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DISCRETE PERTURBED GRADIENT FLOW AND ITS APPLICATION*

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Abstract. We study discrete dynamical system with perturbed gradient flow structure and its related applications. We prove that states with uniform bound will eventually converge to an equilibrium state, where Łojasiewicz inequality plays an important role. Moreover, the convergence rate is uniform with respect to the mesh size, which implies uniform transition from discrete time model to continuous time model. As direct applications, we use this theory to prove the emergent dynamics in discrete thermodynamic Kuramoto model and swarmalator model.

Keywords. Discrete perturbed gradient flow; Łojasiewicz inequality; Discrete swarmalator model.

AMS subject classifications. 39A10; 39A12; 34D05; 68M10.

1. Introduction

Collective behaviors are common in our world and daily life. For instance, the aggregation of bacteria, swarming of fish, flocking of birds, and synchronous flash of fireflies, etc. It is very interesting and important to study the emergence of these collective behaviors, since then people can apply this natural mechanism into various areas in industry and academic research [3,16,34–36,38,44]. To this end, different kinds of dynamic models have been proposed, to name a few, Winfree model [43], Kuramoto model [8,28], Vicsek model [42], Cucker-Smale model [10], Motsch-Tadmor model [32,33] etc. These models have been extensively studied in recent decades, including particle model at the microscopic level [2, 6, 15, 40, 41], mean field limit equation at the kinetic level [4,5], hydrodynamic limit equation at the macroscopic level [11,12,24], models with general digraph [9, 13, 29, 30, 37, 39], random environment and stochastic perturbations [1, 26], discrete time models [18, 25].

In this fruitful research, one of the most important issue is to interpret the dissipation in these models, since it is the dissipation mechanism that drives the agents to a particular formation. Then, it is found in many models that, the dissipation structure can be captured from the view of gradient flow [17, 45]. More precisely, the system with gradient flow structure reads

$$\frac{d}{dt}x(t) = -\nabla_x P(x(t)).$$

Then, one considers the dynamics of potential $P(x)$, and immediately obtains the decreasing of the potential along the flow $x(t)$, i.e.,

$$\frac{d}{dt}P(x(t)) = -(\nabla_x P(x(t)))^2. \tag{1.1}$$

The decreasing of potential only shows the convergence of $x(t)$ to an equilibrium in a weak sense. Then, according to [17], one only needs to prove the uniform boundedness of $x(t)$ to yield the strong convergence, due to the gradient flow structure. On the other

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hand, in more general and complex models, one can only obtain a perturbed gradient flow as below,

$$\frac{d}{dt}x(t) = -\nabla_x P(x(t)) + f(t). \tag{1.2}$$

Note the decreasing of $P(x)$ in (1.1) is not true due to the inhomogeneous term. Instead, a hypo-coercive type estimate has been obtained in [17] for dissipative $f(t)$, which shows $P(x)$ is bounded by a decreasing quantity. Then, uniform boundedness of $x(t)$ still implies the strong convergence of $x(t)$ to an equilibrium state. Please refer to Section 2 for more details about previous results.

In the present paper, we mainly focus on the discrete version of gradient type flow and its applications to collective dynamical models. Since the data collections in real world and simulations in computer are all discrete, the continuous model can be viewed as an approximation of the real in some sense. Thus, it makes sense to study the discrete model directly, and verify the consistency between the discrete and continuous models. This is the natural motivation of our work. Now, the discrete perturbed gradient (DPG for short) flow reads

$$x(n+1) - x(n) = -h\nabla_x P(x(n)) + hf(n). \tag{1.3}$$

In [45], the authors studied the discrete gradient flow without perturbation, and successfully proved the emergence of synchronization of discrete Kuramoto model. However, the DPG flow has not been studied before. Different from the continuous model, the discrete model has no derivative, and thus we have to do careful estimates on the higher order error to yield the dissipation, which draws many complicated calculations. Moreover, to apply the theory to particular models is also nontrivial, because it is usually difficult to obtain the uniform bound of the agents.

Based on above discussions and observations, our main results in this paper are two-fold. First, we assume the uniform boundedness of the agents in DPG flow (1.3), and show the convergence of the agents to an equilibrium asymptotically. This extends the results of continuous model to discrete case. More precisely, we have the following theorem.

THEOREM 1.1. *Let $x(n)$ be a solution to discrete perturbed gradient (DPG) flow (1.3), and suppose the following three assertions hold:*

(1) *$P(x)$ is analytic in an open domain $U \subseteq \mathbb{R}^n$.*

(2) *For any n, $x(n)$ is uniformly bounded in a convex compact domain $D \subseteq U$.*

(3) *The perturbation $f(n)$ decays to zero exponentially fast, i.e.,*

$$|f(n)| \leq \bar{C}e^{-\lambda(n+1)h},$$

where $\bar{C}$, h and λ are positive constants, and $h \ll 1$.

Then for sufficiently small h, there exists a state $x^\infty \in D$ such that

$$\lim_{n\to+\infty} x(n) = x^\infty, \quad \nabla_x P(x^\infty) = 0.$$

Next, we apply the result to two collective dynamic models, i.e., discrete thermodynamic Kuramoto (DTK for short) model (see (2.4)) and discrete swarmalator (DS for short) model (see (1.4)), and show the emergence of synchronization and swarming. As the DTK model has been studied in [23] with other methods, we will only briefly explain how to recover the results by using DPG flow in Section 2. Then, we will mainly focus on DS model, which is written as below,

$$\begin{cases} x_i^h(n+1)=x_i^h(n)+h\omega_i+\dfrac{h}{N}\displaystyle\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\Gamma_a(\theta_j^h(n)-\theta_i^h(n))\dfrac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\alpha} \\ \qquad\qquad\qquad\qquad -\dfrac{h}{N}\displaystyle\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\Gamma_r(\theta_j^h(n)-\theta_i^h(n))\dfrac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\beta}, \\ \theta_i^h(n+1)=\theta_i^h(n)+h\nu_i+\dfrac{h}{N}\displaystyle\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\dfrac{\kappa}{|x_j^h(n)-x_i^h(n)|^\gamma}\sin(\theta_j^h(n)-\theta_i^h(n)), \\ n=0,1,\ldots,\quad 1\le i\le N,\quad \mathcal{N}=\{1,2,\ldots,N\}, \\ x_i^h(0)=x_{i0},\theta_i^h(0)=\theta_{i0}. \end{cases} \tag{1.4}$$

We will provide more introduction of the DS model (1.4) in Section 2, in which we will also show the connection between the DS model (1.4) and the DPG flow (1.3). Then we apply Theorem 1.1 to obtain the our second main theorem.

THEOREM 1.2 (Swarming in DS model). *For initial data without collisions, we have the following two conclusions,*

(1) *there will be no collisions between particles for any step, and thus the iteration scheme is well defined for any n. Moreover, the minimal inter-particle distance has a uniformly lower bound for any n.*

(2) *For identical case $\nu_i=0$, let $(x_i^h(n),\theta_i^h(n))_{i=1}^N$ be the solution to (1.4). Suppose the following two assertions hold,*

  (i) *$|x_i^h-x_j^h|$ has positive upper bound uniformly with respect to n, i.e.,*

$$\sup_{0\le n<+\infty}\sup_{i.j}|x_i^h(n)-x_j^h(n)|\le C_2<+\infty,$$

  where C_2 is a positive constant.

  (ii) *θ_i satisfy small initial condition*

$$\sum_{i=1}^N\theta_i^h(0)^2<\frac{\pi^2}{64}.$$

Then, the emergence of complete synchronization of θ_i^h and swarming of x_i^h will occur asymptotically for sufficiently small mesh size. In other words, there exist constants $\bar{C}$, λ, h_0 and an equilibrium state x^∞ such that, the following asymptotical behaviors occur for $h\le h_0$,

$$\lim_{n\to+\infty}|\theta_i^h(n)-\theta_j^h(n)|\le\bar{C}e^{-\lambda(n+1)h},\qquad \lim_{n\to+\infty}|x^h(n)-x^\infty|=0.$$

REMARK 1.1. Theorem 1.2 requires a priori assumption that the diameter of $x_i^h(n)$ is uniformly bounded. Similar to the continuous time model studied in [17], the proof of the uniform upper bound is nontrivial and requires well prepared initial configuration. Since we focus on the application of the discrete perturbed gradient flow theory, we will only show how to capture the gradient flow structure of the DS model (1.4) in this paper, and the verification of the assumption will not be contained.

The rest of the paper is organized as follows. In Section 2, we will review some well known preliminary results and show the gradient flow structure in DTK model and DS model. Next, in Section 3, we will show the detailed proof of Theorem 1.1. Moreover, we will provide an immediate corollary which shows a uniform transition from DPG flow to continuous perturbed gradient flow (1.2). In Section 4, we first show collision avoidance in DS model (1.4), which guarantees the iteration scheme holds for all n. Then we provide the uniform lower bound and upper bound of the agents, which together with Theorem 1.1 imply the emergence of swarming, and thus finish the proof of Theorem 1.2. Finally, Section 5 is contributed as a summary.

2. Preliminaries

Previous results and preliminary lemmas will be provided in this part, and we will mainly introduce related results in continuous time model. Then, we will discuss the DTK model and DS model respectively, and show the DPG flow structure contained in each model.

2.1. Previous results.

Firstly, we introduce the Łojasiewicz inequality which plays an important role in the study of gradient flow.

Lemma 2.1 ([31] Łojasiewicz inequality). *Suppose that $P:D\subseteq\mathbb{R}^n\to\mathbb{R}$ is analytic in the open set D. Let $\bar{x}$ be a critical point of P, i.e., $\nabla P(\bar{x})=0$. Then there exist $r>0$, $q>0$, and $\eta\in[\frac{1}{2},1)$ such that*

$$|\nabla P(x)|\geq q|P(x)-P(\bar{x})|^{\eta},\quad \forall x\in B(\bar{x},r).$$

Then, applying Łojasiewicz inequality, one can obtain the following convergence result in dynamic systems with gradient flow structure.

Lemma 2.2 ([14]). *Suppose $P(x)$ is an analytic function. Let $x(t)$ be uniformly bounded and follow a gradient flow with $P(x)$ to be the potential i.e. $\dot{x}=-\nabla_x P(x)$. Then $x(t)$ converges to a limit x^{∞}.*

For more general and complicated coupled systems, usually there is no gradient flow structure. Instead, these systems may contain a perturbation of gradient flow, then similar results as in Lemma 2.2 can be obtained.

Lemma 2.3 ([17]). *Suppose $P(x)$ is an analytic function. Let $x(t)$ be uniformly bounded and follow a gradient flow with $P(x)$ to be the potential i.e. $\dot{x}=-\nabla_x P(x)+f(t)$, where $f(t)$ is a continuous vector-valued function and $|f(t)|\leq C_1e^{-C_2t}$. Then $x(t)$ converges to a limit x^{∞}.*

Remark 2.1. In [17], the authors added the requirement that $|\nabla_x P(x(t))|^2$ is uniformly continuous with respect to t, due to the application of Barbalat's lemma in the proof. But since $P(x)$ is analytic and $x(t)$ is in a compact domain, all the derivatives of P are uniformly bounded along the flow $x(t)$. Moreover, as $f(t)$ is continuous with exponential decay, $f(t)$ is also uniformly bounded with respect to t. Then, we obtain the uniform boundedness of $\frac{d}{dt}|\nabla_x P(x(t))|^2$, which is sufficient to imply the uniform continuity of $|\nabla_x P(x(t))|^2$. Therefore, we get rid of the uniform continuity requirement in Lemma 2.3.

