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BOUNDEDNESS IN A THREE-DIMENSIONAL CHEMOTAXIS-STOKES SYSTEM INVOLVING A SUBCRITICAL SENSITIVITY AND INDIRECT SIGNAL PRODUCTION*

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Abstract. This paper is concerned with the Keller-Segel-Stokes system

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\nabla\cdot(D(n)n)-\nabla\cdot(S(n)\nabla v), \\ v_t+\mathbf{u}\cdot\nabla v=\Delta v-v+w, \\ w_t+\mathbf{u}\cdot\nabla w=\Delta w-w+n, \\ \mathbf{u}_t=\Delta\mathbf{u}+\nabla P+n\nabla\phi, \quad \nabla\cdot u=0, \end{cases} \qquad (*)$$

under no-flux/no-flux/no-flux/Dirichlet boundary conditions in smoothly bounded domains $\Omega\subset\mathbb{R}^3$, with given suitably regular functions D, S and ϕ.

Under the assumption that there exist $m_0\in\mathbb{R}$, $m\geq m_0$, $k_D>0$ and $K_D>0$ such that

$$k_D s^{m_0-1}\leq D(s)\leq K_D s^{m-1} \quad \text{for all } s>1,$$

and that $S(0)=0$ as well as

$$\frac{|S(s)|}{D(s)}\leq K_0 s^{\alpha} \quad \text{for all } s>1$$

with $K_0>0$, it is shown that for all suitably regular initial data an associated initial-boundary value problem $(*)$ possesses a globally defined bounded classical solution provided $\alpha<\frac{8}{9}$. We underline that the same results were established for the corresponding system with direct signal production in a well-known result for $\alpha<\frac{2}{3}$ in [X. Cao, Z. Angew. Math. Phys., 71:61, 2020] and [M. Winkler, Appl. Math. Lett., 112:106785, 2021]. Our result rigorously confirms that the indirect signal production mechanism genuinely contributes to the global solvability of the three-dimensional Keller-Segel-Stokes system.

Keywords. Keller-Segel-Stokes; Blow-up prevention; Indirect signal production.

AMS subject classifications. 92C17; 35K65; 35Q92.

1. Introduction

Processes of directed movement of cells in response to a chemical stimulus, referred to as chemotaxis, plays an essential role in the interaction of cells with their environment in various biological processes such as embryonic development, wound healing, disease progression, the location of food sources, avoidance of predators, attracting mates, slime mold aggregation, tumour angiogenesis, and primitive steak formation [16]. The pioneering works of the chemotaxis model was introduced by Keller and Segel in [19], describing the aggregation of cellular slime mold toward a higher concentration of a chemical signal, which reads

$$\begin{cases} n_t=\Delta n-\chi\nabla\cdot(n\nabla c), & x\in\Omega, \quad t>0, \\ c_t=\Delta c-c+n, & x\in\Omega, \quad t>0. \end{cases} \tag{1.1}$$

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The mathematical analysis of (1.1) and the variants thereof mainly concentrates on the boundedness and blow-up of the solutions (refer to e.g. [13, 22, 40, 43] and the references therein).

However, from a physical point of view, the equation modeling the migration of cells should rather be regarded as nonlinear diffusion [28], especially the slow diffusion with finite propagation property, and other reason is the bacterial cells have positive sizes which are not negligible, a so-called volume-filling effect is introduced by Hillen and Painter in the Keller-Segel model [14]. The associated system accounting for this is then quasilinear and involves more general functions $D(\cdot)$ and $S(\cdot)$,

$$\begin{cases} n_t = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(S(n)\nabla c), & x\in\Omega, \quad t>0, \\ c_t = \Delta c - c + n, & x\in\Omega, \quad t>0, \end{cases} \tag{1.2}$$

where D and S fulfill

$$D\in C^2([0,\infty)),\ D>0 \text{ and } S\in C^2([0,\infty)) \text{ with } S(0)=0, \tag{1.3}$$

as well as some $m_0\in\mathbb{R}$, $m\geq m_0$, $k_D>0$ and $K_D>0$ such that

$$k_D s^{m_0-1}\leq D(s)\leq K_D s^{m-1} \quad \text{for all } s>1. \tag{1.4}$$

If for all $s>1$, we have

$$\frac{S(s)}{D(s)}\leq K_{DS}s^{\alpha} \quad \text{with some } K_{DS}>0 \text{ and } \alpha<\frac{2}{N}, \tag{1.5}$$

then (1.2) possesses a bounded solution if the initial data are sufficiently regular [29]. Whereas, if (1.5) is replaced by

$$\frac{S(s)}{D(s)}\geq \widetilde{K}_{DS}s^{\alpha} \quad \text{with some } \widetilde{K}_{DS}>0 \text{ and } \alpha>\frac{2}{N}, \tag{1.6}$$

provided some technical conditions are satisfied, the authors in [3–5, 41] prove that the solution blows up at finite time. That is, when $N\geq 2$, $\Omega\subset\mathbb{R}^N$ be a ball, Winkler [41] proved that if $\frac{D(u)}{S(u)}$ grows faster than $u^{\frac{2}{N}}$ as $u\to\infty$ and some further technical conditions are fulfilled, then there exist solutions that blow up in either finite or infinite time. When $N\geq 3$, $D,S\in C^2([0,\infty))$ and there is $\beta\in C^2([0,\infty))$ such that $D(s)>0$, $S(s)=s\beta(s)$ and $\beta(s)\geq c_0>0$ for all $s\geq 0$, if $G(s)\leq \alpha s^{2-\alpha}$, $s\geq s_0$ with some $\alpha>\frac{2}{N}$ as well as $H(s)\leq \gamma G(s)+b(s+1)$, $s>0$ with some $\gamma\in(0,\frac{N-2}{N})$, where $G(s)=\int_{s_0}^{s}\int_{s_0}^{\sigma}\frac{D(\eta)}{S(\eta)}d\eta d\sigma$ and $H(s)=\int_0^s \frac{\sigma D(\sigma)}{S(\sigma)}d\sigma$, Ciéslak and Stinner [3] showed that for any initial data satisfying appropriate condition, the corresponding solution of (1.2) blows up at the finite time. Furthermore, Ciéslak and Stinner [4] found critical exponents on the growth of S distinguishing between the possibility of finite-time blowup and the lack of it when D and S satisfy the supercritical relation. For more related results, we refer to previous studies e.g. [9, 15, 17, 29] and the references therein.

Models (1.1) and (1.2) assume that there is no interplay between cells/chemicals and their ambient surroundings. However, some experimental observations have shown that the motion of cells also can be substantially influenced by the surrounding fluid [33]. Tuval et al. [33] proposed a chemotaxis-fluid model by considering the bacteria-induced motion of fluid through buoyant forces and the fluid-driven transport of bacteria and signal where the signal is consumed. Since then, considerable efforts have been made in addressing the global existence, boundedness and asymptotic behavior of solutions to the associated initial-boundary value problem of models (refer to e.g. [30,42,44–47] and the references therein). As in the classical Keller-Segel model, where the chemoattractant is produced rather than consumed by bacteria, the relevant Keller-Segel-Navier-Stokes system with rotational effect of the form is given by:

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\nabla\cdot(D(n)n)-\nabla\cdot(nS(x,n,v)\cdot\nabla v), & x\in\Omega,t>0,\\ c_t+\mathbf{u}\cdot\nabla c=\Delta c-c+n, & x\in\Omega,t>0,\\ \mathbf{u}_t+\kappa(\mathbf{u}\cdot\nabla)\mathbf{u}=\Delta\mathbf{u}+\nabla P+n\nabla\phi,\ \nabla\cdot u=0, & x\in\Omega,t>0. \end{cases} \tag{1.7}$$

