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THE GLOBAL GENERALIZED SOLUTION OF THE CHEMOTAXIS-NAVIER–STOKES SYSTEM WITH LOGISTIC SOURCE*

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Abstract. In this paper, we consider the initial boundary value problem of the chemotaxis-Navier–Stokes system with low regularity, and we show that the system has a global generalized solution, which was first introduced by M. Winkler [SIAM J. Math. Anal., 47(4):3092–3115, 2015].

Keywords. Chemotaxis-Navier–Stokes system; Logistic source; Global generalized solution; Low regularity.

AMS subject classifications. 35A01; 35K51; 35Q92; 92C17.

1. Introduction and main results

1.1. Introduction. In this paper, we consider the following chemotaxis-Navier–Stokes system:

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n-\nabla\cdot(\chi n\nabla m)+f(n), & x\in\Omega, t>0,\\ m_t+u\cdot\nabla m=\Delta m+g(n,m), & x\in\Omega, t>0,\\ u_t+(u\cdot\nabla)u=\Delta u+\nabla P+n\nabla\omega, & x\in\Omega, t>0,\\ \nabla\cdot u=0, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a smooth bounded domain, $n=n(x,t)$ denotes the cell density, $m=m(x,t)$ is the chemical (oxygen) concentration, $u=u(x,t)$ is the fluid velocity, $P=P(x,t)$ is the pressure of the fluid, $\omega=\omega(x)$ represents the potential function, χ represents chemotaxis sensitivity, the function $f\in C^1([0,+\infty))$ describes the growth and death of the cell, and $g(n,m)$ represents the production and consumption of chemical substances.

Chemotaxis refers to the kinetic response of biological individuals (e.g. bacteria, insects) or biological tissues (e.g. cells, tubes) to chemical substances. Generally, the movement of an organism or cell from a lower concentration of chemoattractant to a higher concentration of chemoattractant is called positive chemotaxis. Similarly, the opposite movement of an organism is called negative chemotaxis. In 1970, Keller and Segel [7] established a mathematical model (called the Keller–Segel model) to describe the chemotaxis phenomenon of amoeba through macroscopic analysis, which is not only used in mathematics but also plays an important role in biology and pharmacology. When the flow of culture medium is not considered, (1.1) becomes the classical Keller–Segel model

$$\begin{cases} n_t=\Delta n-\nabla\cdot(\chi n\nabla m)+f(n),\\ m_t=\Delta m+g(n,m). \end{cases} \tag{1.2}$$

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In recent years, there has been much research on the properties of (1.2).

When the substance m is produced by bacteria n and participates in chemical reactions and is consumed, and the production and consumption rates are both linearly related to n, we generally consider $g(n,m)=-m+n$. When $f(n)\equiv 0$, there are many results of the Keller–Segel model regarding the global existence of the solution and the blow-up of the solution (see, e.g., [1, 6, 13, 14, 21, 24, 32]). When $f(n)=\rho n-\mu n^{\alpha}$, Winkler [22] proved that under the Neumann boundary conditions, when μ is large enough, for sufficiently smooth initial values, (1.2) has a unique global-in-time classical solution with $\alpha=2$. Next, Lankeit [8] proved the existence of a global weak solution for the chemotactic system for any small $\mu>0$. Furthermore, Viglialoro [20] proved the existence of a very weak solution of the chemotactic system when $\alpha>2-\frac{1}{d}$. Recently, [28] and [34] extended the results of [20] to $\alpha\geq\frac{2d+4}{d+4}$ and $\alpha\geq\min\{\frac{2d+4}{d+4},\frac{2d-2}{d}\}$ when $d\geq 2$, respectively. Particularly, Winkler [29] showed that logistic source $f(n)$ can rule out the occurrence of persistent Dirac-type singularities.

When the substance m is consumed by bacteria n, and no new substances are produced in the area, we generally choose $g(n,m)=-nm$. Similarly, when $f(n)=0$, Tao [15] proved that under suitable initial value assumptions, if $0\leq\chi\leq\frac{1}{6(n+1)\|m(\cdot,0)\|_{L^\infty(\Omega)}}$, then the corresponding initial boundary value problem of system (1.2) has a unique uniformly bounded global solution. At the same time, Tao and Winkler [16] proved that the three-dimensional chemotaxis model has at least one global weak solution with large data. In [36], the authors studied the asymptotic stability and decay rate of the classical solution of (1.2) based on [15]. Next, Winkler [26] defined the concept of a generalized solution, which requires the solution to satisfy only very mild regularity assumptions. Then, Lyu [12] generalized the result of [26] to the Keller–Segel model with logistic source $f(n)$. For the global existence and boundedness results of the classical solution of (1.2) with logistic source, we refer the reader to [2, 11].

In addition, there are many results for the chemotaxis-Navier–Stokes system (1.1). When $f\equiv 0$, Winkler [23] proved that in the absence of initial value smallness, the chemotaxis-Navier–Stokes system has a unique two-dimensional global classical solution and at least one three-dimensional global weak solution. In addition, the stability of the solution of the two-dimensional chemotaxis-Navier–Stokes system was obtained in [25]. When $f(n)=\rho n-\mu n^{\alpha},\alpha=2$, Lankeit [9] constructed a weak solution of (1.3) and proved its long-time behavior. Next, Winkler [27] showed that the chemotaxis-Navier–Stokes system has at least one appropriate generalized global solution, and obtained its asymptotic stability using appropriate assumptions about ρ and μ. Recently, when $\alpha>1,d=2,3$, Wang [30, 31] and Ding [5] obtained the global solvability and eventual smoothness of chemotaxis-(Navier–)Stokes, respectively. For more results on the chemotaxis-Navier–Stokes system, we refer the reader to [3, 10, 17–19, 33, 35].

In this article, we consider the initial boundary value problem of (1.1) in the case $g(n,m)=-nm,\chi>0$. That is, we consider (without loss of generality, we take $\chi=1$):

$$\begin{cases} n_t+(u\cdot\nabla)n=\Delta n-\nabla\cdot(n\nabla m)+f(n), & x\in\Omega, t>0\\ m_t+(u\cdot\nabla)m=\Delta m-nm, & x\in\Omega, t>0\\ u_t+(u\cdot\nabla)u=\Delta u+\nabla P+n\nabla\omega, & x\in\Omega, t>0\\ \nabla\cdot u=0, & x\in\Omega, t>0\\ \nabla n\cdot\nu=\nabla m\cdot\nu=0, u=0, & x\in\partial\Omega, t>0\\ n(x,0)=n_0(x), m(x,0)=m_0(x), u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where $\omega \in W^{1,\infty}(\Omega)$, ν is the unit outer normal vector of $\partial\Omega$, and the logistic source $f \in C^1([0,+\infty))$ satisfies

$$f(0)=0, \quad f(s) \le \rho s - \mu s^\alpha, \quad \rho \in \mathbb{R}, \quad \mu > 0, \quad \alpha > 1, \tag{1.4}$$

for any $s \ge 0$.

1.2. Main result. In this paper, we consider the existence of generalized solutions for the initial boundary value problem (1.3). To state the main result of this paper, we first introduce the definition of generalized solutions as follows.

