by Zhao [25], and the stability of the rarefaction wave, contact wave, and stationary solution were investigated by Yin-Zhu [24], Luo-Yin-Zhu [17], and Luo-Yin-Zhu [18]. More recently, Chen-Hong-Shi [7] have extended the result of [25] to the general case which allows the strength of initial phase field to vary arbitrarily within its physical meaning. Compared with the well-posedness of the solutions, there are few results for sharp interface limit problem, and the results mainly focus on numerical analysis. Witterstein [23] points out that, formally, the sharp-interface limit of compressible NSAC system is the standard two-phase compressible Navier-Stokes equations.

The motivation of this paper is to explore the sharp interface limit for compressible NSAC system. Considering the complexity and difficulty of this problem, we first analyze the disturbance near the shock wave solution. By introducing the scaling transform, we know that the sharp interface limit problem is equivalent to the large-time behavior of the solutions, the details are as follows. Without causing any confusion, the coordinate system, after the transformation, we still call it (x,t):

$$x \Rightarrow \frac{x}{\epsilon}, \qquad t \Rightarrow \frac{t}{\epsilon}, \tag{1.13}$$

then the system (1.10)–(1.11) can be rewritten as the following:

$$\begin{cases} v_t - u_x = 0, & x\in\mathbb{R}, t>0,\\ u_t + p_x(v) = \left(\frac{u_x}{v}\right)_x - \frac{1}{2}\left(\frac{\chi_x^2}{v^2}\right)_x, & x\in\mathbb{R}, t>0,\\ \chi_t = -v\mu, & x\in\mathbb{R}, t>0,\\ \mu = (\chi^3-\chi) - \left(\frac{\chi_x}{v}\right)_x, & x\in\mathbb{R}, t>0,\\ (v,u,\chi)(x,0) = (v_0,u_0,\chi_0)(x), & x\in\mathbb{R}, \end{cases} \tag{1.14}$$

with

$$\lim_{x\to\pm\infty}(v_0,u_0,\chi_0)(x) = (v_\pm, u_\pm, \pm 1), \quad -1 \le \chi_0 \le 1. \tag{1.15}$$

Without loss of generality, $p(v)$ is assumed to be a smooth function of v satisfying

$$p'(v) < 0, \quad p''(v) \ge 0, \quad p''(v) \not\equiv 0. \tag{1.16}$$

and $v_\pm, u_\pm$ satisfy the following entropy conditions

$$\rho_- = \frac{1}{v_-} > \rho_+ = \frac{1}{v_+} > 0, \quad u_- > u_+. \tag{1.17}$$

The left state (v_-,u_-) and the right state (v_+,u_+) are connected by the 2-shock with the speed $s>0$, where s is determined by the following R–H conditions

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0,\\ -s(u_+ - u_-) + p(v_+) - p(v_-) = 0, \end{cases} \tag{1.18}$$

and has the following expression

$$s=\sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}. \tag{1.19}$$

Now we begin to give our main results. Considering the following Riemann problem for Euler system

$$\begin{cases} v_t-u_x=0, \\ u_t+p_x=0, \\ (v,u)(x,0)=\begin{cases}(v_-,u_-),\ x<0,\\ (v_+,u_+),\ x>0,\end{cases} \end{cases} \tag{1.20}$$

Note that the eigenvalues of system (1.20) are $\lambda_1=-\sqrt{-p'(v)}<0,\ \lambda_2=\sqrt{-p'(v)}>0$, from the entropy condition (1.17), one has

$$0<\lambda_2(v_+,u_+)<s<\lambda_2(v_-,u_-). \tag{1.21}$$

The strength of initial specific volume v_0 is defined below

$$\delta_1=|v_+-v_-|. \tag{1.22}$$

From the theory of hyperbolic equations, we know that the unique entropy solution (v^s,u^s) of (1.20) is

$$(v^s,u^s)(x,t)=\begin{cases}(v_-,u_-),\ x<st,\\ (v_+,u_+),\ x>st.\end{cases} \tag{1.23}$$

As we know, the 2-viscous shock wave of the Cauchy problem for Navier-Stokes system

$$\begin{cases} v_t-u_x=0, & x\in\mathbb{R},t>0,\\ u_t+p_x(v)=\left(\frac{u_x}{v}\right)_x, & x\in\mathbb{R},t>0,\\ (v,u)(x,0)=(v_0,u_0)(x), & x\in\mathbb{R}, \end{cases} \tag{1.24}$$

has the form $(V,U)(x-st)$ which connects (v_-,u_-) on the left and (v_+,u_+) on the right uniquely up to a shift and satisfies

$$\begin{cases} -sV_y-U_y=0, & y\in\mathbb{R},\\ -sU_y+p(V)_y=\left(\frac{U_y}{V}\right)_y, & y\in\mathbb{R},\\ \lim\limits_{y\to\pm\infty}(V,U)=(v_\pm,u_\pm), \end{cases} \tag{1.25}$$

where $t=t, y=x-st$. The existence and decay properties of this 2-viscous shock wave is given below

LEMMA 1.1 (cf. Matsumura-Nishihara [19]). *Assume that (1.16),(1.17)–(1.19), there exists a unique smooth solution $(V,U)(y)$ of the system (1.25) up to a shift, and satisfies*

$$\begin{aligned} &V_y=\frac{V}{s}\left[p(v_\pm)+s^2(v_\pm-V)-p(V)\right],\\ &0<V_y\le\frac{v_+}{s}\left(p(v_-)+s^2v_-\right),\\ &U_y=-sV_y\le 0, \qquad v_-<V<v_+. \end{aligned} \tag{1.26}$$

Moreover, there are positive constants $c_\pm$ *such that*

$$\begin{array}{l} |V(x-st)-v_\pm| = O(1)\delta_1 e^{-c_\pm\delta_1|x-st|}, \\ |U(x-st)-u_\pm| = O(1)\delta_1 e^{-c_\pm\delta_1|x-st|}, \\ |V_y,U_y| = O(1)\delta_1^2 e^{-c_\pm\delta_1|x-st|}. \end{array} \tag{1.27}$$

For the convenience of obtaining derivative estimates of v, similar as He-Huang [14] and Vasseur-Yao [22], the following effective velocity is introduced, which was proposed by Shelukhin [21] and Bresch-Desjardins [1], [2] to obtain the entropy estimates.