When $D(n)\equiv 1$ and $S(x,n,v)$ is a tensor-valued sensitivity satisfying some dampening condition, such as $S(x,n,v)\le C_S(1+n)^{-\alpha}$, Wang and Xiang [37] proved that the Keller-Segel-Stokes system (1.7) with $\kappa=0$ possesses a global boundedness solution in a two-dimensional smoothly bounded domain. To the best of our knowledge, this is the first result on global existence and boundedness in a Keller-Segel-Stokes system with tensor-valued sensitivity. Wang and Xiang [38] further showed that if $\alpha>\frac{1}{2}$, the Keller-Segel-Stokes system (1.7) with $\kappa=0$ also admits a global classical solution which is uniformly bounded in three-dimensional smoothly bounded domain. Parallel to the case of the corresponding Keller-Segel-Navier-Stokes system, Wang [34] proved that the system (1.7) possesses at least one global very weak solution if $\alpha>\frac{1}{3}$ in three-dimensional smoothly bounded domains. More recently, when $S(x,n,v)\equiv 1$ and $\kappa=1$, Winkler [50] showed that if $\|n_0\|_{L^1(\Omega)}<2\pi$, the system (1.7) admits a globally defined generalized solution; in particular, this hypothesis is fully explicit and independent of the initial size of further solution components. Moreover, the obtained solution is seen to enjoy a certain temporally averaged boundedness property which, inter alia, rules out any finite-time collapse into persistent Dirac-type measures, as well as convergence to such singular profiles in the large-time limit. When $D(n)$ is replaced by Δn^m $(m\ge 1)$, $S(x,n,v)\equiv 1$, Black [1] proved that if $m>\frac{4}{3}$, the system (1.7) possesses at least one global very weak solution. Moreover, if $m>\frac{5}{3}$, the system (1.7) admits at least one global weak solution. When the system (1.7) has a logistic source $rn-\mu n^2$, Tao and Winkler [31] showed that the corresponding initial-boundary problem possesses a global classical solution which is bounded in three-dimensional smoothly bounded domains under the explicit condition $\mu\ge 23$. In two-dimensional smoothly bounded domains, Tao and Winkler [32] proved that the Keller-Segel-Navier-Stokes possesses a global classical bounded solution for each $\mu>0$. Liu et al. [21] showed that if $m\ge\frac{1}{3}$ and $\alpha>\frac{6}{5}-m$, the corresponding initail-boundary problem possesses at least one global bounded weak solution for the Keller-Segel-Stokes system with nonlinear diffusion and logistic source in the three-dimensional bounded domains. Jin [18] improved the results in [21], and established the global existence and boundedness of weak solutions for any $m>0$ and $\alpha>0$. For more related results, we refer to previous studies e.g. [6,20,23,35,36,48,49,55] and the references therein.

The chemotactic signal is produced directly by cells in the classical Keller-Segel system, yet the signal generation undergoes intermediate stages in some realistic biological processes [27]. In recent years, much attention has been focused on the following Keller-Segel system with indirect signal production

$$\begin{cases} u_t=\Delta u-\chi\nabla\cdot(u\nabla v), & x\in\Omega,t>0,\\ \tau v_t=\Delta v-v+w, & x\in\Omega,t>0,\\ \tau w_t=\Delta w-w+u, & x\in\Omega,t>0. \end{cases} \tag{1.8}$$

For the system (1.8), when $f \equiv 0$, Fujie and Senba [11] proved that for all reasonable initial data the solution of the system (1.8) in the case $N \leq 3$ is global in the classical sense. In the case $N = 4$, they construct a Lyapunov functional of the system (1.8) and use the Adams-type inequality to derive that if there exists a constant $64\pi^2$ such that $\|u_0\|_{L^1(\Omega)} < \frac{64\pi^2}{\chi}$ is radially symmetric, then the solution to the Neumann boundary value problem of system (1.8) exists globally in time and remains bounded. When the system (1.8) has a logistic source $\mu(u-u^\alpha)$, Zhang et al. [57] showed that if $\alpha > \frac{N}{4} + \frac{1}{2}$, the system (1.8) possesses a global bounded classical solution. Moreover, if $\mu > 0$ is sufficiently large, the global bounded solution (u,v,w) converges to (1,1,1). When the system (1.8) with rotational sensitivity, that is, $\chi \nabla \cdot (u \nabla v)$ is replaced by $\nabla \cdot (S(x,u,v,w)\nabla v)$, and the rotational sensitivity function $S = (S_{ij})_{i,j \in \{1,2,\ldots,N\}}$ satisfies for all $i,j \in \{1,2,\ldots,N\}$, $S_{ij} \in C^2(\overline{\Omega} \times [0,\infty))$, and that $|S_{ij}(x,u,v,w)| \leq Ku$ with constant $K > 0$, $f(0) \geq 0$ and $f(s) \leq k - \mu s^\alpha$, Dong and Peng [8] proved that if $\alpha > \frac{N}{4} + \frac{1}{2}$, the system (1.8) possesses a unique global bounded classical solution. This implies that the rotational flux in indirect signal production mechanism maintains the regularity of the system. For the nonlinear diffusion case, when $\Delta u - \chi \nabla \cdot (u \nabla v) + f(u)$ is replaced by $\nabla \cdot (D(u)\nabla u - S(u)\nabla v)$ and $D(u) \geq a_0(u+1)^{-\alpha}$, $0 \leq S(u) \leq b_0(u+1)^\beta$ with $a_0, b_0 > 0$, $\alpha, \beta \in \mathbb{R}$, Ding and Wang [7] proved that if $\alpha + \beta < \min\{1 + \frac{2}{N}, \frac{4}{N}\}$, the system (1.8) possesses a globally bounded and classical solution.

As is observed above, on one hand, the mathematical studies on the fluid-free Keller-Segel system (1.2) indicated that subcritical sensitivity is sufficient to suppress any unboundedness phenomenon. To be more precise, Tao and Winkler [29] proved that the system (1.2) admits global bounded classical solutions for all suitably regular but arbitrarily large initial data whenever $\alpha < \alpha_*(N) := \frac{2}{N}$. When the Keller-Segel(-Navier)-Stokes system with subcritical sensitivity of the form in (1.6), that is, $nS(x,n,v)$ is replaced by $S(n)$, Winkler [51] showed the global existence of bounded classical solutions to (1.7) for widely arbitrary initial data actually within the entire range $\alpha < \frac{2}{3}$ in 3D smoothly bounded domain with $\kappa = 0$. As D and S satisfy (1.3)-(1.6) with $\frac{S(s)}{D(s)} \leq K_{DS} s^{1-\alpha}$, Cao [2] proved that if $\alpha > \frac{1}{3}$ and either $m > \frac{1}{3}$ or $m \leq \frac{1}{3}$ and $m + 4\alpha > 1$, the Keller-Segel-Stokes system (1.7) admits a global bounded classical solution. On the other hand, we noticed that Yu [56] investigated a 2D Keller-Segel-Stokes system with indirect signal production, and showed that suitable saturation of chemotactic sensitivity can prevent the blow-up of solution. Following [56], some recent results rigorously revealed that the mechanism of indirect signal production is conductive to the global solvability of two-dimensional Keller-Segel(-Navier)-Stokes system and 3D Keller-Segel-Stokes system. Wang and Yang [39] claimed that global boundedness of classical solution can be derived for the 2D Keller-Segel-Stokes system without any saturation effect on sensitivity and for the 3D Keller-Segel-Stokes system with $\alpha > \frac{1}{9}$ when the signal production is indirect. Winkler [51] showed the global existence of bounded classical solutions to Keller-Segel-Stokes system (1.7) for widely arbitrary initial data actually within the entire range $\alpha < \frac{2}{3}$ in 3D smoothly bounded domain. This inspires us to ask the following interesting and significant question: **Will the indirect signal production mechanism genuinely contribute to the global solvability of the three-dimensional Keller-Segel-Stokes system**?

