COMMUN. MATH. SCI.
Vol. 21, No. 6, pp. 1609–1623

STABILITY OF CONTACT DISCONTINUITY FOR AN ISENTROPIC VISCOSITY SYSTEM WITH CHAPLYGIN GAS*

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Abstract. This paper is devoted to studying the large-time behavior of solutions for an isentropic viscosity system with Chaplygin gas. Since all the characteristic fields of the corresponding inviscid Euler equations are linearly degenerate, the classical Riemann solutions only contain contact discontinuities. It is proved that for the isentropic viscosity system with Chaplygin gas, the viscous contact wave which approximates the corresponding contact discontinuity is asymptotically stable. The proof is given by elementary energy methods without anti-derivative technique.

Keywords. Stability; viscous contact wave; isentropic viscosity system; Chaplygin gas.

AMS subject classifications. 35B40; 76N10; 35B45; 76N30.

1. Introduction

In this paper, we study the large-time behavior toward viscous contact wave of the 1-D compressible isentropic viscosity system in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu\left(\dfrac{u_x}{v}\right)_x, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

where v is the specific volume, u is the velocity, $\mu > 0$ is the viscosity constant and the pressure $p(v)$ is given by the Chaplygin gas [21]:

$$p(v) = -v. \tag{1.2}$$

(1.1) is supplemented with the initial data

$$(v,u)(x,0) = (v_0,u_0)(x), \tag{1.3}$$

and the far field condition

$$\lim_{x\to\pm\infty}(v,u)(x,t) = (v_\pm,u_\pm), \tag{1.4}$$

where $v_\pm > 0$ and $u_\pm$ are constant states. In aerodynamics, the gas state Equation (1.2) was introduced by Chaplygin [5], Tsien [25] and von Karman [17] as an appropriate mathematical model to calculate the lifting force on a wing of an airplane. A Chaplygin gas owns a negative pressure, which has been advertised as a possible model for dark energy [3, 22, 23].

The qualitative behavior of the solutions to (1.1)-(1.4) is closely related to that of the corresponding inviscid system

$$\begin{cases} v_t - u_x = 0, \\ u_t - v_x = 0, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.5}$$

*Received: April 06, 2022; Accepted (in revised form): December 03, 2022. Communicated by Feimin Huang.

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(1.5) is also called the Chaplygin gas equations which have attracted extensive attention. Brenier [4] solved the Riemann problem of one-dimensional Chaplygin gas by considering the effects of concentration. Serre [21] studied multidimensional shock interaction for Chaplygin gas, and further research is given in [9]. Chen-Qu [7] gave global self-similar solutions to the Y-type Riemann problem of two-dimensional Chaplygin gas equations.

Now we turn to recall the Riemann problem of system (1.5), see also [7,9]. The two characteristic speeds of (1.5) are

$$\lambda_1=-1, \qquad \lambda_2=1. \tag{1.6}$$

The corresponding right eigenvectors and left eigenvectors are

$$r_1=(1,1)^T, \qquad r_2=(1,-1)^T, \tag{1.7}$$

and

$$l_1=\Big(\frac{1}{2},\frac{1}{2}\Big), \qquad l_2=\Big(\frac{1}{2},-\frac{1}{2}\Big), \tag{1.8}$$

respectively. We can check that

$$\nabla\lambda_i\cdot r_i=0, \quad i=1,2, \tag{1.9}$$

which means that all the characteristic fields are linearly degenerate. Moreover, structural conditions of left-right eigenvectors are satisfied, i.e.,

$$\nabla l_i\cdot r_i=0, \quad \nabla r_i\cdot r_i=0, \quad i=1,2. \tag{1.10}$$

Thus, solving the Riemann problem of (1.5) with the initial data

$$(v,u)(x,t=0)=\begin{cases}(v_-,u_-), & x<0,\\ (v_+,u_+), & x>0,\end{cases} \tag{1.11}$$

we can find that the bounded elementary waves only contain contact discontinuities. Without loss of generality, we only consider the 2-contact discontinuity which satisfies the R-H condition

$$\begin{cases}-s(v_+-v_-)-(u_+-u_-)=0,\\ -s(u_+-u_-)-(v_+-v_-)=0,\end{cases} \tag{1.12}$$

where $s=\lambda_2=1$ is the speed of the 2-contact discontinuity. We study the large-time behavior of solutions to (1.1)-(1.4) under the condition (1.12). Here the solutions are expected to tend toward the viscous version of 2-contact discontinuity ("2-viscous contact wave") as time goes to infinity.

The investigation of the stability of viscous contact wave dates back to Xin [27] who obtained the meta-stability of viscous contact wave for the compressible Euler system with uniform viscosity. Later Liu-Xin [18] generalized this result for a class of general systems of nonlinear conservation laws and obtained the pointwise decay estimates of the solutions. Huang-Pan-Wang [13] analysed the stability of contact discontinuity for Jin-Xin relaxation system. Zeng [29] studied the asymptotic stability of a superposition wave, i.e., shock waves and contact waves, for systems of viscous conservation laws. There was also a series of research works on the related problems of full compressible Navier-Stokes system. As for the Cauchy problem, Huang-Matsumura-Xin [12] used the anti-derivative method to obtain not only the stability of the viscous contact wave, but also its convergence rate. Later on, Huang-Xin-Yang [14] improved the results [12] to both Navier-Stokes equations and Boltzmann equations under general perturbation. In the case of the composite waves, Huang-Li-Matsumura [10] showed the asymptotic stability of the combination of contact wave and rarefaction waves. For the related work of stability of contact discontinuity, we also refer to [11,15].

However, to the best of our knowledge, even though there are many studies on the contact discontinuity mentioned above, there are few results on the stability of contact discontinuity for the isentropic viscosity system. It is well-known that if the pressure is taken to be that of the polytropic gas in (1.1), the Riemann solutions of the corresponding inviscid system contain shocks and rarefaction waves, but do not include the contact discontinuities. And there is considerable research on the stability of solutions related to shocks and rarefaction waves for (1.1) with polytropic gas, for example, see [2,6,16,19,20,26,28] and the references cited therein. Since the classical Riemann solutions of (1.5) only contain contact discontinuities, we consider the stability of contact discontinuity to the isentropic system (1.1) with Chaplygin gas.

