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RIEMANN-HILBERT PROBLEM FOR THE FOCUSING HIROTA EQUATION WITH COUNTERPROPAGATING FLOWS*

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Abstract. The focusing Hirota equation is analyzed with a general initial condition via the inverse scattering transform, whose asymptotic behavior at infinity consists of counterpropagating waves. According to some necessary conditions, including jump condition, normalization condition, residue conditions and suitable growth condition near the branch points, the inverse problem is transformed into a matrix Riemann-Hilbert (RH) problem jumping along the branch cuts and real axis, the problem is transformed into a set of linear algebraic integral equations, and the reconstruction formula of potential is successfully obtained. In addition, the zero point of the analytical scattering coefficient on the continuous spectrum is placed on a sufficiently large circle, so a modified piecewise analytical RH problem is further successfully constructed. Finally, the exact expressions of soliton solution and breathing solution of focusing Hirota equation under degenerate initial value conditions are discussed.

Keywords. The focusing Hirota equation; Riemann-Hilbert problem; Counterpropagating flows.

AMS subject classifications. 35Q15; 35Q51; 35C08.

1. Introduction

Since Gardner, Greene, Kruskal and Miura have studied the initial value problem of the fast decay of the well-known Korteweg-de Vries (KdV) equation, an inverse scattering transform (IST) was proposed to solve the initial value problem of nonlinear integrable systems [15]. Later, Zakharov and Shabat extended the IST to the initial value problem of the classical nonlinear Schrödinger (NLS) equation [42]. After that, the IST was further applied to the defocusing NLS equation with nonzero boundary conditions (NZBCs) for the first time in [43], the situation has been studied and summarized by various works [13, 14]. It is worth mentioning that until recently, the only result of IST with NZBCs for the focusing NLS equation can only partially solve the problem, because only the case where the processing potential has no asymptotic phase difference and amplitude difference were studied in [16, 25]. For the assumption of the same amplitude in two infinite spaces, the development of potential energy with arbitrary asymptotic phase difference is considered by Biondini et al. in [6], which was used to improve the study of the soliton solution for the focusing NLS equation via the IST method with NZBCs. The nonlinear Schrödinger equation is

$$iu_t + u_{xx} - 2\ell(|u|^2 - u_0^2)u = 0, \tag{1.1}$$

where $\ell = \pm 1$ denote the defocusing and focusing case. The additional term $-2u_0^2 u$ in (1.1), which not only makes the boundary value condition satisfy the solution of the equation, but also the boundary value is independent of time t. The work in [6] studied the situation with symmetric boundary values (that is, $\lim\limits_{x\to+\infty} u(x,t) = \lim\limits_{x\to-\infty} u(x,t)$)

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[5,13,45,46], which means that the potential has no asymptotic phase difference, while in another work it further studied the situation with non-asymmetric boundary value conditions (that is one-sided boundary conditions) for initial values [12,29]. Additionally, the problem of soliton solutions corresponding to the second-order poles of the scattering coefficients has also been further discussed [28,47]. Since then, the study of soliton solutions and long-time asymptotic behavior of nonlinear evolution equations with NZBCs and ZBCs has attracted extensive attention [8,9,20,21,26,33–38,40,41]. Recently, Charlier and Lenells developed the inverse scattering theory to study the exact expression for the leading asymmetric term of the Boussinesq equation together with a precise error estimate in ten main asymptotic sectors in the (x,t)-plane [10].

The NLS equation is one of the basic models in nonlinear integrable systems. It is widely used in many aspects, such as deep water waves, nonlinear optics, Bose-Einstein condensation, and other phenomena [1,2,27,31], but it is insufficient in describing femtosecond propagation and resonance phenomena. Thus, it is necessary to consider high-order effects and even multi-component systems [22,23,39]. In 1973, Hirota proposed a high-order nonlinear evolution equation which is called the Hirota equation [18]. It can not only be reduced to the classical Schrödinger equation but also to the well-known complex KdV equation.

The focusing Hirota equation [18] is typically written in the form

$$iu_t+\alpha[u_{xx}+2(|u|^2-u_0^2)u]+i\beta(u_{xxx}+6|u|^2u_x)=0,\quad (x,t)\in\mathbb{R}^2, \tag{1.2}$$

where the real constants α and β denote the second-order and third-order dispersions, respectively, and the subscripts x and t denote partial differentiation, which also has been shown in many meaningful works in other documents. The focusing Hirota equation has some applications in physics, which can be used to describe the propagation of ultrashort optical pulses in optical fibers [3,11,17,19,32,44]. This work focuses on the IST and solutions for the focusing Hirota equation [18] with counterpropagating flows for solving the initial value problem with a more general class of initial conditions $u(x,0)$, which reduce to plane waves only as $x\to\pm\infty$, that is,

$$u(x,0)=A_\pm e^{-iW|x|\pm i\epsilon}(1+O(1)),\quad x\to\pm\infty, \tag{1.3}$$

where $W,\epsilon\in\mathbb{R}$ and $A_\pm>0$. Hereafter, the potential $u:\mathbb{R}\times\mathbb{R}^+\to\mathbb{C}$. In [44], the IST and soliton solutions of the focusing and defocusing Hirota equation with NZBCs are studied. Here, we discuss the IST for the focusing Hirota equation with counterpropagating flows. It is worth noting that there are some differences in the following four aspects compared with previous work.

(i) The difference of initial value conditions. Our work is to discuss with a more general initial value condition (1.3), which is a condition with counterpropagating flows (that is, $W\neq 0$). But it is worth mentioning that for the more general initial value condition $u(x,0)=A_\pm e^{iW_\pm x+i\varepsilon_\pm}(1+O(1))$ as $x\to\pm\infty$, it can be simplified to the initial value condition (1.3) by using the phase invariance (that is, $W_\pm=\pm W$ and $\epsilon_\pm=\pm\epsilon$).

(ii) The locations of the branch cuts are different because $W\neq 0$ throughout this work, which makes the branch cuts originally on the imaginary axis shift to other positions. Obviously, if $W=0$, the result of our work can be reduced to that of the work [44].

(iii) The structure of the Jost solutions $\Phi_\pm(x,t,k)$ is different. To begin with, assuming that $\tilde{\Psi}_\pm(x,t,k)$ are the solutions of the first part of the Lax pair (1.4), then the specific expressions of $\tilde{\Psi}_{\pm}(x,t,k)$ are derived, which depend on x, t, and k. Then, suppose that $\tilde{\Phi}_{\pm}(x,t,k)$ are the solutions of both parts of Lax pair (1.4), so that $\tilde{\Psi}_{\pm}(x,t,k)$ and $\tilde{\Phi}_{\pm}(x,t,k)$ are linearly related to the space part of Lax pair, one has $\tilde{\Phi}_{\pm}(x,t,k) = \tilde{\Psi}_{\pm}(x,t,k)B_{\pm}(t,k)$, and then we make time evolution for $B_{\pm}(t,k)$. Finally, the Jost solution is obtained.

(*iv*) There are two differences in the structure of the RH problem. (i) Due to the existence of the branch points, when constructing the RH problem, we not only need to consider jump conditions, normalization conditions and residue conditions, but also need to discuss the growth conditions near the branch points. The existence of the limit of the matrix function $M(x,t,k)$ at the branch points is used as the growth condition of the RH problem in general to ensure the uniqueness of the solution. (ii) By utilizing a sufficiently large circle B_R and the vanishing lemma, the constructed modified RH problem has no singular behavior at the branch points. All the behavior of the matrix function $M(x,t,k)$ near the branch points and the residue conditions generated by the discrete spectrum is encoded along the ∂B_R into the jump matrix $G^{(1)}(k)$ (that is, the jump on the circle), which is defined in RH Problem 1.2.

(*v*) Finally, the initial value condition with counterpropagating flow is degenerated into a finite density initial value. Therefore, the explicit expressions of the soliton solution and breathe solution of the focusing Hirota equation under the corresponding initial value conditions are obtained.