Main results. Motivated by some previous studies [26,39,51,52], in this paper, we are concerned with the following Keller-Segel-Stokes system with subcritical sensitivity and indirect signal production mechanism

$$
\begin{cases}
n_t+\mathbf{u}\cdot\nabla n=\nabla\cdot(D(n)n)-\nabla\cdot(S(n)\nabla v), & x\in\Omega, t>0,\\
v_t+\mathbf{u}\cdot\nabla v=\Delta v-v+w, & x\in\Omega, t>0,\\
w_t+\mathbf{u}\cdot\nabla w=\Delta w-w+n, & x\in\Omega, t>0,\\
\mathbf{u}_t \qquad\qquad =\Delta\mathbf{u}+\nabla P+n\nabla\phi, \quad \nabla\cdot\mathbf{u}=0, & x\in\Omega, t>0,\\
\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial w}{\partial\nu}=0, \quad u=0, & x\in\partial\Omega, t>0,\\
n(x,0)=n_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x),\ \mathbf{u}(x,0)=\mathbf{u}_0(x), & x\in\Omega,
\end{cases}
\tag{1.9}
$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. **Our main goal is to affirmatively answer the above question. Specifically, we shall examine when α is larger than $\frac{2}{3}$, whether the three-dimensional Keller-Segel-Stokes system (1.9) is globally classical solvable.**

To prepare a precise presentation of our main results, throughout this work we assume that the given gravitational potential function ϕ fulfills

$$\phi\in W^{2,\infty}(\Omega), \tag{1.10}$$

and that the quadruple of initial data (n_0,v_0,w_0,u_0) satisfies

$$
\begin{cases}
n_0\in C^0(\overline{\Omega}) \text{ is nonnegative,}\\
v_0\in W^{1,\infty}(\Omega) \text{ is nonnegative,}\\
w_0\in W^{1,\infty}(\Omega) \text{ is nonnegative and}\\
\mathbf{u}_0\in\bigcup_{\beta\in(\frac{3}{4},1)}D(A^\beta),
\end{cases}
\tag{1.11}
$$

where $A=-\mathcal{P}\Delta$ represents the Stokes operator with domain $D(A^\beta):=W^{2,2}(\Omega;\mathbb{R}^3)\cap W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^3)$ with $W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^3):=W^{1,2}_0(\Omega;\mathbb{R}^3)\cap L^2_\sigma(\Omega;\mathbb{R}^3)$ and $L^2_\sigma(\Omega;\mathbb{R}^3):=\{\varphi\in L^2(\Omega)|\nabla\cdot\varphi=0\}$, and $\mathcal{P}$ represents the Helmholtz projection of $L^2(\Omega;\mathbb{R}^3)$ onto $L^2_\sigma(\Omega;\mathbb{R}^3)$.

Now, we state our main results of the present paper.

THEOREM 1.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary, the function ϕ fulfills (1.10). If D and S satisfy (1.3), (1.4) as well as*

$$\frac{|S(s)|}{D(s)}\le K_0 s^\alpha \quad \textit{for all } s>1 \tag{1.12}$$

with some $m_0\in\mathbb{R}$, $m\ge m_0$, $k_D>0$, $K_D>0$, $K_0>0$ and

$$\alpha<\frac{8}{9}. \tag{1.13}$$

Then for each $(n_0,v_0,w_0,\boldsymbol{u}_0)$ fulfilling (1.11), there exist

$$
\begin{cases}
n\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\
v\in\bigcap_{p>3}C^0([0,\infty);W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\
w\in\bigcap_{q>3}C^0([0,\infty);W^{1,q}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\
\boldsymbol{u}\in\bigcap_{\beta\in(\frac{3}{4},1)}C^0([0,\infty);D(A^\beta))\cap C^{2,1}(\overline{\Omega}\times(0,\infty);\mathbb{R}^3)
\end{cases}
$$

such that $n,v,w\geq 0$ *in* $\Omega\times(0,\infty)$*, and that with some* $P\in C^{1,0}(\overline{\Omega}\times(0,\infty))$*, and such that* (n,v,w,u,P) *solves* (1.9) *in the classical sense in* $\Omega\times(0,\infty)$*. Moreover, this solution is bounded in the sense that with some* $\beta\in(\frac{3}{4},1)$ *and* $C>0$*,*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|A^\beta\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)}\leq C \tag{1.14}$$

for all $t>0$*.*

Remark 1.1. Compared with the existing result on the Keller-Segel-Stokes system with direct signal production, Theorem 1.1 rigorously confirms that the indirect signal production mechanism genuinely facilitates the global existence of bounded classical solutions of the three-dimensional Keller-Segel-Stokes system. Indeed, the global existence of bounded classical solutions to the three-dimensional Keller-Segel-Stokes system with $\alpha<\frac{2}{3}$ was obtained in [51, Theorem 1.3] and [2, Theorem 1.1] while Theorem 1.1 in the present work established the global existence of bounded classical solutions in the same sense as that of [2,51] to the three-dimensional Keller-Segel-Stokes system (1.9) with suitably large α. In short, this result affirmatively answers the above question.

Remark 1.2. We leave the open question of how far the explicit condition (1.13) indeed is optimal for the conclusion made in Theorem 1.1.

In this paper, we use symbols C_i and c_i $(i=1,2,\cdots)$ as some generic positive constants which may vary in the context. For simplicity, $u(x,t)$ is written as u, the integral $\int_\Omega u(x)dx$ is written as $\int_\Omega u(x)$ and $\int_0^t\int_\Omega u(x,t)dxdt$ is written as $\int_0^t\int_\Omega u(x,t)$.

In the remaining part of this paper, we will first give the local existence result and some basic regularity estimates as preliminaries in Section 2. In Section 3, we give some elementary estimates for the solution to the system (1.1) and prove the Theorem 1.1.

2. Preliminaries

Firstly, we have the following local existence result as well as a convenient extensibility criterion by means of some well-known arguments in the theory of chemotaxis-fluid system. Since the proof is rather standard, we omit it for simplicity and refer the reader to [42, Lemma 2.1] for more details.

Lemma 2.1. *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with smooth boundary,* $\phi\in W^{2,\infty}(\Omega)$*,* $D\in C^2([0,\infty))$*,* $S\in C^2([0,\infty))$ *and initial data* $(n_0,v_0,w_0,\boldsymbol{u}_0)$ *fulfilling* (1.11)*. Then there exist* $T_{\max}\in(0,\infty]$ *and quintuple* (n,v,w,u,P) *with*

$$\begin{cases} n\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ v\in\bigcap_{p>3}C^0([0,T_{\max});W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ w\in\bigcap_{q>3}C^0([0,T_{\max});W^{1,q}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ \boldsymbol{u}\in\bigcap_{\beta\in(\frac{3}{4},1)}C^0([0,T_{\max});D(A^\beta))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max});\mathbb{R}^3)\end{cases}$$

such that $n,v,w\geq 0$ *in* $\Omega\times(0,T_{\max})$*, and that with some* $P\in C^{1,0}(\overline{\Omega}\times(0,T_{\max}))$*, and such that* (n,v,w,u,P) *solves* (1.9) *in the classical sense in* $\Omega\times(0,T_{\max})$*. Moreover, if* $T_{\max}<\infty$*, then for all* $\beta\in(\frac{3}{4},1)$*,*

$$\limsup_{t\nearrow T_{\max}}(\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|A^\beta\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)})=\infty. \tag{2.1}$$

LEMMA 2.2. *Assume that the conditions of Lemma 2.1 hold, the solution of (1.9) fulfills*

$$\int_\Omega n(\cdot,t) = m := \int_\Omega n_0 \quad \text{for all } t \in (0,T_{\max}) \tag{2.2}$$

and

$$\int_\Omega v(\cdot,t) \le \max\left\{\int_\Omega n_0,\ \int_\Omega v_0,\ \int_\Omega w_0\right\} \quad \text{for all } t \in (0,T_{\max}) \tag{2.3}$$

as well as

$$\int_\Omega w(\cdot,t) \le \max\left\{\int_\Omega n_0,\ \int_\Omega w_0\right\} \quad \text{for all } t \in (0,T_{\max}). \tag{2.4}$$

Proof. Integrating the first three equations, we readily conclude that (2.2)–(2.4) are valid. We refer to [39, Lemma 2.2] for more details. The proof is complete. □

LEMMA 2.3 (Gagliardo-Nirenberg interpolation inequality [24, 25]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume $p,q \in [1,\infty]$, and $r \in (0,p)$ with $p < \infty$ for $q = N$ and $p \le \frac{qN}{N-q}$ for $q < N$. Then, for $\theta \in (0,1]$ given by: $-\frac{N}{p} = (1-\frac{N}{q})\theta - \frac{N}{r}(1-\theta)$ and some $C_{GN} > 0$, we have*

$$\|z\|_{L^p(\Omega)} \le C_{GN} \|z\|^{\theta}_{W^{1,q}(\Omega)} \|z\|^{1-\theta}_{L^r(\Omega)}$$

for any $z \in W^{1,q}(\Omega) \cap L^r(\Omega)$.