DEFINITION 1.1 (Generalized solutions, see [26]). *Let*

$$W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^2) := W^{1,2}_0(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega;\mathbb{R}^2),$$

where $L^2_\sigma(\Omega;\mathbb{R}^2) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) | \nabla \cdot \varphi = 0 \ \text{ in } \ \mathcal{D}'(\Omega)\}$ *represents the solenoidal subspace of* $L^2(\Omega;\mathbb{R}^2)$. *Assume that the triple of functions*

$$\begin{cases} n \in L^1_{loc}(\bar\Omega \times [0,+\infty)), \\ m \in L^\infty_{loc}(\bar\Omega \times [0,+\infty)) \cap L^2_{loc}\left([0,+\infty); W^{1,2}(\Omega)\right), \\ u \in L^2_{loc}\left([0,+\infty); W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^2)\right), \end{cases}$$

satisfies $n \ge 0, m \ge 0$ *and*

$$f(n) \in L^1_{loc}(\bar\Omega \times [0,+\infty)).$$

Then (n,m,u) *is said to be a global generalized solution of the initial boundary value problem* (1.3), *if*

$$\int_\Omega n(\cdot,t) \le \int_\Omega n_0 + \int_0^t \int_\Omega f(n) \quad \text{for a.e. } \ t > 0$$

and

$$-\int_0^{+\infty}\int_\Omega m\varphi_t - \int_\Omega m_0 \varphi(\cdot,0) = -\int_0^{+\infty}\int_\Omega \nabla m \cdot \nabla\varphi - \int_0^{+\infty}\int_\Omega nm\varphi + \int_0^{+\infty}\int_\Omega mu \cdot \nabla\varphi \tag{1.5}$$

for all $\varphi \in C^\infty_0(\bar\Omega \times [0,+\infty))$, *if*

$$-\int_0^{+\infty}\int_\Omega u\zeta_t - \int_\Omega u_0 \zeta(\cdot,0) = \int_0^{+\infty}\int_\Omega (u\otimes u) : \nabla\zeta + \int_0^{+\infty}\int_\Omega n\nabla\omega \cdot \zeta - \int_0^{+\infty}\int_\Omega \nabla u \cdot \nabla\zeta \tag{1.6}$$

for all $\zeta \in C^\infty_0(\overline{\Omega} \times [0,+\infty);\mathbb{R}^2)$ *with* $\nabla \cdot \zeta = 0$ *in* $\Omega \times (0,+\infty)$, *and if moreover there exists a function* $\phi \in C^2([0,+\infty))$ *satisfying*

$$\begin{aligned} &\phi(n)u \in L^1_{loc}\left(\bar\Omega \times [0,+\infty);\mathbb{R}^2\right) \\ &\phi(n), \phi''(n)|\nabla n|^2, f(n)\phi'(n) \in L^1_{loc}(\bar\Omega \times [0,+\infty)) \\ &\phi''(n)n\nabla n, \phi'(n)n \in L^2_{loc}(\bar\Omega \times [0,+\infty)) \end{aligned}$$

and

$$\begin{aligned}
&-\int_0^{+\infty}\int_\Omega \phi(n)\varphi_t-\int_\Omega \phi(n_0)\varphi(\cdot,0)\\
\geqslant&-\int_0^{+\infty}\int_\Omega \phi''(n)|\nabla n|^2\varphi-\int_0^{+\infty}\int_\Omega \phi'(n)\nabla n\nabla\varphi\\
&+\int_0^{+\infty}\int_\Omega \phi''(n)n\nabla n\cdot\nabla m\varphi+\int_0^{+\infty}\int_\Omega \phi'(n)n\nabla m\cdot\nabla\varphi\\
&+\int_0^{+\infty}\int_\Omega \phi(n)u\nabla\varphi+\int_0^{+\infty}\int_\Omega f(n)\phi'(n)\varphi
\end{aligned}$$

holds for all nonnegative $\varphi\in C_0^\infty(\bar\Omega\times[0,+\infty))$.

Now, we state our main results as follows.

THEOREM 1.1. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded domain with smooth boundary, and* $f\in C^1([0,+\infty))$ *satisfies* (1.4). *Assume that the initial data* (n_0,m_0,u_0) *satisfies*

$$\begin{aligned}
&n_0\in L^1(\Omega)\ \ \text{is nonnegative with}\ \ n_0\not\equiv 0,\\
&m_0\in W^{1,p}(\Omega)\ \ \text{is nonnegative with}\ \ p>2,\\
&u_0\in L^2\left(\Omega;\mathbb{R}^2\right)\ \ \text{and}\ \ \nabla\cdot u_0=0\ \ \text{in}\ \ \mathcal{D}'(\Omega),
\end{aligned}\tag{1.7}$$

then the initial boundary value problem (1.3) *possesses at least one global generalized solution* (n,m,u) *defined by Definition 1.1.*

REMARK 1.1. In fact, when $f(n)=0$, the existence of global weak solutions and smooth solutions to Equation (1.3) was proved in [23], where the initial values were chosen as

$$\begin{cases}
n_0\in C^0(\bar\Omega),\quad n_0>0 \text{ in } \bar\Omega,\\
m_0\in W^{1,q}(\Omega) \text{ for some } q>N,\quad m_0>0 \text{ in } \bar\Omega,\\
u_0\in D(A^\alpha) \text{ for some } \alpha\in\left(\frac{N}{4},1\right).
\end{cases}$$

Unlike their results, our initial values here only satisfy (1.7), and they have a lower regularity.

1.3. Sketch of the proof. The rest of this paper is organized as follows. In Section 2, we give an approximation system and show that its solutions satisfy the properties similar to those of Definition 1.1. In Section 3, we derive the uniform estimates of the solutions of the approximation system. In the last section, utilizing the estimates established in Section 3, we pass the limit on the approximate solutions and obtain the generalized solutions of the problem (1.3).

2. Regularized problem and basic properties

In order to obtain the generalized solution of (1.3), we introduce an approximation system in this section. First, we introduce a family of functions $\{n_{0\varepsilon}\}_{\varepsilon\in(0,1)}$ and $\{u_{0\varepsilon}\}_{\varepsilon\in(0,1)}$ that satisfy

$$\{n_{0\varepsilon}\}_{\varepsilon\in(0,1)}\subset C^0(\bar\Omega) \text{ with } n_{0\varepsilon}\geq 0 \text{ in } \Omega,\tag{2.1}$$

$$\{u_{0\varepsilon}\}_{\varepsilon\in(0,1)}\subset C^1(\bar\Omega;\mathbb{R}^2) \text{ with } \nabla\cdot u_{0\varepsilon}=0 \text{ in } \Omega,\quad u_{0\varepsilon}=0 \text{ on } \partial\Omega,\tag{2.2}$$

$$n_{0\varepsilon}\to n_0 \text{ in } L^1(\Omega),\quad u_{0\varepsilon}\to u_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon\to 0,\tag{2.3}$$

$$\int_\Omega n_{0\varepsilon} \le 2\int_\Omega n_0 \quad \text{for all} \quad \varepsilon \in (0,1). \tag{2.4}$$

Second, we consider the following approximation system:

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla\cdot(n_\varepsilon \nabla m_\varepsilon) + f(n_\varepsilon), & x\in\Omega, t>0\\ m_{\varepsilon t} + u_\varepsilon\cdot\nabla m_\varepsilon = \Delta m_\varepsilon - n_\varepsilon m_\varepsilon, & x\in\Omega, t>0\\ u_{\varepsilon t} + (u_\varepsilon\cdot\nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon\nabla\omega, & x\in\Omega, t>0\\ \nabla\cdot u_\varepsilon = 0, & x\in\Omega, t>0\\ \nabla n_\varepsilon\cdot\nu = \nabla m_\varepsilon\cdot\nu = 0, u_\varepsilon = 0, & x\in\partial\Omega, t>0\\ n_\varepsilon(x,0) = n_{0\varepsilon}(x), m_\varepsilon(x,0) = m_0(x), u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x\in\Omega, \end{cases} \tag{2.5}$$

for $\varepsilon \in (0,1)$.

Indeed, one can prove that the system (2.5) has a global classical solution.

Lemma 2.1. *Let $\varepsilon\in(0,1)$. Suppose that $(n_{0\varepsilon}, m_0, u_{0\varepsilon})$ satisfies (2.1)–(2.4). Then (2.5) admits a classical solution*

$$\begin{cases} n_\varepsilon \in C^0(\bar\Omega\times[0,+\infty)) \cap C^{2,1}(\bar\Omega\times(0,+\infty)),\\ m_\varepsilon \in \bigcap_{p>2} C^0\left([0,+\infty); W^{1,p}(\Omega)\right) \cap C^{2,1}(\bar\Omega\times(0,+\infty)),\\ u_\varepsilon \in C^0(\bar\Omega\times[0,+\infty)) \cap C^{2,1}(\bar\Omega\times(0,+\infty)),\\ P_\varepsilon \in C^{1,0}(\bar\Omega\times(0,+\infty)). \end{cases} \tag{2.6}$$

Proof. The proof is very similar to the one of Lemma 2.1 in [23], and we omit the details here. ∎

Next we will prove that $(n_\varepsilon, m_\varepsilon, u_\varepsilon)$ satisfies the properties similar to those of Definition 1.1.