REMARK 1.1. For $W=0$, the results of this work can be reduced to the NZBCs with symmetrical amplitude, corresponding to the work [44]; for $\alpha=0,\beta=1$, the focusing Hirota equation with initial conditions (1.3) reduces to the complex modified Korteweg-de Vries (mKdV) equation with initial conditions, while for $\beta=0,\alpha=1$, the focusing Hirota equation with initial conditions (1.3) reduces to the NLS equation [7] with initial conditions (1.1).

1.1. The spectrum problems of the focusing Hirota equation. The inverse scattering method for integrable systems is used to obtain the solution of the equation, such as the focusing Hirota equation. As a special class of nonlinear dynamic systems or nonlinear differential and difference equations, integrable systems have a remarkable characteristic, that is, Lax pairs. It was introduced in [24] after integrating the Korteweg-de Vries equation by Gardner, Greene, Kruskal, and Miura [15]. The Lax pair we use for the focusing Hirota equation is

$$\Phi_x = U\Phi, \tag{1.4a}$$
$$\Phi_t = V\Phi, \tag{1.4b}$$

where

$$U = U(x,t;k) = ik\sigma_3 + Q(x,t), \tag{1.5a}$$
$$V = V(x,t;k) = \alpha V_1(x,t;k) + \beta V_2(x,t;k), \tag{1.5b}$$
$$V_1 = V_1(x,t;k) = -2kU(x,t;k) + i\sigma_3(Q_x - Q^2), \tag{1.5c}$$
$$V_2 = V_2(x,t;k) = -2kV_1(x,t;k) + [Q_x, Q] + 2Q^3 - Q_{xx}, \tag{1.5d}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -u^*(x,t) & 0 \end{pmatrix}.$$

The superscript $*$ denotes complex conjugate. The Equations (1.4) are referred to as the scattering problem, $\Phi(x,t;k)$ inferred to as the eigenfunction, k is the scattering parameter, and $u(x,t)$ is the scattering potential.

As we all know, satisfying the normalization condition is one of the conditions for constructing RH problem. Since $\Phi_\pm(x,t;k)$ are the simultaneous fundamental matrix solutions of Lax pair (1.4), in order to find the relationship between the solution and RH problem later, we introduce a piecewise meromorphic function $M(x,t;k)$ to meet the conditions required for constructing RH problem. The specific preparatory work is given in the spectral analysis in Section 2. In what follows, we only give the definition of piecewise meromorphic function $M(x,t;k)$ and the modified Lax pair it satisfies.

LEMMA 1.1. *The sectionally meromorphic matrix $M(x,t,k)$ holds the modified Lax pair as follows:*

$$M_x(x,t;k)-ik[\sigma_3,M(x,t;k)]=Q(x,t)M(x,t;k), \tag{1.6a}$$

$$M_t(x,t;k)+(2i\alpha k^2-4i\beta k^3)[\sigma_3,M(x,t;k)]=\tilde{V}_1\alpha M(x,t;k)+\tilde{V}_2\beta M(x,t;k), \tag{1.6b}$$

where

$$\tilde{V}_1=-(2kQ-i\sigma_3(Q_x-Q^2)),$$
$$\tilde{V}_2=4k^2Q-2ik\sigma_3(Q_x-Q^2)+[Q_x,Q]-2Q^3-Q_{xx}.$$

The inverse scattering method in our work recovers the solution of the focusing Hirota equation from the scattering data. Here, we first introduce a piecewise meromorphic function $M(x,t,k)$ to eliminate the oscillation of asymptotic exponent, such that $M(x,t;k)=I+O(1/k)$ as $k\to\infty$ with $\det M(x,t;k)=1$ and

$$M(x,t;k)=\Psi(x,t;k)e^{-i\theta_0\sigma_3},\quad k\in\mathbb{C}\setminus\Sigma, \tag{1.7}$$

where $\theta_0=k[x-(2\alpha k-4\beta k^2)t]$ and $\Sigma=\mathbb{R}\cup\Sigma_+\cup\Sigma_-$ is shown in Figure 2.1.

1.2. Riemann-Hilbert problems. The RH Problem 1.1 provides a Riemann-Hilbert problem to solve a matrix with jump condition from segment line and real axis. Different from the previous description, the piecewise meromorphic function $M(x,t;k)$ constructed here not only has discontinuity across Σ^0 (which is defined in Section 2.1), but also breaks the symmetry of $x\mapsto -x$. Moreover, the jump matrix is nonsingular on $\Sigma^0_\pm$ defined in Section 2.1, but it grows infinitely at the branch points. The Riemann-Hilbert approach to inverse scattering was first introduced in [30]. A Riemann-Hilbert problem, or RH problem, for a 2×2 matrix $M(x,t;k)$ consists of finding $M(x,t;k)$ such that:

RH PROBLEM 1.1. *The matrix $M(x,t;k)$ satisfies the properties as follows:*

(i) *Analyticity: $M(x,t;k)$ is analytic in $\mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z})$, where $Z=\{k_1,\ldots,k_N\}\subset\mathbb{C}^+\setminus\Sigma$.*

(ii) *Jump condition:*

$$M_+(x,t;k)=M_-(x,t;k)G(x,t;k),$$

where the jump matrix $G(x,t,k)$ is defined by

$$G(x,t;k)=e^{i\theta_0(x,t;k)\sigma_3}G_0(k)e^{-i\theta_0(x,t;k)\sigma_3}, \tag{1.8}$$

with

$$G_0(k)=\begin{cases}\begin{pmatrix}1 & \rho(k)\\ \rho^*(k) & 1+\rho(k)\rho^*(k)\end{pmatrix}, & k\in\mathbb{R},\\ \begin{pmatrix}i\rho(k)e^{-i\delta} & 0\\ -ie^{-i\delta} & -\frac{ie^{i\delta}}{\rho(k)}\end{pmatrix}, & k\in\Sigma^0_{+,1},\\ \begin{pmatrix}1 & -r(k)\\ 0 & 1\end{pmatrix}, & k\in\Sigma^0_{-,1},\\ \begin{pmatrix}\frac{ie^{-i\delta}}{\rho^*(k^*)} & -ie^{i\delta}\\ 0 & -i\rho^*(k^*)e^{i\delta}\end{pmatrix}, & k\in\Sigma^0_{+,2},\\ \begin{pmatrix}1 & 0\\ r^*(k^*) & 1\end{pmatrix}, & k\in\Sigma^0_{-,2}.\end{cases} \tag{1.9}$$

(iii) *Asymptotic behavior:*

$$M(x,t;k)=\mathbb{I}+O\left(\frac{1}{k}\right).$$

(iv) *Residue conditions:*

$$\underset{k=k_n}{Res}\, M(x,t;k)=\left(0\ \gamma_n e^{2i\theta_0(x,t;k_n)}M_1(x,t;k_n)\right), \tag{1.10a}$$

$$\underset{k=k_n^*}{Res}\, M(x,t;k)=\left(-\gamma_n e^{-2i\theta_0(x,t;k_n^*)}M_2(x,t;\bar{k}_n)\ 0\right). \tag{1.10b}$$

(v) *Growth conditions: If $(u-u_\pm)\in L_x^{1,1}(\mathbb{R}^\pm)$, for all $t\in\mathbb{R}$ with $V\neq 0$, in generic case, $\mu_{+,1}(x,t;k)$ and $\mu_{-,2}(x,t;k)$ are linearly independent at branch points $p_{\pm,m}$ $(m=1,2)$, the function $M(x,t;k)$ satifies growth conditions (3.4) at branch points $p_{\pm,m}$.*

Specifically, $M_1(x,t;k)$ and $M_2(x,t;k)$ denote the first and second column of $M(x,t;k)$, respectively. $r(k)$ and $\rho(k)$ are called reflection coefficients. If the scattering data $s_{22}(k)$ has a finite set of simple zeros, $Z=\{k_1,\dots,k_N\}\subset\mathbb{C}^+\setminus\Sigma$, such that $M(x,t;k)$ is analytic in $\mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z})$. Not only that, $M(x,t,k)$ has simple poles at branch points, and $c_1,c_2,\dots,c_n\in\mathbb{C}$ satify the residue conditions (3.1). We need to discuss not only the normalization condition, jump condition and residue condition of matrix $M(x,t;k)$, but also the appropriate growth condition near the branch points. In fact, we will describe the behavior of the Jost solution at the branch points in Section 2.7 Moreover, it is observed that $\Phi_{+,1}(x,t;k)$ and $\Phi_{-,2}(x,t;k)$ are growing towards their respective branch points $p_{+,m}$ $(m=1,2)$ with the power of $-1/4$, and the scattering data $s_{11}(k)$ and $s_{22}(k)$ are growing towards the branch points $p_{\pm,m}$ with the power of $-1/4$, respectively. It is the behavior near the branch points that determines the growth condition of the inverse problem. Therefore, in generic case, the growth conditions of $M(x,t;k)$ are disscussed according to the above results and the definition of matrix function $M(x,t;k)$.