Next, we introduce a lemma in [23], which shows a uniform convergence from the discrete time model to the continuous time model.

Lemma 2.4 ([23]). *Let $\{a^h(n)\}_{h>0}$ be a one-parameter family of sequences in $\mathbb{R}^N$,* *and let $a(\cdot)$ be a curve on $\mathbb{R}^N$. Suppose that*

$$a^h(\infty):=\lim_{n\to\infty}a^h(n),\quad a(\infty):=\lim_{t\to\infty}a(t)$$

exist and there are two continuous functions p_1 and p_2 satisfying

$$\begin{aligned}&|a^h(n)-a^h(\infty)|\le p_1(nh),\quad &&|a(nh)-a(\infty)|\le p_2(nh),\\ &\forall n,h>0, &&\lim_{t\to\infty}p_1(t)=\lim_{t\to\infty}p_2(t)=0.\end{aligned}$$

If we further assume

$$\limsup_{h\to0}\sup_{0<n<\frac{\tau}{h}}\left(|a(nh)-a^h(n)|\right)=0,\quad \forall\tau>0, \tag{2.1}$$

then $\{a^h(n)\}_{h>0}$ converges to $a(\cdot)$ uniformly in time.

This result has been also applied implicitly in [25,45], etc., to show the uniform-in-time transition from discrete model to continuous time model. Finally, we give a simple lemma about the sub-additive property of concave function.

LEMMA 2.5 ([45]). *Let $g(x)$ be a concave function defined on $[0,+\infty)$ and $g(0)\ge0$, then g is sub-additive on $[0,+\infty)$ i.e.*

$$g(a)+g(b)\ge g(a+b),\quad a,b\in[0,+\infty).$$

This simple property has been also applied in [7,19,21,46], and plays an important role to get the dissipation structure.

2.2. Discrete thermodynamic Kuramoto (DTK) model. Our first application of DPG flow is for DTK model. In [22], the authors derived the thermodynamic Cucker-Smale model to describe the temperature effects on collective behavior. Later on, this idea was extended to DTK model in [20], which reads

$$\begin{cases}\dot\theta_i=\nu_i+\dfrac{\kappa_1}{N}\displaystyle\sum_{j=1}^N\frac{\psi_{ij}}{T_i}\sin(\theta_j-\theta_i),\quad t>0,\\ \dot T_i=\dfrac{\kappa_2}{N}\displaystyle\sum_{j=1}^N\frac{\zeta_{ij}}{1+T_i}\left(\frac{1}{T_i}-\frac{1}{T_j}\right),\quad t>0,\end{cases} \tag{2.2}$$

where θ_i denotes the phase of the i-th oscillator and T_i is the temperature. Then, in [23], the discretization of above model has been addressed as below,

$$\begin{cases}\theta_i(n+1)=\theta_i(n)+\nu_ih+\dfrac{\kappa_1h}{NT_i(n)}\displaystyle\sum_{j=1}^N\psi_{ij}\sin(\theta_j(n)-\theta_i(n)),\\ f(T_i(n+1))=f(T_i(n))+\dfrac{\kappa_2h}{N}\displaystyle\sum_{j=1}^N\zeta_{ij}\left(\frac{1}{T_i(n)}-\frac{1}{T_j(n)}\right),\quad f(x)=x+\dfrac{x^2}{2}.\end{cases} \tag{2.3}$$

This is an implicit scheme, which preserves the conservation of the total energy of T_i. Thus the limit T^∞ can be determined by initial configuration. Then, one can show the exponential decay of the temperature, i.e.,

$$|T_i(n)-T^\infty|\le Ce^{-\lambda(n+1)h}.$$

Then, substituting above estimate into $(2.3)_1$, we apply the uniform upper and lower bounds of the sin function and T_i to obtain the following perturbed gradient flow structure,

$$\begin{cases} \theta_i(n+1)-\theta_i(n)=-h\nabla_\theta P(\theta(n))+hf(n), \\ f(n)=\dfrac{\kappa_1}{N}\displaystyle\sum_{j=1}^N \psi_{ij}\sin(\theta_j(n)-\theta_i(n))\left(\dfrac{1}{T_i(n)}-\dfrac{1}{T^\infty}\right)\le Ce^{-\lambda(n+1)h}, \\ P(\theta)=-\displaystyle\sum_{i=1}^N \nu_i\theta_i+\dfrac{\kappa_1}{2NT^\infty}\sum_{i=1}^N\sum_{j=1}^N \psi_{ij}\left(1-\cos(\theta_j-\theta_i)\right). \end{cases}$$

According to Theorem 1.1, in order to prove the emergence of synchronization, we only need to show the uniform bound of the phase diameter. As the sufficient condition for boundedness has been provided in [23], we will not show the details in the present paper.

On the other hand, one can also apply the Euler one-step scheme to discretize the system (2.2) and obtain that

$$\begin{cases} \theta_i(n+1)=\theta_i(n)+\nu_i h+\dfrac{\kappa_1 h}{NT_i(n)}\displaystyle\sum_{j=1}^N \psi_{ij}\sin(\theta_j(n)-\theta_i(n)), \\ T_i(n+1)=T_i(n)+\dfrac{\kappa_2 h}{N}\displaystyle\sum_{j=1}^N \dfrac{\zeta_{ij}}{1+T_i(n)}\left(\dfrac{1}{T_i(n)}-\dfrac{1}{T_j(n)}\right). \end{cases} \tag{2.4}$$

We can also apply Theorem 1.1 to show the emergence of synchronization. In this case, the total energy of T_i is not conserved, but as the temperature will converge to a common limit $\bar{T}^\infty$ exponentially fast, one may prove that the error between $\bar{T}^\infty$ and T^∞ is of order h. Therefore, when h tends to zero, both solutions to (2.3) and (2.4) will converge to the solution of continuous model uniformly in time.

2.3. Discrete swarmalator (DS) model. Our second application is for the DS model. This model is used to describe the emergence of collective behavior from the competition between attraction and repulsion mechanics [27], which reads

$$\begin{cases} \dfrac{dx_i}{dt}=\omega_i+\dfrac{1}{N}\displaystyle\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\left[\Gamma_a(\theta_j-\theta_i)\dfrac{x_j-x_i}{|x_j-x_i|^\alpha}-\Gamma_r(\theta_j-\theta_i)\dfrac{x_j-x_i}{|x_j-x_i|^\beta}\right], \quad t>0, \\ \dfrac{d\theta_i}{dt}=\nu_i+\dfrac{1}{N}\displaystyle\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\dfrac{\kappa}{|x_j-x_i|^\gamma}\sin(\theta_j-\theta_i), \quad \mathcal{N}=\{1,2,\dots,N\}, \\ (x_i(0),\theta_i(0))=(x_{i0},\theta_{i0}). \end{cases} \tag{2.5}$$

Here ω_i and ν_i are called as natural velocity and frequency of the $i-th$ particle, repectively, and α,β and γ are positive constants satisfying $1\le\alpha<\beta$. The functions Γ_a and Γ_r in $(2.5)_1$ denote the attraction, repulsion strengths between particles, respectively. Moreover, they are assumed to satisfy parity and boundedness conditions:

$$\begin{aligned} &\Gamma_a(\theta)=\Gamma_a(-\theta), \quad \Gamma_r(\theta)=\Gamma_r(-\theta), \quad \theta\in R, \\ &0<m_a\le\Gamma_a(\theta)\le M_a<\infty, \quad 0<m_r\le\Gamma_r(\theta)\le M_r<\infty. \end{aligned} \tag{2.6}$$

For example, in [27], the authors choose the attraction and repulsion forces as below,

$$\Gamma_r = 1, \quad \Gamma_a(\theta) = 1 + J\cos\theta, \quad |J| < 1.$$

Similar as before, with the Euler one-step scheme, the discretized version of the swarmalator model can be written as (1.4). Then, we are going to write $(1.4)_1$ as a perturbed gradient flow. We first define the attraction potential, repulsion potential and the perturbation as below,

$$\begin{aligned}
&V_\alpha := \begin{cases} \displaystyle\sum_{i=1}^N w_i \cdot x_i + \frac{1}{N}\sum_{i\neq j} \log(|x_i - x_j|)\Gamma_a(0), & \alpha = 2, \\ \displaystyle\sum_{i=1}^N w_i \cdot x_i + \frac{1}{N}\sum_{i\neq j} \frac{|x_i - x_j|^{2-\alpha}}{2-\alpha}\Gamma_a(0), & \alpha \geq 1, \quad \alpha \neq 2, \end{cases} \\
&V_\beta := \begin{cases} \displaystyle -\frac{1}{N}\sum_{i\neq j} \log(|x_i - x_j|)\Gamma_r(0), & \beta = 2, \\ \displaystyle \frac{1}{N}\sum_{i\neq j} \frac{|x_i - x_j|^{2-\beta}}{\beta - 2}\Gamma_r(0), & \beta > \alpha \geq 1, \quad \beta \neq 2, \end{cases} \\
&f(n) = \frac{1}{N}\sum_{\substack{j\in\mathcal{N} \\ j\neq i}} \left(\Gamma_a(\theta_j^h(n) - \theta_i^h(n)) - \Gamma_a(0)\right) \frac{x_j^h(n) - x_i^h(n)}{|x_j^h(n) - x_i^h(n)|^\alpha} \\
&\qquad - \frac{1}{N}\sum_{\substack{j\in\mathcal{N} \\ j\neq i}} \left(\Gamma_r(\theta_j^h(n) - \theta_i^h(n)) - \Gamma_r(0)\right) \frac{x_j^h(n) - x_i^h(n)}{|x_j^h(n) - x_i^h(n)|^\beta}, \quad \beta > \alpha \geq 1.
\end{aligned} \tag{2.7}$$

Now, we substitute (2.7) into the equation $(1.4)_1$, and rewrite (1.4) as follows,

$$\begin{cases} x_i^h(n+1) - x_i^h(n) = -h\nabla_x [V_\alpha(x(n)) + V_\beta(x(n))] + hf(n), \\ \displaystyle \theta_i^h(n+1) = \theta_i^h(n) + h\nu_i + \frac{h}{N}\sum_{\substack{j\in\mathcal{N} \\ j\neq i}} \frac{\kappa}{|x_j^h(n) - x_i^h(n)|^\gamma} \sin(\theta_j^h(n) - \theta_i^h(n)), \end{cases} \tag{2.8}$$

where we use $x(n)$ to denote the vector $(x_i(n))$. As $f(n)$ depends on θ_i, we need to first prove exponential synchronization of θ_i and then we can apply Theorem 1.1 to $(1.4)_1$. Therefore, we assumed $\nu_i = 0$ in Theorem 1.2 so that the exponential synchronization of θ_i can be easily proved. Then, according to Theorem 1.1, the Theorem 1.2 can be verified once we prove the uniform boundedness of $x(n)$.