$$h(y,t) = u - \frac{v_y}{v}, \qquad H(y) = U - \frac{V_y}{V}. \tag{1.28}$$

Substituting (1.28) into Equations (1.14) and (1.25) respectively, we have

$$\begin{cases} v_t - sv_y - h_y = \left(\frac{v_y}{v}\right)_y, & y\in\mathbb{R}, t>0, \\ h_t - sh_y + p_y(v) = -\frac{1}{2}\left(\frac{\chi_y^2}{v^2}\right)_y, & y\in\mathbb{R}, t>0, \\ \chi_t - s\chi_y = -v\mu, & y\in\mathbb{R}, t>0, \\ \mu = (\chi^3-\chi) - \left(\frac{\chi_y}{v}\right)_y, & y\in\mathbb{R}, t>0, \\ (v,h,\chi)(y,0) = (v_0,h_0,\chi_0)(y), & y\in\mathbb{R}, \end{cases} \tag{1.29}$$

and

$$\begin{cases} V_t - sV_y - H_y = \left(\frac{V_y}{V}\right)_y, & y\in\mathbb{R}, \\ H_t - sH_y + p_y(V) = 0, & y\in\mathbb{R}, \end{cases} \tag{1.30}$$

with

$$\lim_{y\to\pm\infty}(v_0,h_0,\chi_0)(y) = (v_\pm,u_\pm,\pm1), \qquad \lim_{y\to\pm\infty}(V,H)(y) = (v_\pm,u_\pm), \tag{1.31}$$

where $h_0 = u_0 - \frac{v_{0y}}{v_0}$. The antiderivatives are defined as follows

$$\Phi(y,t) \overset{\text{def}}{=} \int_{-\infty}^{y} \big(v(z,t)-V(z+\alpha)\big)dz, \quad \Psi(y,t) \overset{\text{def}}{=} \int_{-\infty}^{y} \big(h(z,t)-H(z+\alpha)\big)dz, \tag{1.32}$$

and

$$\Phi_0 = \Phi(y,0) = \int_{-\infty}^{y} \big(v_0(z)-V(z+\alpha)\big)dz, \Psi_0 = \Psi(y,0) = \int_{-\infty}^{y} \big(h_0 - H(z+\alpha)\big)dz, \tag{1.33}$$

where the shift α is

$$\alpha = \frac{1}{v_+ - v_-}\int_{-\infty}^{+\infty} \big(v_0(y)-V(y)\big)dy, \tag{1.34}$$

and then

$$\int_{-\infty}^{+\infty} \big(v_0(y)-V(y+\alpha)\big)dy = 0, \quad \int_{-\infty}^{+\infty} \big(h_0(y)-H(y+\alpha)\big)dy = 0. \tag{1.35}$$

Suppose that

$$\Phi_0,\Psi_0 \in L^2(\mathbb{R}),\ (v_0-V, h_0-H)\in H^2(\mathbb{R})\cap L^1(\mathbb{R}),\ \chi_0^2-1\in L^2(\mathbb{R}),\ \chi_{0x}\in H^2(\mathbb{R}), \tag{1.36}$$

and

$$\inf_{\mathbb{R}} v_0 > 0, \qquad |\chi_0| \leq 1, \quad \text{on } \mathbb{R}. \tag{1.37}$$

THEOREM 1.1. *Assume that* (1.33)-(1.37)*, then there exists a positive constant* δ*, such that if*

$$\|(\Phi_0, \Psi_0)\|_{H^3(\mathbb{R})} + \|\chi_{0x}\|_{H^2(\mathbb{R})} + \|\chi_0^2 - 1\|_{L^2(\mathbb{R})} + |v_+ - v_-| \leq \delta, \tag{1.38}$$

the Cauchy problem (1.14)-(1.15) *has a unique strong solution* (v, u, χ) *satisfying*

$$\begin{aligned}
&\big(v - V, u - U\big) \in C\big([0, +\infty); H^2(\mathbb{R})\big), \quad \chi^2 - 1 \in C\big([0, +\infty); L^2(\mathbb{R})\big), \\
&\chi_x \in C\big([0, +\infty); H^2(\mathbb{R})\big), \quad \chi_x \in L^2\big([0, +\infty); H^3(\mathbb{R})\big), \\
&v - V \in L^2\big([0, +\infty); H^3(\mathbb{R})\big), \quad u - U \in L^2\big([0, +\infty); H^2(\mathbb{R})\big),
\end{aligned} \tag{1.39}$$

and $-1 \leq \chi \leq 1$*. Moreover*

$$\lim_{t \to +\infty} \big\|\big(v(x,t) - V(x - st + \alpha), u(x,t) - U(x - st + \alpha)\big)\big\|_{L^\infty(\mathbb{R})} = 0, \tag{1.40}$$

and

$$\lim_{t \to +\infty} \big\|\chi^2 - 1\big\|_{L^\infty(\mathbb{R})} = 0. \tag{1.41}$$

THEOREM 1.2. *Assume that* (1.33)-(1.37)*, then there exists a positive constant* δ*, such that if*

$$\|(\Phi_0, \Psi_0)\|_{L^2(\mathbb{R})} + \|\chi_0^2 - 1\|_{L^2(\mathbb{R})} + |v_+ - v_-| \leq \delta, \tag{1.42}$$

the Cauchy problem (1.10)-(1.11) *admits a family of global smooth solutions* (v, u, χ) *and the following sharp interface limit holds*

$$\lim_{\epsilon \to 0^+} \big\|\big(v - v^s, u - u^s, \chi^2 - 1\big)\big\|_{L^\infty(\Sigma_h)} = 0, \tag{1.43}$$

where

$$\Sigma_h = \Big\{(x,t)\Big| |x - st| \geq h, h \leq t \leq +\infty\Big\},$$

for any positive constant $h > 0$*, and* (v^s, u^s) *is the entropy solution of* (1.20) *with the expression* (1.23)*.*

REMARK 1.2. Theorem 1.1 allows the initial phase field to oscillate between ± 1, therefore, it can be used to explain the phase separation phenomenon.

REMARK 1.3. Theorem 1.2 shows that the phase field jumps and phase separation occurs as the interface thickness approaches zero. Theorem 1.1 and Theorem 1.2 show that, under certain conditions, the sharp interface limit is consistent with the large-time behavior for compressible immiscible two-phase flow.

We now make some comments on the analysis of this paper. One key issue that needs to be addressed is to obtain the higher-order derivative estimate of the specific volume v, this difficulty arises due to the strong coupling between v and χ in the compressible NSAC system (1.10). The key to solving this problem is that, the effective velocity h

(1.28) is introduced, so that the hyperbolic equation $(1.14)_1$ becomes parabolic equation $(1.29)_1$, and this makes it relatively easy to obtain higher-order derivative estimates of v. Another key issue that needs to be addressed is that what we consider here is the shock wave perturbation of specific volume v and the velocity field u, while the phase field is the perturbation near the phase separation of immiscible two-phase flow, the stability analysis of the former requires anti-derivative method, while the latter does not, this is bound to face the disunity of analytical methods. To overcome this difficulty, we adopt the method of using the antiderivative only for v and u, while keeping the phase field χ unchanged. The last key point to resolve is that the initial value of the phase field varies between ± 1, and such large amplitude of initial phase field make it difficult to obtain the energy estimates. Fortunately, the estimate (2.15) is observed, from which we get the uniformly bounded estimation for $\|\chi^2-1\|_{L^2(\mathbb{R})}$. Therefore, the strength of the phase field can vary arbitrarily in its physical meaning.