On the other hand, from the previous work [27], we can learn that the contact discontinuity cannot be an asymptotic state for the viscous system, but a viscous contact wave which approximates the contact discontinuity on any finite time interval as the viscosity tends to zero is nonlinearly stable. Thus, in current paper, a natural and key question is that how to construct the 2-viscous contact wave which approximates the 2-contact discontinuity of (1.5).

Our idea is as follows. Note that in system (1.5), (v,u) can be expressed by the i-Riemann invariants $\omega_i(v,u), i=1,2$. Motivated by [12], we apply Riemann invariants to construct the 2-viscous contact wave which is denoted by (v^{cd},u^{cd}). More precisely, the 2-Riemann invariant $\omega_2(v,u)$ is constant along the 2-contact discontinuity curve [24]. However, due to the physical viscosity of (1.1), $\omega_2(v^{cd},u^{cd})$ is not constant anymore. In order to construct the 2-viscous contact wave, we want $\omega_2(v^{cd},u^{cd})$ to approximate the same constant. For convenience, denote $\omega_1(v^{cd},u^{cd}):=\bar{\omega}_1$ and $\omega_2(v^{cd},u^{cd}):=\bar{\omega}_2$. Calculations show that $\bar{\omega}_2$ seems like a dissipative term, see (2.20) below. This is similar to the case of Navier-Stokes equations, where the 2-Riemann invariant $\bar{u}$ is a function of Θ_x(see (1.11) in [12]). For $\bar{\omega}_1$, it satisfies a nonlinear diffusion equation with convection, see (2.19). Then, the 2-viscous contact wave (v^{cd},u^{cd}) can be constructed successfully in Theorem 2.1. Finally, we prove the asymptotic stability of (v^{cd},u^{cd}) by elementary energy methods without anti-derivative technique in Theorem 3.1.

The paper is organized as follows. In Section 2, we construct the 2-viscous contact wave for system (1.1). In Section 3, we reformulate the problem (1.1)-(1.4) to a new perturbation system and state the stability result of viscous contact wave. In Section 4, a priori estimate is established to prove the stability result.

Notations. Throughout this paper, we use the standard notations $L^p(\mathbb{R})$ and $H^k(\mathbb{R})$ to denote the usual Lebesgue space and Sobolev space in $\mathbb{R}$ with norms $\|\cdot\|_{L^p(\mathbb{R})}=:\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k(\mathbb{R})}=:\|\cdot\|_{H^k}$, respectively. $C,C_i(i=1,2)$ denotes the generic positive constant which is independent of time t unless otherwise stated.

2. The construction of 2-viscous contact wave

Now we start to construct the 2-viscous contact wave (v^{cd},u^{cd}) which approximates the 2-contact discontinuity of (1.5). We denote the i-Riemann invariants of the system (1.5) as $\omega_i(v,u), i=1,2$. Then by

$$\nabla\omega_i\cdot r_i=0,\quad i=1,2, \tag{2.1}$$

we get

$$\omega_1(v,u)=v-u, \quad \omega_2(v,u)=v+u. \tag{2.2}$$

From (1.12), it holds that

$$\omega_{2+}=u_+ + v_+ = u_- + v_- = w_{2-}. \tag{2.3}$$

Without loss of generality, let $w_{2\pm}=0$. Since $w_2(v^{cd},u^{cd})$ is not constant anymore, in order to construct the 2-viscous contact wave (v^{cd},u^{cd}), we want

$$\omega_2(v^{cd},u^{cd})-w_{2+}=J(\omega_1)w_{1x}\to 0 \quad \text{as } x\to\pm\infty, \tag{2.4}$$

where $J(w_1)$ will be determined below. To do this, taking the difference between $(1.1)_1$ and $(1.1)_2$, we get

$$(v-u)_t+(v-u)_x=-\mu\left(\frac{u_x}{v}\right)_x, \tag{2.5}$$

which yields

$$\omega_{1t}+\omega_{1x}=\mu\left(\frac{(\omega_1-\omega_2)_x}{\omega_2+\omega_1}\right)_x. \tag{2.6}$$

Now let

$$\omega_2=J(\omega_1)\omega_{1x}. \tag{2.7}$$

In what follows, we shall show the exact formula of $J(\omega_1)$. By omitting the higher error terms in (2.6), we have

$$\omega_{1t}+\omega_{1x}\approx\mu\left(\frac{\omega_{1x}}{\omega_1}\right)_x. \tag{2.8}$$

From $(1.1)_1$, we obtain

$$(\omega_1+\omega_2)_t-(\omega_2-\omega_1)_x=0. \tag{2.9}$$

Then combining (2.8) and (2.9), we have

$$(\omega_{2t}-\omega_{2x})=-(\omega_{1t}+\omega_{1x})\approx-\mu\left(\frac{\omega_{1x}}{\omega_1}\right)_x. \tag{2.10}$$

Substituting (2.7) into (2.10) gives

$$\begin{cases} J'(\omega_1)\omega_{1t}\omega_{1x}+J(\omega_1)\omega_{1tx}-J'(\omega_1)\omega_{1x}^2-J(\omega_1)\omega_{1xx}\approx-\mu\dfrac{\omega_{1xx}}{\omega_1}+\mu\dfrac{\omega_{1x}^2}{\omega_1^2}, \\ J'(\omega_1)\left(-\omega_{1x}+\mu\left(\dfrac{\omega_{1x}}{\omega_1}\right)_x\right)\omega_{1x}+J(\omega_1)\left(-\omega_{1xx}+\mu\left(\dfrac{\omega_{1x}}{\omega_1}\right)_{xx}\right)-J'(\omega_1)\omega_{1x}^2 \\ -J(\omega_1)\omega_{1xx}\approx-\mu\dfrac{\omega_{1xx}}{\omega_1}+\mu\dfrac{\omega_{1x}^2}{\omega_1^2}. \end{cases} \tag{2.11}$$

Omitting the higher error terms in (2.11), we have

$$-2J'(\omega_1)=\frac{\mu}{\omega_1^2}, \quad -2J(\omega_1)=-\frac{\mu}{\omega_1}. \tag{2.12}$$