Since the matrix function $M(x,t;k)$ constructed above has spectral singularity at the branch points, the existence and uniqueness of the solution for the RH problem involves singular behavior. In a recent work, an alternative matrix and RH problem are defined, which are also regular at the branch points. Thus, a new modified piecewise analytic matrix $M^{(1)}(x,t;k)$ defined by (5.2) is constructed, which satisfies the modified RH problem as follows.

RH PROBLEM 1.2. *The modified, piecewise analytic function matrix $M^{(1)}(x,t;k)$ meets the following conditions:*

(i) *Analyticity: $M^{(1)}(x,t;k)$ is analytic in $\mathbb{C}\setminus\Sigma^{(1)}$, where $\Sigma^{(1)}=(-\infty,-R)\cup B_R\cup(R,\infty)$.*

(ii) *Jump condition:*

$$M_-^{(1)}(x,t;k)=M_+^{(1)}(x,t;k)G^{(1)}(x,t;k),$$

where the jump matrix $G^{(1)}(x,t;k)$ is defined by

$$G^{(1)}(x,t;k)=e^{i\theta_0(x,t;k)\sigma_3}G_0^{(1)}(k)e^{-i\theta_0(x,t;k)\sigma_3}, \tag{1.11}$$

with

$$G_0^{(1)}(k)=\begin{cases}G_0(k), & k\in(-\infty,R)\cup(R,+\infty),\\ C(k), & k\in\partial B_R\cap\mathbb{C}^+,\\ C^{-1}(k), & k\in\partial B_R\cap\mathbb{C}^-,\end{cases}$$

where $C(k)$ is defined in (5.2).

(iii) *Asymptotic behavior: $M^{(1)}(x,t;k)=\mathbb{I}+O\left(\frac{1}{k}\right)$.*

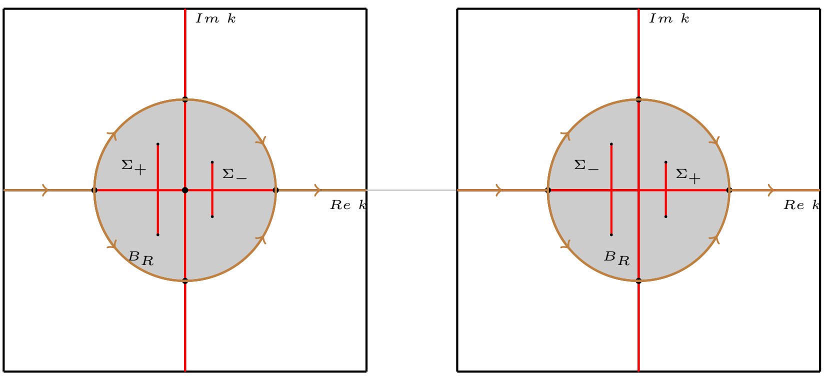

FIG. 1.1. *The branch cuts $\Sigma_\pm$ with $A_-<A_+$ for $W>0$(left) and $W<0$(right).*

The details of function $M^{(1)}(x,t;k)$ can be described as (5.2). It is not difficult to find that the modified RH problem is different from the RH Problem 1.1. To begin with, we choose a large enough ball, which is a sphere B_R with radius R centered on the origin of the complex k-plane, and the sphere contains the zeros generated by scattering coefficients $s_{11}(k)$, $s_{22}(k)$ and branch cutting $\Sigma_\pm$ (see Figure 1.1 for details). Since all zeros of scattering coefficients $s_{11}(k)$, $s_{22}(k)$ are in the interior of B_R, the function $\psi(x,t;k)$ we chose in Proposition 5.2 is analytic, $M^{(1)}(x,t;k)$ here is piecewise analytic function rather than piecewise meromorphic function. It means that the residue condition is not necessary in the modified RH problem (1.2). Because the branch cuts are covered by the big circle, $M^{(1)}(x,t;k)$ is analytic on the branch cuts and at the branch point. In other words, the behavior of $M(x,t;k)$ near the branch points and the residue condition of discrete spectrum are rearranged to the jump matrix $G^{(1)}(x,t;k)$ along ∂B_R.

1.3. Plan of the proof. For the focusing Hirota equation with counterpropogating flows, we study a series of spectral analysis properties at the branch points from the spectral problem, and then recover the potential from the scattering data by means of RH problem.

Section 2 mainly employs the Lax pair of focusing Hirota equation to do spectral analysis to find the necessary conditions for establishing the corresponding RH problem, including analyticity, symmetry, asymptotic behavior, the distribution of discrete spectrum. Note that the existence and uniqueness of the eigenfunction is guaranteed by the conditions $u-u_\pm \in L_x^1(\mathbb{R})$. However, these regions do not include branch points $p_{\pm,m}$ $(m=1,2)$, so the asymptotic behavior of Jost solutions $\Phi_\pm(x,t,k)$ and scattering coefficients at these branch points are further analyzed in detail.

In Section 3, the inverse problem of focusing Hirota Equation (1.2) is discussed. Firstly, in order to regularize the established RH problem (1.1), the residue conditions at discrete spectral points are analyzed in Section 3.1. As discussed in Section 2, Jost functions have singularity at these branch points $p_{\pm,m}$ $(m=1,2)$, so it is necessary to find a condition that makes the established RH problem have a unique solution, that is, the growth condition discussed in Section 3.2. As usual, the solution of the focusing Hirota equation under the condition (1.3) can be transformed into a closed algebraic system by Pelemlj formula combined with residue conditions, which is the content of Section 3.3. Using the technique discussed in [4], we find an alternative solution of Lax pair and further define it in a sufficiently large circle B_R, which contains all discrete spectrum and branch points, so that the newly constructed matrix-valued function $M^{(1)}(x,t,k)$ (5.2) is a piecewise analytical function. The existence and uniqueness of the solution of the modified RH problem satisfied by this new matrix-valued function $M^{(1)}(x,t,k)$ can be guaranteed by the vanishing lemma shown in Section 5.

At last, the exact expressions of one-soliton solution and breathe solution of focusing Hirota equation under special degenerate initial value conditions are given in Section 6. Some conclusions and discussions are presented in Section 7.

2. Direct scattering problem

In this section, we mainly consider the Jost solutions under conditions (2.1) and introduce Riemann surface to study the related properties of eigenvalue functions. The analytical properties, symmetry and asymptotic properties at the branch points $p_{\pm,m}$ $(m=1,2)$, the asymptotic behaviors of the modified eigenfunctions $\mu_\pm(x,t,k)$ are defined in (2.22) as $k\to\infty$ and scattering coefficients are studied.