Notations. Throughout this paper, L^2 denotes the space of measurable functions on $\mathbb{R}$ which are square integrable, with the norm $\|f\|=(\int_{\mathbb{R}}|f|^2dy)^{\frac{1}{2}}$. $H^l(l\geq 0)$ denotes the Sobolev space of L^2-functions f on $\mathbb{R}$ whose derivatives $\partial_y^j f, j=1,\cdots$ are L^2 functions too, with the norm $\|f\|_l=(\sum_{j=0}^l\|\partial_y^j f\|^2)^{\frac{1}{2}}$.

2. The Proof of the Theorem

Subtracting $(1.29)_{1,2}$ from $(1.30)_{1,2}$ and taking the antiderivative, one has

$$\begin{cases} \Phi_t - s\Phi_y - \Psi_y - \dfrac{1}{V}\Phi_{yy} + \dfrac{1}{V^2}V_y\Phi_y = F, & y\in\mathbb{R}, t>0,\\ \Psi_t - s\Psi_y + p'(V)\Phi_y = G - \dfrac{1}{2}\dfrac{\chi_y^2}{(V+\Phi_y)^2}, & y\in\mathbb{R}, t>0,\\ \chi_t - s\chi_y = -(V+\Phi_y)\mu, & y\in\mathbb{R}, t>0,\\ \mu = (\chi^3-\chi) - \left(\dfrac{\chi_y}{V+\Phi_y}\right)_y, & y\in\mathbb{R}, t>0,\\ (\Phi,\Psi,\chi)(y,0) = (\Phi_0,\Psi_0,\chi_0)(y), & y\in\mathbb{R}, \end{cases} \tag{2.1}$$

with

$$\lim_{y\to+\infty}(\Phi_0,\Psi_0,\chi_0) = (0,0,\pm 1), \qquad -1\leq\chi_0\leq 1, \tag{2.2}$$

where

$$\begin{aligned} F &= \left(\frac{1}{v}-\frac{1}{V}\right)\Phi_{yy} + \left(\frac{1}{v}-\frac{1}{V}+\frac{1}{V^2}\Phi_y\right)V_y,\\ G &= -\Big(p(v)-p(V)-p'(V)(v-V)\Big). \end{aligned} \tag{2.3}$$

For any given $m, M>0$, we define the solution space $X_{m,M}(0,T)$ as follows

$$\begin{aligned} X_{m,M}(0,T) = \Big\{(\Phi,\Psi,\chi)\Big|&(\Psi,\Phi)\in C(0,T;H^3), \chi^2-1\in C(0,T;L^2), \chi_y\in C(0,T;H^2),\\ &\sup_{t\in(0,T)}\left(\|(\Phi,\Psi)\|_3+\|\chi_y\|_2+\|\chi^2(t)-1\|\right)\leq M,\\ &\inf_{y\in\mathbb{R},t\in(0,T)} v(y,t)\geq m>0\Big\}. \end{aligned} \tag{2.4}$$

Proposition 2.1. *Assume that* (1.5), (1.16)–(1.19), (1.32)-(1.37). *If*

$$\|(\Phi_0,\Psi_0)\|_3 + \|\chi_{0y}\|_2 + \|\chi_0^2-1\| \leq M, \tag{2.5}$$

and

$$\inf_{y\in\mathbb{R}}\Big(V+\Phi_{0y}\Big)>m>0, \tag{2.6}$$

then there exists T^* small enough, such that, the Cauchy problem (2.1)-(2.2) admits a unique solution $(\Phi,\Psi,\chi)\in X_{\frac{m}{2},2M}\big([0,T^*]\big)$, satisfying

$$\begin{aligned}&\|(\Phi,\Psi)\|_3^2+\|\chi^2-1\|^2+\|\chi_y\|_2^2+\int_0^t\Big(\|\Phi_y\|_3^2+\|\Psi_y\|_2^2+\|\chi_y\|_3^2\Big)d\tau\\ \le& C\Big(\|(\Phi_0,\Psi_0)\|_3^2+\|\chi_0^2-1\|^2+\|\chi_{0y}\|_2^2\Big),\end{aligned} \tag{2.7}$$

and

$$-1\le\chi\le1, \tag{2.8}$$

where C is the positive constant which may depend on (v_-,u_-).

Proposition 2.1 is the conclusion about the existence and uniqueness of local solutions for the system (2.1), it can be can be proved by the usual linearization method and the fixed point theorem, the details are omitted. In order to obtain the existence of global solution, we will establish the a prior estimates in Proposition 2.2 as follows. From the definition (2.4), choosing M small enough, called as δ_0, by using Sobolev embedding theorem, there exist $m_0>0$, such that

$$0<\frac{3}{4}v_-\le V+\Phi_y\le\frac{5}{4}v_-,\qquad \inf_{y\in\mathbb{R},t\in(0,T)}3\chi^2-1\ge m_0>0. \tag{2.9}$$

Thus, the space $X_{m,M}$ can be simplified as follows

$$\begin{aligned}X_{\delta_0}(0,T)=\Big\{(\Phi,\Psi,\chi)\Big|&(\Phi,\Psi)\in C(0,T;H^3),\chi^2-1\in C(0,T;L^2),\chi_y\in C(0,T;H^2),\\ &\sup_{t\in(0,T)}\big(\|(\Phi,\Psi)\|_3+\|\chi_y\|_2+\|\chi^2-1\|\big)\le\delta_0\Big\}.\end{aligned} \tag{2.10}$$

PROPOSITION 2.2. *Suppose that $(\Phi,\Psi,\chi)\in X_\delta\big([0,+\infty)\big)$ is the solution of the Cauchy problem (2.1)-(2.2) for some $T>0$, there exist the positive constants δ_0 independent of T, such that if*

$$\|(\Phi_0,\Psi_0)\|_3+\|\chi_{0y}\|_2+\|\chi_0^2-1\|+|v_+-v_-|\le\delta_0, \tag{2.11}$$

it holds that

$$\begin{aligned}&\|(\Phi,\Psi)(t)\|_3^2+\|\chi^2(t)-1\|^2+\|\chi_y(t)\|_2^2+\textstyle\int_0^{+\infty}\Big(\|\Phi_y\|_3^2+\|\Psi_y\|_2^2+\|\chi_y\|_3^2\Big)d\tau\\ \le\ & C\Big(\|(\Phi_0,\Psi_0)\|_3^2+\|\chi_0^2-1\|^2+\|\chi_{0y}\|_2^2\Big),\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\textstyle\int_0^{+\infty}\left|\frac{d}{dt}\|\Phi_{yy}\|^2\right|+\left|\frac{d}{dt}\|\Psi_{yy}\|^2\right|+\left|\frac{d}{dt}\|\chi_{yy}\|^2\right|d\tau\\ \le\ & C\Big(\|(\Phi_0,\Psi_0)\|_3^2+\|\chi_0^2-1\|^2+\|\chi_{0y}\|_2^2\Big),\end{aligned} \tag{2.13}$$

where C is the positive constant which may depend on (v_-,u_-) but is independent of T.