3. Proof of Theorem 1.1

The aim of this section is to establish a temporally independent L^∞ bound for ∇v. Recalling the arguments pursued in our recent work [26], and also [52], it is important to turn to higher order conditional estimates as compared to $W^{1,\infty}$-topology, which is based on L^p-L^q estimates of the sectorial operator [15]. Although Lemma 3.1 to Lemma 3.5 can be found in our recent work [26], for the convenience of the readers and the integrity of this manuscript, we sketch the main steps of Lemma 3.1 to Lemma 3.5. Here and throughout the sequel, we abbreviate $B := B_k$ denoting the sectorial realization of $-\Delta + 1$ under homogeneous Neumann boundary conditions in $\bigcap_{k>1} L^k(\Omega)$, and let $(B^\mu)_{\mu>0}$ represent the associated family of positive fractional power $B^\mu = B_k^\mu$. Then the respective domains $D(B_k^\mu)$ are continuous embedded into $W^{1,\infty}(\Omega)$ whenever $2\mu - \frac{3}{k} > 1$. In order to express in more concise form, we let

$$H_p(t) := 1 + \sup_{s\in(0,t)} \|n(\cdot,s)\|_{L^p(\Omega)}, \quad t \in (0,T_{\max}) \tag{3.1}$$

and

$$I_{k,\mu}(t) := 1 + \sup_{s\in(0,t)} \|B^\mu(v(\cdot,s) - e^{-sB}v_0)\|_{L^k(\Omega)}, \quad t \in (0,T_{\max}). \tag{3.2}$$

LEMMA 3.1. *Let (n,v,w,u) be the solution constructed in Lemma 2.1. Then for some $\beta \in (\frac{3}{4},1)$, $p \ge 2$, $k > 3$ and $\delta > 0$, there exist $K_1 = K_1(\beta,p,k,\delta) > 0$ and $K_2 = K_2(\beta,p,k,\delta) > 0$ with the properties that*

$$\|A^\beta \boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)} \le K_1 H_p^{\frac{p}{p-1}\cdot\left(\frac{4\beta-1}{6}+\delta\right)}(t) \quad \text{for all } t \in (0,T_{\max}) \tag{3.3}$$

and, whereafter, that

$$\|\boldsymbol{u}(\cdot,t)\|_{L^k(\Omega)} \le K_2 H_p^{\frac{p}{p-1}\cdot\left(\frac{k-3}{3k}+\delta\right)}(t) \quad \text{for all } t\in(0,T_{\max}). \tag{3.4}$$

Proof. Based on the fourth equation in (1.9), it follows from the reasoning of Proposition 1.1 and Corollary 2.1 in [52], we can establish (3.3) and (3.4) immediately. The proof is complete. ∎

LEMMA 3.2. *Let (n,v,w,u) be the solution constructed in Lemma 2.1. Then for any $p\ge 2$, $q\ge 2$ and $k\ge 2$, there exists $K_3=K_3(\beta,p,k)>0$ with the property that*

$$\|v(\cdot,t)\|_{L^k(\Omega)} \le K_3 H_p^{\frac{p}{p-1}\cdot\frac{k-1}{9k}}(t) \quad \text{for all } t\in(0,T_{\max}). \tag{3.5}$$

Proof. Multiplying the third equation in (1.9) by w^{q-1} and integrating by parts, we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega w^q + \frac{4(q-1)}{q^2}\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + \int_\Omega w^q = \int_\Omega w^{q-1}n. \tag{3.6}$$

Owing to $1<\frac{3q}{2q+1}<\frac{3}{2}<p$, by the Hölder inequality and Young's inequality, we deduce

$$\begin{aligned}
&\int_\Omega w^{q-1}n \le \|w^{\frac{q}{2}}\|_{L^6(\Omega)}^{\frac{2(q-1)}{q}} \|n\|_{L^{\frac{3q}{2q+1}}(\Omega)}\\
&\le \frac{q-1}{q}\left\{\frac{1}{q}\left(\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + \int_\Omega w^q\right)\right\} + \frac{1}{q}\cdot\left(q^{\frac{q-1}{q}} c_1^{\frac{q-1}{q}} \|n\|_{L^{\frac{3q}{2q+1}}(\Omega)}\right)^q\\
&\le \frac{q-1}{q^2}\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + \frac{q-1}{q^2}\int_\Omega w^q + q^{q-2}c_1^{q-1}\|n\|_{L^p(\Omega)}^{\frac{p(q-1)}{3(p-1)}}\|n\|_{L^1(\Omega)}^{\frac{2pq+p-3q}{3(p-1)}}\\
&\le \frac{q-1}{q^2}\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + \frac{q-1}{q^2}\int_\Omega w^q + q^{q-2}c_1^{q-1} m^{\frac{2pq+p-3q}{3(p-1)}} H_p^{\frac{p(q-1)}{3(p-1)}}(t)
\end{aligned} \tag{3.7}$$

for all $t\in(0,T_{\max})$, where $c_1>0$ is a constant. Inserting (3.7) into (3.6) and using the fact $1-\frac{q-1}{q^2}\ge 1-\frac{1}{q}\ge\frac{1}{2}\ge\frac{1}{q}$, we thus obtain that for any choice of $t_*\in(0,T_{\max})$,

$$\frac{d}{dt}\int_\Omega w^q + \int_\Omega w^q \le q^{q-2}c_1^{q-1} m^{\frac{2pq+p-3q}{3(p-1)}} H_p^{\frac{p(q-1)}{3(p-1)}}(t) \quad \text{for all } t\in(0,t_*).$$

In accordance with ODE comparison arguments,

$$\int_\Omega w^q \le \max\left\{\int_\Omega w_0^q,\ q^{q-2}c_1^{q-1} m^{\frac{2pq+p-3q}{3(p-1)}} H_p^{\frac{p(q-1)}{3(p-1)}}(t_*)\right\} \quad \text{for all } t\in(0,t_*].$$

When evaluated at $t=t_*$, in view of (1.11) this readily concludes that

$$\|w(\cdot,t)\|_{L^q(\Omega)} \le c_2 H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t) \quad \text{for all } t\in(0,T_{\max}) \tag{3.8}$$

with constant $c_2>0$. Likewise,

$$\|v(\cdot,t)\|_{L^k(\Omega)} \le c_3\left(1+\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^q(\Omega)}\right)^{\frac{q}{q-1}\cdot\frac{k-1}{3k}} \qquad \text{for all } t\in(0,T_{\max}), \tag{3.9}$$

where $c_3>0$ is a constant. Combining (3.8) with (3.9), we have

$$\|v(\cdot,t)\|_{L^k(\Omega)}\le c_3\left(1+c_2H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)\right)^{\frac{q}{q-1}\cdot\frac{k-1}{3k}}\le c_4H_p^{\frac{p}{p-1}\cdot\frac{k-1}{9k}}(t)\ \text{for all}\ t\in(0,T_{\max}).$$

The proof is complete. □

LEMMA 3.3. *Let (n,v,w,u) be the solution constructed in Lemma 2.1. Assume that $\mu\in(\frac{1}{2},1)$, $p\ge 2$ and $k>3$. Then for all $\delta>0$, there exists $K_4=K_4(\mu,k,\delta,p)>0$ such that*

$$\|\nabla(v(\cdot,t)-e^{-tB}v_0)\|_{L^\infty(\Omega)}\le K_4H_p^{\frac{p}{p-1}\cdot\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}}(t)\cdot I_{k,\mu}^{\frac{k+3}{2k\mu}+\delta}(t) \tag{3.10}$$

for all $t\in(0,T_{\max})$.