3. Discrete perturbed gradient flow

In this section, we will rigorously prove Theorem 1.1, which is a discretized version of Lemma 2.3. Then we will show the uniform transition from (1.3) to (1.2). For convenience, we recall that Theorem 1.1 considers the following system:

$$\begin{aligned} &x(n+1) - x(n) = -h\nabla_x P(x(n)) + hf(n), \\ &|f(n)| \leq \bar{C}e^{-\lambda(n+1)h}, \end{aligned} \tag{3.1}$$

where $x(n)$ is supposed to be uniformly bounded for any n. Then, we have the following proof of Theorem 1.1.

Proof of Theorem 1.1.

Proof. Since $x(n)$ is uniformly bounded in D, we immediately obtain that there exists a subsequence $x(n_k)$ and corresponding limit x^∞ such that

$$\lim_{k\to+\infty} x(n_k)=x^\infty. \tag{3.2}$$

In the following, we will prove that this subsequence limit actually is the limit of the whole sequence.

• (Step 1.) Firstly, we show that x^∞ is a critical point of P. As $x(n)$ is uniformly bounded and $P(x)$ is analytic, the second order derivatives of $P(x)$ can reach the maximum and minimum values. More precisely, there exists a positive constant C_2 such that

$$\max_{i,j}\max_{x\in D}\left\{|\partial_{x_i}P(x)|,\ |\partial_{x_i}\partial_{x_j}P(x)|\right\}\le C_2. \tag{3.3}$$

Then let $H(x)$ be the Hessian matrix at x, we apply the Taylor expansion and remainder formula to imply that there exists a value $\xi(n)$ such that

$$\begin{aligned}
&P(x(n+1))-P(x(n))\\
&=\nabla_x P(x(n))(x(n+1)-x(n))+\frac{1}{2}(x(n+1)-x(n))H(\xi(n))(x(n+1)-x(n))\\
&=h\nabla_x P(x(n))(-\nabla_x P(x(n))+f(n))\\
&\qquad+\frac{h^2}{2}(-\nabla_x P(x(n))+f(n))H(\xi(n))(-\nabla_x P(x(n))+f(n)).
\end{aligned} \tag{3.4}$$

As D is convex, we know that $\xi(n)$ also belongs to D and thus we can apply (3.3) to conclude that $|\partial_{x_i}\partial_{x_j}P(\xi(n))|\le C_2$. Therefore, we combine (3.1), (3.3) and (3.4) to obtain

$$P(x(n+1))-P(x(n))\le -|\nabla_x P(x(n))|^2 h+\frac{Ch^2}{2}|\nabla_x P(x(n))|^2+C_1 he^{-\lambda(n+1)h}, \tag{3.5}$$

where C_1 is a constant depending on C_2 and $\bar{C}$ defined in (3.1), and C is a constant depending on C_2 and dimension of the phase space. We now define the perturbed potential $\bar{P}(x(n))$ as

$$\bar{P}(x(n))=P(x(n))+\frac{2C_1e^{-\lambda nh}}{\lambda}. \tag{3.6}$$

Recall that λ is the exponential decay rate of the source term $f(n)$. Then, we substitute the estimate in (3.5) into the perturbed potential (3.6), and apply Taylor expansion to yield following estimates,

$$\begin{aligned}
&\bar{P}(x(n+1))-\bar{P}(x(n))\\
&=P(x(n+1))-P(x(n))+\frac{2C_1e^{-\lambda(n+1)h}}{\lambda}-\frac{2C_1e^{-\lambda nh}}{\lambda}\\
&\le -|\nabla_x P(x(n))|^2h+\frac{Ch^2}{2}|\nabla_x P(x(n))|^2+C_1he^{-\lambda(n+1)h}+\frac{2C_1}{\lambda}e^{-\lambda(n+1)h}(1-e^{\lambda h})\\
&\le -|\nabla_x P(x(n))|^2h+\frac{C}{2}h^2|\nabla_x P(x(n))|^2-C_1he^{-\lambda(n+1)h}-\frac{C_1}{\lambda}e^{-\lambda(n+1)h}\lambda^2\xi^2,
\end{aligned} \tag{3.7}$$

where ξ is the positive constant between zero and h in the remainder of Taylor expansion. Thus for sufficiently small h such that $h<\frac{2}{C}$, (3.7) implies that

$$\bar{P}(x(n+1))-\bar{P}(x(n))\leq|\nabla_x P(x(n))|^2 h(\frac{C}{2}h-1)\leq 0, \tag{3.8}$$

which means $\bar{P}(x(n))$ is monotonically decreasing, and thus $\bar{P}(x(n))$ must approach to a limit when n tends to infinity. On the other hand, as $x(n)$ is uniformly bounded in D and $\bar{P}$ is continuous, we have $\bar{P}(x(n))$ is uniformly bounded on D. Therefore, the limit of $\bar{P}(x(n))$ must be finite as n tends to infinity. More precisely, we can find P^∞ such that

$$\lim_{n\to+\infty}\bar{P}(x(n))=P^\infty. \tag{3.9}$$

Combining (3.9) and the fact that $\lim_{n\to+\infty}\frac{C_1 e^{-\lambda nh}}{\lambda}=0$, we obtain

$$\lim_{n\to+\infty}P(x(n))=P^\infty. \tag{3.10}$$

Especially, when considering the convergent subsequence $x(n_k)$ constructed in the beginning, we apply (3.2), (3.10) and the continuity of P to have that

$$P^\infty=\lim_{n\to+\infty}P(x(n))=\lim_{k\to+\infty}P(x(n_k))=P\left(\lim_{k\to+\infty}x(n_k)\right)=P(x^\infty). \tag{3.11}$$

Now, we add up the inequality (3.7) and apply (3.11) to obtain that

$$\bar{P}(x(0))-P^\infty=-\sum_{n=0}^{+\infty}\bar{P}(x(n+1))-\bar{P}(x(n))\geq\sum_{n=0}^{+\infty}|\nabla_x P(x(n))|^2 h(\frac{Ch}{2}-1). \tag{3.12}$$

As $\bar{P}(x(0))$, P^∞ and h are all finite, the finiteness of sum of the sequence $|\nabla_x P(x(n))|^2$ implies

$$\lim_{n\to+\infty}|\nabla_x P(x(n))|=0.$$

Then, the continuity of $|\nabla_x P(\cdot)|$ implies that

$$|\nabla_x P(x^\infty)|=\lim_{k\to+\infty}|\nabla_x P(x(n_k))|=0,$$

which means x^∞ is a critical point of P.

• (Step 2.) In the following two steps, we will show the convergence of the limit, and we will provide some estimates on $|\nabla_x P(x)|$ in this step. As we already proved that x^∞ is a critical point of P, we apply Lojasiewicz inequality in Lemma 2.1 to imply that there exist $q>0$, $R>0$ and $\eta\in[\frac{1}{2},1)$ such that

$$|\nabla_x P(x)|\geq q|P(x)-P(x^\infty)|^\eta,\quad \forall x\in B(x^\infty,R). \tag{3.13}$$

Moreover, without loss of generality, we may assume $P(x^\infty)=0$ since $P(x^\infty)$ is finite. Now we let

$$g(t)=\frac{t-nh}{h}\bar{P}(x(n+1))+\frac{(n+1)h-t}{h}\bar{P}(x(n)),\quad nh\leq t\leq(n+1)h.$$

Due to (3.9) and (3.11), both $\bar{P}(x(n))$ and $\bar{P}(x(n+1))$ will tend to $P(x^\infty)=0$ asymptotically. Therefore, as a convex combination of $\bar{P}(x(n))$ and $\bar{P}(x(n+1))$, $g(t)$ is Lipschitz continuous and will converge to $P(x^\infty)=0$ asymptotically. Moreover, we combine (3.7) and (3.10) and sufficiently smallness of h to have

$$\begin{aligned}\frac{d}{dt}g(t)=\frac{\bar{P}(x(n+1))-\bar{P}(x(n))}{h} &\le -|\nabla_x P(x(n))|^2(1-\frac{Ch}{2})-C_1e^{-\lambda(n+1)h}\\ &\le -|\nabla_x P(x(n))|^2(1-\frac{Ch}{2})-(1-\frac{Ch}{2})C_1e^{-\lambda h}e^{-\lambda nh}\\ &\le -(1-\frac{Ch}{2})(|\nabla_x P(x(n))|^2+\frac{C_1}{2}e^{-\lambda nh})\\ &\le 0, \end{aligned} \tag{3.14}$$

which shows $g(t)$ is non-increasing. Next we set $w(t)=g(t)^{1-\eta}$ where η is from (3.13), and we can obtain

$$\frac{d}{dt}w(t)=(1-\eta)g(t)^{-\eta}\frac{d}{dt}g(t)\le-(1-\eta)g(t)^{-\eta}(1-\frac{Ch}{2})(|\nabla_x P(x(n))|^2+\frac{C_1}{2}e^{-\lambda nh}). \tag{3.15}$$

It is obvious that $w(t)$ is also non-increasing. Then, the estimate of $|\nabla_x P(x(n))|^2$ can be obtained in the following two cases respectively.

Case 1: If $|\nabla_x P(x(n))|^2\ge e^{-2\lambda\eta nh}$ for some n, we may relax the estimate in (3.15) for $t\in[nh,(n+1)h]$ as below

$$\frac{d}{dt}w(t)\le-(1-\eta)g(t)^{-\eta}(1-\frac{Ch}{2})|\nabla_x P(x(n))|^2.$$

Then, we use the fact $(|a|+|b|)^\eta\le|a|^\eta+|b|^\eta$ for $\frac{1}{2}\le\eta<1$, the decreasing of $\bar{P}$, the assumption $|\nabla_x P(x(n))|^2\ge e^{-2\lambda\eta nh}$ and Lojasiewicz estimate (3.13) to obtain

$$\begin{aligned}|\nabla_x P(x(n))| &\le -\left(\frac{d}{dt}w(t)\right)\left(\frac{g(t)^\eta}{(1-\eta)(1-\frac{Ch}{2})|\nabla_x P(x(n))|}\right)\\ &\le -\left(\frac{d}{dt}w(t)\right)\left(\frac{(P(x(n))+\frac{2C_1e^{-\lambda nh}}{\lambda})^\eta}{(1-\eta)(1-\frac{Ch}{2})|\nabla_x P(x(n))|}\right)\\ &\le -\left(\frac{d}{dt}w(t)\right)\left(\frac{|P(x(n))|^\eta}{(1-\eta)(1-\frac{Ch}{2})|\nabla_x P(x(n))|}+\frac{(\frac{2C_1e^{-\lambda nh}}{\lambda})^\eta}{(1-\eta)(1-\frac{Ch}{2})|\nabla_x P(x(n))|}\right)\\ &\le -\left(\frac{d}{dt}w(t)\right)\left(\frac{1}{q(1-\eta)(1-\frac{Ch}{2})}+\frac{(2C_1)^\eta}{\lambda^\eta(1-\eta)(1-\frac{Ch}{2})}\right).\end{aligned}$$