2.1. Boundary conditions and Jost solutions. In this subsection, we establish IST for the initial value problem of the focusing Hirota equation with NZBCs, which we then use in the subsequent sections to compute the solution of the equation with counterpropagating flows. The following two exact plane wave solutions satisfying the focusing Hirota equation are considered,

$$u_\pm(x,t)=A_\pm e^{-2ih_\pm(x,t)\pm i\epsilon}, \tag{2.1}$$

with

$$h_\pm(x,t)=\frac{1}{2}\left[\pm Wx+((W^2-2A_\pm^2)\alpha\pm W\beta((\pm W)^2-6A_\pm^2))t\right]. \tag{2.2}$$

The symbols $\pm$ indicate the asymptotic behavior as $x\to\pm\infty$. We obtain the following asymptotic matrix spectral problem under the nonzero boundary condition (1.3)

$$\Phi_{\pm,x}=U_\pm\Phi_\pm, \tag{2.3a}$$

$$\Phi_{\pm,t}=V_{\pm}\Phi_{\pm}, \tag{2.3b}$$

where

$$U_{\pm}(x,t)=ik\sigma_3+Q_{\pm}(x,t), \tag{2.4a}$$
$$V_{\pm}(x,t)=-2ik^2\sigma_3+i\sigma_3((Q_{\pm,x}(x,t)-Q_{\pm}^2(x,t))-2kQ_{\pm}(x,t), \tag{2.4b}$$

with

$$Q_{\pm}(x,t)=e^{-ih_{\pm}(x,t)\sigma_3}(A_{\pm}\sigma_3e^{\pm i\epsilon\sigma_3}\sigma_1)e^{ih_{\pm}(x,t)\sigma_3}. \tag{2.5}$$

The purpose of the following is to obtain the solution $\widetilde{\psi}_{\pm}(x,t;k)$ of Lax pair (1.4) about the first part, a simple calculation reveals that

$$\begin{aligned}U_{\pm}(x,t;k)&=e^{-ih_{\pm}\sigma_3}[ik\sigma_3+A_{\pm}\sigma_3e^{\pm i\epsilon\sigma_3}\sigma_1]e^{ih_{\pm}\sigma_3}\\&\triangleq e^{-ih_{\pm}\sigma_3}\hat{U}_{\pm}(x,t;k)e^{ih_{\pm}\sigma_3}.\end{aligned} \tag{2.6}$$

Therefore, Lax pair (1.4) can rewrite the space part as

$$\left(e^{ih_{\pm}\sigma_3}\widetilde{\psi}_{\pm}(x,t;k)\right)_x=\hat{U}_{\pm}\left(k\pm\frac{W}{2}\right)e^{ih_{\pm}\sigma_3}\widetilde{\psi}_{\pm}(x,t;k). \tag{2.7}$$

At this time, the matrix $\Gamma_{\pm}(k)$ composed of the eigenvector corresponding to the eigenvalue are

$$\Gamma_{\pm}(k)=\begin{pmatrix}1 & \frac{iA_{\pm}e^{i\epsilon\sigma_3}}{\lambda_{\pm}(k)+(k+W/2)}\\ \frac{iA_{\pm}e^{-i\epsilon\sigma_3}}{\lambda_{\pm}(k)+(k+W/2)} & 1\end{pmatrix}, \tag{2.8}$$

$$\det\Gamma_{\pm}(k)=\frac{2\lambda_{\pm}(k)}{\lambda_{\pm}(k)+(k+W/2)}\triangleq D_{\pm}(k)\triangleq(d_{\pm}(k))^2, \tag{2.9}$$

where

$$\lambda_{\pm}(k)=\sqrt{(k+W/2)^2+A_{\pm}^2}. \tag{2.10}$$

Therefore, the fundamental matrix solution to the first part of the Lax pair (1.4) is derived as

$$\widetilde{\psi}_{\pm}(x,t;k)=e^{-ih_{\pm}\sigma_3}\Gamma_{\pm}(k)e^{i\lambda_{\pm}\sigma_3x}. \tag{2.11}$$

Our purpose is to seek the simultaneous solutions $\widetilde{\Phi}_{\pm}(x,t;k)$ of Lax pair. We know that the original Lax pair is a first-order linear equation system. For this reason, $\widetilde{\psi}_{\pm}(x,t;k)$ and $\widetilde{\Phi}_{\pm}(x,t;k)$ are linearly related, they have

$$\widetilde{\Phi}_{\pm}(x,t;k)=\widetilde{\psi}_{\pm}(x,t;k)H_{\pm}(t;k). \tag{2.12}$$

We further make time evolution so that $H(0,k)$ is not lost in generality, letting $H(0,k)=I$, the fundamental simultaneous matrix solutions are obtained by $\widetilde{\Phi}_{\pm}(x,t)$

$$\widetilde{\Phi}_{\pm}(x,t)=e^{-ih_{\pm}(x,t)\sigma_3}\Gamma_{\pm}(k)e^{i\theta_{\pm}(x,t)}, \tag{2.13}$$

where

$$\theta_{\pm}(x,t;k)=\lambda_{\pm}(k)\left\{x-2\left[\alpha(k\mp W/2)-\beta[2(k+W/2)^2-A_{\pm}^2]\right]t\right\}. \tag{2.14a}$$

Some notations are explained as follows in this work:

- $\mathbb{R}^{\pm}=\{k\in\mathbb{R}:\pm Rek>0\}$, $\mathbb{C}^{\pm}=\{k\in\mathbb{C}:\pm Imk>0\}$, where the superscripts $\pm$ represent the limit taken from the left (right) side of the positive (negative) side of the oriented contour in k-plane, respectively.
- The branch cuts Σ_+ and Σ_- are introduced with $p_{\pm,1}=\mp W/2+iA_\pm$, $p_{\pm,2}=\mp W/2-iA_\pm$ (see Figure 2.1 for details).
- $\Sigma_+=[p_{+,1},p_{+,2}]=\Sigma_{+,1}\cup\Sigma_{+,2}$ and $\Sigma_-=[p_{-,1},p_{-,2}]=\Sigma_{-,1}\cup\Sigma_{-,2}$ (see Figure 2.1 for details).
- $\Sigma_{\pm,1}=\Sigma_\pm\cap(\mathbb{C}^+\cup\mp W/2)=\mp W/2+[0,iA_\pm]$ and $\Sigma_{\pm,2}=\Sigma_\pm\cap(\mathbb{C}^-\cup\mp W/2)=\mp W/2+[-iA_\pm,0]$ (see Figure 2.1 for details).
- $\Sigma^0_{+,1}=\Sigma_{+,1}\setminus\{p_{+,1},-W/2\}$, $\Sigma^0_{+,2}=\Sigma_{+,2}\setminus\{p_{+,2},-W/2\}$, $\Sigma^0_{-,1}=\Sigma_{-,1}\setminus\{p_{-,1},-W/2\}$, $\Sigma^0_{-,2}=\Sigma_{-,2}\setminus\{\bar{p}_{-,2},W/2\}$.

Inspired by (2.13), the Jost solutions $\Phi_\pm(x,t,k)$ that satisfy the simultaneous solution of the Lax pair (1.4), which are defined by

$$\Phi_\pm(x,t;k)\sim\frac{1}{d_\pm}e^{-ih_\pm(x,t)\sigma_3}\Gamma_\pm(k)e^{i\theta_\pm(x,t)\sigma_3},\quad x\to\pm\infty. \tag{2.15}$$

An application of Abel's theorem yields $\det\Phi_\pm(x,t;k)=1$.

Attention is now turned to investigating some properties and asymptotic behavior of the Jost solutions $\Phi_\pm(x,t;k)$ at the branch points $p_{\pm,m}$ $(m=1,2)$.