The proof of Proposition 2.2 is given below, which is divided into the following lemmas.

Lemma 2.1. *Suppose $(\Phi,\Psi,\chi)\in X_\delta(0,T)$ is the solution of system (2.1), then, for $t\in[0,T]$, the following inequalities hold*

$$|\chi(x,t)|\le 1,\qquad \forall (x,t)\in(-\infty,+\infty)\times[0,T], \tag{2.14}$$

and

$$\begin{aligned}&\|\big(\Phi,\Psi,\chi^2-1\big)(t)\|^2+\textstyle\int_0^t\Big(\|\Phi_y\|^2+\|\sqrt{V_y}\Psi\|^2+\|\chi_y\|^2+\|\chi^3-\chi\|^2\Big)d\tau\\ \le\ & C\big\|(\Phi_0,\Psi_0,\chi_0^2-1)\big\|^2+C\textstyle\int_0^t\|\Phi_{yy}\|^2d\tau,\end{aligned} \tag{2.15}$$

where C is the positive constant which may depend on (v_-,u_-) but is independent of T.

Proof. By using the maximum principle for parabolic equation (see Lemma 2.1 in [20]) and (2.9), one obtains

$$\chi^2\le 1, \tag{2.16}$$

which yields (2.14). Multiplying $(2.1)_1$ by Φ, $(2.1)_2$ by $-\frac{\Psi}{p'(V)}$, $(2.1)_3$ by $\chi^3-\chi$, adding them together, one has

$$\begin{aligned}&\Big(\frac{\Phi^2}{2}-\frac{\Psi^2}{2p'(V)}+\frac{(\chi^2-1)^2}{4}\Big)_t+\Big(\frac{s\Psi^2}{2p'(V)}-\frac{s\Phi^2}{2}-\Phi\Psi-\frac{\Phi\Phi_y}{V}-\frac{s}{4}(\chi^2-1)^2\Big)_y\\ &\quad-\big(\chi_y(\chi^3-\chi)\big)_y+(3\chi^2-1)\chi_y^2+(V+\Phi_y)(\chi^3-\chi)^2+\frac{1}{V}\Phi_y^2+\frac{s}{2}\frac{p''(V)}{(p'(V))^2}V_y\Psi^2\\ =&-\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi(\chi^2-1)+\frac{1}{2}\frac{\chi_y^2}{(V+\Phi_y)^2}\frac{\Psi}{p'(V)}+F\Phi-\frac{G}{p'(V)}\Psi.\end{aligned} \tag{2.17}$$

Integrating (2.17) over $(-\infty,+\infty)\times[0,t)$ by parts, one gets

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big(\frac{\Phi^2}{2}-\frac{\Psi^2}{2p'(V)}+\frac{(\chi^2-1)^2}{4}\Big)dy\Big|_0^t+\int_{-\infty}^{+\infty}(3\chi^2-1)\chi_y^2dyd\tau\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}\Big[(V+\Phi_y)(\chi^3-\chi)^2+\frac{1}{V}\Phi_y^2+\frac{s}{2}\frac{p''(V)}{(p'(V))^2}V_y\Psi^2\Big]dyd\tau\\ =&\int_0^t\int_{-\infty}^{+\infty}\Big(\frac{\chi_y^2}{2(V+\Phi_y)^2}\frac{\Psi}{p'(V)}-\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi(\chi^2-1)+F\Phi-\frac{G\Psi}{p'(V)}\Big)dyd\tau.\end{aligned} \tag{2.18}$$

Since

$$\begin{aligned}|F|&=\Big|\big(\frac{1}{v}-\frac{1}{V}\big)\Phi_{yy}+\big(\frac{1}{v}-\frac{1}{V}+\frac{1}{V^2}\Phi_y\big)V_y\Big|\\ &\le C\Big(|\Phi_y||\Phi_{yy}|+|V_y||\Phi_y|^2\Big)\\ &\le C\Big(|\Phi_y||\Phi_{yy}|+\delta_1^2e^{-c\delta_1|y|}|\Phi_y|^2\Big),\end{aligned} \tag{2.19}$$

and

$$|G|\le C\Phi_y^2, \tag{2.20}$$

combining with (2.9) and (2.4), one obtains

$$\begin{aligned}\Big|\int_0^t\int_{-\infty}^{+\infty}F\Phi dyd\tau\Big| \le & C\delta\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau+C\delta_1^2\int_0^t\int_{-\infty}^{+\infty}e^{-c\delta_1|y|}|\Phi||\Phi_y|^2dyd\tau\\ \le & C\delta\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau+C\delta_0\delta_1^2\int_0^t\int_{-\infty}^{+\infty}e^{-c\delta_1|y|}|\Phi_y|^2dyd\tau\\ \le & C\delta\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau+C\delta_0\delta_1\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)dxd\tau\\ \le & C\delta\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau,\end{aligned}\tag{2.21}$$

and

$$\Big|\int_0^t\int_{-\infty}^{+\infty}\frac{G\Psi}{p'(V)}dyd\tau\Big|\le C\delta\int_0^t\|\Phi_y\|^2d\tau.\tag{2.22}$$

Moreover, by using (2.9) and (2.4) again, one gets

$$\Big|\int_0^t\int_{-\infty}^{+\infty}\frac{\chi_y^2}{2(V+\Phi_y)^2}\frac{\Psi}{p'(V)}dyd\tau\Big|\le C\delta\int_0^t\|\chi_y\|^2d\tau,\tag{2.23}$$

and

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{+\infty}\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi(\chi^2-1)dyd\tau\Big|\\ \le&\frac{\delta}{2}\int_0^t\|\chi_y\|^2d\tau+C\delta_1\int_0^t\|\chi^3-\chi\|^2d\tau+C\int_0^t\|\Phi_{yy}\|^2d\tau.\end{aligned}\tag{2.24}$$

From (2.21)-(2.24), combining with Lemma 1.1, choosing δ_1, and δ small enough, (2.15) is achieved. Thus, the proof of Lemma 2.1 is completed. □

LEMMA 2.2. *Suppose* $(\Phi,\Psi,\chi)\in X_\delta(0,T)$ *is the solution of system* (2.1), *then, for* $t\in[0,T]$, *the following inequality holds*

$$\begin{aligned}&\|(\Phi_y,\Psi_y,\chi_y)(t)\|^2+\int_0^t\Big(\|\Phi_{yy}\|^2+\|\Psi_y\|^2+\|\chi_{yy}\|^2\Big)d\tau\\ \le& C\Big\|(\Phi_{0y},\Psi_{0y},\chi_{0y})\Big\|^2,\end{aligned}\tag{2.25}$$

where C *is the positive constant which may depend on* (v_-,u_-) *but is independent of* T.