Case 2: If $|\nabla_x P(x(n))|^2\le e^{-2\lambda\eta nh}$ for some n, then for any $t\in[nh,(n+1)h]$, we apply the fact $\frac{1}{2}\le\eta<1$, $(|a|+|b|)^\eta\le|a|^\eta+|b|^\eta$ for $\frac{1}{2}\le\eta<1$, the decreasing of $\bar{P}$, the assumption $|\nabla_x P(x(n))|^2\le e^{-2\lambda\eta nh}$ and the Lojasiewicz estimate (3.13) to yield that

$$\begin{aligned}|\nabla_x P(x(n))| &\le -\left(\frac{d}{dt}w(t)\right)\left(\frac{g(t)^\eta}{(1-\eta)(1-\frac{Ch}{2})(|\nabla_x P(x(n))|+\frac{C_1}{2}e^{-\lambda(1-\eta)nh})}\right)\\ &\le -\left(\frac{d}{dt}w(t)\right)\Big(\frac{|P(x(n))|^\eta}{(1-\eta)(1-\frac{Ch}{2})(|\nabla_x P(x(n))|+\frac{C_1}{2}e^{-\lambda(1-\eta)nh})}+\end{aligned}$$

$$\left.\frac{(\frac{2C_1e^{-\lambda nh}}{\lambda})^\eta}{(1-\eta)(1-\frac{Ch}{2})(|\nabla_x P(x(n))|+\frac{C_1}{2}e^{-\lambda(1-\eta)nh})}\right)$$

$$\leq -\left(\frac{d}{dt}w(t)\right)\left(\frac{1}{q(1-\eta)(1-\frac{Ch}{2})}+\frac{(2C_1)^\eta e^{-\lambda\eta nh}}{\lambda^\eta(1-\eta)(1-\frac{Ch}{2})(\frac{C_1}{2}e^{-\lambda(1-\eta)nh})}\right)$$

$$\leq -\left(\frac{d}{dt}w(t)\right)\left(\frac{1}{q(1-\eta)(1-\frac{Ch}{2})}+\frac{2^{\eta+1}e^{-\lambda(2\eta-1)nh}}{\lambda^\eta(1-\eta)(1-\frac{Ch}{2})C_1^{1-\eta}}\right)$$

$$\leq -\left(\frac{d}{dt}w(t)\right)\left(\frac{1}{q(1-\eta)(1-\frac{Ch}{2})}+\frac{2^{\eta+1}}{\lambda^\eta(1-\eta)(1-\frac{Ch}{2})C_1^{1-\eta}}\right).$$

Thus, we combine two estimates in Case 1 and Case 2 to conclude that

$$|\nabla_x P(x(n))|\leq -l\frac{d}{dt}w(t),\quad nh\leq t\leq (n+1)h, \tag{3.16}$$

where $l:=\frac{1}{q(1-\eta)(1-\frac{Ch}{2})}+\max\left\{\frac{(2C_1)^\eta}{(\lambda)^\eta(1-\eta)(1-\frac{Ch}{2})},\frac{2^{\eta+1}}{\lambda^\eta(1-\eta)(1-\frac{Ch}{2})C_1^{1-\eta}}\right\}$.

• (Step 4.) Now, we are ready to finish the proof of the theorem. We will prove that $\lim_{n\to+\infty} x(n)=x^\infty$ by contradiction. Suppose not, then there exists a positive constant r such that, for any M there exists an integer $n_M\geq M$ satisfying

$$|x(n_M)-x^\infty|\geq r. \tag{3.17}$$

Without loss of generality, we can assume r is sufficiently small such that $r\leq R$, where R is in (3.13). Therefore, the Lojasiewicz inequality in (3.13) still holds in $B(x^\infty,r)$, and we can apply (3.2), (3.10) and (3.15) to find a sufficiently large n_0 such that

$$|x(n_0)-x^\infty|<\frac{r}{2},\quad \bar{C}\frac{e^{-\lambda n_0 h}}{\lambda}\leq\frac{r}{4},\quad l(w(n_0h)-w(mh))\leq\frac{r}{4}\ \text{for}\ \forall m\geq n_0. \tag{3.18}$$

Now, accroding to (3.17), we can find $n^*>n_0$ such that

$$|x(n^*)-x^\infty|\geq r. \tag{3.19}$$

On the other hand, we can estimate the difference between $x(n^*)$ and $x(n_0)$ via the iteration scheme. In fact we have

$$x(n^*)-x(n_0)=\sum_{i=n_0}^{n^*-1}(x(i+1)-x(i))=\sum_{i=n_0}^{n^*-1}(-h\nabla_x P(x(i))+hf(i)),$$

which together with (3.16) and (3.18) imply that

$$\begin{aligned}|x(n^*)-x(n_0)|&\leq\sum_{i=n_0}^{n^*-1}|(x(i+1)-x(i))|\\&\leq\sum_{i=n_0}^{n^*-1}\int_{ih}^{(i+1)h}|-\nabla_x P(x(i))|dt+\sum_{i=n_0}^{n^*-1}\int_{ih}^{(i+1)h}\bar{C}e^{-\lambda(i+1)h}dt\\&\leq\int_{n_0h}^{n^*h}-l\frac{d}{dt}w(t)dt+\bar{C}\int_{n_0h}^{n^*h}e^{-\lambda t}dt\end{aligned}$$

$$\begin{aligned}&\le l(w(n_0h)-w(n^*h))+\bar{C}\frac{e^{-\lambda n_0 h}}{\lambda}\\&\le \frac{r}{2}.\end{aligned}\tag{3.20}$$

Finally, we combine (3.18) and (3.20) together to find that

$$|x(n^*)-x^\infty|\le |x(n^*)-x(n_0)|+|x(n_0)-x^\infty|<\frac{r}{2}+\frac{r}{2}=r,$$

which is a contradiction to (3.19). Therefore, we can conclude that $\lim_{n\to+\infty} x(n)=x^\infty$. ∎

Then, we apply Lemma 2.3, Lemma 2.4 and Theorem 1.1 to obtain that the discrete perturbed gradient flow (3.1) converges to the following continuous gradient flow uniformly in time,

$$\begin{cases}\dfrac{dx}{dt}=-\nabla_x P(x)+f(t)\\ f(t)\le \bar{C}e^{-\lambda t}.\end{cases}\tag{3.21}$$

Corollary 3.1. *Suppose $x^h(n)$ is the solution to (3.1), $x(t)$ is the solution to (3.21), and the two solutions have common initial data. Moreover, we assume $P(x)$ is analytic in an open domain $U\subseteq\mathbb{R}^n$, and $f(t)$ is continuous with respect to t, where $f(n)$ in (3.1) now denotes the value of $f(t)$ at $t=nh$. Then, if the diameter of $x^h(n)$ and $x(t)$ are uniformly bounded in a common convex compact domain $D\in U$ independent of h, we have*

$$\lim_{h\to 0}\sup_{0\le n<+\infty}|x^h(n)-x(nh)|=0.$$

Proof. We only need to verify the conditions in Lemma 2.4. Firstly, under the assumptions in the corollary, we apply Lemma 2.3 and Theorem 1.1 to obtain the existence of the limit of both discrete and continuous flows. Without loss of generality, we denote them by $x^h(+\infty)$ and $x(+\infty)$, respectively.

Next, according to (3.20), we have

$$|x^h(n)-x^h(+\infty)|\le l(w(nh)-w(+\infty))+\bar{C}\frac{e^{-\lambda nh}}{\lambda}=lw(nh)+\bar{C}\frac{e^{-\lambda nh}}{\lambda},$$

where we use $w(+\infty)=P(x^\infty)=0$. Therefore, we may choose $p_1(t)$ to be

$$p_1(t)=lw(t)+\bar{C}\frac{e^{-\lambda t}}{\lambda},$$

where l is defined in (3.16) and $\bar{C}$ is defined in (3.1). Then $p_1(t)$ obviously tends to zero due to the decreasing of $w(t)$ to zero. Then, for continuous time model, we can follow [17] to construct $p_2(t)$ as below,

$$p_2(t)=\bar{l}(\bar{P})^{1-\eta}+\bar{C}\frac{e^{-\lambda t}}{\lambda},\quad \bar{l}=\frac{1}{q(1-\eta)}+\max\left\{\frac{(2C_1)^\eta}{(\lambda)^\eta(1-\eta)},\frac{2^{\eta+1}}{\lambda^\eta(1-\eta)C_1^{1-\eta}}\right\}.$$

According to the proof in [17], $|x(nh)-x(+\infty)|\le p_2(nh)$, and $p_2(t)$ is decreasing to zero asymptotically.

Finally, the consistency of one-step Euler scheme guarantees the convergence from x^h to x in any finite time period, since the two solutions are assigned same initial data. Therefore, all the requirements in Lemma 2.4 are fulfilled, and thus we conclude the uniform-in-time convergence

$$\lim_{h\to 0}\sup_{0\le n<+\infty}|x^h(n)-x(nh)|=0.$$

□

4. Discrete swarmalator model

In this section, we will show details of proof of Theorem 1.2. According to (2.8) and the discussions in Section 2, in order to show the convergence, we actually only need to verify that all the conditions in Theorem 1.1 are fulfilled. We will split the proof into three steps in the following.

4.1. Minimal inter-particle distance. In this part, we show that the distances between particles are uniformly bounded from below. We will first treat the diameter $\mathcal{D}^h(n)$, and then we finish the proof by induction. In the following, we assume

$$\min_{i,j}|x_i^h(0)-x_j^h(0)|>0,\quad \mathcal{D}^h(n):=\max_{i,j}|x_i^h(n)-x_j^h(n)|,\quad \mathcal{D}(\omega):=\max_{i,j}|\omega_i-\omega_j|.$$

Then even if there exists a first collision step n_c, we must have $n_c\ge 1$. Therefore, the iteration scheme is well defined before n_c, and we can do estimates before n_c. Actually, we have the following lemma for diameter $\mathcal{D}^h(n)$ before n_c.