Lemma 2.2. *Let $\phi\in C^2([0,+\infty))$, then*

$$\begin{aligned} \int_\Omega \partial_t\phi(n_\varepsilon)\varphi = &-\int_\Omega \phi''(n_\varepsilon)|\nabla n_\varepsilon|^2\varphi - \int_\Omega \phi'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi\\ &+\int_\Omega \phi''(n_\varepsilon)n_\varepsilon\varphi\nabla n_\varepsilon\cdot\nabla m_\varepsilon + \int_\Omega n_\varepsilon\phi'(n_\varepsilon)\nabla m_\varepsilon\cdot\nabla\varphi\\ &+\int_\Omega \phi(n_\varepsilon)u_\varepsilon\cdot\nabla\varphi + \int_\Omega f(n_\varepsilon)\phi'(n_\varepsilon)\varphi \end{aligned} \tag{2.7}$$

holds for any $\varepsilon\in(0,1)$ and $\varphi\in C^\infty(\bar\Omega\times(0,+\infty))$.

Proof. Multiplying the first equation of (2.5) by $\phi'(n_\varepsilon)\varphi$ and integrating the result over Ω, we have

$$\begin{aligned} \int_\Omega \partial_t\phi(n_\varepsilon)\varphi &= \int_\Omega \phi'(n_\varepsilon)\varphi(\Delta n_\varepsilon - u_\varepsilon\cdot\nabla n_\varepsilon - \nabla\cdot(n_\varepsilon\nabla m_\varepsilon) + f(n_\varepsilon))\\ &= -\int_\Omega \nabla n_\varepsilon\cdot\nabla(\phi'(n_\varepsilon)\varphi) + \int_\Omega n_\varepsilon\nabla m_\varepsilon\cdot\nabla(\phi'(n_\varepsilon)\varphi)\\ &\quad -\int_\Omega \phi'(n_\varepsilon)\varphi u_\varepsilon\cdot\nabla n_\varepsilon + \int_\Omega f(n_\varepsilon)\phi'(n_\varepsilon)\varphi. \end{aligned} \tag{2.8}$$

This, combined with the Green identity, gives

$$-\int_\Omega \nabla n_\varepsilon\cdot\nabla(\phi'(n_\varepsilon)\varphi)=-\int_\Omega \phi''(n_\varepsilon)|\nabla n_\varepsilon|^2\varphi-\int_\Omega \phi'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi, \tag{2.9}$$

$$\int_\Omega n_\varepsilon\nabla m_\varepsilon\cdot\nabla(\phi'(n_\varepsilon)\varphi)=\int_\Omega \phi''(n_\varepsilon)n_\varepsilon\varphi\nabla n_\varepsilon\cdot\nabla m_\varepsilon+\int_\Omega n_\varepsilon\phi'(n_\varepsilon)\nabla m_\varepsilon\cdot\nabla\varphi, \tag{2.10}$$

$$-\int_\Omega \phi'(n_\varepsilon)\varphi u_\varepsilon\cdot\nabla n_\varepsilon=-\int_\Omega \varphi\nabla\phi(n_\varepsilon)\cdot u_\varepsilon=\int_\Omega \phi(n_\varepsilon)u_\varepsilon\cdot\nabla\varphi \tag{2.11}$$

where we have used $\nabla\cdot u_\varepsilon=0$ in $\Omega\times(0,+\infty)$ and $u_\varepsilon=0$ on $\partial\Omega\times(0,+\infty)$. Putting (2.9), (2.10) and (2.11) into (2.8), we get (2.7) directly. ∎

3. Uniform estimates

In this section, we derive some estimates about the solutions $(n_\varepsilon,m_\varepsilon,u_\varepsilon)$ of (2.5).

3.1. Estimate of $\{n_\varepsilon\}_{\varepsilon\in(0,1)}$.

LEMMA 3.1. *Let $(n_\varepsilon,m_\varepsilon,u_\varepsilon)$ be a smooth solution of (2.5). Then*

- *For any $\varepsilon\in(0,1)$, $\{n_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies*

$$\int_\Omega n_\varepsilon(\cdot,t)=\int_\Omega n_{0\varepsilon}+\int_0^t\int_\Omega f(n_\varepsilon)\quad \text{for all}\quad t\geq 0. \tag{3.1}$$

- *Let $T>0$, then there exists $C=C(T)>0$ satisfying*

$$\int_0^T\int_\Omega |f(n_\varepsilon)|\leq C\quad \text{for all}\quad \varepsilon\in(0,1). \tag{3.2}$$

- *Furthermore,*

$$\sup_{\varepsilon\in(0,1)}\sup_{t\geq 0}\int_\Omega n_\varepsilon(\cdot,t)<\infty. \tag{3.3}$$

Proof. Integrating the first equation of (2.5) over Ω, we can directly obtain (3.1) due to the no-flux boundary conditions for n_ε and m_ε and homogeneous Dirichlet boundary conditions for u_ε. It is deduced from (1.4) and [12, Lemma 3.3, Lemma 3.4] that

$$f(n_\varepsilon)\leq C_1-n_\varepsilon,\quad f_+(n_\varepsilon)\leq C_2 \tag{3.4}$$

with $C_1>0$, $C_2>0$. Thus, (3.2) follows directly from the decomposition

$$|f(n_\varepsilon)|=2f_+(n_\varepsilon)-f(n_\varepsilon).$$

Finally, by the first equation of (2.5) and (3.4), we obtain

$$\frac{d}{dt}\int_\Omega n_\varepsilon=\int_\Omega f(n_\varepsilon)\leq C_1|\Omega|-\int_\Omega n_\varepsilon,$$

that is

$$\frac{d}{dt}(e^t\int_\Omega n_\varepsilon)\leq C_1|\Omega|e^t.$$

Integrating the above result with respect to t, we deduce

$$\int_\Omega n_\varepsilon(\cdot,t) < C_1|\Omega| + \int_\Omega n_{0\varepsilon}.$$

Combining (2.4), we get (3.3). This completes the proof. ∎

We also have the uniform integrability of n_ε, which is useful in the other estimates in the rest of this paper. Indeed, we have

LEMMA 3.2. *Let $T>0$, then*

$$\{n_\varepsilon\}_{\varepsilon\in(0,1)} \text{ and } \{(n_\varepsilon+1)^{-1}f(n_\varepsilon)\}_{\varepsilon\in(0,1)}$$

are uniformly integrable on $\Omega\times(0,T)$.

Proof. It is similar to the proof of de La Vallée Poussin's theorem [4, Lemma 1.2], we omit the details here. ∎

3.2. Estimate of $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$.

LEMMA 3.3. *Let $(n_\varepsilon, m_\varepsilon, u_\varepsilon)$ be a smooth solution of (2.5). Assume $T>0$, then for any $\varepsilon\in(0,1)$, there exists $C=C(T)>0$ such that*

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C \tag{3.5}$$

and

$$\int_0^T\int_\Omega |\nabla u_\varepsilon(x,t)|^2 \le C \tag{3.6}$$

for all $t\in(0,T)$.