Proof. Due to $\mu\in(\frac{1}{2},1)$, one can find $k>3$ sufficiently large and $\delta>0$ arbitrary small such that

$$1-\frac{k+3}{2k\mu}>0\quad\text{and}\quad\delta<1-\frac{k+3}{2k\mu},$$

we can thus take

$$\vartheta(\delta):=\frac{k+3}{2k\mu}+\delta\mu<\mu.$$

By the interpolation inequality in [10, Theorem 2.14.1] for fractional powers of sectorial operators, there exist $c_1=c_1(\mu,k,\delta)>0$ and $c_2=c_2(\mu,k,\delta)>0$ such that

$$\begin{aligned}
&\|B^\vartheta(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)}\\
&\le c_1\|B^\mu(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)}^{\frac{\vartheta}{\mu}}\|v(\cdot,t)-e^{-tB}v_0\|_{L^k(\Omega)}^{1-\frac{\vartheta}{\mu}}\\
&\le c_2H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}-\frac{(k-1)\delta}{9k}\right]}(t)\cdot I_{k,\mu}^{\frac{k+3}{2k\mu}+\delta}(t)\\
&\le c_2H_p^{\frac{p}{p-1}\cdot\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}}(t)\cdot I_{k,\mu}^{\frac{k+3}{2k\mu}+\delta}(t)\quad\text{for all}\ t\in(0,T_{\max})
\end{aligned}$$

with $c_2:=c_1(K_3+\|v_0\|_{L^k(\Omega)})^{\frac{p}{p-1}\cdot\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}}$, which combined with the continuous embedding $D(B^\vartheta)\hookrightarrow W^{1,\infty}(\Omega)$ [12], implies that there is $c_3=c_3(\mu,k,\delta)>0$ such that

$$\begin{aligned}
\|\nabla(v(\cdot,t)-e^{-tB}v_0)\|_{L^\infty(\Omega)}&\le c_3\|B^\vartheta(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)}\\
&\le c_2c_3H_p^{\frac{p}{p-1}\cdot\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}}(t)\cdot I_{k,\mu}^{\frac{k+3}{2k\mu}+\delta}(t)
\end{aligned}$$

for all $t\in(0,T_{\max})$, and thus, (3.10) follows with $K_4=c_2c_3$. The proof is complete. □

LEMMA 3.4. *Let (n,v,w,u) be the solution constructed in Lemma 2.1. Assume that $k\ge 3$, $p\ge 2$ and $\mu\in(\frac{1}{2},1)$ are such that $k(2\mu-1)>3$. Then for all $\delta>0$ one can find $K_5=K_5(\mu,k,p,\delta)>0$ fulfilling*

$$\|B^\mu(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)}\le K_5H_p^{\frac{p}{p-1}\cdot\left(\frac{6\mu k+k-1}{9k}+\delta\right)}(t)\quad\textit{for all}\ t\in(0,T_{\max}). \tag{3.11}$$

Proof. Taking $\delta>0$ sufficiently small such that

$$\delta<\min\left\{1-\frac{k+3}{2k\mu},\ 2k(1-\mu)\right\}, \tag{3.12}$$

we let

$$l := \frac{3k}{3+2k(1-\mu)-\delta}, \tag{3.13}$$

(3.12) asserts that $\delta < 2k(1-\mu)$ and $k > 3+2k(1-\mu)+2k\mu\delta > 3+2k(1-\mu)$, and thus,

$$k = \frac{3k}{3+2k-2k\mu-2k(1-\mu)} > l = \frac{3k}{3+2k(1-\mu)-\delta} > \frac{3k}{3+2k(1-\mu)} > 3. \tag{3.14}$$

By the variation-of-constants formula for v and applying B^μ, we have

$$\begin{aligned} \|B^\mu(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)} \le & \int_0^t \|B^\mu e^{-(t-s)B}w(\cdot,s)\|_{L^k(\Omega)}ds \\ & + \int_0^t \|B^\mu e^{-B(t-s)}u(\cdot,s)\nabla v(\cdot,s)\|_{L^k(\Omega)}ds \end{aligned} \tag{3.15}$$

for all $t\in(0,T_{\max})$, according to the L^p-L^q estimates for the corresponding semigroup [15], we get

$$\begin{aligned} \int_0^t \|B^\mu e^{-(t-s)B}w(\cdot,s)\|_{L^k(\Omega)}ds & \le c_1 \int_0^t (1+(t-s)^{-\mu})e^{-(t-s)}\|w(\cdot,s)\|_{L^k(\Omega)}ds \\ & \le c_2 H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)\int_0^\infty (1+\sigma^{-\mu})e^{-\sigma}d\sigma \\ & \le c_3 H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t) \quad \text{for all } t\in(0,T_{\max}) \end{aligned} \tag{3.16}$$

with $c_1,c_2>0$ and $c_3 := c_2\int_0^\infty(1+\sigma^{-\mu})e^{-\sigma}d\sigma<\infty$ due to $\mu\in(\frac{1}{2},1)$, here we used (3.8) in Lemma 3.2, and that for all $t\in(0,T_{\max})$,

$$\begin{aligned} & \int_0^t \|B^\mu e^{-(t-s)B}u(\cdot,s)\nabla v(\cdot,s)\|_{L^k(\Omega)}ds \\ \le & c_4\int_0^t \left(1+(t-s)^{-\mu-\frac{3}{2}\left(\frac{1}{l}-\frac{1}{k}\right)}\right)e^{-(t-s)}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^l(\Omega)}ds \\ \le & c_4\int_0^t \left(1+(t-s)^{-\mu-\frac{3}{2}\left(\frac{1}{l}-\frac{1}{k}\right)}\right)e^{-(t-s)}\|u(\cdot,s)\|_{L^l(\Omega)} \\ & \times\left(\|\nabla(v(\cdot,t)-e^{-tB}v_0)\|_{L^\infty(\Omega)}+\|\nabla e^{-tB}v_0)\|_{L^\infty(\Omega)}\right)ds \\ \le & c_5 H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{l-3}{3l}+\delta\right]}(t)\cdot I_{k,\mu}^{\frac{k+3}{2k\mu}+\delta}(t) \end{aligned} \tag{3.17}$$

where $c_4,c_5>0$. Therefore, inserting (3.16) and (3.17) into (3.15), and using (3.12), Young's inequality, there exists $c_6>0$ such that

$$\begin{aligned} \|B^\mu(v(\cdot,t)-e^{-tB}v_0)\|_{L^k(\Omega)} \le & c_3 H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)+\frac{1}{2}I_{k,\mu}(t) \\ & + c_6 H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{l-3}{3l}+\delta\right]\cdot\frac{2\mu k}{2\mu k-k-3-2\mu k\delta}}(t) \end{aligned}$$

for all $t\in(0,T_{\max})$, which implies

$$I_{k,\mu}(t) \le c_3 H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)+\frac{1}{2}I_{k,\mu}(t)+c_6 H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{l-3}{3l}+\delta\right]\cdot\frac{2\mu k}{2\mu k-k-3-2\mu k\delta}}(t)$$

for all $t\in(0,T_{\max})$, and thus,

$$I_{k,\mu}(t)\le 2+2c_3H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)$$
$$+2c_6H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{l-3}{3l}+\delta\right]\cdot\frac{2\mu k}{2\mu k-k-3-2\mu k\delta}}(t) \tag{3.18}$$

for all $t\in(0,T_{\max})$. Letting

$$\varphi(\widetilde{\delta}):=\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{2\mu k-k-3+\widetilde{\delta}}{3k}+\widetilde{\delta}\right]\cdot\frac{2\mu k}{2\mu k-k-3-2\mu k\widetilde{\delta}},$$

we know that $\varphi(\widetilde{\delta})\searrow\frac{p}{p-1}\cdot\frac{6k\mu+k-1}{9k}$ as $\widetilde{\delta}\searrow 0$, hence for some chosen $\widetilde{\delta}>0$, it is possible to select $\widehat{\delta}\in(0,\min\{1-\frac{k+3}{2k\mu},\ 2k(1-\mu)\})$ such that

$$\varphi(\widehat{\delta})\le\frac{p}{p-1}\cdot\left(\frac{6k\mu+k-1}{9k}+\delta\right).$$