Then we get

$$J(\omega_1)=\frac{\mu}{2\omega_1}. \tag{2.13}$$

Thus, we determine $\bar{\omega}_1,\bar{\omega}_2$ as follows:

$$\begin{cases}\bar{\omega}_{1t}+\bar{\omega}_{1x}=\mu\Big(\dfrac{\bar{\omega}_{1x}}{\bar{\omega}_1}\Big)_x,\\ \bar{\omega}_2=\dfrac{\mu}{2\bar{\omega}_1}\bar{\omega}_{1x},\end{cases} \tag{2.14}$$

where $\bar{\omega}_1=\omega_1(v^{cd},u^{cd})=v^{cd}-u^{cd}$ and $\bar{\omega}_2=\omega_2(v^{cd},u^{cd})=v^{cd}+u^{cd}$. In order to show the properties of the 2-viscous contact wave (v^{cd},u^{cd}), we need to analyse (2.14). Specifically, $\bar{\omega}_1$ satisfies the following nonlinear convective diffusion equation with far field constant states:

$$\begin{cases}\bar{\omega}_{1t}+\bar{\omega}_{1x}=\mu\Big(\dfrac{\bar{\omega}_{1x}}{\bar{\omega}_1}\Big)_x,\\ \bar{\omega}_1(\pm\infty,t)=\omega_{1\pm}=v_\pm-u_\pm>0.\end{cases} \tag{2.15}$$

Assume $\xi=x-t$, then by using the coordinates transformation $(x,t)\to(\xi,t)$, (2.15) turns to

$$\begin{cases}\bar{\omega}_{1t}=\mu\Big(\dfrac{\bar{\omega}_{1\xi}}{\bar{\omega}_1}\Big)_\xi,\\ \bar{\omega}_1(\pm\infty,t)=\omega_{1\pm}>0.\end{cases} \tag{2.16}$$

According to [1,8], the nonlinear Equation (2.16) admits a unique self similarity solution $\bar{\omega}_1(\eta),\eta=\frac{\xi}{\sqrt{1+t}}$. Moreover, $\bar{\omega}_1(\eta)$ is a monotone function, increasing if $\omega_{1-}<\omega_{1+}$ and decreasing if $\omega_{1-}>\omega_{1+}$. Let $\delta=|\omega_{1+}-\omega_{1-}|$, then $\bar{\omega}_1$ satisfies

$$|\bar{\omega}_{1\xi}|=O(1)\delta(1+t)^{-\frac{1}{2}}e^{-\frac{\omega_{1\pm}\xi^2}{4\mu(1+t)}},\quad \text{as } \xi\to\pm\infty. \tag{2.17}$$

Then we state the following properties of 2-viscous contact wave.

THEOREM 2.1. *The 2-viscous contact wave of* (1.1) *is*

$$(v^{cd},u^{cd})=\Big(\frac{1}{2}(\bar{\omega}_1+\bar{\omega}_2),\frac{1}{2}(\bar{\omega}_2-\bar{\omega}_1)\Big), \tag{2.18}$$

here $\bar{\omega}_1,\bar{\omega}_2$ *satisfies*

$$\begin{cases}\bar{\omega}_{1t}+\bar{\omega}_{1x}=\mu\Big(\dfrac{\bar{\omega}_{1x}}{\bar{\omega}_1}\Big)_x,\\ \bar{\omega}_1(\pm\infty,t)=\omega_{1\pm}>0,\end{cases} \tag{2.19}$$

and

$$\begin{cases}\bar{\omega}_2=\dfrac{\mu}{2\bar{\omega}_1}\bar{\omega}_{1x},\\ \bar{\omega}_2(\pm\infty,t)=\omega_{2\pm}=0.\end{cases} \tag{2.20}$$

Moreover, (v^{cd},u^{cd}) *satisfies*

$$\begin{cases}v^{cd}_t-u^{cd}_x=g_1,\\ u^{cd}_t-v^{cd}_x=\mu\Big(\dfrac{u^{cd}_x}{v^{cd}}\Big)_x+g_2,\end{cases} \tag{2.21}$$

where

$$\begin{aligned} g_1 &= \Big(\frac{\mu}{4\bar\omega_1}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x\Big)_x, \\ g_2 &= \Big(\frac{\mu}{4\bar\omega_1}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x\Big)_x + \mu\Big(\frac{(\bar\omega_1-\bar\omega_2)_x}{\bar\omega_1+\bar\omega_2} - \frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x, \end{aligned} \tag{2.22}$$

and

$$|(g_1,g_2)| = O(1)\delta(1+t)^{-\frac{3}{2}}e^{-\frac{c(x-t)^2}{1+t}}, \quad as \quad x\to\pm\infty. \tag{2.23}$$

Proof. By previous analysis, we immediately get that

$$|(\bar\omega_{1x}, u_x^{cd}, v_x^{cd})| = O(1)\delta(1+t)^{-\frac{1}{2}}e^{-\frac{c(x-t)^2}{1+t}}, \quad \text{as} \quad x\to\pm\infty. \tag{2.24}$$

Now we only need to verify (2.21)-(2.23). At first, we have

$$\begin{aligned} v_t^{cd} - u_x^{cd} &= \frac{1}{2}(\bar\omega_{1t}+\bar\omega_{2t}) - \frac{1}{2}(\bar\omega_{2x}-\bar\omega_{1x}) \\ &= \frac{\mu}{2}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x + \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_t - \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x \\ &= \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x + \frac{\mu}{4}\Big(\frac{\bar\omega_{1t}}{\bar\omega_1}\Big)_x = \Big(\frac{\mu}{4\bar\omega_1}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x\Big)_x = g_1. \end{aligned} \tag{2.25}$$

Secondly, it holds that

$$\begin{aligned} u_t^{cd} - v_x^{cd} - \mu\Big(\frac{u_x^{cd}}{v^{cd}}\Big)_x &= \frac{1}{2}(\bar\omega_{2t}-\bar\omega_{1t}) - \frac{1}{2}(\bar\omega_{1x}+\bar\omega_{2x}) + \mu\Big(\frac{(\bar\omega_1-\bar\omega_2)_x}{\bar\omega_1+\bar\omega_2}\Big)_x \\ &= \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_t - \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x - \frac{\mu}{2}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x + \mu\Big(\frac{(\bar\omega_1-\bar\omega_2)_x}{\bar\omega_1+\bar\omega_2}\Big)_x \\ &= \frac{\mu}{4}\Big(\frac{\bar\omega_{1t}}{\bar\omega_1}\Big)_x - \frac{\mu}{4}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x - \frac{\mu}{2}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x + \mu\Big(\frac{(\bar\omega_1-\bar\omega_2)_x}{\bar\omega_1+\bar\omega_2}\Big)_x \\ &= \Big(\frac{\mu}{4\bar\omega_1}\Big(\frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x\Big)_x + \mu\Big(\frac{(\bar\omega_1-\bar\omega_2)_x}{\bar\omega_1+\bar\omega_2} - \frac{\bar\omega_{1x}}{\bar\omega_1}\Big)_x = g_2. \end{aligned} \tag{2.26}$$