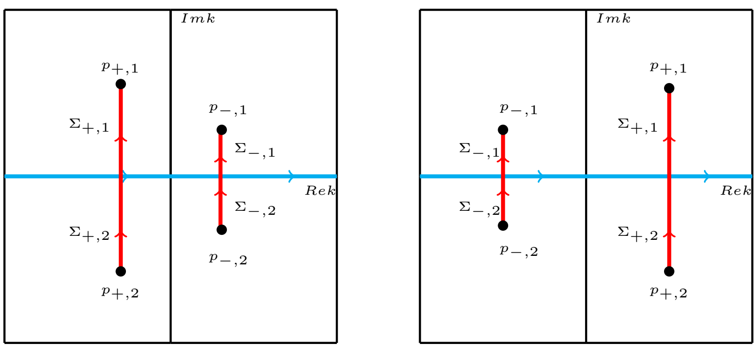

FIG. 2.1. *The contours $\Sigma_{\pm,1}$ and $\Sigma_{\pm,2}$ for $W>0$ (left) and $W<0$ (right), where $A_+>A_-$.*

2.2. Riemann surface. For all $k\in\mathbb{R}$, since the eigenvalues are doubly branched, we introduce the two-sheeted Riemann surface defined by

$$\lambda(k;A)=\sqrt{k^2+A^2}=(k-iA)^{\frac{1}{2}}(k+iA)^{\frac{1}{2}}. \tag{2.16}$$

It is worth noting that $\lambda(k;A)\in\mathbb{R}$ as $k\in\mathbb{R}\cup[-iA,iA]$ shown in Figure 2.2. We take the branch cut of $\lambda(k;A)$ to lie along $[-iA,iA]$ oriented upward, and define $\lambda(k;A)$ as continuous from the right. The branch points are $k=\pm iA$. Letting

$$\begin{cases}k+iA=r_1e^{i\varphi_1},\\ k-iA=r_2e^{i\varphi_2},\end{cases} \tag{2.17}$$

where $\varphi_1,\ \varphi_2\in(-\frac{\pi}{2},\frac{3\pi}{2})$, then the single-valued analytic function on the Riemann

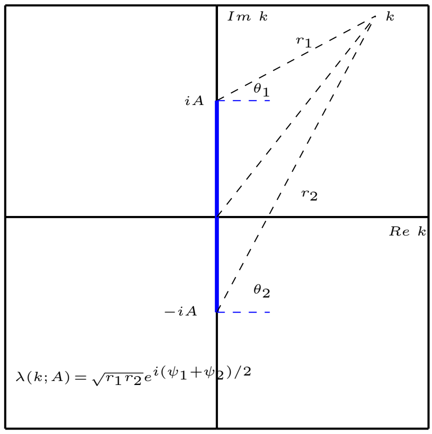

FIG. 2.2. *The Riemann surface defined by* $\lambda^2 = k^2 + A^2$*, where* $\lambda(k;A) = \sqrt{r_1 r_2}e^{i(\psi_1+\psi_2)/2}$ *with angles* $\psi_1, \psi_2 \in [-\pi/2, 3\pi/2]$.

surface can be obtained. Moreover, $\lim\limits_{x\to\infty} \lambda(k;A) = k$ in any direction.

LEMMA 2.1. *The function* $\lambda(k;A)$ *defined by* (2.16) *admits the properties as follows:*

$$\operatorname{Im}\lambda(k;A) \gtrless 0, \qquad k \in \mathbb{C}^{\pm} \setminus [-iA, iA], \qquad \operatorname{Res}\lambda(k;A) \gtrless 0, \quad k \in \mathbb{R}^{\pm} \cup i\mathbb{R}, \tag{2.18a}$$
$$\lambda(-k;A) = -\lambda(k;A), \quad k \in \mathbb{C} \setminus [-iA, iA], \qquad \lambda(k^*;A) = \lambda^*(k;A), \quad k \in \mathbb{C}. \tag{2.18b}$$

As usual, define

$$\lambda^{\pm}(k;A) = \lim_{\epsilon\to 0} \lambda(k+\epsilon;A) = \mp\lambda(k;A), \quad k \in [-iA, iA], \tag{2.19}$$
$$\lambda^{\pm}_{\pm}(k;A) = \mp\lambda_{\pm}(k;A), \quad k \in \Sigma_{\pm}. \tag{2.20}$$

After that, when $\lambda_{\pm}(k)$ do not produce ambiguity, we will suppress their dependence on k. To be clear, it is easy to prove that

$$D^{+}_{\pm}(k) = \frac{4\lambda^2_{\pm}}{A^2_{\pm} D_{\pm}(k)}, \quad k \in \Sigma_{\pm}, \tag{2.21}$$

and $D_{\pm}(k)$ are analytic in $k \in \mathbb{C} \setminus \Sigma_{\pm}$. Similarly, we have to understand the discontinuity of $d_{\pm}(k)$.

2.3. Analyticity properties. In this subsection, we introduce the modified eigenfunctions $\mu_{\pm}(x,t;k)$ and derive the integral equation. Similar to [6, 7], by analyzing the Neumann series of the integral equation, the existence and uniqueness of the eigenfunctions $\mu_{\pm}(x,t;k)$ for all $k \in \Sigma_{\pm}$ are proved. Providing that $u(x,t) - u_{\pm} \in L^1_x(\mathbb{R}^{\pm})$ the analytic property of the modified eigenfunctions $\mu_{\pm}(x,t;k)$ are obtained.

In order to eliminate the asymptotic exponential oscillation and poles in factor $d_{\pm}(k)$, the modified eigenfunctions are introduced

$$\mu_{\pm}(x,t;k) = d_{\pm}(k) e^{ih_{\pm}(x,t)\sigma_3} \Phi_{\pm}(x,t;k) e^{-i\theta_{\pm}(x,t)\sigma_3}. \tag{2.22}$$

Using standard methods, we then obtain linear integral equations of Volterra type for

$\mu_\pm(x,t;k)$:

$$\mu_\pm(x,t;k)=\Gamma_\pm(k)+\int_{\pm\infty}^{x}\Gamma_\pm(k)e^{i\lambda_\pm(x-y)\hat{\sigma}_3}\Gamma_\pm^{-1}(k)e^{2ih_\pm(y,t)\sigma_3}\Delta Q_\pm(y,t)\mu_\pm(y,t;k)dy, \tag{2.23}$$

where $\Gamma_\pm$ are defined by (2.14), $e^{\hat{\sigma}_3}\mathcal{A}=e^{\sigma_3}\mathcal{A}e^{-\sigma_3}$ and $\Delta Q_\pm(x,t;k)=Q(x,t;k)-Q_\pm(x,t;k)$. In addition, $\mu_{\pm,1}(x,t;k)$, $\Phi_{\pm,1}(x,t;k)$ and $\mu_{\pm,2}(x,t;k)$, $\Phi_{\pm,2}(x,t;k)$ represtent the first and second column of $\mu_\pm(x,t;k)$, $\Phi_\pm(x,t;k)$, respectively. The existence and uniqueness of the eigenfunctions $\Phi_\pm(x,t;k)$ can be guaranteed by introducing the Neumann series for the integral Equations (2.23) for all $k\in\Sigma$ provided that $u-u_\pm\in L_x^1(\mathbb{R}^\pm)$ and $t\in\mathbb{R}$.

THEOREM 2.1. *Provided that $u-u_\pm\in L_x^1(\mathbb{R}^\pm)$ for $t\in\mathbb{R}$, the eigenfunctions $\Phi_\pm(x,t;k)$ can be analytically extended onto the corresponding regions of the complex k-plane as shown in Table 2.1.*

$\Phi_{+,1}(x,t;k)$	$\Phi_{+,2}(x,t;k)$	$\Phi_{-,1}(x,t;k)$	$\Phi_{-,2}(x,t;k)$
$\mathbb{C}^+\setminus\Sigma_{+,1}$	$\mathbb{C}^-\setminus\Sigma_{+,2}$	$\mathbb{C}^-\setminus\Sigma_{-,2}$	$\mathbb{C}^+\setminus\Sigma_{-,1}$