LEMMA 4.1. *Suppose Γ_a and Γ_r are smooth functions with property (2.6), and all the agents are collisionless initially, i.e.,*

$$\min_{i,j}|x_i^h(0)-x_j^h(0)|>0.$$

Then, for sufficiently small h, the diameter $\mathcal{D}^h(n)$ has the following lower bound before the first collision step n_c,

$$\mathcal{D}^h(n)>\frac{\delta_{\mathcal{Q}}}{2},\quad 0\le n\le n_c,$$

where

$$\delta_{\mathcal{Q}}=\min\left\{\mathcal{D}^h(0),\ \left(\frac{m_r}{2M_a}\right)^{\frac{1}{\beta-\alpha}},\ \left(\frac{m_r}{N\mathcal{D}(\omega)}\right)^{\frac{1}{\beta-1}}\right\}.$$

Proof. Without loss of generality, we let (i,j) be a pair of indices such that $\mathcal{D}^h(n)=|x_i^h(n)-x_j^h(n)|$ at the step n, where $0\le n<n_c$. Then, the next iteration is well defined since $n+1\le n_c$. Thus we have

$$\begin{aligned}&\mathcal{D}^h(n+1)^2-\mathcal{D}^h(n)^2\\ &\ge (x_i^h(n+1)-x_j^h(n+1))^2-(x_i^h(n)-x_j^h(n))^2\\ &=(x_i^h(n+1)^2-x_i^h(n)^2)+(x_j^h(n+1)^2-x_j^h(n)^2)+2(-x_i^h(n+1)x_j^h(n+1)+x_i^h(n)x_j^h(n)).\end{aligned}\tag{4.1}$$

Substituting (1.4) into the above formula (4.1), then only first-order and second-order terms remain. More precisely, directly calculation simplifies the second-order terms as follows,

$$h^2\Big(\omega_i+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n))\Big)^2$$

$$
\begin{aligned}
&+h^2(\omega_j+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n)))^2\\
&-2h^2(\omega_i+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n)))\\
&\qquad(\omega_j+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n)))\\
&=h^2\Big[(\omega_i-\omega_j)+\tfrac{1}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n))-\tfrac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n))\Big]^2,
\end{aligned}
$$

where we make use of the following notations for simplicity,

$$
\begin{aligned}
&\Phi(x_j^h(n)-x_i^h(n),\theta_j^h(n)-\theta_i^h(n))\\
:=&\Gamma_a(\theta_j^h(n)-\theta_i^h(n))\frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\alpha}-\Gamma_r(\theta_j^h(n)-\theta_i^h(n))\frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\beta}.
\end{aligned}
\tag{4.2}
$$

Similarly, the first-order term in (4.1) can be rewritten as follows,

$$
\begin{aligned}
&2hx_i^h(n)(\omega_i+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n)))\\
&\quad+2hx_j^h(n)(\omega_j+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n)))\\
&\quad-2hx_i^h(n)(\omega_j+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n)))\\
&\quad-2hx_j^h(n)(\omega_i+\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n)))\\
=&2h(x_i^h(n)-x_j^h(n))(\omega_i-\omega_j)\\
&\quad+2h(x_i^h(n)-x_j^h(n))\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n))\\
&\quad-2h(x_i^h(n)-x_j^h(n))\frac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n)).
\end{aligned}
$$

Combining (4.1) and the above calculations of first-order and second-order terms, we have the following estimate of the diameter,

$$
\begin{aligned}
&\mathcal{D}^h(n+1)^2-\mathcal{D}^h(n)^2\\
\geq& h\left[2(x_i^h(n)-x_j^h(n))(\omega_i-\omega_j)+\frac{4\Gamma_r(\theta_j^h(n)-\theta_i^h(n))}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}}-\frac{4\Gamma_a(\theta_j^h(n)-\theta_i^h(n))}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}}\right]\\
&+\tfrac{2h}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ k\neq i,j}}\left[(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_i^h(n))\left(\frac{\Gamma_a(\theta_k^h(n)-\theta_i^h(n))}{|x_k^h(n)-x_i^h(n)|^\alpha}-\frac{\Gamma_r(\theta_k^h(n)-\theta_i^h(n))}{|x_k^h(n)-x_i^h(n)|^\beta}\right)\right]\\
&-\tfrac{2h}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ k\neq i,j}}\left[(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))\left(\frac{\Gamma_a(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^\alpha}-\frac{\Gamma_r(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^\beta}\right)\right]
\end{aligned}
$$

$$+h^2\Big[(\omega_i-\omega_j)+\tfrac{1}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ i\neq k}}\Phi(x_k^h(n)-x_i^h(n),\theta_k^h(n)-\theta_i^h(n))-\tfrac{1}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ j\neq k}}\Phi(x_k^h(n)-x_j^h(n),\theta_k^h(n)-\theta_j^h(n))\Big]^2,$$
$$=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.$$

• (Step 1.) In this step, we will estimate $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ respectively, and then construct the iteration error of the diameter.

⋄ (Estimate of $\mathcal{I}_1$): We split the last term in $\mathcal{I}_1$ into two equal parts and use the boundedness of Γ_a and Γ_r to get

$$\begin{aligned}\mathcal{I}_1=&2h\left(\frac{\Gamma_r(\theta_j^h(n)-\theta_i^h(n))}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}}+(x_i^h(n)-x_j^h(n))(\omega_i-\omega_j)\right)\\&+\frac{2h}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}}\left(\frac{\Gamma_r(\theta_j^h(n)-\theta_i^h(n))}{|x_j^h(n)-x_i^h(n)|^{\beta-\alpha}}-2\Gamma_a(\theta_j^h(n)-\theta_i^h(n))\right)\\ \geq&2h\left(\frac{m_r}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}}-|x_i^h(n)-x_j^h(n)||\omega_i-\omega_j|\right)\\&+\frac{2h}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}}\left(\frac{m_r}{|x_j^h(n)-x_i^h(n)|^{\beta-\alpha}}-2M_a\right).\end{aligned}\tag{4.3}$$

⋄ (Estimate of $\mathcal{I}_2$): Since the pair (i,j) is the maximal indices, then we have

$$|x_i^h(n)-x_k^h(n)|\leq|x_i^h(n)-x_j^h(n)|\quad and\quad|x_j^h(n)-x_k^h(n)|\leq|x_i^h(n)-x_j^h(n)|,$$

which immediately implies

$$\begin{aligned}&(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))\\=&(x_i^h(n)-x_j^h(h))(x_i^h(n)-x_j^h(n)+x_j^h(n)-x_k^h(n))\\=&|x_i^h(n)-x_j^h(n)|^2-(x_i^h(n)-x_j^h(n))(x_j^h(n)-x_k^h(n))\geq0.\end{aligned}\tag{4.4}$$

Substituting (4.4) into $\mathcal{I}_2$, we have the following estimate,

$$\begin{aligned}\mathcal{I}_2=&\frac{2h}{N}\sum_{\substack{k\in\mathcal{N}\\k\neq i,j}}\left[\frac{(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))}{|x_k^h(n)-x_i^h(n)|^{\alpha}}\left(\frac{\Gamma_r(\theta_k^h(n)-\theta_i^h(n))}{|x_k^h(n)-x_i^h(n)|^{\beta-\alpha}}-\Gamma_a(\theta_k^h(n)-\theta_i^h(n))\right)\right]\\ \geq&\frac{2h}{N}\sum_{\substack{k\in\mathcal{N}\\k\neq i,j}}\left[\frac{(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))}{|x_k^h(n)-x_i^h(n)|^{\alpha}}\left(\frac{m_r}{|x_k^h(n)-x_i^h(n)|^{\beta-\alpha}}-M_a\right)\right].\end{aligned}\tag{4.5}$$

⋄ (Estimate of $\mathcal{I}_3$): For $\mathcal{I}_3$, there is a negative sign in front. Therefore, we may rewrite it as follows,

$$\begin{aligned}\mathcal{I}_3=&-\frac{2h}{N}\sum_{\substack{k\in\mathcal{N}\\k\neq i,j}}\left[(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))\left(\frac{\Gamma_a(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^{\alpha}}-\frac{\Gamma_r(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^{\beta}}\right)\right]\\=&\frac{2h}{N}\sum_{\substack{k\in\mathcal{N}\\k\neq i,j}}\left[(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))\left(\frac{\Gamma_r(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^{\beta}}-\frac{\Gamma_a(\theta_k^h(n)-\theta_j^h(n))}{|x_k^h(n)-x_j^h(n)|^{\alpha}}\right)\right].\end{aligned}$$

Similar to the estimate of $\mathcal{I}_2$, we use the relation

$$(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))\geq0,$$

and the same argument as before to find

$$\mathcal{I}_3 \geq \frac{2h}{N} \sum_{\substack{k\in\mathcal{N}\\ k\neq i,j}} \left[\frac{(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))}{|x_k^h(n)-x_i^h(n)|^\alpha} \left(\frac{m_r}{|x_k^h(n)-x_i^h(n)|^{\beta-\alpha}} - M_a \right) \right]. \tag{4.6}$$

◇ (Estimate of $\mathcal{I}_4$): It's easy to see $\mathcal{I}_4 \geq 0$.

Combining (4.3), (4.5), (4.6) and the fact $\mathcal{I}_4 \geq 0$, we obtain that the following inequality holds for all $n < n_c$:

$$\begin{aligned}
&\mathcal{D}^h(n+1)^2 - \mathcal{D}^h(n)^2 \\
&\geq 2h \left(\frac{m_r}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}} - |x_i^h(n)-x_j^h(n)||\omega_i-\omega_j| \right) \\
&\quad + \frac{2h}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}} \left(\frac{m_r}{|x_j^h(n)-x_i^h(n)|^{\beta-\alpha}} - 2M_a \right) \\
&\quad + \frac{2h}{N} \sum_{\substack{k\in\mathcal{N}\\ k\neq i,j}} \left[\frac{(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))}{|x_k^h(n)-x_i^h(n)|^\alpha} \left(\frac{m_r}{|x_k^h(n)-x_i^h(n)|^{\beta-\alpha}} - M_a \right) \right].
\end{aligned} \tag{4.7}$$

• (Step 2.) In this step, we will apply (4.7) to find the lower bound of the diameter. According to (4.7), as $\beta > \alpha$, we know $\mathcal{D}^h(n)$ is increasing if $|x_k^h(n)-x_i^h(n)|$ is small. Now we set

$$\delta_{\mathcal{Q}} = \min\left\{ \mathcal{D}^h(0),\ \left(\frac{m_r}{2M_a}\right)^{\frac{1}{\beta-\alpha}},\ \left(\frac{m_r}{N\mathcal{D}(\omega)}\right)^{\frac{1}{\beta-1}} \right\}.$$

Then, we claim that

$$\mathcal{D}^h(n) > \frac{\delta_{\mathcal{Q}}}{2}, \quad 0 \leq n \leq n_c. \tag{4.8}$$

We will apply inductive criteria to verify the claim. Firstly, it is obvious that the claim holds for initial step. Then we assume $\mathcal{D}^h(n) > \frac{\delta_{\mathcal{Q}}}{2}$ at step n where $n < n_c$, and check the claim for step $n+1$. In the following, we will consider two different cases.

▲ (Case 1.) We first consider the case when the following estimates hold,

$$\frac{\delta_{\mathcal{Q}}}{2} < \mathcal{D}^h(n) = |x_i^h(n)-x_j^h(n)| \leq \delta_{\mathcal{Q}}, \quad n < n_c.$$

Then, according to the definition of $\delta_{\mathcal{Q}}$ and the fact $\mathcal{D}^h(n) = |x_i^h(n)-x_j^h(n)| \leq \delta_{\mathcal{Q}}$, we have

$$\mathcal{D}^h(n) = |x_i^h(n)-x_j^h(n)| \leq \delta_{\mathcal{Q}} \leq (\frac{m_r}{N\mathcal{D}(\omega)})^{\frac{1}{\beta-1}}.$$

By $\mathcal{D}(\omega) \geq |\omega_i - \omega_j|$, we have

$$\frac{m_r}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}} - |x_i^h(n)-x_j^h(n)||\omega_i-\omega_j| \geq 0.$$

On the other hand, we apply similar arguments to have

$$\mathcal{D}^h(n) = |x_i^h(n)-x_j^h(n)| \leq \delta_{\mathcal{Q}} \leq (\frac{m_r}{2M_a})^{\frac{1}{\beta-\alpha}},$$

which implies

$$\frac{m_r}{|x_i^h(n)-x_j^h(n)|^{\beta-\alpha}}-2M_a\geq 0.$$

Substituting the above estimates into (4.7), we get

$$\mathcal{D}^h(n+1)^2-\mathcal{D}^h(n)^2\geq 0,$$

which verify the claim (4.8) for step $n+1$ whenever $n<n_c$.