Finally, we estimate the terms $\bar\omega_{1xx}, \bar\omega_{1xxx}$. From (2.24), we get

$$\begin{aligned} |\bar\omega_{1xx}| &= O(1)\delta(1+t)^{-\frac{3}{2}}|(x-t)|e^{-\frac{\omega_\pm(x-t)^2}{4\mu(1+t)}} = O(1)\delta(1+t)^{-1}e^{-\frac{\omega_\pm(x-t)^2}{4\mu(1+t)}}, \\ |\bar\omega_{1xxx}| &= O(1)\delta(1+t)^{-\frac{3}{2}}e^{-\frac{\omega_\pm(x-t)^2}{4\mu(1+t)}} + O(1)\delta(1+t)^{-\frac{5}{2}}(x-t)^2e^{-\frac{\omega_\pm(x-t)^2}{4\mu(1+t)}} \\ &= O(1)\delta(1+t)^{-\frac{3}{2}}e^{-\frac{\omega_\pm(x-t)^2}{4\mu(1+t)}}. \end{aligned} \tag{2.27}$$

Using (2.24), (2.27), and from (2.22), we immediately get (2.23). □

3. The stability theorem

Now, we are in the position to state our main result. Let

$$\phi = v - v^{cd}, \quad \psi = u - u^{cd}. \tag{3.1}$$

Taking the coordinate transformation $(x,t)\to(\xi,t)=(x-t,t)$, from (1.1) and (2.21), we can deduce that

$$\begin{cases} \phi_t-\phi_\xi-\psi_\xi=-g_1, \\ \psi_t-\psi_\xi-\phi_\xi=\mu\Big(\dfrac{\psi_\xi}{v}-\dfrac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi-g_2, \end{cases} \tag{3.2}$$

with the initial data

$$(\phi,\psi)(\xi,0)=(\phi_0,\psi_0) \tag{3.3}$$

and far field states

$$(\phi,\psi)(\pm\infty,t)=0. \tag{3.4}$$

Define the function space

$$\begin{aligned}\mathbb{X}(0,T)=\Big\{&(\phi,\psi)\,|\,(\phi,\psi)\in C([0,T];H^1(\mathbb{R})),\phi_\xi\in L^2([0,T];L^2(\mathbb{R})),\\ &\psi_\xi\in L^2([0,T];H^1(\mathbb{R}))\Big\}.\end{aligned}$$

Our main result is as follows:

THEOREM 3.1. *For any given (v_-,u_-), suppose that (v_+,u_+) satisfies (1.12). Let (v^{cd},u^{cd}) be the 2-viscous contact wave defined in (2.21) with strength $\delta=|v_+-v_-|+|u_+-u_-|$. Then there exist positive constants δ_0 and ε_0 such that if $\delta\le\delta_0$ and $\|(\phi_0,\psi_0)\|_{H^1}\le\varepsilon_0$, the Cauchy problem (1.1)-(1.4) admits a unique global solution (v,u) satisfying $(\phi,\psi)\in\mathbb{X}(0,+\infty)$ and*

$$\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}}|(v-v^{cd},u-u^{cd})(\xi,t)|=0. \tag{3.5}$$

Several remarks concerning the above theorem go as follows.

REMARK 3.1. Our stability result of viscous contact wave is concerning the isentropic viscosity system. The 1-viscous contact wave can be constructed similarly and the stability result also holds. Compared with the non-isentropic compressible Navier-Stokes equations [10, 12, 14], the speed of viscous contact wave in our case is not zero. This yields a convection term, i.e., $\bar{\omega}_{1x}$ in (2.19) in the construction of the viscous contact wave. The appearance of the convection term gives rise to a new difficulty in the Heat Kernel estimates in Lemma 4.2. In order to overcome this difficulty, we do the coordinate transformation $(x,t)\to(\xi,t)$ for the system (3.2). Finally, we can use the elementary energy methods without anti-derivative technique to get our stability result.

REMARK 3.2. In contrast with the previous work where the pressure is taken to be that of the polytropic gas, e.g., [10, 12, 14], the pressure $p(v^{cd})=-v^{cd}$ for the viscous contact wave in our case is not constant anymore and it is increasing or decreasing when $v_->v_+$ or $v_-<v_+$. This phenomenon is very different from the case of polytropic gas, which also leads to the different construction of the viscous contact wave.

REMARK 3.3. The structural conditions (1.10) are trivially verified for system (1.1). In addition, in contrast with [18, 27], the viscosity of (1.1) is not artificial but physical.

4. Stability analysis

In this section, we are going to prove Theorem 3.1. The existence of local solution can be obtained by a standard argument (e.g., see [10,12]), and we are mainly concerned about the following *a priori estimate*.

PROPOSITION 4.1 (A priori estimate). *There exist positive constants $\varepsilon_0 \le 1$, $\delta_0 \le 1$ such that if $T>0$, $(\phi,\psi)\in \mathbb{X}(0,T)$ be a solution of (3.2)-(3.4) and*

$$N(T)=\sup_{0\le t\le T}\|(\phi,\psi)(t)\|_{H^1}\le \varepsilon_0,\quad |v_+-v_-|+|u_+-u_-|=\delta\le\delta_0, \tag{4.1}$$

then for $t\in[0,T]$, it holds that

$$\sup_{0\le t\le T}\|(\phi,\psi)(t)\|_{H^1}^2+\int_0^t \|\phi_\xi(\tau)\|_{L^2}^2+\|\psi_\xi(\tau)\|_{H^1}^2 d\tau\le C(\|(\phi_0,\psi_0)\|_{H^1}^2+\delta). \tag{4.2}$$

Once Proposition 4.1 is proved, we can extend the local solution to the global solution in $t\in[0,+\infty)$.