Therefore, by an elementary calculation, we make use of (3.13) having

$$\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{l-3}{3l}+\widehat{\delta}\right]\cdot\frac{2\mu k}{2\mu k-k-3-2\mu k\widehat{\delta}}$$
$$=\varphi(\widehat{\delta})\le\frac{p}{p-1}\cdot\left(\frac{6k\mu+k-1}{9k}+\delta\right),$$

which in conjunction with (3.18) and (3.1) implies (3.11). The proof is complete. ∎

LEMMA 3.5. *Let (n,v,w,u) be the solution constructed in Lemma 2.1. Assume that $p\ge 2$ and $\delta>0$. Then there exists $K_6=K_6(p,\delta)>0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\le K_6H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\quad \textit{for all } t\in(0,T_{\max}). \tag{3.19}$$

Proof. Given $\delta>0$, we thus choose $k=k(\mu)>\frac{3}{2\mu-1}$ suitably large satisfying

$$\frac{4(k+1)}{9k}<\frac{4}{9}+\delta,$$

which ensures that

$$\varphi(\widetilde{\delta}):=\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\left(\frac{6\mu k+k-1}{9k}+\widetilde{\delta}\right)\cdot\left(\frac{k+3}{2k\mu}+\widetilde{\delta}\right),\quad \widetilde{\delta}>0 \tag{3.20}$$

fulfills

$$\varphi(\widetilde{\delta})\searrow\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\frac{(k+3)(6\mu k+k-1)}{18k^2\mu}=\frac{4(k+1)}{9k}<\frac{4}{9}+\delta,$$

and thus, we can find $\breve{\delta}=\breve{\delta}(p,\delta)>0$ in such a way that

$$\varphi(\breve{\delta})\le\frac{4}{9}+\delta. \tag{3.21}$$

Consider Lemma 3.3 with Lemma 3.4, there exists $c_1=c_1(p,k,\mu,\breve{\delta})>0$ satisfying

$$\|\nabla(v(\cdot,t)-e^{-tB}v_0)\|_{L^\infty(\Omega)}\le c_1H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\left(\frac{6\mu k+k-1}{9k}+\widetilde{\delta}\right)\cdot\left(\frac{k+3}{2k\mu}+\widetilde{\delta}\right)\right]}(t) \tag{3.22}$$

for all $t\in(0,T_{\max})$. Furthermore, by the Neumann heat semigroup theory [40, Lemma 1.3], we get

$$\|\nabla e^{-tB}v_0\|_{L^\infty(\Omega)}\le c_2\|v_0\|_{W^{1,\infty}(\Omega)}\quad \text{for all } t\in(0,T_{\max}) \tag{3.23}$$

with $c_2>0$ being a constant. Thus, by combination of (3.1) and (3.21)–(3.23), we have

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\\
&\le \|\nabla(v(\cdot,t)-e^{-tB}v_0)\|_{L^\infty(\Omega)}+\|\nabla e^{-tB}v_0\|_{L^\infty(\Omega)}\\
&\le c_1 H_p^{\frac{p}{p-1}\cdot\left[\frac{(k-1)(2\mu k-k-3)}{18k^2\mu}+\left(\frac{6\mu k+k-1}{9k}+\tilde{\delta}\right)\cdot\left(\frac{k+3}{2k\mu}+\tilde{\delta}\right)\right]}(t)+c_2\|v_0\|_{L^\infty(\Omega)}(t)\\
&\le c_3 H_p^{\frac{p}{p-1}\cdot\varphi(\breve{\delta})}(t)\le c_3 H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\quad \text{for all } t\in(0,T_{\max})
\end{aligned}$$

with $c_3=c_1+c_2K_0$ and K_0 is as in (1.11), which implies (3.19). The proof is complete. ∎

LEMMA 3.6. *Let the conditions of Theorem 1.1 hold, for all $p>1$ there exists $C=C(p)>0$ such that*

$$\zeta_p(s):=\int_0^s\int_0^\sigma \frac{\tau^{m+p-3}}{D(\tau)}d\tau d\sigma,\quad s\ge 0$$

fulfills

$$\frac{1}{C}s^p-1\le\zeta_p(s)\le Cs^{p+m-m_0}+C\quad \text{for all } s\ge 0.$$

Proof. The result can be obtained by a straightforward calculation. ∎

LEMMA 3.7. *Assume that $\phi\in W^{2,\infty}(\Omega)$, that D and S satisfy (1.3), (1.4), (1.12) as well as (1.13) with some $m_0\in\mathbb{R}$, $m\ge m_0$, $k_D>0$, $K_D>0$, $K_0>0$ and $\alpha<\frac{8}{9}$, and that (1.11) holds. Let $(n,v,w,\boldsymbol{u},P)$ be as in Lemma 2.1. Then for all $p_*>1$ there exists $p\ge p_*$ such that*

$$\sup_{t\in(0,T_{\max})}\|n(\cdot,t)\|_{L^p(\Omega)}<\infty. \tag{3.24}$$

Proof. Given $p_*>1$, owing to $8-9\alpha>0$, we can choose $p\ge p_*$ such that $\eta:=m-m_0\ge 0$ and

$$p>\max\left\{\frac{8-9\alpha+4\eta}{8-9\alpha},\ 4-2\alpha-m,\ 3-m,\ \frac{3+\eta-3m}{2}\right\}. \tag{3.25}$$

By a simple calculation, it is easy to see $\frac{3(p+\eta-1)}{(p-1)(4-3\alpha)}\cdot\frac{4}{9}<1$, so that we can choose $\delta>0$ such that

$$\rho:=\frac{3(p+\eta-1)}{(p-1)(4-3\alpha)}\cdot\left(\frac{4}{9}+\delta\right)<1. \tag{3.26}$$

From Lemma 3.5, there exists $c_1>0$ such that

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\le c_1 H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\quad \text{for all } t\in(0,T_{\max}). \tag{3.27}$$

Inspired by [29, Lemma 3.3] and using $\zeta_p(n)$ in Lemma 3.6, Young's inequality, we have

$$\frac{d}{dt}\int_\Omega\zeta_p(n)=-\int_\Omega n^{m+p-3}|\nabla n|^2+\int_\Omega n^{m+p-3}\frac{S(n)}{D(n)}\nabla n\cdot\nabla v$$

$$
\begin{aligned}
&\le -\frac{1}{2}\int_\Omega n^{m+p-3}|\nabla n|^2+\frac{1}{2}\int_\Omega n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2\\
&=-\frac{2}{(m+p-1)^2}\int_\Omega \left|\nabla n^{\frac{m+p-1}{2}}\right|^2+\frac{1}{2}\int_\Omega n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2 \qquad (3.28)
\end{aligned}
$$

for all $t\in(0,T_{\max})$. By (1.12), we deduce

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2\\
=&\frac{1}{2}\int_{n\le 1} n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2+\frac{1}{2}\int_{n>1} n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2\\
\le& c_2\int_\Omega|\nabla v|^2+\frac{K_0^2}{2}\int_\Omega n^{m+p-3+2\alpha}|\nabla v|^2 \quad \text{for all } t\in(0,T_{\max}) \qquad (3.29)
\end{aligned}
$$

with constant $c_2>0$. Combining (3.27) with (3.29), we get

$$
\frac{1}{2}\int_\Omega n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2\le c_1^2c_2|\Omega|H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)+\frac{K_0^2c_1^2}{2}H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\int_\Omega n^{m+p-3+2\alpha} \qquad (3.30)
$$

for all $t\in(0,T_{\max})$. Thanks to (3.25), by a simple calculation, we derive $3(m+p-1)-(m+p-3+2\alpha)>0$, and thus, $\frac{2}{m+p-1}<\frac{2(m+p-3+2\alpha)}{m+p-1}<6$. Let $r_1:=\frac{2(m+p-3+2\alpha)}{m+p-1}$, by the Gagliardo-Nirenberg inequality, there exists $c_3>0$ such that