▲ (Case 2.) We next consider the case when $\mathcal{D}^h(n)=|x_i^h(n)-x_j^h(n)|>\delta_{\mathcal{Q}}$. In this case, the right-hand side of (4.7) may not be positive. But fortunately, if there is any negative term on the right-hand side of (4.7), we must have

$$|x_k^h(n)-x_i^h(n)|\geq \min\left\{(\frac{m_r}{M_a})^{\frac{1}{\beta-\alpha}},\ (\frac{m_r}{N\mathcal{D}(\omega)})^{\frac{1}{\beta-1}}\right\}:=\mathcal{R}.$$

Then, combining together (4.7) and the fact that $\mathcal{D}^h(n)>\delta_{\mathcal{Q}}$, we have the following estimates of the negative terms,

$$\begin{aligned}
&-|x_i^h(n)-x_j^h(n)||\omega_i-\omega_j|\geq -\mathcal{D}^h(n)\mathcal{D}(\omega),\\
&-\frac{2M_a}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}}\geq -\frac{2M_a}{N\delta_{\mathcal{Q}}^{\alpha-2}},\\
&-\frac{(x_i^h(n)-x_j^h(n))(x_i^h(n)-x_k^h(n))}{|x_k^h(n)-x_i^h(n)|^{\alpha}}M_a\geq -\frac{\mathcal{D}^h(n)M_a}{|x_k^h(n)-x_i^h(n)|^{\alpha-1}}\geq -\frac{\mathcal{D}^h(n)M_a}{\mathcal{R}^{\alpha-1}}.
\end{aligned}$$

Now, substituting above estimates into (4.7), we finally obtain that

$$\begin{aligned}
\mathcal{D}^h(n+1)^2&\geq \mathcal{D}^h(n)^2-2h\left(\mathcal{D}^h(n)\mathcal{D}(\omega)+\frac{2M_a}{N\delta_{\mathcal{Q}}^{\alpha-2}}+\frac{\mathcal{D}^h(n)M_a}{\mathcal{R}^{\alpha-1}}\right)\\
&\geq \mathcal{D}^h(n)^2-h\mathcal{D}^h(n)^2-h\left(4\mathcal{D}^2(\omega)+\frac{4M_a}{N\delta_{\mathcal{Q}}^{\alpha-2}}+\frac{4M_a^2}{\mathcal{R}^{2(\alpha-1)}}\right)\geq \frac{\delta_{\mathcal{Q}}^2}{2}>\frac{\delta_{\mathcal{Q}}^2}{4},
\end{aligned}\tag{4.9}$$

where we set h to be sufficiently small so that

$$h\leq \min\left\{\frac{\delta_{\mathcal{Q}}^2}{4\left(4\mathcal{D}^2(\omega)+\frac{4M_a}{N\delta_{\mathcal{Q}}^{\alpha-2}}+\frac{4M_a^2}{\mathcal{R}^{2(\alpha-1)}}\right)},\quad \frac{1}{4}\right\},\quad \mathcal{R}=\min\left\{(\frac{m_r}{M_a})^{\frac{1}{\beta-\alpha}},\ (\frac{m_r}{N\mathcal{D}(\omega)})^{\frac{1}{\beta-1}}\right\}.$$

Therefore, we conclude $\mathcal{D}^h(n+1)>\frac{\delta_{\mathcal{Q}}}{2}$, which means claim (4.8) holds for $n+1$ where $n<n_c$.

Now we combine all above estimates in Case 1 and Case 2, and apply the inductive criteria to finish the verification of claim (4.8). Thus we finish the proof of the lemma. □

Next, for any fixed $n\leq n_c$, we naturally have an order of the quantities

$$\mathcal{D}_{ij}^h:=|x_i^h(n)-x_j^h(n)|.$$

Since there are $\frac{N(N-1)}{2}$ pairs (i,j), without loss of generality, we may assume the following order at a fixed step n

$$\mathcal{D}_1^h(n)\leq \mathcal{D}_2^h(n)\leq ...\leq \mathcal{D}_{\mathcal{Q}}^h(n)=\mathcal{D}^h(n),\quad \mathcal{Q}=\frac{N(N-1)}{2}.$$

Note $\mathcal{D}_k^h(n)=\mathcal{D}_{ij}$ for some i,j, but this correspondence may change when n grows. In next lemma, we assume $\mathcal{D}_q^h(n)$ are uniformly bounded from below for all $q>p$, and then prove the uniform lower bound for $\mathcal{D}_p^h(n)$.

LEMMA 4.2. *Given $p\in[1,Q)$ and $p<q\le Q$. Suppose all the conditions in Lemma 4.1 are fulfilled, and there exists a sequence of positive constants such that*

$$\inf_{0\le n\le n_c} \mathcal{D}_q^h(n)\ge \frac{\delta_q}{2},\quad \delta_{p+1}\le\delta_{p+2}\le\ldots\le\delta_Q,\quad 0\le n\le n_c.$$

Then, we claim that there exists a positive constant δ_p such that, the following assertions hold for sufficiently small h,

$$\inf_{0\le n\le n_c} \mathcal{D}_p^h(n)>\frac{\delta_p}{2},\quad \delta_p\le\delta_{p+1},\quad 0\le n\le n_c.$$

Proof. As the proof is rather long and similar to that of Lemma 4.1, we put it in the Appendix. ∎

Finally, combining Lemma 4.1 and Lemma 4.2 together, we apply the inductive argument again to obtain the following lemma.

LEMMA 4.3 (Collision avoidance). *Suppose all the conditions in Lemma 4.1 and Lemma 4.2 are fulfilled. Then, for sufficiently small h, the inter-particle distance has a uniform lower bound, which means the system (1.4) is collisionless. More precisely, there exists a positive constant δ_1 such that*

$$\inf_{0\le n<+\infty}\min_{i,j}|x_i^h(n)-x_j^h(n)|>\frac{\delta_1}{2}.$$

Proof. According to Lemma 4.1 and Lemma 4.2, we apply inductive criteria to obtain that, there exists a positive constant δ_1 such that,

$$\inf_{0\le n\le n_c} \mathcal{D}_1^h(n)>\frac{\delta_1}{2}. \tag{4.10}$$

Now, if the first collision step n_c is finite, we have

$$\min_{i,j}|x_i^h(n_c)-x_j^h(n_c)|=0.$$

But this is an obvious contradiction to (4.10). Thus we know n_c is infinity, which means the system is collisionless. Moreover, from the proof of Lemma 4.1 and Lemma 4.2, the constant δ_1 is independent of n. Thus the estimate (4.10) is uniform with respect to n, and we finish the proof. ∎

4.2. Proof of Theorem 1.2.

Proof. Now, we provide a proof of Theorem 1.2. The first assertion directly follows from Lemma 4.3, and thus we only need to prove the second assertion. As we assume identical frequencies $\nu_i=0$ in the second assertion, we can rewrite (1.4) as below:

$$\begin{aligned}&x_i^h(n+1)-x_i^h(n)\\&=h\omega_i+\frac{h}{N}\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\left[(1+J(\cos\theta_j^h(n)-\theta_i^h(n)))\frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\alpha}-\frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\beta}\right],\end{aligned}$$

$$\theta_i^h(n+1)-\theta_i^h(n)=\frac{h}{N}\sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\frac{\kappa}{|x_j-x_i|^\gamma}\sin(\theta_j^h(n)-\theta_i^h(n)). \tag{4.11}$$

Next, we need to find a proper potential to rewrite (4.11) into a perturbed gradient flow. For convenience, we will consider the case where $\alpha\neq 2$, $\beta\neq 2$, and the other cases can be treated similarly. Then, we have the following constructions,

$$\begin{aligned}
&X(n+1)-X(n)=-h\nabla_X P(X(n))+hf(n),\\
&P(X(n)):=\sum_{i=1}^N\omega_i x_i^h(n)+\frac{1}{N}\sum_{i=1}^N\sum_{i\neq j}(1+J)\frac{|x_i^h(n)-x_j^h(n)|^{2-\alpha}}{2-\alpha}+\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}\frac{|x_i^h(n)-x_j^h(n)|^{2-\beta}}{\beta-2},\\
&f(n):=\frac{1}{N}\sum_{i=1}^N\sum_{j\neq i}J(\cos(\theta_j^h(n)-\theta_i^h(n))-1)\frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^\alpha}.
\end{aligned} \tag{4.12}$$

Now, in order to show that (4.12) is a perturbed gradient flow as introduced in (1.3) and apply Theorem 1.1, we need to show exponential decay of $f(n)$, which relates to the asymptotical behavior of $\theta_i^h(n)$. Now, as $\nu_i=0$, we have from (1.4) that

$$\theta_i^h(n+1)=\theta_i^h(n)+\frac{\kappa h}{N}\sum_{j=1}^N\frac{\sin(\theta_j^h(n)-\theta_i^h(n))}{|x_j^h(n)-x_i^h(n)|^\gamma}.$$

It is obviously the conservation of mean of θ_i. Without loss of generality, we may assume $\sum_{i=1}^n\theta_i^h(0)=0$. Now the main difficulty is that the order of θ_i is not preserved when n grows, and it is not convenient to show the estimates of the diameter. Therefore, we switch to do energy estimates as below,

$$\begin{aligned}
&\sum_{i=1}^N\theta_i^h(n+1)^2\\
&=\sum_{i=1}^h\left(\theta_i^h(n)+\frac{\kappa h}{N}\sum_{j=1}^N\frac{\sin(\theta_j^h(n)-\theta_i^h(n))}{|x_j^h(n)-x_i^h(n)|^\gamma}\right)^2\\
&=\sum_{i=1}^N\theta_i^h(n)^2+\frac{2h\kappa}{N}\sum_{i=1}^N\sum_{j=1}^N\frac{\sin(\theta_j^h(n)-\theta_i^h(n))}{|x_j^h(n)-x_i^h(n)|^\gamma}\theta_i^h(n)+\frac{h^2\kappa^2}{N^2}\sum_{j=1}^N\sum_{i=1}^h\frac{\left[\sin(\theta_j^h(n)-\theta_i^h(n))\right]^2}{|x_j^h(n)-x_i^h(n)|^{2\gamma}}\\
&\le\sum_{i=1}^h\theta_i^h(n)^2-\frac{h\kappa}{N}\sum_{i=1}^N\sum_{j=1}^N\frac{\sin(\theta_j^h(n)-\theta_i^h(n))(\theta_j^h(n)-\theta_i^h(n))}{|x_j^h(n)-x_i^h(n)|^\gamma}\\
&\qquad+\frac{h^2\kappa^2}{N^2C_1^{2\gamma}}\sum_{j=1}^N\sum_{i=1}^N\left[\sin(\theta_j^h(n)-\theta_i^h(n))\right]^2\\
&\le\sum_{i=1}^h\theta_i^h(n)^2-\frac{h\kappa}{NC_2^\gamma}\sum_{i=1}^N\sum_{j=1}^N\sin(\theta_j^h(n)-\theta_i^h(n))(\theta_j^h(n)\\
&\qquad-\theta_i^h(n))+\frac{h^2\kappa^2}{N^2C_1^{2\gamma}}\sum_{j=1}^N\sum_{i=1}^N\left[\sin(\theta_j^h(n)-\theta_i^h(n))\right]^2.\\
&=\mathcal{I}_{11}+\mathcal{I}_{12}+\mathcal{I}_{13}.
\end{aligned} \tag{4.13}$$