LEMMA 4.1. *Under the assumptions of Proposition 4.1, it holds that*

$$\|(\phi,\psi)(t)\|_{L^2}^2+\int_0^t\|\psi_\xi(\tau)\|_{L^2}^2 d\tau\le C\big(\|(\phi_0,\psi_0)\|_{L^2}^2+\delta+\delta\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau$$
$$+\delta\int_0^t(1+\tau)^{-1}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}\phi^2 d\xi d\tau\big). \tag{4.3}$$

Proof. Multiplying $(3.2)_1$ by ϕ, $(3.2)_2$ by ψ, summing them up, we obtain

$$\frac{1}{2}(\phi^2+\psi^2)_t-(\frac{1}{2}\phi^2+\frac{1}{2}\psi^2+\phi\psi)_\xi+\mu\frac{\psi_\xi^2}{v}$$
$$=-g_1\phi+\Big(\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)\psi\Big)_\xi+\mu\Big(\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)\psi_\xi-g_2\psi. \tag{4.4}$$

By the Sobolev inequality, (4.1) indicates that

$$\sup_{t\in[0,T]}\|(\phi,\psi)(t)\|_{L^\infty}\le\varepsilon_0, \tag{4.5}$$

and $\frac{1}{2}\max(v_-,v_+)<v<\frac{3}{2}\max(v_-,v_+)$.

Integrating (4.4) over $\mathbb{R}\times[0,t]$ and using Cauchy's inequality, we have

$$\|(\phi,\psi)(t)\|_{L^2}^2+\int_0^t\|\psi_\xi(\tau)\|_{L^2}^2d\tau$$
$$\le C\|(\phi_0,\psi_0)\|_{L^2}^2+C\int_0^t\int_{\mathbb{R}}|(g_1\phi,g_2\psi)|(\xi,\tau)d\xi d\tau+C\int_0^t\int_{\mathbb{R}}\Big(\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)^2d\xi d\tau. \tag{4.6}$$

We estimate the terms on the right-hand side of (4.6) as follows.

$$\int_0^t\int_{\mathbb{R}}|(g_1\phi,g_2\psi)|(\xi,\tau)d\xi d\tau$$
$$\le C\int_0^t\int_{\mathbb{R}}\delta(1+\tau)^{-\frac{3}{2}}e^{-\frac{c\xi^2}{1+\tau}}|(\phi,\psi)|d\xi d\tau$$

$$\leq C\delta\int_0^t\Big(\int_{\mathbb{R}}(1+\tau)^{-\frac{3}{2}}e^{-\frac{c\xi^2}{1+\tau}}d\xi\Big)^{\frac{4}{3}}d\tau+C\delta\int_0^t\|(\phi,\psi)\|_{L^\infty}^4d\tau$$
$$\leq C\big(\delta+\delta\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau\big). \tag{4.7}$$

By Theorem 2.1, we can obtain

$$\int_0^t\int_{\mathbb{R}}\Big(\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)^2d\xi d\tau\leq C\int_0^t\int_{\mathbb{R}}(u_\xi^{cd}\phi)^2d\xi d\tau\leq C\delta\int_0^t(1+\tau)^{-1}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}\phi^2d\xi d\tau. \tag{4.8}$$

Thus, substituting (4.7)-(4.8) into (4.6), we get the desired result (4.3). □

LEMMA 4.2 (Heat Kernel estimates [10]). *Let*

$$f=\int_{-\infty}^{\xi}\omega^2(y,t)dy, \tag{4.9}$$

where

$$\omega(y,t)=(1+t)^{-\frac{1}{2}}e^{-\frac{cy^2}{1+t}}. \tag{4.10}$$

Then it holds that

$$\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2d\xi d\tau\leq C+C\|(\phi,\psi)(t)\|_{L^2}^2+C\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau. \tag{4.11}$$

Proof. The proof of (4.11) is divided into the following two parts:

$$\int_0^t\int_{\mathbb{R}}(\phi+\psi)^2\omega^2d\xi d\tau\leq C_1+C_1\Big(\|(\phi,\psi)(\tau)\|_{L^2}^2+\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau\Big) \tag{4.12}$$

and

$$\int_0^t\int_{\mathbb{R}}(\phi-\psi)^2\omega^2d\xi d\tau$$
$$\leq C_2+\frac{C_2}{\epsilon}\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau+2C_2(\epsilon+\delta)\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2d\xi d\tau, \tag{4.13}$$

where ϵ is a positive constant which will be determined later.

Step 1: $(3.2)_1+(3.2)_2$ yields that

$$(\phi+\psi)_t-2(\phi+\psi)_\xi=-g_1+\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi-g_2. \tag{4.14}$$

Multiplying (4.14) by $(\phi+\psi)f$ yields

$$\frac{1}{2}\big((\phi+\psi)^2f\big)_t-\frac{1}{2}(\phi+\psi)^2f_t-((\phi+\psi)^2f)_\xi+(\phi+\psi)^2f_\xi$$
$$=-(g_1+g_2)(\phi+\psi)f+\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi(\phi+\psi)f. \tag{4.15}$$

Noticing that

$$\|f(\cdot,t)\|_{L^\infty}\leq C(1+t)^{-\frac{1}{2}},\quad \|f_t(\cdot,t)\|_{L^\infty}\leq C(1+t)^{-\frac{3}{2}},\quad f_\xi=\omega^2(\xi,t), \tag{4.16}$$

then integrating (4.15) over $\mathbb{R}\times[0,t]$, we can deduce that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}(\phi+\psi)^2\omega^2d\xi dt=&-\int_0^t\int_{\mathbb{R}}\frac{1}{2}((\phi+\psi)^2f)_td\xi dt+\int_0^t\int_{\mathbb{R}}\frac{1}{2}(\phi+\psi)^2f_td\xi dt\\&-\int_0^t\int_{\mathbb{R}}(g_1+g_2)(\phi+\psi)fd\xi dt-\int_0^t\int_{\mathbb{R}}\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)((\phi+\psi)f)_\xi d\xi dt.\end{aligned}\tag{4.17}$$

The terms on the right-hand side of (4.17) can be estimated as follows.