TABLE 2.1. *The analyticity of $\Phi_\pm(x,t;k)$.*

Moreover, $\Phi_\pm(x,t;k)$ are continuous on $k\in\mathbb{R}$, which implies that $\Phi_\pm(x,t;k)$ are a continuous real differentiable function of k, i.e., $C^1(\mathbb{R})$. Since $\Phi_\pm(x,t;k)$ are two fundamental matrix solutions of the scattering problem, hence, there exists a 2×2 constant scattering matrix $S(k)=(s_{ij}(k))$ $(i,j=1,2)$ such that

$$\Phi_-(x,t;k)=\Phi_+(x,t;k)S(k),\quad k\in\mathbb{R}, \tag{2.24}$$

where $s_{ij}(k)$ are called the scattering coefficients. Similarly, the scattering relationship on the left-side is defined as

$$\Phi_+(x,t;k)=\Phi_-(x,t;k)R(k),\quad k\in\mathbb{R}, \tag{2.25}$$

where $R(k)=(r_{ij})$ $(i,j=1,2)$. It follows from (2.24) that $s_{ij}(k)$ have the Wronskian representations:

$$s_{11}(k)=r_{22}(k)=|\Phi_{-,1}(x,t;k),\Phi_{+,2}(x,t;k)|,\quad k\in\mathbb{R}\cup\mathbb{C}^-\setminus\{p_{\pm,2}\}, \tag{2.26a}$$
$$s_{22}(k)=r_{11}(k)=|\Phi_{+,1}(x,t;k),\Phi_{-,2}(x,t;k)|,\quad k\in\mathbb{R}\cup\mathbb{C}^+\setminus\{p_{\pm,1}\}, \tag{2.26b}$$
$$s_{12}(k)=r_{12}(k)=|\Phi_{-,2}(x,t;k),\Phi_{+,2}(x,t;k)|,\quad k\in\mathbb{R}\cup\Sigma_{+,1}^0\cup\Sigma_{-,2}^0, \tag{2.26c}$$
$$s_{21}(k)=r_{21}(k)=|\Phi_{+,1}(x,t;k),\Phi_{-,1}(x,t;k)|,\quad k\in\mathbb{R}\cup\Sigma_{-,1}^0\cup\Sigma_{+,2}^0, \tag{2.26d}$$

where $|f,g|$ denotes the determinant of the solutions f, g.

THEOREM 2.2. *Suppose that $u-u_\pm\in L_x^1(\mathbb{R}^\pm)$, then $s_{11}(k)$ and $r_{22}(k)$ can be analytically extended to $\mathbb{C}^-\setminus\Sigma$, while $s_{22}(k)$ and $r_{11}(k)$ can be analytically extended to $\mathbb{C}^+\setminus\Sigma$.*

As usual, we define the reflection coefficients as

$$\rho(k)=\frac{s_{12}(k)}{s_{22}(k)},\qquad k\in\mathbb{R}\cup\Sigma_{+,1}^0, \tag{2.27a}$$

$$r(k)=\frac{1}{r_{21}(k)s_{22}(k)}, \quad k\in\mathbb{R}\cup\Sigma^0_{-,1}. \tag{2.27b}$$

To sum up what has been stated above, if $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$, and for $t\in\mathbb{R}$, the scattering matrix $S(k)$ is a continuous real differentiable function, that is $C^1(\mathbb{R})$. But the reflection coefficient $\rho(k)\in C^1(\mathbb{R}\setminus Z)$, where Z is the set of zero points of $s_{22}(k)$, which will be introduced below.

Based on the scattering relationship and in order to facilitate the expression of the inverse problem in the future, it is still possible to introduce the piecewise function matrix solution $\Psi(x,t;k)$ for $k\in\mathbb{C}\setminus\Sigma$. It is worth noting that $\Psi(x,t;k)$, the fundamental matrix solution of Lax pair (1.4) with $\det\Psi(x,t;k)=1$, is defined by

$$\Psi(x,t;k)=\begin{cases}\begin{pmatrix}\Phi_{+,1}(x,t;k) & \dfrac{\Phi_{-,2}(x,t;k)}{s_{22}}\end{pmatrix}, & k\in\mathbb{C}^+\setminus\Sigma,\\ \begin{pmatrix}\dfrac{\Phi_{-,1}(x,t;k)}{s_{11}} & \Phi_{+,2}(x,t;k)\end{pmatrix}, & k\in\mathbb{C}^-\setminus\Sigma.\end{cases} \tag{2.28}$$

2.4. Symmetry properties. The symmetries

$$X^*(x,t;k^*)=-\sigma X(x,t;k)\sigma, \quad T^*(x,t;k^*)=-\sigma T(x,t;k)\sigma, \quad k\in\mathbb{C}, \tag{2.29}$$

where $\sigma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, from which some symmetry properties can be derived as follows.

The first symmetry. If $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$, for all $t\in\mathbb{R}$, the Jost solutions and scattering coefficients have the symmetries:

$$\phi^*_{+,1}(x,t;k^*)=\sigma\phi_{+,2}(x,t;k), \qquad k\in\mathbb{R}\cup\mathbb{C}^-\cup\Sigma^0_{+,1}\setminus\{p_{+,2}\}, \tag{2.30a}$$
$$\phi^*_{+,2}(x,t;k^*)=-\sigma\phi_{+,1}(x,t;k), \qquad k\in\mathbb{R}\cup\mathbb{C}^+\cup\Sigma^0_{+,2}\setminus\{p_{+,1}\}, \tag{2.30b}$$
$$\phi^*_{-,1}(x,t;k^*)=\sigma\phi_{-,2}(x,t;k), \qquad k\in\mathbb{R}\cup\mathbb{C}^+\cup\Sigma^0_{-,2}\setminus\{p_{-,1}\}, \tag{2.30c}$$
$$\phi^*_{-,2}(x,t;k^*)=-\sigma\phi_{-,1}(x,t;k), \qquad k\in\mathbb{R}\cup\mathbb{C}^-\cup\Sigma^0_{-,1}\setminus\{p_{-,2}\}, \tag{2.30d}$$

which in turns yields the symmetry conditions

$$\begin{cases}s^*_{22}(k^*)=s_{11}(k),\\ r^*_{11}(k^*)=r_{22}(k),\end{cases} \quad k\in\mathbb{R}\cup\mathbb{C}^-\setminus\{p_{\pm,2}\}, \begin{cases}s^*_{12}(k^*)=-s_{21}(k),\\ r^*_{21}(k^*)=-r_{12}(k),\end{cases} \quad k\in\mathbb{R}\cup\Sigma^0_{+,2}\cup\Sigma^0_{-,1}.$$

Lemma 2.2. *If $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$, for all $t\in\mathbb{R}$, one admits the condition*

$$\Psi^\dagger(x,t;k)=\Psi(x,t;k)^{-1}, \quad k\in\mathbb{C}\setminus\Sigma, \tag{2.31}$$

where the subscript $\dagger$ means the Schwarz conjugate-transpose.

In addition to the symmetries mentioned above, another symmetry about Jost eigenfunction and scattering coefficients is also derived. But this symmetry will change because $\Phi_\pm(x,t;k)$ are defined by the scaling factors $d_\pm(k)$, which will cause $\Phi_\pm(x,t;k)$ to contain the factors $\frac{iA_\pm}{\lambda_\pm+(k\pm W/2)}$.