$$
\begin{aligned}
\int_\Omega n^{m+p-3+2\alpha}=&\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{r_1}(\Omega)}^{r_1}\\
&\le(2C_{GN})^{r_1}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{r_1\iota_1}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{r_1(1-\iota_1)}\\
&\quad+(2C_{GN})^{r_1}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{r_1}\\
&\le c_3\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{2\cdot\frac{3(m+p-4+2\alpha)}{3m+3p-4}}+c_3 \qquad (3.31)
\end{aligned}
$$

for all $t\in(0,T_{\max})$, where $\iota_1=\frac{3(m+p-1)(m+p-4+2\alpha)}{(3m+3p-4)(m+p-3+2\alpha)}\in(0,1)$ and $\frac{3(m+p-4+2\alpha)}{3m+3p-4}<1$ thanks to $\alpha<\frac{4}{3}$, C_{GN} is as in Lemma 2.3. Let $\gamma:=\frac{3m+3p-4}{3(m+p-4+2\alpha)}$, it is easy to see that $\gamma>1$, once more employing Young's inequality, there exists $c_4>0$ such that

$$
\begin{aligned}
&\frac{K_0^2c_1^2}{2}H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\cdot c_3\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{2\cdot\frac{m+p-4+2\alpha}{3m+3p-4}}\\
\le&\frac{1}{(m+p-1)^2}\int_\Omega\left|\nabla n^{\frac{m+p-1}{2}}\right|^2+c_4H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)\cdot\frac{\gamma}{\gamma-1}}(t)
\end{aligned}
$$

for all $t\in(0,T_{\max})$, and thus,

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega n^{m+p-3}\frac{S^2(n)}{D^2(n)}|\nabla v|^2\\
\le&\frac{K_0^2c_1^2}{2}H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)\cdot\left(c_3\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{2\cdot\frac{m+p-4+2\alpha}{3m+3p-4}}+c_3\right)\\
&+c_1^2c_2|\Omega|H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)
\end{aligned}
$$

$$\leq \frac{1}{(m+p-1)^2}\int_\Omega \left|\nabla n^{\frac{m+p-1}{2}}\right|^2 + c_4 H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)\cdot\frac{\gamma}{\gamma-1}}(t)$$
$$+\frac{K_0^2c_1^2c_3}{2}H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)+c_1^2c_2|\Omega|H_p^{\frac{2p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)}(t)$$
$$\leq \frac{1}{(m+p-1)^2}\int_\Omega \left|\nabla n^{\frac{m+p-1}{2}}\right|^2 + c_5 H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)\cdot\frac{3m+3p-4}{4-3\alpha}}(t) \tag{3.32}$$

for all $t\in(0,T_{\max})$, where $c_5:=c_4+c_1^2c_2|\Omega|+\frac{K_0^2c_1^2c_3}{2}$. From Lemma 3.6, there exists $c_6>0$ such that

$$\int_\Omega \zeta_p(n)\leq c_6\int_\Omega n^{p+\eta}+c_6 \quad \text{for all } t\in(0,T_{\max}). \tag{3.33}$$

Let $r_2:=\frac{2(3m+3p-4)(p+\eta)}{3(m+p-1)(p+\eta-1)}$. Once more employing the Gagliardo-Nirenberg inequality, there exist $c_7:=(2c_6)^{\frac{3m+3p-4}{3(p+\eta-1)}}\cdot(2C_{GN})^{r_2}>0$ and $c_8:=c_7\|n_0\|_{L^1(\Omega)}^{r_2\cdot\frac{m+p-1}{2}}+(2c_6)^{\frac{3m+3p-4}{3(p+\eta-1)}}>0$ such that

$$\begin{aligned}\left(\int_\Omega\zeta_p(n)\right)^{\frac{3m+3p-4}{3(p+\eta-1)}} &\leq (2c_6)^{\frac{3m+3p-4}{3(p+\eta-1)}}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(p+\eta)}{m+p-1}}(\Omega)}^{r_2}+(2c_6)^{\frac{3m+3p-4}{3(p+\eta-1)}}\\
&\leq c_7\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{r_2\iota_2}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{r_2(1-\iota_2)}\\
&\quad +c_7\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{r_2}+(2c_6)^{\frac{3m+3p-4}{3(p+\eta-1)}}\\
&\leq c_8\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{r_2\iota_2}+c_8\\
&=c_8\int_\Omega\left|\nabla n^{\frac{m+p-1}{2}}\right|^2+c_8,\end{aligned}$$

and thus,

$$\int_\Omega\left|\nabla n^{\frac{m+p-1}{2}}\right|^2\geq\frac{1}{c_8}\cdot\left(\int_\Omega\zeta_p(n)\right)^{\frac{3m+3p-4}{3(p+\eta-1)}}-1 \quad \text{for all } t\in(0,T_{\max}), \tag{3.34}$$

where C_{GN} is as in Lemma 2.3. By combination of (3.34) with (3.28), there exist $c_9:=\frac{1}{(m+p-1)^2c_8}$ and $c_{10}:=c_5+\frac{1}{(m+p-1)^2}$ such that

$$\frac{d}{dt}\int_\Omega\zeta_p(n)+c_9\cdot\left(\int_\Omega\zeta_p(n)\right)^{\frac{3m+3p-4}{3(p+\eta-1)}}\leq c_{10}H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)\cdot\frac{3m+3p-4}{4-3\alpha}}(t_*)$$

for all $t\in(0,t_*)$. By an ODE comparison argument, we have

$$\int_\Omega\zeta_p(n(\cdot,t))\leq\max\left\{\int_\Omega\zeta_p(n_0),\ \left(\frac{c_{10}}{c_9}H_p^{\frac{p}{p-1}\cdot\left(\frac{4}{9}+\delta\right)\cdot\frac{3m+3p-4}{4-3\alpha}}(t_*)\right)^{\frac{3(p+\eta-1)}{3m+3p-4}}\right\}$$

for all $t\in(0,t_*)$, and thus,

$$\int_\Omega\zeta_p(n(\cdot,t))\leq c_{11}H_p^{p\cdot\frac{3(p+\eta-1)}{(p-1)(4-3\alpha)}\cdot\left(\frac{4}{9}+\delta\right)}(t_*)=c_{11}H_p^{p\rho}(t_*)$$

for all $t\in(0,t_*)$ with $c_{11}:=\max\left\{\int_\Omega \zeta_p(n_0),\left(\frac{c_{10}}{c_9}\right)^{\frac{3(p+\eta-1)}{3m+3p-4}}\right\}$. By Lemma 3.6, there exists $c_{12}>0$ such that

$$\int_\Omega n^p \le c_{12}\int_\Omega \zeta_p(n)+c_{12} \quad \text{for all } t\in(0,T_{\max})$$

this implies

$$\int_\Omega n^p(\cdot,t)\le c_{11}c_{12}H_p^{p\rho}(t_*)+c_{12} \quad \text{for all } t_*\in(0,T_{\max}) \text{ and } t\in(0,t_*).$$

In accordance with the definition of H_p, we get

$$H_p(t_*)\le 1+[c_{12}(c_{11}+1)]^{\frac{1}{p}}H_p^\rho(t_*)=:c_{13}H_p^\rho(t_*) \quad \text{for all } t_*\in(0,T_{\max}).$$

Due to $\rho<1$, we readily obtain

$$H_p(t_*)\le c_{13}^{\frac{1}{1-\rho}} \quad \text{for all } t_*\in(0,T_{\max}).$$

The proof is complete. ◻

From a combination of Lemma 3.1, Lemma 3.5 and Lemma 3.7, we immediately obtain the quantities v and w. For w, the corresponding boundedness needs to be verified by similar means as performed in Lemma 3.4 and Lemma 3.5. Finally, on basis of the boundedness of v,w and u, the temporally independent L^∞ bounds of n can be obtained through a suitable application of heat semigroup theories as done in [52].