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\frac{1}{2}((\phi+\psi)^2f)_td\xi dt&\le\|(\phi,\psi)(t)\|_{L^2}^2\|f(\cdot,t)\|_{L^\infty}+\|(\phi_0,\psi_0)\|_{L^2}^2\|f(\cdot,0)\|_{L^\infty}\\&\le\frac{C_1}{5}(\|(\phi,\psi)(t)\|_{L^2}^2+\|(\phi_0,\psi_0)\|_{L^2}^2),\end{aligned}\tag{4.18}$$

$$\begin{aligned}\frac{1}{2}\int_0^t\int_{\mathbb{R}}(\phi+\psi)^2f_td\xi dt&\le\int_0^t(\|\phi\|_{L^2}^2+\|\psi\|_{L^2}^2)\|f_t\|_{L^\infty}dt\\&\le\frac{C_1}{5}\int_0^t(1+t)^{-\frac{3}{2}}(\|\phi\|_{L^2}^2+\|\psi\|_{L^2}^2)dt\\&\le\frac{C_1}{5}\|(\phi,\psi)(t)\|_{L^2}^2,\end{aligned}\tag{4.19}$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}(g_1+g_2)(\phi+\psi)fd\xi dt&\le C_1\delta\int_0^t\|(\phi,\psi)\|_{L^\infty}^4dt+C_1\int_0^t\Big(\int_{\mathbb{R}}|(g_1,g_2)|d\xi\Big)^{\frac{4}{3}}dt\\&\le C_1\delta\textstyle\int_0^t\|(\phi,\psi)\|_{L^\infty}^4dt+C_1\delta\int_0^t\Big(\int_{\mathbb{R}}(1+\tau)^{-\frac{3}{2}}e^{-\frac{c\xi^2}{1+\tau}}d\xi\Big)^{\frac{4}{3}}dt\\&\le C_1\delta\|(\phi,\psi)(t)\|_{L^2}^2\int_0^t\|(\phi_\xi,\psi_\xi)\|_{L^2}^2dt+C_1\delta,\end{aligned}\tag{4.20}$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\mu\frac{\psi_\xi}{v}(\phi_\xi f+\psi_\xi f)d\xi dt&\le\frac{C_1}{5}\int_0^t\|\psi_\xi\|_{L^2}\|\phi_\xi\|_{L^2}(1+t)^{-\frac{1}{2}}dt\\&\le\frac{C_1}{5}\int_0^t\|(\phi_\xi,\psi_\xi)\|_{L^2}^2dt,\end{aligned}\tag{4.21}$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\mu\frac{\psi_\xi}{v}(\phi+\psi)f_\xi d\xi dt&=\int_0^t\int_{\mathbb{R}}\mu\frac{\psi_\xi}{v}(\phi+\psi)\omega^2d\xi dt\\&\le\frac{C_1}{5}\int_0^t\|\psi_\xi\|_{L^2}\|(\phi,\psi)\|_{L^2}(1+t)^{-1}dt\\&\le\frac{C_1}{5}\int_0^t\|\psi_\xi\|_{L^2}^2dt+\frac{C_1}{5}\int_0^t(1+t)^{-2}\|(\phi,\psi)\|_{L^2}^2dt,\end{aligned}\tag{4.22}$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\mu\frac{u_\xi^{cd}\phi}{vv^{cd}}(\phi_\xi f+\psi_\xi f)d\xi dt&\le\frac{C_1}{5}\int_0^t(1+t)^{-1}(\|\phi\|_{L^2}\|\phi_\xi\|_{L^2}+\|\phi\|_{L^2}\|\psi_\xi\|_{L^2})dt\\&\le\frac{C_1}{5}\int_0^t\|(\phi_\xi,\psi_\xi)\|_{L^2}^2dt+\frac{C_1}{5}\int_0^t(1+t)^{-2}\|\phi\|_{L^2}^2dt\end{aligned}\tag{4.23}$$

and

$$\int_0^t\int_{\mathbb{R}}\mu\frac{u_\xi^{cd}\phi}{vv^{cd}}(\phi+\psi)f_\xi d\xi dt\leq\frac{C_1}{5}\int_0^t(1+t)^{-\frac{3}{2}}\|(\phi,\psi)\|_{L^2}^2dt. \tag{4.24}$$

Substituting (4.18)-(4.24) into (4.17), choosing ε_0, δ appropriately small lead to

$$\int_0^t\int_{\mathbb{R}}(\phi+\psi)^2\omega^2d\xi d\tau\leq C_1+C_1\Big(\|(\phi,\psi)(\tau)\|_{L^2}^2+\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau\Big). \tag{4.25}$$

Therefore, (4.12) is proved.

Step 2: $(3.2)_1-(3.2)_2$ yields that

$$(\phi-\psi)_t=-g_1-\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi+g_2. \tag{4.26}$$

Remember the heat kernel estimates in [10] (see Lemma 1 of [10], let $h(\xi,t)=(\phi-\psi)(\xi,t))$. Then

$$h_\xi\in L^2(0,t;L^2(\mathbb{R})),\ h_t\in L^2(0,t;H^{-1}(\mathbb{R})) \tag{4.27}$$

gives that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(\phi-\psi)^2\omega^2d\xi d\tau\\ \leq&\frac{C_2}{2}+\frac{C_2}{2}\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau+\frac{C_2}{2}\int_0^t\langle(\phi-\psi)_t,(\phi-\psi)G^2\rangle_{H^{-1}\times H^1}d\tau,\end{aligned} \tag{4.28}$$

where G is

$$G(\xi,t)=\int_{-\infty}^{\xi}\omega(y,t)dy. \tag{4.29}$$

The last term on the right-hand side of (4.28) can be estimated as follows. From (4.26), we have

$$\int_0^t\langle(\phi-\psi)_t,(\phi-\psi)G^2\rangle_{H^{-1}\times H^1}dt=\int_0^t\int_{\mathbb{R}}\Big(-g_1-\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi+g_2\Big)(\phi-\psi)G^2d\xi dt. \tag{4.30}$$

Similar as (4.20), we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}|(g_1,g_2)||(\phi,\psi)|G^2d\xi dt\leq&\frac{C_2\delta}{2}\int_0^t\|(\phi,\psi)\|_{L^\infty}^4dt+\frac{C_2\delta}{2}\int_0^t\Big(\int_{\mathbb{R}}(1+t)^{-\frac{3}{2}}e^{-\frac{c\xi^2}{1+\tau}}d\xi\Big)^{\frac{4}{3}}dt\\ \leq&\frac{C_2\delta}{2}\|(\phi,\psi)\|_{L^2}^2\int_0^t\|(\phi_\xi,\psi_\xi)\|_{L^2}^2dt+\frac{C_2\delta}{2}.\end{aligned} \tag{4.31}$$

For the rest of the terms in (4.30), by Young's inequality with ϵ, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\mu\Big(\frac{\psi_\xi}{v}-\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)((\phi-\psi)G^2)_\xi d\xi dt\\ \leq&\frac{C_2}{2\epsilon}\int_0^t\|(\phi_\xi,\psi_\xi)\|_{L^2}^2dt+2C_2(\epsilon+\delta)\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2d\xi dt.\end{aligned} \tag{4.32}$$

Plugging (4.30)-(4.32) back into (4.28), we get

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(\phi-\psi)^2\omega^2 d\xi d\tau\\ \leq& C_2+\frac{C_2}{\epsilon}\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2 d\tau+2C_2(\epsilon+\delta)\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2 d\xi d\tau.\end{aligned}\tag{4.33}$$

Therefore, (4.13) is proved.