The second symmetry. If $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$, for all $t\in\mathbb{R}$ and cyclic indices j,ℓ, the Jost eigenfunctions have the following relations:

$$\Phi^+_{+,j}(x,t;k)=-ie^{(-1)^j i\epsilon}\Phi_{+,\ell}(x,t;k), \quad k\in\Sigma^0_{+,j}, \tag{2.32a}$$

$$\Phi^{+}_{-,j}(x,t;k)=-ie^{(-1)^{\ell}i\epsilon}\Phi_{-,\ell}(x,t;k), \quad k\in\Sigma^{0}_{+,\ell}. \tag{2.32b}$$

Proof. With the symmetries of $\lambda^{+}_{\pm}(k)=-\lambda_{\pm}(k)$, the expressions of $\Gamma_{\pm}(\lambda_{\pm}\mapsto-\lambda_{\pm})$ are calculated directly, then we are combining with the expression of eigenfunctions $\Phi_{\pm}(x,t;k)$. Using the above results, the calculation of $\Phi_{\pm}(x,t;k)$ $(\lambda_{\pm}\mapsto-\lambda_{\pm})$ can be directly derived. ∎

Under the assumption of (2.1) and $W\neq0$, the symmetries of the scattering coefficients satisfy

$$s^{+}_{22}(k)=ie^{-i\epsilon}s_{12}(k), \qquad r^{+}_{11}(k)=-ie^{-i\epsilon}r_{12}(k), \qquad k\in\Sigma^{0}_{+,1}, \tag{2.33a}$$
$$s^{+}_{22}(k)=-ie^{-i\epsilon}s_{21}(k), \qquad r^{+}_{11}(k)=ie^{-i\epsilon}r_{21}(k), \qquad k\in\Sigma^{0}_{-,1}, \tag{2.33b}$$
$$s^{+}_{11}(k)=ie^{i\epsilon}s_{21}(k), \qquad r^{+}_{22}(k)=-ie^{i\epsilon}r_{21}(k), \qquad k\in\Sigma^{0}_{+,2}, \tag{2.33c}$$
$$s^{+}_{11}(k)=-ie^{i\epsilon}s_{12}(k), \qquad r^{+}_{22}(k)=ie^{i\epsilon}r_{12}(k), \qquad k\in\Sigma^{0}_{-,2}. \tag{2.33d}$$

2.5. Continuous spectrum and discrete eigenvalues. As the scalar case and the two-component case, the continuous spectrum is composed of all the values of k such that the eigenvalue $\lambda(k)\in\mathbb{R}$, that is, all $k\in\mathbb{R}\cup\Sigma_{\pm}$. A discrete set $Z\cup\bar{Z}$ of eigenvalues is called the discrete spectrum. In the unusual case, the set of two columns of $\Phi_{+}(x,t;k)$ is consistent with the set of two columns of $\Phi_{-}(x,t;k)$, and this set contains the continuous spectrum of the scattering problem. As a matter of fact, $\Phi_{\pm}(x,t;k)$ are defined only on $k\in\mathbb{R}$ at the same time. Here, considering the domain of definition and analytic region of Jost solution, for $k\in\Sigma^{0}_{+}$, the analytic column of $\Phi_{-}(x,t;k)$ can be represented by the linear combination of the columns of $\Phi_{+}(x,t;k)$. Similarly, the same is true for $k\in\Sigma^{0}_{-}$.

Lemma 2.3. *Provided that $u-u_{\pm}\in L^{1}_{x}(\mathbb{R}^{\pm})$ and for all $t\in\mathbb{R}$, $\Phi_{+,1}(x,t;k)$, $\Phi_{+,2}(x,t;k)$, $\Phi_{-,1}(x,t;k)$ and $\Phi_{-,2}(x,t;k)$ are bounded, in $\mathbb{R}\cup\Sigma^{0}_{-,1}$, $\mathbb{R}\cup\Sigma^{0}_{-,2}$, $\mathbb{R}\cup\Sigma^{0}_{+,2}$ and $\mathbb{R}\cup\Sigma^{0}_{+,1}$, respectively.*

Afterwards, we define $\Sigma^{0}:=\mathbb{R}\cup\Sigma^{0}_{+}\cup\Sigma^{0}_{-}$, which constitutes a continuous spectrum. The existence of eigenfunctions with those values of $k\in\mathbb{C}\setminus\Sigma$ constitutes the discrete spectrum of the scattering problem. If $u-u_{\pm}\in L^{1}_{x}(\mathbb{R}^{\pm})$ and for all $t\in\mathbb{R}$, these are the values of k in $\mathbb{C}^{+}\setminus\Sigma$ precisely, where $s_{22}(k)=0$, and the values of $s_{11}(\bar{k})=0$ in $\mathbb{C}^{-}\setminus\Sigma$.

The set $Z\cup\bar{Z}$ of discrete eigenvalues consists of an infinite number of isolated points in $\mathbb{C}\setminus\Sigma$, where $Z\subset\mathbb{C}^{+}\setminus\Sigma$. Specially, we must also pay closer attention to the fact that the set of discrete eigenvalues may have one or more accumulation points in $\Sigma=\mathbb{R}\cup\Sigma_{+}\cup\Sigma_{-}$. Typically, $s_{11}(k)$ and $s_{22}(k)$ are not analytic at branch points, and the spectral singularities are the zeros of $s_{11}(k)$ and $s_{22}(k)$ along the continuous spectrum; the scattering coefficients do not vanish on Σ^{0}_{+} and Σ^{0}_{-}. In what follows, we discuss the relations of the scattering coefficients at the branch points.

In what follows, we obtain the residue conditions that will be needed for the inverse problem.

Theorem 2.3. *Provided that $u-u_{\pm}\in L^{1}_{x}(\mathbb{R}^{\pm})$ and for all $t\in\mathbb{R}$, if $s_{22}(k)$ has a finite set of simple zeros, $Z=\{k_{1},\dots,k_{N}\}\subset\mathbb{C}^{+}\setminus\Sigma$, there exists norming constants $\{\gamma_{1},\dots,\gamma_{n}\}\in\mathbb{C}$ such that*

$$\operatorname*{Res}_{k=k_{n}}\Phi(x,t;k)=\big(0\ \gamma_{n}\Phi_{1}(x,t;k_{n})\big), \quad n=1,\dots,N, \tag{2.34a}$$
$$\operatorname*{Res}_{k=k^{*}_{n}}\Phi(x,t;k)=\big(-\gamma^{*}_{n}\Phi_{2}(x,t;k^{*}_{n})\ 0\big), \quad n=1,\dots,N. \tag{2.34b}$$

2.6. Asymptotic behavior as $k \to \infty$. In order to set up the inverse problem correctly, we need to consider asymptotic propertities of the Jost eigenfunctions $\Phi_\pm(x,t;k)$ and scattering coefficients as $k \to \infty$, to recover the potential from the scattering data.

THEOREM 2.4. *If u is continuously differentiable with $u-u_\pm \in L^1_x(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$, the following equality can be derived as*

$$\Phi_\pm(x,t;k) = e^{i\theta_0\sigma_3}(I+O(1)), \quad k \to \infty, \tag{2.35}$$

and the reconstruction formula of the solution can be expressed as

$$u(x,t) = -2i \lim_{k\to\infty} k(e^{-i\theta_0\sigma_3}\Phi_-(x,t;k))_{12}. \tag{2.36}$$

Proof. The modified eigenfunctions $\mu_\pm(x,t;k)$ admit the behaviors

$$\mu_\pm(x,t;k) = I + O(\frac{1}{k}), \quad k \to \infty. \tag{2.37}$$

Furthermore, we have

$$u(x,t) = -2i \lim_{k\to\infty} e^{-2ih_-(x,t)} k(\mu_-(x,t;k))_{12}. \tag{2.38}$$

Using the asymptotic properties of each factors as well as the expression for $\Phi_\pm(x,t,k)$ defined in (2.22), one has

$$\Phi_\pm(x,t;k) = e^{i(\theta_\pm - h_\pm)\sigma_3}(I+O(1)), \quad k \to \infty,$$

with

$$\lambda_\pm(k) = (k \pm \frac{W}{2}) + \frac{(A_\pm)^2}{2(k\pm\frac{W}{2})} + O(k^{-3}), \quad k \to \infty, \tag{2.39}$$

and

$$\theta_\pm(x,t;k) = \theta_0(x,t;k) + h_\pm(x,t) + O(\frac{1}{k}), \quad k \to \infty, \tag{2.40}$$

where $\theta_0(x,t;k) = k[x-(2\alpha k - \beta k^2)t]$. This completes the proof of Theorem 2.4. ◻