Now, as $\sum_{i=1}^N \theta_i^h(0)^2 < \frac{\pi^2}{64}$ for initial step, we immediately have $|\theta_i^h(0)| < \frac{\pi}{8}$ and thus $\theta_j^h(0) - \theta_i^h(0) \in (-\frac{\pi}{4}, \frac{\pi}{4})$. This implies that

$$\sin(\theta_j^h(0) - \theta_i^h(0))(\theta_j^h(0) - \theta_i^h(0)) \geq \left(\sin(\theta_j^h(0) - \theta_i^h(0))\right)^2 \geq 0.$$

Therefore, we have $\mathcal{I}_{12} \leq 0$ in (4.13). Moreover, as $\mathcal{I}_{12}$ is first order term of h and $\mathcal{I}_{13}$ is second order term, we can choose sufficiently small h so that

$$\mathcal{I}_{12} + \mathcal{I}_{13} \leq 0.$$

Substituting above estimate into (4.13), we obtain the following decreasing property for sufficiently small h,

$$\sum_{i=1}^N \theta_i^h(1)^2 \leq \sum_{i=1}^N \theta_i^h(0)^2.$$

Then, we may repeat this procedure and apply principle of induction to conclude that

$$\sum_{i=1}^N \theta_i^h(n)^2 \leq \frac{\pi^2}{64}, \quad \forall n \geq 0.$$

Therefore, we have $|\theta_i^h(n)| < \frac{\pi}{8}$ and thus $\theta_j^h(n) - \theta_i^h(n) \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Combining the fact that $\frac{\sin x}{x}$ is even and monotonically decreasing in $(0, \frac{\pi}{4})$, we can obtain the following inequality:

$$\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} < \left|\frac{\sin(\theta_j^h(n) - \theta_i^h(n))}{\theta_j^h(n) - \theta_i^h(n)}\right| < 1.$$

Now, we substitute above estimates into (4.13), and combine the assumption $\sum_{i=1}^n \theta_i^h(0) = 0$ to obtain that

$$\begin{aligned}
&\sum_{i=1}^N \theta_i^h(n+1)^2 \\
&\leq \sum_{i=1}^N \theta_i^h(n)^2 - \frac{4h\kappa\sin\frac{\pi}{4}}{\pi N C_2^\gamma} \sum_{i=1}^N \sum_{j=1}^N (\theta_j^h(n) - \theta_i^h(n))^2 + \frac{h^2\kappa^2}{N^2 C_1^{2\gamma}} \sum_{j=1}^N \sum_{i=1}^N (\theta_j^h(n) - \theta_i^h(n))^2 \\
&\leq \sum_{i=1}^N \theta_i^h(n)^2 - \frac{8h\kappa\sin\frac{\pi}{4}}{\pi N C_2^\gamma} \sum_{i=1}^N \theta_i^h(n)^2 + \frac{2h^2\kappa^2}{N^2 C_1^{2\gamma}} \sum_{i=1}^N \theta_i^h(n)^2 \\
&= \left(1 - hE + h^2 F\right) \sum_{i=1}^N \theta_i^h(n)^2.
\end{aligned}$$

Where $E = \frac{8\kappa\sin\frac{\pi}{4}}{\pi N C_2^\gamma}$ and $F = \frac{2\kappa^2}{N^2 C_1^{2\gamma}}$. Therefore, we can easily obtain the following estimate for all $n \geq 0$,

$$\sum_{i=1}^N \theta_i^h(n+1)^2 \leq (1 - hE + h^2 F)^{n+1} \sum_{i=1}^N \theta_i^h(0)^2$$

$$= e^{(n+1)\ln(1-hE+h^2F)} \sum_{i=1}^{N} \theta_i^h(0)^2$$
$$\leq e^{-h(n+1)E} \sum_{i=1}^{N} \theta_i^h(0)^2. \tag{4.14}$$

According to (4.14) and the zero mean assumption, we can find positive constants C and λ such that the following estimate holds for $|\theta_j^h(n)-\theta_i^h(n)|$,

$$|\theta_j^h(n)-\theta_i^h(n)| \leq \sqrt{(\theta_j^h(n)-\theta_i^h(n))^2}$$
$$\leq \sqrt{\theta_j^h(n)^2+\theta_i^h(n)^2} \leq \sqrt{\sum_{i=1}^{N} \theta_i^h(n)^2} \leq Ce^{-\lambda(n+1)h}. \tag{4.15}$$

This shows the emergence of complete synchronization for θ_i^h. Finally, recall the definition of $f(n)$ in (4.12), we directly apply the uniform lower bound in Lemma 4.3, the assumption of uniform upper bound, Taylor expansion and (4.15) to have

$$f(n) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j\neq i} J(\cos(\theta_j^h(n)-\theta_i^h(n))-1) \frac{x_j^h(n)-x_i^h(n)}{|x_j^h(n)-x_i^h(n)|^{\alpha}}$$
$$\leq O(1) \left|\cos(\theta_j^h(n)-\theta_i^h(n))-1\right|$$
$$\leq O(1) \left|\theta_j^h(n)-\theta_i^h(n)\right|$$
$$\leq \bar{C}e^{-\lambda(n+1)h},$$

where $O(1)$ denotes some positive constants. This shows the hypotheses in Theorem 1.1 are fulfilled and hence there exists X^∞ such that $X(n)$ converges asymptotically to X^∞, which completes the proof of Theorem 1.2. ∎

5. Summary

In this paper, we first provide a discrete version of the perturbed gradient flow theory. And then we used discrete pseudo-gradient flow to prove the asymptotic behavior of discrete DTK model and discrete DS model. Since there has been a detailed introduction for DTK model in [23], in this paper, we just briefly explain the mainly results by using discrete perturbed gradient flow. For discrete DS model, we first show that there exists a lower bound between inter-particle for all steps n. And then we made use of the lower bound between particles to show that there is no collision between particles which guarantees the global existence of the solution. Next, we suppose the inter-particle has positive upper bound and then we used the uniformly boundedness of relative distence between particles to show the practical synchronization when the phases of particles are initially confined in the quarter-circle. Finally, we used the result for phase synchronization with the discrete perturbed gradient flow structure of DS model to show that the particle converges to the asymptotic position which implies the flocking phenomena. However, there still remain some problems to be solved in the discrete DS model. For example, the rigorous proof of the uniform upper bound is still unknown, moreover the initial condition of θ_i^h is hopefully to be relaxed. These issues will be pursued in the future.

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Appendix. Proof of Lemma 4.2.

Proof. Similar as in Lemma 4.1, we will consider $n \le n_c$ where n_c denotes the possible first collision step. Without loss of generality, we assume there exists a pair (i,j) such that

$$\mathcal{D}_p^h(n) = |x_i^h(n) - x_j^h(n)|.$$

Then we can collect all pairs of particles whose distance is larger than $\mathcal{D}_p^h(n)$ at step n. More precisely, we define a set of indices as follows

$$\mathcal{C}_{ij} := \{ s \in \mathcal{N} \,|\, |x_s^h(n) - x_t^h(n)| = \mathcal{D}_q^h(n) > \mathcal{D}_p^h(n) = |x_i^h(n) - x_j^h(n)| \quad \text{for some} \quad t \in \mathcal{N} \}.$$

Now, for any $\mathcal{D}_q^h(n) > \mathcal{D}_p^h(n)$, $\mathcal{D}_q^h(n)$ must correspond to $|x_s^h(n) - x_t^h(n)|$ for some s,t. Then, we consider two cases below:

Case A: Either $i \notin \mathcal{C}_{ij}$ or $j \notin \mathcal{C}_{ij}$,

Case B: $i,j \in \mathcal{C}_{ij}$.

In the following, we will study the lower bound of $\mathcal{D}_p^h(n)$ for the above two cases respectively.

• Case A: Without loss of generality, we may assume $i \notin \mathcal{C}_{ij}$. Then we pick out $\mathcal{D}_q^h(n)$ for some $q > p$ such that $\mathcal{D}_q^h(n) = |x_s^h(n) - x_t^h(n)|$. Since $i \notin \mathcal{C}_{ij}$, we conclude that $s \neq i$ and $t \neq i$. Thus we have

$$|x_s^h(n) - x_i^h(n)| \le \mathcal{D}_p^h(n), \quad |x_t^h(n) - x_i^h(n)| \le \mathcal{D}_p^h(n).$$

We use the triangle inequality to have

$$2\mathcal{D}_p^h(n) \ge |x_s^h(n) - x_i^h(n)| + |x_t^h(n) - x_i^h(n)| \ge |x_s^h(n) - x_t^h(n)| = \mathcal{D}_q^h(n) \ge \frac{\delta_q}{2}.$$

As above estimates hold for all $0 \le n \le n_c$, we have

$$\mathcal{D}_p^h(n) \ge \frac{\delta_q}{4} \ge \frac{\delta_{p+1}}{4}, \quad n \le n_c.$$

• Case B: In this case, both i and j belong to $\mathcal{C}_{ij}$. Then we consider the following two subcases.

⋄ (Case B-1): Suppose there exists some common index s satisfying the following condition:

$$\begin{aligned} &|x_s^h(n) - x_i^h(n)| > \mathcal{D}_p^h(n), \quad |x_s^h(n) - x_j^h(n)| \le \mathcal{D}_p^h(n). \\ \text{or} \quad &|x_s^h(n) - x_i^h(n)| \le \mathcal{D}_p^h(n), \quad |x_s^h(n) - x_j^h(n)| > \mathcal{D}_p^h(n). \end{aligned} \tag{A.1}$$

Without loss of generality, we assume the first inequality holds. Then we again use the triangle inequality to have

$$\mathcal{D}_p^h(n) \ge \frac{|x_j^h(h) - x_i^h(n)| + |x_s^h(n) - x_j^h(n)|}{2} \ge \frac{|x_s^h(n) - x_i^h(n)|}{2} \ge \frac{\delta_{p+1}}{2}.$$

In the second inequality, we use the same argument to obtain the same conclusion.