Proof. Taking the L^2 scalar product of the third equation of (2.5) with u_ε, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\omega.$$

If $\alpha\ge 2$, thanks to the Hölder inequality, Young's inequality and the Poincaré inequality, there exist positive constants $C_1 := \|\nabla\omega\|_{L^\infty}$ and C_2 such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\omega\\ &\le \|u_\varepsilon\|_{L^2(\Omega)}\|n_\varepsilon\|_{L^2(\Omega)}\|\nabla\omega\|_{L^\infty(\Omega)}\\ &\le C_1C_2\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|n_\varepsilon\|_{L^2(\Omega)}\\ &\le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}C_3^2\|n_\varepsilon\|^2_{L^2(\Omega)}\end{aligned} \tag{3.7}$$

with $C_3 := \max\{C_1, C_2\}$. Recalling the definition of f in (1.4), we have

$$f(n_\varepsilon) \le \rho n_\varepsilon - \mu n_\varepsilon^\alpha, \quad \rho\in\mathbb{R}, \mu>0, \alpha>1.$$

According to Lemma 3.1, there exists $C_4 = C_4(T)$ such that

$$\int_0^T\int_\Omega |n_\varepsilon|^\alpha \lesssim \int_0^T\int_\Omega n_\varepsilon + \int_0^T\int_\Omega |f(n_\varepsilon)| \le C_4.$$

Integrating (3.7) with respect to t yields

$$\begin{aligned}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 &\le \int_0^t \|n_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |u_{0\varepsilon}|^2 \\ &\le \int_0^T \|n_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |u_{0\varepsilon}|^2 \end{aligned} \tag{3.8}$$

with $t\in(0,T)$ and for all $\varepsilon\in(0,1)$.
If $\alpha\in(1,2)$, by the Hölder inequality, we get

$$\int_\Omega n_\varepsilon^{\frac{2}{3-\alpha}} = \int_\Omega n_\varepsilon^{\frac{2-\alpha}{3-\alpha}} n_\varepsilon^{\frac{\alpha}{3-\alpha}} \le \|n_\varepsilon\|_{L^1(\Omega)}^{\frac{2-\alpha}{3-\alpha}} \|n_\varepsilon^\alpha\|_{L^1(\Omega)}^{\frac{1}{3-\alpha}}.$$

According to Lemma 3.1 and Lemma 3.2, there exists $C_5=C_5(T)$ such that

$$\begin{aligned}\int_0^T \left(\int_\Omega n_\varepsilon^{\frac{2}{3-\alpha}}\right)^{3-\alpha} &\le \int_0^T \left(\|n_\varepsilon\|_{L^1(\Omega)}^{\frac{2-\alpha}{3-\alpha}} \|n_\varepsilon^\alpha\|_{L^1(\Omega)}^{\frac{1}{3-\alpha}}\right)^{3-\alpha} \\ &\le \left(\sup_{t\in(0,T)} \|n_\varepsilon\|_{L^1(\Omega)}\right)^{2-\alpha} \int_0^T \int_\Omega |n_\varepsilon|^\alpha \\ &\le C_5. \end{aligned} \tag{3.9}$$

Employing the embedding inequality $W_0^{1,2}(\Omega)\hookrightarrow L^{\frac{2}{\alpha-1}}(\Omega)$ and the Poincaré inequality, we obtain that

$$\|u_\varepsilon\|_{L^{\frac{2}{\alpha-1}}(\Omega)} \le C_6 \|\nabla u_\varepsilon\|_{L^2(\Omega)}$$

with $C_6=C_6(\alpha,\Omega)>0$, and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\omega \\ &\le \|u_\varepsilon\|_{L^{\frac{2}{\alpha-1}}(\Omega)} \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)} \|\nabla\omega\|_{L^\infty(\Omega)} \\ &\le C_1 \|u_\varepsilon\|_{L^{\frac{2}{\alpha-1}}(\Omega)} \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)} \\ &\le C_1 C_6 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2} C_7^2 \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)}^2, \end{aligned}$$

with $C_7:=\max\{C_1,C_6\}$. Integrating above result with respect to t, we obtain

$$\begin{aligned}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 &\le \int_0^t \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)}^2 + \int_\Omega |u_{0\varepsilon}|^2 \\ &\le \int_0^T \|n_\varepsilon\|_{L^{\frac{2}{3-\alpha}}(\Omega)}^2 + \int_\Omega |u_{0\varepsilon}|^2 \end{aligned} \tag{3.10}$$

for all $t\in(0,T)$. Therefore, combining (2.3), (3.8), (3.9) with (3.10) gives

$$\sup_{t\in(0,T)} \int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le C_8 \quad \text{for all } \varepsilon\in(0,1)$$

with $C_8 = C_8(\alpha,\Omega,T) > 0$. ∎

LEMMA 3.4. *Let $T>0$, then $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);L^2(\Omega;\mathbb{R}^2))$ with respect to the strong topology.*

Proof. Multiplying the third equation of (2.5) by $\zeta \in C_0^\infty(\Omega;\mathbb{R}^2)$ with $\nabla\cdot\zeta = 0$ and integrating by parts, we have

$$\begin{aligned}
\left|\int_\Omega u_{\varepsilon t}\zeta\right| &= \left|-\int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\zeta + \int_\Omega \Delta u_\varepsilon\cdot\zeta + \int_\Omega \nabla P_\varepsilon\cdot\zeta + \int_\Omega n_\varepsilon\nabla\omega\cdot\zeta\right| \\
&= \left|-\int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\zeta - \int_\Omega \nabla u_\varepsilon : \nabla\zeta + \int_\Omega n_\varepsilon\nabla\omega\cdot\zeta\right| \\
&\le \|u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\zeta\|_{L^\infty(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\zeta\|_{L^2(\Omega)} \\
&\quad + \|n_\varepsilon\|_{L^1(\Omega)}\|\nabla\omega\|_{L^\infty(\Omega)}\|\zeta\|_{L^\infty(\Omega)} \\
&\lesssim (\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|n_\varepsilon\|_{L^1(\Omega)})\|\zeta\|_{W_0^{s,2}(\Omega)}
\end{aligned}$$

for any $s\in\mathbb{N}$ and $s>1$, where we have used $W_0^{s,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ and the Hölder inequality. Combining (3.3), (3.5) with (3.6), and using the Young inequality, we get

$$\|u_{\varepsilon t}\|_{(W_0^{s,2}(\Omega))^*} \lesssim \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 + \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + \|n_\varepsilon\|_{L^1(\Omega)} + 1.$$

Moreover, we have

$$\{u_{\varepsilon t}\}_{\varepsilon\in(0,1)} \text{ is bounded in } L^1((0,T);(W_0^{s,2}(\Omega)\cap L_\sigma^2(\Omega))^*).$$

Finally, by using the Aubin–Lions lemma, we complete the proof. ∎

3.3. Estimate of $\{m_\varepsilon\}_{\varepsilon\in(0,1)}$.

LEMMA 3.5. *Let $(n_\varepsilon,m_\varepsilon,u_\varepsilon)$ be a smooth solution of (2.5). Then*

$$\|m_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|m_0\|_{L^\infty(\Omega)}$$

for all $t>0$.

Proof. It can be directly obtained by using the maximum principle, because n_ε is nonnegative, $m_\varepsilon \ge 0$ and $\nabla m_\varepsilon\cdot\nu = 0$ on $\partial\Omega$. ∎

LEMMA 3.6. *Let $\varepsilon\in(0,1)$, then there exists $C>0$, such that the solution (2.6) satisfies*

$$\int_0^{+\infty}\int_\Omega |\nabla m_\varepsilon|^2 \le C, \quad \int_0^{+\infty}\int_\Omega n_\varepsilon m_\varepsilon \le C. \tag{3.11}$$

Proof. Taking the L^2 scalar product of the second equation of (2.5) with m_ε, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |m_\varepsilon|^2 + \int_\Omega |\nabla m_\varepsilon|^2 = -\int_\Omega n_\varepsilon m_\varepsilon \quad \text{for all } t>0.$$

Since n_ε and m_ε are all nonnegative, by integrating with respect to t, we obtain (3.11). Similarly, by integrating the second equation of (2.5) by parts over Ω, we have

$$\frac{d}{dt}\int_\Omega m_\varepsilon + \int_\Omega u_\varepsilon\cdot\nabla m_\varepsilon = \int_\Omega \Delta m_\varepsilon - \int_\Omega n_\varepsilon m_\varepsilon,$$

since

$$\nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad \Omega \times (0,+\infty)$$

and

$$\nabla n_\varepsilon \cdot \nu = \nabla m_\varepsilon \cdot \nu = 0, u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty).$$