Proof. Multiplying $(2.1)_1$ by $-\Phi_{yy}$, $(2.1)_2$ by $-\frac{\Psi_{yy}}{p'(V)}$, $(2.1)_3$ by χ_{yy}, adding them together, one has

$$\begin{aligned}&\Big(\frac{\Phi_y^2}{2}-\frac{\Psi_y^2}{2p'(V)}+\chi_y^2\Big)_t+\Big(\frac{s\Phi_y^2}{2}-\Phi_t\Phi_y+\Phi_y\Psi_y+\frac{\Psi_t\Psi_y}{p'(V)}-\frac{s\Psi_y^2}{2p'(V)}-\chi_t\chi_y-\frac{G\Psi_y}{p'(V)}\Big)_y\\ &-\Big((V+\Phi_y)(\chi^3-\chi)\chi_y\Big)_y+(V+\Phi_y)(3\chi^2-1)\chi_y^2+\chi_{yy}^2+\frac{\Phi_{yy}^2}{V}+\frac{sp''(V)V_y}{(p'(V))^2}\Psi_y^2\\ =&F\Phi_{yy}-\frac{V_y}{V^2}\Phi_y\Phi_{yy}-\Big(\frac{1}{p'(V)}\Big)_yp'(V)\Phi_y\Psi_y-\frac{1}{p'(V)}G_y\Psi_y\end{aligned}$$

$$-\Big(V_y+\Phi_{yy}\Big)(\chi^3-\chi)\chi_y+\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi_{yy}. \tag{2.26}$$

Integrating (2.26) over $(-\infty,+\infty)\times[0,t)$ by parts, one gets

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big(\frac{\Phi_y^2}{2}-\frac{\Psi_y^2}{2p'(V)}+\chi_y^2\Big)dy\Big|_0^t+\int_0^t\int_{-\infty}^{+\infty}(V+\Phi_y)(3\chi^2-1)\chi_y^2dyd\tau\\&+\int_0^t\int_{-\infty}^{+\infty}\Big[\chi_{yy}^2+\frac{\Phi_{yy}^2}{V}+\frac{sp''(V)V_y}{(p'(V))^2}\Psi_y^2\Big]dyd\tau\\=&\int_0^t\int_{-\infty}^{+\infty}\Big[F\Phi_{yy}-\frac{V_y}{V^2}\Phi_y\Phi_{yy}-\Big(\frac{1}{p'(V)}\Big)_yp'(V)\Phi_y\Psi_y-\frac{1}{p'(V)}G_y\Psi_y\Big]dyd\tau\\&\int_0^t\int_{-\infty}^{+\infty}\Big[-(V_y+\Phi_{yy})(\chi^3-\chi)\chi_y+\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi_{yy}\Big]dyd\tau.\end{aligned} \tag{2.27}$$

Noting that (2.4), (2.19)-(2.20), one has

$$\Big|\int_0^t\int_{-\infty}^{+\infty}(F\Phi_{yy}-\frac{V_y}{V^2}\Phi_y\Phi_{yy})dyd\tau\Big|\le C\delta\int_0^t\|\Phi_{yy}\|^2d\tau+C\int_0^t\|\Phi_y\|^2d\tau, \tag{2.28}$$

and

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{+\infty}[-\Big(\frac{1}{p'(V)}\Big)_yp'(V)\Phi_y\Psi_y-\frac{1}{p'(V)}G_y\Psi_y]dyd\tau\Big|\\\le&\int_0^t\int_{-\infty}^{+\infty}\frac{s}{2}\frac{p''(V)V_y}{(p'(V))^2}\Psi_y^2dyd\tau+\int_0^t\|\Psi_y\|^2d\tau+C\delta\int_0^t\big(\|\Psi_y\|^2+\|\Psi_{yy}\|^2\big)d\tau.\end{aligned} \tag{2.29}$$

Moreover,

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{+\infty}[-(V_y+\Phi_{yy})(\chi^3-\chi)\chi_y+\frac{V_y+\Phi_{yy}}{V+\Phi_y}\chi_y\chi_{yy}]dyd\tau\Big|\\\le&\frac{1}{2}\int_0^t\|\chi_{yy}\|^2d\tau+\frac{1}{2v_+}\int_0^t\|\Phi_{yy}\|^2d\tau+C(\delta+|v_+-v_-|)\int_0^t\|\chi^3-\chi\|^2d\tau.\end{aligned} \tag{2.30}$$

From (2.28)-(2.30), combining with Lemma 1.1, choosing δ, and $|v_+-v_-|$ small enough, one obtains

$$\begin{aligned}&\int_{-\infty}^{+\infty}(\Phi_y^2+\Psi_y^2+\chi_y^2)dy+\int_0^t\int_{-\infty}^{+\infty}(\Phi_{yy}^2+V_y\Psi_y^2+\chi_{yy}^2+\chi_y^2)dyd\tau\\\le&C\int_{-\infty}^{+\infty}(\Phi_{0y}^2+\Psi_{0y}^2+\chi_{0y}^2)dy+C\int_0^t\int_{-\infty}^{+\infty}\Phi_y^2dyd\tau.\end{aligned} \tag{2.31}$$

Combining with (2.15), choosing δ small enough, one gets

$$\begin{aligned}&\|(\Phi_y,\Psi_y,\chi_y)(t)\|^2+\int_0^t\Big(\|\Phi_{yy}\|^2+\|\sqrt{V_y}\Psi_y\|^2+\|\chi_{yy}\|^2\Big)d\tau\\\le&C\|(\Phi_{0y},\Psi_{0y},\chi_{0y})\|^2.\end{aligned} \tag{2.32}$$

Multiplying $(2.1)_1$ by Ψ_y, differentiating $(2.1)_2$ with respect to y, and multiplying the result by by Φ, adding them together, one has

$$\Psi_y^2=(\Phi\Psi_y)_t+\Big(\big(p(v)-p(V)\big)\Phi-s\Phi\Psi_y+\frac{1}{2}\frac{\Phi\chi_y^2}{(V+\Phi_y)^2}\Big)_y$$

$$-\big(p(v)-p(V)\big)\Phi_y-\frac{\Phi_{yy}\Psi_y}{V}-F\Psi_y-\frac{1}{2}\frac{\Phi_y\chi_y^2}{(V+\Phi_y)^2},\tag{2.33}$$

integrating (2.26) over $(-\infty,+\infty)\times[0,t)$, by using (2.32) and Lemma 2.1, then, (2.25) is achieved, and the proof of Lemma 2.2 is completed. ∎

LEMMA 2.3. *Suppose $(\Phi,\Psi,\chi)\in X_\delta(0,T)$ is the solution of system (2.1), then, for $t\in[0,T]$, the following inequality holds:*

$$\begin{aligned}&\|(\Phi_{yy},\Psi_{yy},\chi_{yy})(t)\|^2+\int_0^t\left(\|\Phi_{yyy}\|^2+\|\Psi_{yy}\|^2+\|\chi_{yyy}\|^2\right)d\tau\\&\le C\big\|(\Phi_{0yy},\Psi_{0yy},\chi_{0yy})\big\|^2,\end{aligned}\tag{2.34}$$

where C is the positive constant which may depend on (v_-,u_-) but is independent of T.