LEMMA 3.8. *Assume that $\phi\in W^{2,\infty}(\Omega)$, that D and S satisfy* (1.3), (1.4), (1.12) *as well as* (1.13), *and that* (1.11) *holds. Then there exists $\beta>\frac{3}{4}$ such that $T_{\max}$ as well as the functions v,w and u from Lemma 2.1 fulfill*

$$\sup_{t\in(0,T_{\max})}\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}<\infty \tag{3.35}$$

and

$$\sup_{t\in(0,T_{\max})}\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}<\infty \tag{3.36}$$

as well as

$$\sup_{t\in(0,T_{\max})}\|A^\beta \boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)}<\infty. \tag{3.37}$$

Proof. From a combination of Lemmas 3.1, 3.5 and 3.7, both (3.35) and (3.37) are immediately derived. In particular, owing to $\beta\in(\frac{3}{4},1)$, the continuous embedding together with (3.37) provides $c_1>0$ and $c_2>0$ such that

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)}\le c_1\|A^\beta\mathbf{u}(\cdot,t)\|_{L^2(\Omega)}\le c_2 \quad \text{for all } t\in(0,T_{\max}). \tag{3.38}$$

For any $q>3$, we let $\varpi\in(\frac{1}{2},1)$ satisfying $\varpi>\frac{q+3}{2q}$, then there exists $\varsigma\in(\frac{1}{2},1)$ fulfilling

$$\varpi>\varsigma>\frac{q+3}{2q}. \tag{3.39}$$

Applying B^{ϖ} on both sides of the variation-of constants formula of w and employing the L^q estimates of the sectorial operator [15], we deduce

$$\begin{aligned}\left\|B^{\varsigma}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)} \le & c_3\int_0^t \left(1+(t-s)^{-\varpi}\right)e^{-(t-s)}\|\mathbf{u}(\cdot,s)\nabla w(\cdot,s)\|_{L^q(\Omega)}ds\\ &+c_3\int_0^t \left(1+(t-s)^{-\varpi}\right)e^{-(t-s)}\|n(\cdot,s)\|_{L^q(\Omega)}ds \end{aligned}\tag{3.40}$$

for all $t\in(0,T_{\max})$ with $c_3>0$ is a constant. From Lemma 3.7, there exists $c_4>0$ such that $\|n(\cdot,t)\|_{L^q(\Omega)}\le c_4$ for all $t\in(0,T_{\max})$. Due to the embedding $D(B^{\varsigma})\hookrightarrow W^{1,q}(\Omega)$ asserted by (3.39), we have

$$\left\|\nabla(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}\le c_5\left\|B^{\varsigma}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}\quad \text{for all } t\in(0,T_{\max}),$$

where $c_5>0$ is a constant. Applying the following interpolation inequality of the fractional power of sectorial operators [10, Theorem 2.14.1],

$$\left\|B^{\varsigma}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}\le\left\|B^{\varpi}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}\left\|w(\cdot,t)-e^{-tB}w_0\right\|_{L^q(\Omega)}^{1-\frac{\varsigma}{\varpi}}$$

for all $t\in(0,T_{\max})$, and thus,

$$\begin{aligned}&\|\mathbf{u}(\cdot,t)\nabla w(\cdot,t)\|_{L^q(\Omega)}\\ \le&\|\mathbf{u}(\cdot,t)\|_{L^\infty(\Omega)}\left(\left\|\nabla(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}+\left\|\nabla e^{-tB}w_0\right\|_{L^q(\Omega)}\right)\\ \le& c_2c_5\left\|B^{\varpi}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}\left(\|w(\cdot,t)\|_{L^q(\Omega)}+\|w_0\|_{L^q(\Omega)}\right)^{1-\frac{\varsigma}{\varpi}}\\ &+c_2c_6\|\nabla w_0\|_{L^q(\Omega)}\\ \le& c_7\left\|B^{\varpi}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}H_p^{\frac{p}{p-1}\cdot\frac{q-1}{3q}}(t)+c_7\\ \le& c_8\left\|B^{\varpi}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}+c_8\quad\text{for all } t\in(0,T_{\max})\end{aligned}\tag{3.41}$$

with some $c_6,c_7,c_8>0$ are constants, here we used (3.1), (3.8) and (3.24). Inserting (3.41) into (3.40) and again using (3.24), we get

$$\begin{aligned}&\left\|B^{\varpi}(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^q(\Omega)}\\ \le& c_3(c_4+c_8)\int_0^t\left(1+(t-s)^{-\varpi}\right)e^{-(t-s)}ds\\ &+c_3c_8\cdot\sup_{\sigma\in(0,t)}\left\|B^{\varpi}(w(\cdot,\sigma)-e^{-\sigma B}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}\cdot\int_0^t\left(1+(t-s)^{-\varpi}\right)e^{-(t-s)}ds\\ \le& c_9+c_9\cdot\sup_{\sigma\in(0,t)}\left\|B^{\varpi}(w(\cdot,\sigma)-e^{-\sigma B}w_0)\right\|_{L^q(\Omega)}^{\frac{\varsigma}{\varpi}}\end{aligned}\tag{3.42}$$

with $c_9:=c_3(c_4+c_8)\int_0^\infty(1+\varrho^{-\varpi})e^{-\varrho}d\varrho<\infty$ thanks to $\varpi<1$ for all $t\in(0,T_{\max})$. Define

$$\Psi_{\varpi,q}(t):=1+\sup_{s\in(0,t)}\left\|B^{\varpi}(w(\cdot,s)-e^{-sB}w_0)\right\|_{L^q(\Omega)}\quad\text{for all } t\in(0,T_{\max}).$$

Hence, (3.42) implies

$$\Psi_{\varpi,q}(t)\le 1+2c_9\Psi_{\varpi,q}^{\frac{\varsigma}{\varpi}}(t)\le c_{10}\Psi_{\varpi,q}^{\frac{\varsigma}{\varpi}}(t)\quad\text{for all } t\in(0,T_{\max}),$$

where $c_{10}=1+2c_9$. Since $\varpi>\varsigma$, it is easy to see that $\frac{\varsigma}{\varpi}<1$, and thus,

$$\Psi_{\varpi,q}(t)\le c_{10}^{\frac{\varpi}{\varpi-\varsigma}}\quad \text{for all } t\in(0,T_{\max}), \tag{3.43}$$

which combined with the embedding $D(B^{\varpi})\hookrightarrow W^{1,\infty}(\Omega)$ provides $c_{11}=c_{11}(q,\zeta,K_0,|\Omega|)>0$ such that

$$\left\|\nabla(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^\infty(\Omega)}\le c_{11}\quad \text{for all } t\in(0,T_{\max}).$$

Combining (1.11) with the heat semigroup estimates [40, Lemma 1.3], we derive

$$\begin{aligned}\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)}&\le\left\|\nabla(w(\cdot,t)-e^{-tB}w_0)\right\|_{L^\infty(\Omega)}+\left\|\nabla e^{-tB}w_0\right\|_{L^\infty(\Omega)}\\&\le c_{11}+c_{12}\|\nabla w_0\|_{L^\infty(\Omega)}\\&\le c_{11}+c_{12}K_0\quad \text{for all } t\in(0,T_{\max})\end{aligned}$$

with some $c_{12}>0$ being a constant. The proof is complete. □

LEMMA 3.9. *Assume that $\phi\in W^{2,\infty}(\Omega)$, that D and S satisfy (1.3), (1.4), (1.12) as well as (1.13), and that (1.11) holds. Then with $T_{\max}$ and n taken from Lemma 2.1, we have*

$$\sup_{t\in(0,T_{\max})}\|n(\cdot,t)\|_{L^\infty(\Omega)}<\infty. \tag{3.44}$$

Proof. Let $J_1:=S(n)\nabla v+nu$. Then for each $q>3$, Lemmas 3.7 and 3.8 assert the existence of $c_1=c_1(q)>0$ such that

$$\|J_1(\cdot,t)\|_{L^q(\Omega)}\le c_1\quad \text{for all } t\in(0,T_{\max}).$$

From a Moser-type iterative argument on the basis of the identity $n_t=\nabla\cdot(D(n)\nabla n)-\nabla\cdot J_1$ [29, Lemma A.1], we get (3.44). The proof is complete. □

Proof. **(Proof of Theorem 1.1.)** Theorem 1.1 is a direct consequence of Lemma 2.1, Lemma 3.8 and Lemma 3.9. The proof is complete. □

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