⋄ (Case B-2): Suppose there is no index s satisfying the condition (A.1). Then, we only need to consider the cases such that,

$$\begin{aligned} \text{Either} \quad &|x_s^h(n) - x_i^h(n)| > \mathcal{D}_p^h(n), \quad \text{and} \quad |x_s^h(n) - x_j^h(n)| > \mathcal{D}_p^h(n). \\ \text{or} \quad &|x_s^h(n) - x_i^h(n)| \le \mathcal{D}_p^h(n), \quad \text{and} \quad |x_s^h(n) - x_j^h(n)| \le \mathcal{D}_p^h(n). \end{aligned} \tag{A.2}$$

Therefore, we can split an index set into two subsets defined as

$$\begin{cases} \mathcal{A}_1 := \{k \,|\, |x_k^h(n) - x_i^h(n)| > \mathcal{D}_p^h(n), |x_k^h(n) - x_j^h(n)| > \mathcal{D}_p^h(n)\}, \\ \mathcal{A}_2 := \{k \,|\, |x_k^h(n) - x_i^h(n)| \le \mathcal{D}_p^h(n), |x_k^h(n) - x_j^h(n)| \le \mathcal{D}_p^h(n)\}. \end{cases}$$

According to (A.2), we have $\mathcal{N} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then, we follow similar analysis as in Lemma 4.1 to obtain for any $n < n_c$ that

$$\begin{aligned}
&\mathcal{D}_p^h(n+1)^2 - \mathcal{D}_p^h(n)^2 \\
&\ge h\left[2(x_i^h(n) - x_j^h(n))(\omega_i - \omega_j) + \frac{4\Gamma_r(\theta_j^h(n) - \theta_i^h(n))}{N|x_j^h(n) - x_i^h(n)|^{\beta-2}} - \frac{4\Gamma_a(\theta_j^h(n) - \theta_i^h(n))}{N|x_j^h(n) - x_i^h(n)|^{\alpha-2}}\right] \\
&\quad + \frac{2h}{N}\sum_{k\in\mathcal{A}_1}(x_i^h(n) - x_j^h(n))(x_k^h(n) - x_i^h(n))\left[\frac{\Gamma_a(\theta_k^h(n) - \theta_i^h(n))}{|x_k^h(n) - x_i^h(n)|^{\alpha}} - \frac{\Gamma_r(\theta_k^h(n) - \theta_i^h(n))}{|x_k^h(n) - x_i^h(n)|^{\beta}}\right] \\
&\quad - \frac{2h}{N}\sum_{k\in\mathcal{A}_1}(x_i^h(n) - x_j^h(n))(x_k^h(n) - x_j^h(n))\left[\frac{\Gamma_a(\theta_k^h(n) - \theta_j^h(n))}{|x_k^h(n) - x_j^h(n)|^{\alpha}} - \frac{\Gamma_r(\theta_k^h(n) - \theta_j^h(n))}{|x_k^h(n) - x_j^h(n)|^{\beta}}\right] \\
&\quad + \frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\ k\ne i,j}}(x_i^h(n) - x_j^h(n))(x_k^h(n) - x_i^h(n))\left[\frac{\Gamma_a(\theta_k^h(n) - \theta_i^h(n))}{|x_k^h(n) - x_i^h(n)|^{\alpha}} - \frac{\Gamma_r(\theta_k^h(n) - \theta_i^h(n))}{|x_k^h(n) - x_i^h(n)|^{\beta}}\right] \\
&\quad - \frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\ k\ne i,j}}(x_i^h(n) - x_j^h(n))(x_k^h(n) - x_j^h(n))\left[\frac{\Gamma_a(\theta_k^h(n) - \theta_j^h(n))}{|x_k^h(n) - x_j^h(n)|^{\alpha}} - \frac{\Gamma_r(\theta_k^h(n) - \theta_j^h(n))}{|x_k^h(n) - x_j^h(n)|^{\beta}}\right] \\
&\quad + h^2\left[(\omega_i - \omega_j) + \tfrac{1}{N}\textstyle\sum_{\substack{k\in\mathcal{N}\\ i\ne k}}\Phi(x_k^h(n) - x_i^h(n), \theta_k^h(n) - \theta_i^h(n)) - \tfrac{1}{N}\sum_{\substack{k\in\mathcal{N}\\ j\ne k}}\Phi(x_k^h(n) - x_j^h(n), \theta_k^h(n) - \theta_j^h(n))\right]^2 \\
&= \sum_{i=1}^{6}\mathcal{I}_{7i}.
\end{aligned}$$

In the sequel, we estimate the terms $\mathcal{I}_{7i}$ separately. Firstly, it is obvious that $\mathcal{I}_{76} \ge 0$. Next, for $\mathcal{I}_{71}$, we use the Cauchy-Schwarz inequality to obtain that

$$\begin{aligned}
\mathcal{I}_{71} &\ge -2h|x_i^h(n) - x_j^h(n)||\omega_i - \omega_j| + \frac{4hm_r}{N|x_j^h(n) - x_i^h(n)|^{\beta-2}} - \frac{4hM_a}{N|x_j^h(n) - x_i^h(n)|^{\alpha-2}} \\
&= -2h|x_i^h(n) - x_j^h(n)|\mathcal{D}(\omega) + \frac{4hm_r}{N|x_j^h(n) - x_i^h(n)|^{\beta-2}} - \frac{4hM_a}{N|x_j^h(n) - x_i^h(n)|^{\alpha-2}}.
\end{aligned} \tag{A.3}$$

The estimates of $\mathcal{I}_{72}$ and $\mathcal{I}_{73}$ can be combined together. Actually, according to previous analysis, we know that

$$|x_k^h(n) - x_j^h(n)| > |x_i^h(n) - x_j^h(n)|, \quad |x_k^h(n) - x_i^h(n)| > |x_i^h(n) - x_j^h(n)|, \quad \text{for} \quad k \in \mathcal{A}_1,$$

which implies that

$$\begin{aligned}
\mathcal{I}_{72} + \mathcal{I}_{73} &\ge -\frac{2h}{N}\sum_{k\in\mathcal{A}_1}\left[\frac{M_a|x_i^h(n) - x_j^h(n)|}{|x_k^h(n) - x_i^h(n)|^{\alpha-1}} + \frac{M_r|x_i^h(n) - x_j^h(n)|}{|x_k^h(n) - x_i^h(n)|^{\beta-1}}\right] \\
&\quad - \frac{2h}{N}\sum_{k\in\mathcal{A}_1}\left[\frac{M_a|x_i^h(n) - x_j^h(n)|}{|x_k^h(n) - x_j^h(n)|^{\alpha-1}} + \frac{M_r|x_i^h(n) - x_j^h(n)|}{|x_k^h(n) - x_j^h(n)|^{\beta-1}}\right] \\
&\ge -4h|x_i^h(n) - x_j^h(n)|\left(\frac{M_a}{(\frac{\delta_{p+1}}{2})^{\alpha-1}} + \frac{M_r}{(\frac{\delta_{p+1}}{2})^{\beta-1}}\right).
\end{aligned} \tag{A.4}$$

Similarly, the estimates of $\mathcal{I}_{74}$ and $\mathcal{I}_{75}$ are also made together. Since $k \in \mathcal{A}_2$ in these two terms, we apply previous analysis to have

$$(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_i^h(n)) \le 0 \quad \text{and} \quad (x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n)) \ge 0.$$

Then, we obtain that

$$\begin{aligned}\mathcal{I}_{74}+\mathcal{I}_{75} \ge & \frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\k\neq i,j}} (x_i^h(n)-x_j^h(n))(x_k^h(n)-x_i^h(n))\left[\frac{M_a}{|x_k^h(n)-x_i^h(n)|^\alpha}-\frac{m_r}{|x_k^h(n)-x_i^h(n)|^\beta}\right]\\ &-\frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\k\neq i,j}} (x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))\left[\frac{M_a}{|x_k^h(n)-x_j^h(n)|^\alpha}-\frac{m_r}{|x_k^h(n)-x_j^h(n)|^\beta}\right].\end{aligned} \tag{A.5}$$

Now, we combine above estimates (A.3), (A.4) and (A.5) to obtain the following estimate of $\mathcal{D}_p$,

$$\begin{aligned}&\mathcal{D}^h(n+1)^2-\mathcal{D}^h(n)^2\\ &\ge 2h\left(\frac{m_r}{N|x_j^h(n)-x_i^h(n)|^{\beta-2}}-|x_i^h(n)-x_j^h(n)|\left(\mathcal{D}(\omega)+\frac{2M_a}{(\frac{\delta_{p+1}}{2})^{\alpha-1}}+\frac{2M_r}{(\frac{\delta_{p+1}}{2})^{\beta-1}}\right)\right)\\ &\quad+\frac{2h}{N|x_j^h(n)-x_i^h(n)|^{\alpha-2}}\left(\frac{m_r}{|x_j^h(n)-x_i^h(n)|^{\beta-\alpha}}-2M_a\right)\\ &\quad+\frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\k\neq i,j}}\left[\frac{(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_i^h(n))}{|x_k^h(n)-x_i^h(n)|^\alpha}\left(\frac{m_r}{|x_k^h(n)-x_i^h(n)|^{\beta-\alpha}}-M_a\right)\right]\\ &\quad+\frac{2h}{N}\sum_{\substack{k\in\mathcal{A}_2\\k\neq i,j}}\left[\frac{(x_i^h(n)-x_j^h(n))(x_k^h(n)-x_j^h(n))}{|x_k^h(n)-x_j^h(n)|^\alpha}\left(\frac{m_r}{|x_k^h(n)-x_j^h(n)|^{\beta-\alpha}}-M_a\right)\right].\end{aligned} \tag{A.6}$$

This is very similar to (4.7), therefore, we can follow the criteria in Lemma 4.1 to find the following estimate,

$$\mathcal{D}_p^h(n)\ge\frac{C_p^*}{2},\quad 0\le n\le n_c,$$

$$C_p^*=\min\left\{\mathcal{D}_p(0),\left(\frac{m_r}{2M_a}\right)^{\frac{1}{\beta-\alpha}},\left(\frac{m_r}{N\left(\mathcal{D}(\omega)+\frac{2M_a}{(\frac{\delta_{p+1}}{2})^{\alpha-1}}+\frac{2M_r}{(\frac{\delta_{p+1}}{2})^{\beta-1}}\right)}\right)^{\frac{1}{\beta-1}}\right\}.$$

Finally, we combine (Case A) and (Case B) to conclude that $\mathcal{D}_p^h(n)\ge\min\left\{\frac{\delta_{p+1}}{4},\frac{C_p^*}{2}\right\}$ for $n\le n_c$. More precisely, we have

$$\mathcal{D}_p^h(n)\ge\frac{\delta_p}{2},\quad \delta_p=\min\left\{\frac{\delta_{p+1}}{4},\frac{C_p^*}{2}\right\},\quad 0\le n\le n_c,$$

$$C_p^*=\min\left\{\mathcal{D}_p(0),\left(\frac{m_r}{2M_a}\right)^{\frac{1}{\beta-\alpha}},\left(\frac{m_r}{N\left(\mathcal{D}(\omega)+\frac{2M_a}{(\frac{\delta_{p+1}}{2})^{\alpha-1}}+\frac{2M_r}{(\frac{\delta_{p+1}}{2})^{\beta-1}}\right)}\right)^{\frac{1}{\beta-1}}\right\}.$$

□

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