Proof. Multiplying $(2.1)_2$ by $\frac{1}{p'(V)}$, differentiating the resultant, $(2.1)_1$, $(2.1)_3$ with respect to y twice respectively, one has

$$\begin{cases}\Phi_{yyt}-s\Phi_{yyy}-\Psi_{yyy}-\left(\dfrac{1}{V}\Phi_{yy}\right)_{yy}+\left(\dfrac{1}{V^2}V_y\Phi_y\right)_{yy}=F_{yy},\\ \left(\dfrac{1}{p'(V)}\Psi_t\right)_{yy}-s\left(\dfrac{\Psi_y}{p'(V)}\right)_{yy}+\Phi_{yyy}=\left(\dfrac{G}{p'(V)}\right)_{yy}-\dfrac{1}{2}\left(\dfrac{\chi_y^2}{(V+\Phi_y)^2p'(V)}\right)_{yy},\\ \chi_{tyy}-s\chi_{yyy}=-\big((V+\Phi_y)\mu\big)_{yy},\\ \mu=(\chi^3-\chi)-\left(\dfrac{\chi_y}{V+\Phi_y}\right)_y.\end{cases}\tag{2.35}$$

Multiplying $(2.35)_1$ by Φ_{yy}, $(2.35)_2$ by $-\Psi_{yy}$, $(2.35)_3$ by χ_{yy}, adding up these results, one obtains

$$\begin{aligned}&\left(\frac{1}{2}\Phi_{yy}^2-\frac{1}{2p'(V)}\Psi_{yy}^2+\frac{\chi_{yy}^2}{2}\right)_t+\frac{1}{V}\Phi_{yyy}^2+\frac{sp''(V)V_y}{2p'^2(V)}\Psi_{yy}^2+\chi_{yyy}^2\\&\quad+\left(\frac{s\Psi_{yy}^2}{2p'(V)}-\frac{s\Phi_{yy}^2}{2}-(\frac{\Phi_{yy}}{V})_y\Phi_{yy}-\Phi_{yy}\Psi_{yy}-\frac{s\chi_{yy}^2}{2}+\big((V+\Phi_y)\mu\big)_y\chi_{yy}-F_y\Phi_{yy}\right)_y\\&=-F_y\Phi_{yyy}+\left(\frac{1}{V}\right)_{yyy}\Phi_y\Phi_{yy}+\left(\frac{2}{V}\right)_{yy}\Phi_{yy}^2-\frac{G_{yy}\Psi_{yy}}{p'(V)}-\left(\frac{1}{p'(V)}\right)_{yy}p'(V)\Phi_y\Psi_{yy}\\&\quad-2\left(\frac{1}{p'(V)}\right)_y(p'(V)\Phi_y)_y\Psi_{yy}+\frac{1}{p'(V)}\left(\frac{\chi_y^2}{(V+\Phi_y)^2}\right)_{yy}\Psi_{yy}\\&\quad-(V_y+\Phi_{yy})(\chi^3-\chi)\chi_{yyy}+(V_y+\Phi_{yy})\Big[(\frac{1}{V+\Phi_y})_y\chi_y\chi_{yyy}+\frac{\chi_{yy}\chi_{yyy}}{V+\Phi_y}\Big]\\&\quad-(V+\Phi_y)(3\chi^2-1)\chi_y\chi_{yyy}+(V+\Phi_y)\left(\frac{1}{V+\Phi_y}\right)_{yy}\chi_y\chi_{yyy}\\&\quad+2(V+\Phi_y)\left(\frac{1}{V+\Phi_y}\right)_y\chi_{yy}\chi_{yyy}.\end{aligned}\tag{2.36}$$

Integrating (2.36) over $(-\infty,+\infty)\times[0,t)$ by parts, one gets

$$\begin{aligned}&\int_{-\infty}^{+\infty}(\frac{\Phi_{yy}^2}{2}-\frac{\Psi_{yy}^2}{2p'(V)}+\frac{\chi_{yy}^2}{2})dy\Big|_0^t+\int_0^t\int_{-\infty}^{+\infty}(\frac{\Phi_{yyy}^2}{V}+\frac{sp''(V)V_y\Psi_{yy}^2}{2p'^2(V)}+\chi_{yyy}^2)dyd\tau\\&=-\int_0^t\int_{-\infty}^{+\infty}F_y\Phi_{yyy}dyd\tau+\int_0^t\int_{-\infty}^{+\infty}[(\frac{1}{V})_{yyy}\Phi_y\Phi_{yy}+(\frac{2}{V})_{yy}\Phi_{yy}^2]dyd\tau\\&\quad-\int_0^t\int_{-\infty}^{+\infty}\frac{G_{yy}\Psi_{yy}}{p'(V)}dyd\tau-\int_0^t\int_{-\infty}^{+\infty}(\frac{1}{p'(V)})_{yy}p'(V)\Phi_y\Psi_{yy}dyd\tau\end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t\int_{-\infty}^{+\infty}\big(\frac{1}{p'(V)}\big)_y\big(p'(V)\Phi_y\big)_y\Psi_{yy}dyd\tau+\int_0^t\int_{-\infty}^{+\infty}\frac{\Psi_{yy}}{p'(V)}\big(\frac{\chi_y^2}{(V+\Phi_y)^2}\big)_{yy}dyd\tau\\
&-\int_0^t\int_{-\infty}^{+\infty}\big[(V_y+\Phi_{yy})(\chi^3-\chi)\chi_{yyy}dyd\tau+(V+\Phi_y)(3\chi^2-1)\chi_y\chi_{yyy}\big]dyd\tau\\
&+\int_0^t\int_{-\infty}^{+\infty}(V_y+\Phi_{yy})\big[\big(\frac{1}{V+\Phi_y}\big)_y\chi_y\chi_{yyy}+\frac{\chi_{yy}\chi_{yyy}}{V+\Phi_y}\big]dyd\tau\\
&+\int_0^t\int_{-\infty}^{+\infty}\big[(V+\Phi_y)\big(\frac{1}{V+\Phi_y}\big)_{yy}\chi_y\chi_{yyy}+2(V+\Phi_y)\big(\frac{1}{V+\Phi_y}\big)_y\chi_{yy}\chi_{yyy}\big]dyd\tau.
\end{aligned}
$$

By using

$$
\begin{aligned}
|F_y|=&\Big|\big(\frac{1}{v}-\frac{1}{V}\big)\Phi_{yyy}+\big(\frac{1}{v}-\frac{1}{V}+\frac{1}{V^2}\Phi_y\big)V_{yy}\\
&+\Big[\big(\frac{1}{V^2}-\frac{1}{v^2}\big)\Phi_{yy}+\big(\frac{1}{V^2}-\frac{1}{v^2}\big)V_y-\frac{1}{V^2}\Phi_{yy}\Big]\Phi_{yy}\\
&+\Big[\big(\frac{1}{V^2}-\frac{1}{v^2}\big)\Phi_{yy}+\big(\frac{1}{V^2}-\frac{1}{v^2}+\frac{1}{V^3}\Phi_y\big)V_y\Big]V_y\Big|\\
\le& C\Big(|\Phi_y||\Phi_{yyy}|+|\Phi_{yy}|^2+|\Phi_y|^2+|\Phi_y||\Phi_{yy}|^2\Big),
\end{aligned}
\tag{2.37}
$$

making use of (2.4), (2.37) and Sobolev's inequality, yields

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}F_y\Phi_{yyy}dyd\tau\Big|\le C\delta\int_0^t\int_{-\infty}^{+\infty}\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2+\|\Phi_{yyy}\|^2\big)d\tau,
$$