Moreover, by adding (4.12) and (4.13), we get

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}2(\phi^2+\psi^2)\omega^2 d\xi d\tau\leq&(C_1+C_2)+C_1\|(\phi,\psi)(\tau)\|_{L^2}^2\\ &+(C_1+\frac{C_2}{\epsilon})\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2 d\tau+2C_2(\epsilon+\delta)\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2 d\xi d\tau.\end{aligned}\tag{4.34}$$

Taking $\epsilon=\frac{1}{2C_2}$ in (4.34), we have

$$\int_0^t\int_{\mathbb{R}}(\phi^2+\psi^2)\omega^2 d\xi d\tau\leq C+C\|(\phi,\psi)(t)\|_{L^2}^2+C\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2 d\tau.\tag{4.35}$$

Then we immediately get (4.11). □

Now, applying Lemma 4.2 to the last term in the right-hand side of (4.3) and choosing δ appropriately small, we can obtain the following Lemma 4.3.

Lemma 4.3. *Under the assumptions of Proposition 4.1, it holds that*

$$\|(\phi,\psi)(\tau)\|_{L^2}^2+\int_0^t\|\psi_\xi(\tau)\|_{L^2}^2 d\tau\leq C\Big(\|(\phi_0,\psi_0)\|_{L^2}^2+\delta+\delta^{\frac{1}{2}}\int_0^\tau\|\phi_\xi(\tau)\|_{L^2}^2 d\tau\Big).\tag{4.36}$$

Lemma 4.4. *Under the assumptions of Proposition 4.1, it holds that*

$$\|\frac{\phi_\xi}{v}(\tau)\|_{L^2}^2+\int_0^t\|\phi_\xi(\tau)\|_{L^2}^2 d\tau\leq C(\|(\phi_0,\psi_0)\|_{H^1}^2+\delta).\tag{4.37}$$

Proof. Multiplying $(3.2)_2$ by $\frac{\phi_\xi}{v}$, we have

$$\begin{aligned}\psi_t\frac{\phi_\xi}{v}=&\frac{\psi_\xi\phi_\xi}{v}+\frac{\phi_\xi^2}{v}+\frac{\mu}{2}\Big(\frac{\phi_\xi}{v}\Big)_t^2-\frac{\mu}{2}\Big(\frac{1}{v^2}\Big)_t\phi_\xi^2+\mu\Big(\frac{1}{v}\Big)_\xi\phi_t\frac{\phi_\xi}{v}-\frac{\mu}{2}\Big(\frac{\phi_\xi}{v}\Big)_\xi^2\\ &+\bigg(\mu\Big(\frac{g_1}{v}\Big)_\xi-\mu\Big(\frac{u_\xi^{cd}\phi}{vv^{cd}}\Big)_\xi-g_2\bigg)\frac{\phi_\xi}{v},\end{aligned}\tag{4.38}$$

where

$$\begin{aligned}\frac{\psi_t\phi_\xi}{v}=&\Big(\frac{\psi\phi_\xi}{v}\Big)_t+\frac{\psi\phi_\xi v_t}{v^2}-\Big(\frac{\psi\phi_t}{v}\Big)_\xi+\Big(\frac{\psi}{v}\Big)_\xi\phi_t\\ =&\Big(\frac{\psi\phi_\xi}{v}\Big)_t+\frac{\psi\phi_\xi(v_\xi+u_\xi)}{v^2}-\Big(\frac{\psi\phi_t}{v}\Big)_\xi+\Big(\frac{\psi}{v}\Big)_\xi(\phi_\xi+\psi_\xi-g_1)\\ =&\Big(\frac{\psi\phi_\xi}{v}\Big)_t+\frac{\psi\phi_\xi u_\xi^{cd}}{v^2}-\Big(\frac{\psi\phi_t}{v}\Big)_\xi+\frac{\psi_\xi\phi_\xi}{v}+\frac{\psi_\xi^2}{v}-\frac{\psi\psi_\xi v_\xi^{cd}}{v^2}-\Big(\frac{\psi}{v}\Big)_\xi g_1.\end{aligned}\tag{4.39}$$

Substituting (4.39) into (4.38), we have

$$
\begin{aligned}
&\left(\frac{\mu}{2}\left(\frac{\phi_\xi}{v}\right)^2-\frac{\psi\phi_\xi}{v}\right)_t+\frac{\phi_\xi^2}{v}\\
=&-\frac{\psi_\xi\phi_\xi}{v}-\mu\left(\frac{v_t^{cd}}{v^3}\right)\phi_\xi^2+\mu\left(\frac{v_\xi^{cd}}{v^2}\right)\phi_t\frac{\phi_\xi}{v}+\frac{\mu}{2}\left(\frac{\phi_\xi}{v}\right)_\xi^2-\left(\mu\left(\frac{g_1}{v}\right)_\xi-\mu\left(\frac{u_\xi^{cd}\phi}{vv^{cd}}\right)_\xi-g_2\right)\frac{\phi_\xi}{v}\\
&+\frac{\psi\phi_\xi u_\xi^{cd}}{v^2}-\left(\frac{\psi\phi_t}{v}\right)_\xi+\frac{\psi_\xi\phi_\xi}{v}+\frac{\psi_\xi^2}{v}-\frac{\psi\psi_\xi v_\xi^{cd}}{v^2}-\left(\frac{\psi}{v}\right)_\xi g_1. \qquad (4.40)
\end{aligned}
$$