Therefore

$$\frac{d}{dt}\int_\Omega m_\varepsilon = -\int_\Omega n_\varepsilon m_\varepsilon.$$

Integrating the above result with respect to t, we have

$$\int_0^{+\infty}\int_\Omega n_\varepsilon m_\varepsilon \le \int_\Omega m_0$$

for any $\varepsilon \in (0,1)$. This completes the proof. ∎

LEMMA 3.7. *Let $T>0$, then $\{m_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.*

Proof. Let $l \in \mathbb{N}$, then we multiply the second equation of (2.5) by $\zeta \in W_0^{l,2}(\Omega)$ with $l>1$, and integrate by parts, from the Hölder inequality, we have

$$\begin{aligned}\left|\int_\Omega m_{\varepsilon t}(x,t)\zeta\right| &= \left|-\int_\Omega u_\varepsilon \cdot \nabla m_\varepsilon \cdot \zeta + \int_\Omega \Delta m_\varepsilon \cdot \zeta - \int_\Omega n_\varepsilon m_\varepsilon \cdot \zeta\right| \\ &\le \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\|\nabla m_\varepsilon\|_{L^2(\Omega)}\|\zeta\|_{L^\infty(\Omega)} + \|\nabla m_\varepsilon\|_{L^2(\Omega)}\|\nabla\zeta\|_{L^2(\Omega)} \\ &\quad + \|n_\varepsilon m_\varepsilon\|_{L^1(\Omega)}\|\zeta\|_{L^\infty(\Omega)}.\end{aligned}$$

By the embedding inequality $W_0^{l,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality, we get

$$\|m_{\varepsilon t}\|_{(W_0^{l,2}(\Omega))^*} \lesssim \|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla m_\varepsilon\|^2_{L^2(\Omega)} + \|n_\varepsilon m_\varepsilon\|_{L^1(\Omega)} + 1.$$

According to (3.5) and Lemma 3.6, we have

$$\int_0^T \|m_{\varepsilon t}\|_{(W_0^{l,2}(\Omega))^*} \lesssim T+1.$$

Therefore, by the Aubin–Lions lemma and $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow (W_0^{l,2}(\Omega))^*$, we obtain Lemma 3.7. ∎

3.4. Estimate of $\{\ln(n_\varepsilon+1)\}_{\varepsilon\in(0,1)}$.

LEMMA 3.8. *Let $T>0$. Assume that $(n_\varepsilon, m_\varepsilon, u_\varepsilon)$ is a smooth solution of (2.5). Then for any $p>2$, there exists $C=C(T)>0$ such that*

$$\int_0^T\int_\Omega |\nabla \ln(n_\varepsilon+1)|^2 \le C \tag{3.12}$$

and

$$\int_0^T \|\partial_t \ln(n_\varepsilon+1)\|_{(W_0^{p,2}(\Omega))^*} \le C \tag{3.13}$$

for all $\varepsilon\in(0,1)$.

Proof. Letting $\varphi\equiv 1$, $\phi(n_\varepsilon)=\ln(n_\varepsilon+1)$ in Lemma 2.2. First, we have

$$\int_0^T\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2=\int_\Omega\ln(n_\varepsilon(\cdot,T)+1)-\int_\Omega\ln(n_{0\varepsilon}+1)\\+\int_0^T\int_\Omega\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon-\int_0^T\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}\tag{3.14}$$

for all $\varepsilon\in(0,1)$. Due to $\frac{n_\varepsilon}{n_\varepsilon+1}\le 1$ and Young's inequality, we have

$$\int_0^T\int_\Omega\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon\le\int_0^T\int_\Omega\frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla m_\varepsilon\\\le\frac12\int_0^T\int_\Omega\frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}+\frac12\int_0^T\int_\Omega|\nabla m_\varepsilon|^2\tag{3.15}$$

and

$$-\int_0^T\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}\le\int_0^T\int_\Omega|f(n_\varepsilon)|.\tag{3.16}$$

Plugging (3.15), (3.16) into (3.14), thanks to (3.2) and (3.11), we get (3.12). Second, we prove (3.13). By Lemma 2.2 we know that

$$\left|\int_\Omega\partial_t\ln(n_\varepsilon+1)\varphi\right|\le\left|\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2\varphi\right|+\left|\int_\Omega\nabla\ln(n_\varepsilon+1)\cdot\nabla\varphi\right|\\+\left|\int_\Omega\ln(n_\varepsilon+1)u_\varepsilon\cdot\nabla\varphi\right|+\left|\int_\Omega\varphi\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon\right|\\+\left|\int_\Omega\frac{n_\varepsilon}{n_\varepsilon+1}\nabla m_\varepsilon\cdot\nabla\varphi\right|+\left|\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}\varphi\right|,$$

where $\varphi\in C_0^\infty(\Omega)$. According to Hölder's inequality, we get

$$\left|\int_\Omega\partial_t\ln(n_\varepsilon+1)\varphi\right|\le\|\nabla\ln(n_\varepsilon+1)\|_{L^2(\Omega)}^2\|\varphi\|_{L^\infty(\Omega)}+\|\nabla\ln(n_\varepsilon+1)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\+\|\ln(n_\varepsilon+1)\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega)}\\+\|\nabla\ln(n_\varepsilon+1)\|_{L^2(\Omega)}\|\nabla m_\varepsilon\|_{L^2(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\\+\|\nabla m_\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}+\|f(n_\varepsilon)\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\Omega)}.$$

Furthermore, owing to the Sobolev embedding inequality, the Poincaré inequality and Young's inequality, we obtain

$$\|\partial_t\ln(n_\varepsilon+1)\|_{(W_0^{p,2}(\Omega))^*}\lesssim\|\nabla\ln(n_\varepsilon+1)\|_{L^2(\Omega)}^2+\|u_\varepsilon\|_{L^2(\Omega)}^2\\+\|\nabla m_\varepsilon\|_{L^2(\Omega)}^2+\int_\Omega|f(n_\varepsilon)|+1$$

with $p>2$. By integrating the result with respect to t and using (3.12), (3.5), (3.11) and (3.2), we obtain (3.13). ∎

LEMMA 3.9. *Let $T>0$, then $\{\ln(n_\varepsilon+1)\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology, and relatively compact in $L^2((0,T);W^{1,2}(\Omega))$ with respect to the weak topology.*

Proof. According to (3.1), (3.12) and the Poincaré inequality, we can get the weak precompactness property immediately. Meanwhile, thanks to Lemma 3.8 and $W^{1,2}(\Omega)\hookrightarrow\hookrightarrow L^2(\Omega)\hookrightarrow(W_0^{p,2}(\Omega))^*$, we can obtain the above lemma by the Aubin–Lions lemma. ∎

3.5. Estimate of $\{n_\varepsilon m_\varepsilon\}_{\varepsilon\in(0,1)}$. In order to get the strong precompactness of $\{\nabla m_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^2(\Omega\times(0,+\infty))$, we prove the following property of $\{n_\varepsilon m_\varepsilon\}_{\varepsilon\in(0,1)}$.