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}\big[\big(\frac{1}{V}\big)_{yyy}\Phi_y\Phi_{yy}+\big(\frac{2}{V}\big)_{yy}\Phi_{yy}^2\big]dyd\tau\Big|\le C\int_0^t\int_{-\infty}^{+\infty}\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau,
$$

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}\frac{G_{yy}\Psi_{yy}}{p'(V)}dyd\tau\Big|\le C\delta\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2+\|\Phi_{yyy}\|^2\big)d\tau,
$$

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}\big(\frac{1}{p'(V)}\big)_y\big(p'(V)\Phi_y\big)_y\Psi_{yy}dyd\tau\Big|\le\frac{sp''(V)V_y\Psi_{yy}^2}{4p'^2(V)}+C\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2\big)d\tau,
$$

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}\big(\frac{1}{p'(V)}\big)_{yy}p'(V)\Phi_y\Psi_{yy}dyd\tau\Big|\le C\int_0^t\big(\|\Phi_y\|^2+\|\Phi_{yy}\|^2+\|\Psi_y\|^2\big)d\tau,
$$

$$
\begin{aligned}
&\Big|\int_0^t\int_{-\infty}^{+\infty}\frac{\Psi_{yy}}{p'(V)}\big(\frac{\chi_y^2}{(V+\Phi_y)^2}\big)_{yy}dyd\tau\Big|\\
\le& C\int_0^t\big(\|\chi_y\|^2+\|\chi_{yy}\|^2\big)d\tau+\frac{1}{8}\int_0^t\|\chi_{yyy}\|^2d\tau,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_0^t\int_{-\infty}^{+\infty}\big[(V_y+\Phi_{yy})(\chi^3-\chi)\chi_{yyy}dyd\tau+(V+\Phi_y)(3\chi^2-1)\chi_y\chi_{yyy}\big]dyd\tau\Big|\\
\le& C\int_0^t\big(\|\chi^3-\chi\|^2+\|\chi_y\|^2\big)d\tau+\frac{1}{8}\int_0^t\|\chi_{yyy}\|^2d\tau,
\end{aligned}
$$

$$
\Big|\int_0^t\int_{-\infty}^{+\infty}(V_y+\Phi_{yy})\big[\big(\frac{1}{V+\Phi_y}\big)_y\chi_y\chi_{yyy}+\frac{\chi_{yy}\chi_{yyy}}{V+\Phi_y}\big]dyd\tau
$$

$$+\int_0^t\int_{-\infty}^{+\infty}\big[(V+\Phi_y)(\frac{1}{V+\Phi_y})_{yy}\chi_y\chi_{yyy}+2(V+\Phi_y)(\frac{1}{V+\Phi_y})_y\chi_{yy}\chi_{yyy}\big]dyd\tau\Big|$$
$$\le C\int_0^t(\|\chi_y\|^2+\|\chi_{yy}\|^2)d\tau+\frac{1}{8}\int_0^t\|\chi_{yyy}\|^2d\tau,$$

combinations of the estimates above and Lemmas 2.1-2.2, under (2.10) and choosing δ_1, δ small enough, one has

$$\|(\Phi_{yy},\Psi_{yy},\chi_{yy})(t)\|^2+\int_0^t(\|\Phi_{yyy}\|^2+\|\sqrt{V_y}\Psi_{yy}\|^2+\|\chi_{yyy}\|^2)d\tau$$
$$\le C\|(\Phi_{0yy},\Psi_{0yy},\chi_{0yy})\|^2. \tag{2.38}$$

Multiplying $(2.1)_1$ by Ψ_{yyy}, combining with $(2.1)_2$, we get

$$\Psi_{yy}^2=\Big(\Phi_y\Psi_{yy}\Big)_t+\Big(-\Phi_t\Psi_{yy}+\Psi_y\Psi_{yy}+(\frac{\Phi_y}{V})_y\Psi_{yy}+(\frac{1}{2}\frac{\chi_y^2}{(V+\Phi_y)^2})_y\Phi_y\Big)_y$$
$$+\big(p(v)-p(V)\big)_{yy}\Phi_y-(\frac{\Phi_y}{V})_{yy}\Psi_{yy}+F\Psi_{yyy}-\Big(\frac{1}{2}\frac{\chi_y^2}{(V+\Phi_y)^2}\Big)_y\Phi_{yy}, \tag{2.39}$$

making use of (2.38), (2.4), and Sobolev's inequality, yields

$$\int_0^t\|\Psi_{yy}\|^2d\tau\le C\|(\Phi_0,\Psi_0,\chi_0)\|_0^2, \tag{2.40}$$

combining with (2.38), (2.34) is achieved, and the proof of Lemma 2.3 is completed. □

The estimates of the third derivative are given by Lemma 2.4 in a similar way, the details of the proof are omitted.

Lemma 2.4. *Suppose $(\Phi,\Psi,\chi)\in X_\delta(0,T)$ is the solution of system (2.1), then, for $t\in[0,T]$, the following inequality holds*

$$\|(\Phi_{yyy},\Psi_{yyy},\chi_{yyy})(t)\|^2+\int_0^t(\|\Phi_{yyyy}\|^2+\|\Psi_{yyy}\|^2+\|\chi_{yyyy}\|^2)d\tau$$
$$\le C\Big\|(\Phi_{0yyy},\Psi_{0yyy},\chi_{0yyy})\Big\|^2, \tag{2.41}$$

where C is the positive constant which may depend on (v_-,u_-) but is independent of T.

Now only (2.13) remains unproved in the proof of Proposition 2.2. To do this, differentiating $(2.1)_1$ twice with respect to y, multiplying it by Φ_{yy}, and integrating the resulting equality over $\mathbb{R}$ with respect to y, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\Phi_{yy}\|^2=-\int_{-\infty}^{+\infty}\Big(\Psi_{yy}\Phi_{yyy}dy+(\frac{1}{V}\Phi_{yy})_y\Phi_{yyy}\Big)dy$$
$$+\int_{-\infty}^{+\infty}\Big(\frac{1}{V^2}V_y\Phi_y\Big)_y\Phi_{yyy}dy-\int_{-\infty}^{+\infty}F_y\Phi_{yyy}dy. \tag{2.42}$$

Then, by using (2.12), we have

$$\int_0^{+\infty}\Big|\frac{d}{dt}\|\Phi_{yy}\|^2\Big|d\tau\le C_0\Big(\|(\Phi_0,\Psi_0)\|_3^2+\|\chi_0^2-1\|^2+\|\chi_{0y}\|_2^2\Big). \tag{2.43}$$

Similarly, $(2.1)_{2,3}$ and the estimate (2.12) give us

$$\int_0^{+\infty} \left|\frac{d}{dt}\|\Psi_{yy}\|^2\right| + \left|\frac{d}{dt}\|\chi_y\|^2\right| d\tau \leq C_0\Big(\|(\Phi_0,\Psi_0)\|_3^2 + \|\chi_0^2-1\|^2 + \|\chi_{0y}\|_2^2\Big). \tag{2.44}$$

And therefore, (2.43)-(2.44) yield the a priori estimate (2.13). The proof of Proposition 2.2 is completed.