Taking ε_0, δ appropriately small and integrating (4.40) over $\mathbb{R}\times[0,t]$ gives

$$
\begin{aligned}
&\|(\frac{\phi_\xi}{v})(\tau)\|_{L^2}^2+\int_0^t\|\phi_\xi(\tau)\|_{L^2}^2d\tau\\
\leq&C\|(\psi_0,\psi_0)\|_{H^1}^2+C\int_0^t\|\psi_\xi(\tau)\|_{L^2}^2d\tau\\
&+C\delta\int_0^t(1+\tau)^{-1}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}(\phi^2+\psi^2)d\xi d\tau+C\delta\int_0^t(1+\tau)^{-3}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}d\xi d\tau\\
\leq&C(\|(\phi_0,\psi_0)\|_{H^1}^2+\delta). \qquad (4.41)
\end{aligned}
$$

Here we use the previous lemmas and get the desired result (4.37). □

LEMMA 4.5. *Under the assumptions of Proposition 4.1, it holds that*

$$\|\psi_\xi(\tau)\|_{L^2}^2+\int_0^t\|\psi_{\xi\xi}(\tau)\|_{L^2}^2d\tau\leq C(\|(\phi_0,\psi_0)\|_{H^1}^2+\delta). \qquad (4.42)$$

Proof. Multiplying $(3.2)_2$ by $-\psi_{\xi\xi}$, we obtain

$$-(\psi_t\psi_\xi)_\xi+\left(\frac{1}{2}\psi_\xi^2\right)_t+\left(\frac{1}{2}\psi_\xi^2\right)_\xi+\phi_\xi\psi_{\xi\xi}+\mu\left(\frac{\psi_\xi}{v}\right)_\xi\psi_{\xi\xi}=\mu\left(\frac{u_\xi^{cd}\phi}{vv^{cd}}\right)_\xi\psi_{\xi\xi}+g_2\psi_{\xi\xi}. \qquad (4.43)$$

Integrating (4.43) over $\mathbb{R}\times[0,t]$ gives

$$
\begin{aligned}
&\|\psi_\xi(\tau)\|_{L^2}^2+\int_0^t\|\psi_{\xi\xi}(\tau)\|_{L^2}^2d\tau\\
\leq&C\|\psi_0\|_{H^1}^2+\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_{L^2}^2d\tau+\int_0^t\int_{\mathbb{R}}\frac{\psi_\xi\phi_\xi\psi_{\xi\xi}}{v^2}d\xi d\tau\\
&+\delta\int_0^t(1+\tau)^{-2}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}\phi^2d\xi d\tau+\delta\int_0^t(1+\tau)^{-3}\int_{\mathbb{R}}e^{-\frac{c\xi^2}{1+\tau}}d\xi d\tau\\
\leq&C\|(\phi_0,\psi_0)\|_{H^1}^2+\delta, \qquad (4.44)
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}}\frac{\psi_\xi\phi_\xi\psi_{\xi\xi}}{v^2}d\xi d\tau\leq&C\int_0^t\|\psi_\xi\|_{L^\infty}\|\phi_\xi\|_{L^2}\|\psi_{\xi\xi}\|_{L^2}d\tau\\
\leq&C\int_0^t\|\psi_\xi\|_{L^2}^{\frac{1}{2}}\|\phi_\xi\|_{L^2}\|\psi_{\xi\xi}\|_{L^2}^{\frac{3}{2}}d\tau\\
\leq&C\varepsilon_0\left(\int_0^t\|\psi_{\xi\xi}(\tau)\|_{L^2}^2d\tau+\int_0^t\|\psi_\xi(\tau)\|_{L^2}^2d\tau\right). \qquad (4.45)
\end{aligned}
$$

Thus, we can complete the proof of this lemma. ∎

From Lemma 4.1-Lemma 4.5, if we choose $C\|(\phi_0,\psi_0)\|_2^2 \le \varepsilon_0^2$, we can obtain (4.2), which proves Proposition 4.1.

Now we turn to prove Theorem 3.1. Differentiating $(3.2)_1$ with respect to ξ, multiplying the resulting equation by ϕ_ξ, we have

$$(\phi_\xi^2)_t - (\phi_\xi^2)_\xi = 2\phi_\xi\psi_{\xi\xi} - 2\phi_\xi g_{1\xi}. \tag{4.46}$$

Integrating (4.46) with respect to ξ over $(-\infty,+\infty)$ yields

$$\left|\frac{d}{dt}(\|\phi_\xi\|_{L^2}^2)\right| \le C(\|\phi_\xi\|_{L^2}^2 + \|\psi_{\xi\xi}\|_{L^2}^2 + \int_{\mathbb{R}} g_{1\xi}^2 d\xi). \tag{4.47}$$

This together with (4.2) implies that

$$\int_0^{+\infty} \left|\frac{d}{dt}(\|\phi_\xi\|_{L^2}^2)\right| dt \le C. \tag{4.48}$$

Then, combining (4.48) with (4.2), we have

$$\|\phi_\xi(\cdot,t)\|_{L^2} \to 0, \quad \text{as} \quad t\to+\infty. \tag{4.49}$$

Similarly, we can deduce that

$$\|\psi_\xi(\cdot,t)\|_{L^2} \to 0, \quad \text{as} \quad t\to+\infty. \tag{4.50}$$

On the other hand, $\|(\phi_\xi,\psi_\xi)(t)\|_{L^2}$ is uniformly bounded for $t\ge 0$ due to (4.2). Thus, for all $\xi\in\mathbb{R}$, it holds that

$$\sup_{\xi\in\mathbb{R}} |\phi(\xi,t)|^2 \le C\|\phi(t)\|_{L^2}\|\phi_\xi(t)\|_{L^2} \to 0, \quad \text{as} \quad t\to+\infty. \tag{4.51}$$

Similarly,

$$\sup_{\xi\in\mathbb{R}} |\psi(\xi,t)| \to 0, \quad \text{as} \quad t\to+\infty. \tag{4.52}$$

Therefore, (3.5) is proved. The proof of Theorem 3.1 is completed.

Acknowledgments. The authors would like to thank the referees for their valuable comments and suggestions which greatly improved the presentation of this paper. The research of Jinjing Liu was supported by National Natural Science Foundation of China grant 12261099, 11801444. The research of Meichen Hou was supported by Natural Science Basic Research Program of Shaanxi (Program No. 2021JQ-428) and partially supported by Scientific Research Program of Shaanxi Provincial Department of Education (Program No. 22JK0582).

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