LEMMA 3.10. *For all $\varepsilon\in(0,1),T>0$, there exists $C>0$ such that*

$$\int_0^T\int_\Omega n_\varepsilon\ln(n_\varepsilon+1)m_\varepsilon\le C. \tag{3.17}$$

Proof. Noting that

$$\begin{aligned}\frac{d}{dt}\int_\Omega m_\varepsilon\ln(n_\varepsilon+1)=&\int_\Omega m_{\varepsilon t}\ln(n_\varepsilon+1)+\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}n_{\varepsilon t}\\ =&-\int_\Omega u_\varepsilon\cdot\nabla m_\varepsilon\ln(n_\varepsilon+1)+\int_\Omega\Delta m_\varepsilon\cdot\ln(n_\varepsilon+1)\\ &-\int_\Omega n_\varepsilon m_\varepsilon\ln(n_\varepsilon+1)-\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}u_\varepsilon\cdot\nabla n_\varepsilon+\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}\Delta n_\varepsilon\\ &-\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}\nabla\cdot(n_\varepsilon\nabla m_\varepsilon)+\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}f(n_\varepsilon)\\ \triangleq&I_1+I_2+I_3+I_4+I_5-\int_\Omega n_\varepsilon m_\varepsilon\ln(n_\varepsilon+1)+\int_\Omega\frac{m_\varepsilon}{n_\varepsilon+1}f(n_\varepsilon),\end{aligned}$$

for all $t>0$. According to the no-flux boundary conditions for n_ε and m_ε, and the homogeneous Dirichlet boundary conditions for u_ε, a direct calculation gives

$$I_1=\int_\Omega\nabla\cdot(u_\varepsilon\ln(n_\varepsilon+1))\cdot m_\varepsilon=\int_\Omega m_\varepsilon u_\varepsilon\cdot\nabla\ln(n_\varepsilon+1),$$

$$I_2=-\int_\Omega\nabla m_\varepsilon\cdot\nabla\ln(n_\varepsilon+1),$$

$$I_3=-\int_\Omega m_\varepsilon u_\varepsilon\cdot\nabla(\ln(n_\varepsilon+1))=-I_1,$$

$$\begin{aligned}I_4=&-\int_\Omega\nabla\left(\frac{m_\varepsilon}{n_\varepsilon+1}\right)\cdot\nabla n_\varepsilon\\ =&-\int_\Omega\frac{\nabla n_\varepsilon}{n_\varepsilon+1}\cdot\nabla m_\varepsilon+\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2m_\varepsilon,\end{aligned}$$

and

$$\begin{aligned}I_5=&\int_\Omega n_\varepsilon\nabla m_\varepsilon\cdot\nabla\left(\frac{m_\varepsilon}{n_\varepsilon+1}\right)\\ =&\int_\Omega\frac{n_\varepsilon}{n_\varepsilon+1}|\nabla m_\varepsilon|^2-\int_\Omega n_\varepsilon m_\varepsilon\nabla m_\varepsilon\cdot\frac{\nabla n_\varepsilon}{(n_\varepsilon+1)^2}\end{aligned}$$

$$=\int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1}|\nabla m_\varepsilon|^2-\int_\Omega \frac{m_\varepsilon n_\varepsilon}{n_\varepsilon+1}\nabla \ln(n_\varepsilon+1)\cdot\nabla m_\varepsilon.$$

Thus

$$\begin{aligned}&\frac{d}{dt}\int_\Omega m_\varepsilon \ln(n_\varepsilon+1)+\int_\Omega n_\varepsilon m_\varepsilon \ln(n_\varepsilon+1)\\ =&-2\int_\Omega \nabla m_\varepsilon\cdot\nabla\ln(n_\varepsilon+1)+\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2 m_\varepsilon\\ &+\int_\Omega\frac{n_\varepsilon}{n_\varepsilon+1}|\nabla m_\varepsilon|^2+\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}m_\varepsilon-\int_\Omega\frac{m_\varepsilon n_\varepsilon}{n_\varepsilon+1}\nabla\ln(n_\varepsilon+1)\cdot\nabla m_\varepsilon.\end{aligned}\tag{3.18}$$

Integrating (3.18) with respect to t, we get

$$\begin{aligned}&\int_0^T\int_\Omega n_\varepsilon m_\varepsilon\ln(n_\varepsilon+1)+\int_\Omega m_\varepsilon(\cdot,T)\ln(n_\varepsilon(\cdot,T)+1)\\ =&\int_\Omega m_0\ln(n_{0\varepsilon}+1)-2\int_0^T\int_\Omega\nabla m_\varepsilon\cdot\nabla\ln(n_\varepsilon+1)\\ &+\int_0^T\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2m_\varepsilon+\int_0^T\int_\Omega\frac{n_\varepsilon}{n_\varepsilon+1}|\nabla m_\varepsilon|^2\\ &-\int_0^T\int_\Omega\frac{m_\varepsilon n_\varepsilon}{n_\varepsilon+1}\nabla\ln(n_\varepsilon+1)\cdot\nabla m_\varepsilon+\int_0^T\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}m_\varepsilon\\ \triangleq&\int_\Omega m_0\ln(n_{0\varepsilon}+1)+J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

By Young's inequality and the Hölder inequality, we have

$$J_1\le\int_0^T\int_\Omega|\nabla m_\varepsilon|^2+\int_0^T\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2,$$

$$J_2\le\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2.$$

Due to $0<\frac{n_\varepsilon}{n_\varepsilon+1}<1$, we get

$$J_3\le\int_0^T\int_\Omega|\nabla m_\varepsilon|^2,$$

$$\begin{aligned}J_4&\le\int_0^T\int_\Omega m_\varepsilon\nabla\ln(n_\varepsilon+1)\cdot\nabla m_\varepsilon\\ &\le\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega\nabla\ln(n_\varepsilon+1)\cdot\nabla m_\varepsilon\\ &\le\frac12\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2+\frac12\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2,\end{aligned}$$

and

$$J_5\le\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|f(n_\varepsilon)|.$$

Thus

$$\begin{aligned}&\int_0^T\int_\Omega n_\varepsilon m_\varepsilon \ln(n_\varepsilon+1)+\int_\Omega m_\varepsilon(\cdot,T)\ln(n_\varepsilon(\cdot,T)+1)\\ \lesssim &\int_0^T\int_\Omega |\nabla m_\varepsilon|^2+\int_0^T\int_\Omega|\nabla \ln(n_\varepsilon+1)|^2+\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega |\nabla\ln(n_\varepsilon+1)|^2\\ &+\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2+\|m_\varepsilon\|_{L^\infty}\int_0^T\int_\Omega|f(n_\varepsilon)|.\end{aligned}$$

According to Lemma 3.1, Lemma 3.5, Lemma 3.6 and Lemma 3.8, we obtain (3.17). This completes the proof. ☐

4. Proof of Theorem 1.1

First, we introduce two useful lemmas.

Lemma 4.1 ([26]). *Let $n\geq 1$ and $A\subseteq\mathbb{R}^n$ be measurable, and suppose that $\{c_j\}_{j\in\mathbb{N}}\subset L^1(A)$ is such that $c_j\geq 0$ a.e. in A for all $j\in\mathbb{N}$ and*

$$c_j\rightharpoonup c \text{ in } L^1(A)\quad \text{and}\, c_j\to c \text{ a.e. in } A$$

with some $c\in L^1(A)$ as $j\to\infty$. Then

$$c_j\to c \text{ in } L^1(A)\quad \text{as}\, j\to+\infty.$$

Lemma 4.2 ([26]). *Let $n\geq 1$ and $A\subseteq\mathbb{R}^n$ be measurable, and suppose that $\{c_j\}_{j\in\mathbb{N}}\subset L^\infty(A)$ and $\{k_j\}_{j\in\mathbb{N}}\subset L^2(A)$ satisfy*

$$\begin{aligned}&|c_j|\leq C \text{ in } A\quad \text{for all}\, j\in\mathbb{N},\\ &c_j\to c \text{ a.e. in } A,\\ &k_j\rightharpoonup k\quad \text{in}\, L^2(A),\end{aligned}$$

as $j\to+\infty$ for some $C>0$, $c\in L^\infty(A)$ and $k\in L^2(A)$. Then

$$c_jk_j\rightharpoonup ck\quad \text{in}\, L^2(A)\quad \text{as}\, j\to+\infty.$$

Second, we have the convergence of solutions of (2.5) as follows.