Proof. **(The proof of Theorems 1.1-1.2.)** Combining with (1.28), by using Sobolev inequality, the asymptotic stability of the solution is obtained

$$\lim_{t\to 0} \big\|\Phi_y, \Psi_y, \chi^2-1\big\|_{L^\infty(\mathbb{R})} = 0, \tag{2.45}$$

i.e. (1.40)-(1.41). Thus, the Theorem 1.1 is achieved. Further, the sharp interface limit (1.43) is a direct consequence of the Theorem 1.1. Note that due to the scaling transformation, only small energy disturbance is required for the initial conditions, and no restriction on oscillations of the initial data are needed. Moreover, the constraint $h>0$ in (1.43) is necessary, and then, the proof of the theorem is completed. □

Acknowledgments. This paper is supported by the National Natural Science Foundation of China (Nos. 12171024, 11971217, 11971020), Natural Science Foundation of Jiangxi Province (No. 20202ACBL211002), Academic and Technical Leaders Training Plan of Jiangxi Province (No. 20212BCJ23027).

REFERENCES

[1] D. Bresch and B. Desjardins, *Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model*, Commun. Math. Phys., 238(1-2):211–223, 2003. 1

[2] D. Bresch and B. Desjardins, *On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models*, J. Math. Pures Appl., 86(4):362–368, 2006. 1

[3] T. Blesgen, *A generalization of the Navier-Stokes equations to two-phase flows*, J. Phys. D, 32:1119–1123, 1999. 1

[4] Y. Chen, Q. He, B. Huang, and X. Shi, *Global strong solution to a thermodynamic compressible diffuse interface model with temperature-dependent heat conductivity in 1D*, Math. Meth. Appl. Sci., 44:12945–12962, 2021. 1

[5] Y. Chen, Q. He, B. Huang, and X. Shi, *The Cauchy problem for non-isentropic compressible Navier-Stokes/Allen-Cahn system with degenerate heat-conductivity*, arXiv preprint, arXiv:2005.11205v4, 2021. 1

[6] Y. Chen, Q. He, B. Huang, and X. Shi, *Navier-Stokes/Allen-Cahn system with generalized Navier boundary condition*, Acta Math. Appl. Sin. Engl. Ser., 38(1):98–115, 2022. 1

[7] Y.-Z.Chen, H. Hong, and X. Shi, *Stability of the phase separation state for compressible Navier-Stokes/Allen-Cahn system*, arXiv preprint, arXiv:2105.07098. 1

[8] M. Chen and X. Guo, *Global large solutions for a coupled compressible Navier-Stokes/Allen-Cahn system with initial vacuum*, Nonlinear Anal. Real World Appl., 37:350–373, 2017. 1

[9] X. Chen, X. Wang, and X. Xu, *Analysis of the Cahn-Hilliard equation with a relaxation boundary condition modeling the contact angle dynamics*, Arch. Ration. Mech. Anal., 213(1):1–24, 2014. 1

[10] S. Chen, H. Wen, and C. Zhu, *Global existence of weak solution to compressible Navier-Stokes/Allen-Cahn system in three dimensions*, J. Math. Anal. Appl., 477:1265–1295, 2019. 1

[11] S. Ding, Y. Li, and W. Luo, *Global solutions for a coupled compressible Navier-Stokes/Allen-Cahn system in 1D*, J. Math. Fluid Mech., 15:335–360, 2013. 1

[12] E. Feireisl, H. Petzeltová, E. Rocca, and G. Schimperna, *Analysis of a phase-field model for two-phase compressible fluids*, Math. Models Meth. Appl. Sci., 20(7):1129–1160, 2010. 1

[13] M. Heida, J. Malek, and K.R. Rajagopal, *On the development and generalizations of Allen-Cahn and Stefan equations within a thermodynamic framework*, Z. Angew. Math. Phys., 63:759–776, 2012. 1, 1.1

[14] L. He and F. Huang, *Nonlinear stability of large amplitude viscous shock wave for general viscous gas*, J. Differ. Equ., 269:1226–1242, 2020. 1

[15] M. Kotschote, *Strong solutions of the Navier-Stokes equations for a compressible fluid of Allen-Cahn type*, Arch. Ration. Mech. Anal., 206:489–514, 2012. 1

[16] J. Lowengrub and L. Truskinovsky, *Quasi-incompressible Cahn-Hilliard fluids and topological transitions*, Proc. Royal Soc. A: Math. Phys. Eng. Sci., 454:2617–2654, 1998. 1.1

[17] T. Luo, H. Yin, and C. Zhu, *Stability of the rarefaction wave for a coupled compressible Navier-Stokes/Allen-Cahn system*, Math. Meth. Appl. Sci., 41(12):4724–4736, 2018. 1

[18] T. Luo, H. Yin, and C. Zhu, *Stability of the composite wave for compressible Navier-Stokes/Allen-Cahn system*, Math. Model. Meth. Appl. Sci., 30(2):343–385, 2020. 1

[19] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan. J. Appl. Math., 2:17–25, 1985. 1.1

[20] P. Poláčik, *Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$: I. Asymptotic symmetry for the Cauchy problem*, Commun. Partial Differ. Equ., 30:1567–1593, 2005. 2

[21] V. Shelukhin, *On the structure of generalized solutions of the one-dimensional equations of a polytropic viscous gas*, J. Appl. Math. Mech., 48(6):665–672, 1984. 1

[22] A. Vasseur and L. Yao, *Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic Navier-Stokes equations with density dependent viscous coefficient*, Commun. Math. Sci., 14(8):2215–2228, 2016. 1

[23] G. Witterstein, *Sharp interface limit of phase change flows*, Adv. Math. Sci. Appl., 20(2):585–629, 2010. 1

[24] H. Yin and C. Zhu, *Asymptotic stability of superposition of stationary solutions and rarefaction waves for 1D Navier–Stokes/Allen–Cahn system*, J. Differ. Equ., 266(11):7291–7326, 2019. 1

[25] X. Zhao, *Global well-posedness and decay estimates for three-dimensional compressible Navier-Stokes-Allen-Cahn system*, Proc. Roy. Soc. Edinb. A: Math., 152(5):1291–1322, 2022. 1