Lemma 4.3. *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset(0,1)$ and*

$$\begin{cases} n\in L^1_{loc}(\bar\Omega\times[0,+\infty))\\ m\in L^2_{loc}\big([0,+\infty);W^{1,2}(\Omega)\big)\\ u\in L^2_{loc}\Big([0,+\infty);W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^2)\Big)\end{cases}$$

with $n,m\geq 0$, such that $\varepsilon=\varepsilon_j\searrow 0$ as $j\to+\infty$ and

$$n_\varepsilon\to n \text{ in } L^1_{loc}\,(\bar\Omega\times[0,+\infty)) \text{ and a.e. in } \Omega\times(0,+\infty), \tag{4.1}$$

$$\ln(n_\varepsilon+1)\to\ln(n+1) \text{ in } L^2_{loc}\,(\bar\Omega\times[0,+\infty)), \tag{4.2}$$

$$\nabla\ln(n_\varepsilon+1)\rightharpoonup\nabla\ln(n+1) \text{ in } L^2_{loc}\,(\bar\Omega\times[0,+\infty)), \tag{4.3}$$

$$m_\varepsilon\to m \text{ in } L^2_{loc}\,(\bar\Omega\times[0,+\infty)) \text{ and a.e. in } \Omega\times(0,+\infty), \tag{4.4}$$

$$\nabla m_\varepsilon \rightharpoonup \nabla m \text{ in } L^2_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.5}$$

$$m_\varepsilon \overset{\star}{\rightharpoonup} m \text{ in } L^\infty_{loc}(\bar\Omega\times[0,+\infty)) \text{ and a.e. in } \Omega\times(0,+\infty), \tag{4.6}$$

$$u_\varepsilon \to u \text{ in } L^2_{loc}(\bar\Omega\times[0,+\infty)) \text{ and a.e. in } \Omega\times(0,+\infty), \tag{4.7}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.8}$$

$$u_\varepsilon\otimes u_\varepsilon \to u\otimes u \text{ in } L^1_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.9}$$

$$n_\varepsilon m_\varepsilon \to nm \text{ in } L^1_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.10}$$

$$u_\varepsilon m_\varepsilon \to um \text{ in } L^1_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.11}$$

$$\frac{f(n_\varepsilon)}{n_\varepsilon+1} \to \frac{f(n)}{n+1} \text{ in } L^1_{loc}(\bar\Omega\times[0,+\infty)). \tag{4.12}$$

Proof. According to Lemma 3.7, Lemma 3.6, and Lemma 3.5, we can directly get (4.4), (4.5) and (4.6), respectively. Furthermore, (4.2) and (4.3) can be obtained by Lemma 3.9. Thus

$$n_\varepsilon \to n \quad \text{a.e. in } \Omega\times(0,+\infty). \tag{4.13}$$

Thanks to the Vitali convergence theorem, combining (4.13) with Lemma 3.2, we get $n_\varepsilon \to n$ in $L^1_{\text{loc}}(\bar\Omega\times[0,+\infty))$. Similarly, due to the compactness of u_ε, Lemma 3.4 shows that (4.7) and (4.8) hold in further subsequences. (4.13) and (4.4) ensure that $n_\varepsilon m_\varepsilon \to nm$ a.e. in $\Omega\times(0,+\infty)$, as $\varepsilon=\varepsilon_j \searrow 0$. Next, according to (4.1) and (4.6), we can prove that

$$n_\varepsilon m_\varepsilon \rightharpoonup nm \text{ in } L^1_{\text{loc}}(\bar\Omega\times[0,+\infty)),$$

in the sense of subsequence (we do not use new notation). Combining the above results, we can directly obtain (4.10) from Lemma 4.1. Moreover, we can get

$$\frac{f(n_\varepsilon)}{n_\varepsilon+1} \to \frac{f(n)}{n+1} \quad \text{a.e. in } \Omega\times(0,+\infty). \tag{4.14}$$

In view of the uniform integrability of $\frac{f(n_\varepsilon)}{n_\varepsilon+1}$ and (4.14), we can conclude that (4.12) holds by the Vitali convergence theorem, the averment (4.9) and (4.11) can be obtained by (4.4) and (4.7). ∎

Now, assuming that φ and ζ satisfy the properties listed in Definition 1.1. Multiplying the second and the third equation of (2.5) by φ,ζ respectively, we have

$$\int_0^{+\infty}\int_\Omega m_\varepsilon\varphi_t + \int_\Omega m_0\varphi(\cdot,0) = \int_0^{+\infty}\int_\Omega \nabla m_\varepsilon\cdot\nabla\varphi + \int_0^{+\infty}\int_\Omega n_\varepsilon m_\varepsilon\varphi - \int_0^{+\infty}\int_\Omega m_\varepsilon u_\varepsilon\cdot\nabla\varphi \tag{4.15}$$

and

$$\int_0^{+\infty}\int_\Omega u_\varepsilon\zeta_t + \int_\Omega u_{0\varepsilon}\zeta(\cdot,0) = -\int_0^{+\infty}\int_\Omega (u_\varepsilon\otimes u_\varepsilon):\nabla\zeta - \int_0^{+\infty}\int_\Omega n_\varepsilon\nabla\omega\cdot\zeta$$

$$+\int_0^{+\infty}\int_\Omega \nabla u_\varepsilon\cdot\nabla\zeta. \tag{4.16}$$

According to (4.4), (4.5), (4.10) and (4.11), we can get (1.5) by taking $\varepsilon=\varepsilon_j\searrow 0$ in each integral respectively. Similarly, we can obtain (1.6). Furthermore, owing to (3.1), (4.1) and Fatou's lemma, we have

$$\int_\Omega n(\cdot,t)\le \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_\Omega n_\varepsilon(\cdot,t)=\int_\Omega n_0+\int_0^t\int_\Omega f(n). \tag{4.17}$$

We also have the following L^2 strong compactness of ∇m_ε (see [12, Lemma 4.2]).

LEMMA 4.4. *Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be as provided by Lemma 4.3. Then there exists a subsequence, such that for a.e. $T>0$,*

$$\nabla m_\varepsilon\to\nabla m \ \text{in}\ L^2(\bar\Omega\times(0,T)),$$

as $\varepsilon=\varepsilon_j\searrow 0$.

Proof. According to (4.5), we get

$$\int_0^T\int_\Omega|\nabla m|^2\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2$$

for all $T>0$. Next, we only need to prove

$$\int_0^T\int_\Omega|\nabla m|^2\ge\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2.$$

In fact, multiplying $(2.5)_2$ by m_ε, we get

$$\int_0^T\int_\Omega|\nabla m_\varepsilon|^2=-\int_0^T\int_\Omega n_\varepsilon m_\varepsilon^2+\frac12\int_\Omega m_0^2-\frac12\int_\Omega m_\varepsilon^2(\cdot,T),$$

Thanks to (4.6) and (4.10), we have

$$\int_0^T\int_\Omega n_\varepsilon m_\varepsilon^2\to\int_0^T\int_\Omega nm^2. \tag{4.18}$$

In fact,

$$\left|\int_0^T\int_\Omega n_\varepsilon m_\varepsilon^2-nm^2\right|\le\left|\int_0^T\int_\Omega (n_\varepsilon m_\varepsilon-nm)m_\varepsilon\right|+\left|\int_0^T\int_\Omega nm(m_\varepsilon-m)\right|\to 0,$$

as $\varepsilon=\varepsilon_j\searrow 0$.
Similarly, according to (4.4), we have

$$\int_\Omega m_\varepsilon^2(\cdot,T)\to\int_\Omega m^2(\cdot,T)\quad\text{for all } T\in(0,+\infty)\backslash M, \tag{4.19}$$

with some null set $M\subset(0,+\infty)$. Therefore, combining with (4.18) and (4.19) gives

$$\lim_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2=-\int_0^T\int_\Omega nm^2+\frac12\int_\Omega m_0^2-\frac12\int_\Omega m^2(\cdot,T)$$

for all $T\in(0,+\infty)\backslash M$, since

$$\nabla\cdot u_\varepsilon=0 \ \text{ in } \Omega\times(0,+\infty) \ \text{ and } \ u_\varepsilon=0 \ \text{ on } \partial\Omega\times(0,+\infty),$$

along with a similar method [26, Lemma 8.1], we get

$$-\int_0^T\int_\Omega nm^2+\frac{1}{2}\int_\Omega m_0^2-\frac{1}{2}\int_\Omega m^2(\cdot,T)\leqslant\int_0^T\int_\Omega|\nabla m|^2$$

with a null set $M_1\subset(0,+\infty)$, for all $T\in(0,+\infty)\backslash M_1$, i.e.,

$$\int_0^T\int_\Omega|\nabla m|^2\geq\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_\Omega|\nabla m_\varepsilon|^2.$$

□

LEMMA 4.5. *Let (n,m,u) be given as in Lemma 4.3 and Lemma 4.4, then there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset(0,1)$ such that*

$$(n_\varepsilon+1)^{-2}n_\varepsilon\nabla n_\varepsilon\cdot\nabla m_\varepsilon\rightharpoonup(n+1)^{-2}n\nabla n\cdot\nabla m \quad \text{in } L^1_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.20}$$

$$(n_\varepsilon+1)^{-1}n_\varepsilon\nabla m_\varepsilon\to(n+1)^{-1}n\nabla m \quad \text{in } L^2_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.21}$$

$$\ln(n_\varepsilon+1)u_\varepsilon\to\ln(n+1)u \quad \text{in } L^1_{loc}(\bar\Omega\times[0,+\infty)), \tag{4.22}$$

as $\varepsilon=\varepsilon_j\searrow 0$.

Proof. Since $|n_\varepsilon(n_\varepsilon+1)^{-1}|\leq 1$, by (4.1), we can infer that $n_\varepsilon(n_\varepsilon+1)^{-1}\to n(n+1)^{-1}$ a.e. in $\Omega\times(0,+\infty)$. According to Lemma 4.2 and Lemma 4.4, we can directly get (4.21). Due to

$$(n_\varepsilon+1)^{-2}n_\varepsilon\nabla n_\varepsilon\cdot\nabla m_\varepsilon=\nabla\ln(n_\varepsilon+1)\cdot(n_\varepsilon+1)^{-1}n_\varepsilon\nabla m_\varepsilon,$$

the convergence (4.20) is an immediate result of (4.3) and (4.21). Similarly, we can get (4.22) by (4.2) and (4.7). □

Finally, we prove Theorem 1.1 based on the above lemmas.

Proof. **(Proof of Theorem 1.1.)** Let (n,m,u) be as constructed in Lemma 4.3. Then (n,m,u) is a global generalized solution of (1.3) in the sense of Definition 1.1. In fact, let

$$\phi(n_\varepsilon)=\ln(n_\varepsilon+1)$$

in Lemma 2.2, for any $\varphi\in C_0^\infty(\bar\Omega\times[0,+\infty))$, we take $T>0$ such that $\operatorname{supp}\varphi\subset\Omega\times[0,T]$. Integrating over $t\in(0,+\infty)$, we get

$$\begin{aligned}
&-\int_0^{+\infty}\int_\Omega\ln(n_\varepsilon+1)\varphi_t-\int_\Omega\ln(n_{0\varepsilon}+1)\varphi(\cdot,0)\\
&=\int_0^{+\infty}\int_\Omega|\nabla\ln(n_\varepsilon+1)|^2\varphi-\int_0^{+\infty}\int_\Omega\frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla\varphi+\int_0^{+\infty}\int_\Omega\ln(n_\varepsilon+1)u_\varepsilon\cdot\nabla\varphi\\
&\quad-\int_0^{+\infty}\int_\Omega\varphi\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon+\int_0^{+\infty}\int_\Omega\frac{n_\varepsilon}{n_\varepsilon+1}\nabla m_\varepsilon\cdot\nabla\varphi+\int_0^{+\infty}\int_\Omega\frac{f(n_\varepsilon)}{n_\varepsilon+1}\varphi
\end{aligned}$$

for all $\varepsilon\in(0,1)$. Taking the advantage of (4.2), (4.3), (4.21) and (4.22), we obtain

$$-\int_0^{+\infty}\int_\Omega \ln(n_\varepsilon+1)\varphi_t\to-\int_0^{+\infty}\int_\Omega \ln(n+1)\varphi_t, \tag{4.23}$$

$$-\int_0^{+\infty}\int_\Omega \frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla\varphi\to-\int_0^{+\infty}\int_\Omega \nabla\ln(n+1)\cdot\nabla\varphi, \tag{4.24}$$

$$\int_0^{+\infty}\int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1}\nabla m_\varepsilon\cdot\nabla\varphi\to\int_0^{+\infty}\int_\Omega \frac{n}{n+1}\nabla m\cdot\nabla\varphi \tag{4.25}$$

and

$$\int_0^{+\infty}\int_\Omega \ln(n_\varepsilon+1)u_\varepsilon\cdot\nabla\varphi\to\int_0^{+\infty}\int_\Omega \ln(n+1)u\cdot\nabla\varphi \tag{4.26}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Similarly, using (4.20) and (4.12), we get

$$-\int_0^{+\infty}\int_\Omega \varphi\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon\to-\int_0^{+\infty}\int_\Omega \varphi\frac{n}{(n+1)^2}\nabla n\cdot\nabla m, \tag{4.27}$$

$$\int_0^{+\infty}\int_\Omega \frac{f(n_\varepsilon)}{n_\varepsilon+1}\varphi\to\int_0^{+\infty}\int_\Omega \frac{f(n)}{n+1}\varphi \tag{4.28}$$

as $\varepsilon=\varepsilon_j\searrow 0$. According to a lower semicontinuity argument, we have

$$\int_0^{+\infty}\int_\Omega |\nabla\ln(n+1)|^2\varphi\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^{+\infty}\int_\Omega |\nabla\ln(n_\varepsilon+1)|^2\varphi. \tag{4.29}$$

Due to the nonnegativity of φ, it follows

$$\begin{aligned}\int_0^{+\infty}\int_\Omega |\nabla\ln(n+1)|^2\varphi\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\Big\{&-\int_0^{+\infty}\int_\Omega \ln(n_\varepsilon+1)\varphi_t-\int_\Omega \ln(n_{0\varepsilon}+1)\varphi(\cdot,0)\\ &+\int_0^{+\infty}\int_\Omega \frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla\varphi-\int_0^{+\infty}\int_\Omega \ln(n_\varepsilon+1)u_\varepsilon\cdot\nabla\varphi\\ &+\int_0^{+\infty}\int_\Omega \varphi\frac{n_\varepsilon}{(n_\varepsilon+1)^2}\nabla n_\varepsilon\cdot\nabla m_\varepsilon-\int_0^{+\infty}\int_\Omega \frac{n_\varepsilon}{n_\varepsilon+1}\nabla m_\varepsilon\cdot\nabla\varphi\\ &-\int_0^{+\infty}\int_\Omega \frac{f(n_\varepsilon)}{n_\varepsilon+1}\varphi\Big\}.\end{aligned} \tag{4.30}$$

Plugging (4.23)–(4.29) into (4.30), we conclude that

$$\begin{aligned}\int_0^{+\infty}\int_\Omega |\nabla\ln(n+1)|^2\varphi\le&-\int_0^{+\infty}\int_\Omega \ln(n+1)\varphi_t-\int_\Omega \ln(n_0+1)\varphi(\cdot,0)\\ &+\int_0^{+\infty}\int_\Omega \frac{1}{n+1}\nabla n\cdot\nabla\varphi-\int_0^{+\infty}\int_\Omega \ln(n+1)u\cdot\nabla\varphi\\ &+\int_0^{+\infty}\int_\Omega \frac{n}{(n+1)^2}\varphi\nabla n\cdot\nabla m-\int_0^{+\infty}\int_\Omega \frac{n}{n+1}\nabla m\cdot\nabla\varphi\\ &-\int_0^{+\infty}\int_\Omega \frac{f(n)}{n+1}\varphi,\end{aligned}$$

for any function φ, we deduce that n satisfies the condition required in Definition 1.1. Combining (4.15), (4.16) with (4.17), we complete the proof of Theorem 1.1. ∎

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