2.7. Asymptotic behavior at the branch points. We ensure the existence, analyticity and continuity of the Jost eigenfunctions in appropriate open regions on the complex plane with the conditions $u-u_\pm \in L^1_x(\mathbb{R}^\pm)$ and for all $t \in \mathbb{R}$. However, the regions covered in the course of these discussions do not include branch points $p_{\pm,1}, p_{\pm,2}$. It is clear that a complete characterisation of the inverse problem is inseparable from the discussions of the behaviours of the scattering coefficients, the Jost eigenfunctions $\Phi_\pm(x,t;k)$ and the modified Jost eigenfunctions $\mu_\pm(x,t;k)$ at the branch points $p_{\pm,1}, p_{\pm,2}$. Introducing the weighted L^1 spaces

$$L^{1,j}(\mathbb{R}^\pm) := \left\{ f : \mathbb{R} \to \mathbb{C} | (1+|x|)^j f \in L^1(\mathbb{R}^\pm) \right\}, \quad j = 1,2.$$

LEMMA 2.4. *If $u - u_\pm \in L^{1,1}_x(\mathbb{R}^\pm)$ and for all $t \in \mathbb{R}$, the modified eigenfunctions $\mu_\pm(x,t;k)$ are continuous at the branch points $p_{\pm,1}, p_{\pm,2}$ and the asymptotic behavior of the modified eigenfunctions $\mu_\pm(x,t;k)$ has the following form*

$$\mu_{+,1}(x,t;k) = \omega^{(0)}_{p_{+,1}}(x,t) + O(1), \quad k \to p_{+,1}, \tag{2.41a}$$

$$\mu_{+,2}(x,t;k)=\omega_{p_{+,2}}^{(0)}(x,t)+O(1), \quad k\to p_{+,2}, \tag{2.41b}$$

$$\mu_{-,2}(x,t;k)=\omega_{p_{-,1}}^{(0)}(x,t)+O(1), \quad k\to p_{-,1}, \tag{2.41c}$$

$$\mu_{-,1}(x,t;k)=\omega_{p_{-,2}}^{(0)}(x,t)+O(1), \quad k\to p_{-,2}, \tag{2.41d}$$

where some vectors $\omega_{p_{\pm,1}}^{(0)}(x,t)$ and $\omega_{p_{\pm,2}}^{(0)}(x,t)$ are never zero.

LEMMA 2.5. *If $u-u_\pm\in L_x^{1,2}(\mathbb{R}^\pm)$ and for all $t\in\mathbb{R}$, the modified eigenfunctions $\mu_\pm(x,t,k)$ are continuous at the branch points $p_{\pm,1},p_{\pm,2}$ and the asymptotic behavior of the modified eigenfunctions $\mu_\pm(x,t;k)$ has the following form*

$$\mu_{+,1}(x,t;k)=\omega_{p_{+,1}}^{(0)}(x,t)+\omega_{p_{+,1}}^{(1)}(x,t)(k-p_{+,1})^{\frac{1}{2}}+O(k-p_{+,1})^{\frac{1}{2}}, \quad k\to p_{+,1}, \tag{2.42a}$$

$$\mu_{+,2}(x,t;k)=\omega_{p_{+,2}}^{(0)}(x,t)+\omega_{p_{+,2}}^{(1)}(x,t)(k-p_{+,2})^{\frac{1}{2}}+O(k-p_{+,2})^{\frac{1}{2}}, \quad k\to p_{+,2}, \tag{2.42b}$$

$$\mu_{-,2}(x,t;k)=\omega_{p_{-,1}}^{(0)}(x,t)+\omega_{p_{-,1}}^{(1)}(x,t)(k-p_{-,1})^{\frac{1}{2}}+O(k-p_{-,1})^{\frac{1}{2}}, \quad k\to p_{-,1}, \tag{2.42c}$$

$$\mu_{-,1}(x,t;k)=\omega_{p_{-,2}}^{(0)}(x,t)+\omega_{p_{-,2}}^{(1)}(x,t)(k-p_{-,2})^{\frac{1}{2}}+O(k-p_{-,2})^{\frac{1}{2}}, \quad k\to p_{-,2}, \tag{2.42d}$$

where some vectors $\omega_{p_{\pm,1}}^{(0)}(x,t)$, $\omega_{p_{\pm,1}}^{(1)}(x,t)$, $\omega_{p_{\pm,2}}^{(0)}(x,t)$ and $\omega_{p_{\pm,2}}^{(1)}(x,t)$ are never zero.

Proof. The behavior of $\mu(x,t;k)$ at the branch points is studied in a more rigorous space $u-u_\pm\in L_x^{1,2}(\mathbb{R}^\pm)$. Although the derivatives of the eigenfunctions with respect to k at the branch points are not well-defined, the derivative of z can be asymptotically estimated at the branch points. Therefore, we introduce the variable z

$$z(k)=\lambda_+(k)+(k+\frac{V}{2}), \tag{2.43}$$

such that

$$k+\frac{W}{2}=\frac{1}{2}(z-\frac{A_+^2}{z}), \quad \lambda_+(k)=\frac{1}{2}(z+\frac{A_+^2}{z}), \tag{2.44}$$

and write the dependence of the integral equation on k as the dependence on z. The limits of $K(\xi,z)$ and $\frac{\partial K}{\partial z}$ at $z\to\pm iA_+$ are further obtained. Once again, the Neumann series is used for the integral equation to discuss the properties of $\frac{\partial\mu_{+,1}}{\partial z}$, then one has

$$\frac{\partial\mu_{+,1}}{\partial z}(x,t;z)=\frac{\partial\mu_{+,1}}{\partial z}(x,t;iA_+)+O(1), \quad z\to iA_+. \tag{2.45}$$

However,

$$\mu_{+,1}(x,t;z)=\mu_{+,1}(x,t;iA_+)+\int_{iA_+}^{z}\frac{\partial\mu_{+,1}}{\partial z}(x,t;s)ds. \tag{2.46}$$

Finally, returning to k, we have

$$\mu_{+,1}(x,t;z)=\omega_{p_{+,1}}^{(0)}(x,t)+\omega_{p_{+,1}}^{(1)}(x,t)(\lambda_+(k)-p_{+,1})+O(\lambda_+(k)-p_{+,1}), \quad k\to p_{+,1}.$$

The final results are derived obviously. □

In order to facilitate the inscriptions of the behaviour of the Jost solutions $\mu_\pm(x,t;k)$ near the branch points $p_{\pm,1}$ and $p_{\pm,2}$, we need to know the behaviours of $d_\pm(k)$ in advance.

Lemma 2.6. *The asymptotic behaviors of $d_\pm(k)$ at the branch points with $m=1,2$ are given by*

$$d_\pm(k)=\left(\frac{8(k-p_{\pm,m})}{iA_\pm}\right)^{\frac{1}{4}}+O(1), \quad k\to p_{\pm,m}. \tag{2.47a}$$

Lemma 2.7. *Under the hypothesis of Lemma 2.4, one has*

$$\Phi_{+,1}(x,t;k)=\frac{\zeta^{(0)}_{p_{+,1}}(x,t)}{(k-p_{+,1})^{\frac{1}{4}}}+O\left((k-p_{+,1})^{-\frac{1}{4}}\right), \quad k\to p_{+,1}, \tag{2.48a}$$

$$\Phi_{+,2}(x,t;k)=\frac{\zeta^{(0)}_{p_{+,2}}(x,t)}{(k-p_{+,2})^{\frac{1}{4}}}+O\left((k-p_{+,2})^{-\frac{1}{4}}\right), \quad k\to p_{+,2}, \tag{2.48b}$$

$$\Phi_{-,2}(x,t;k)=\frac{\zeta^{(0)}_{p_{-,1}}(x,t)}{(k-p_{-,1})^{\frac{1}{4}}}+O\left((k-p_{-,1})^{-\frac{1}{4}}\right), \quad k\to p_{-,1}, \tag{2.48c}$$

$$\Phi_{-,1}(x,t;k)=\frac{\zeta^{(0)}_{p_{-,2}}(x,t)}{(k-p_{-,2})^{\frac{1}{4}}}+O\left((k-p_{-,2})^{-\frac{1}{4}}\right), \quad k\to p_{-,2}, \tag{2.48d}$$

for some vectors $\zeta^{(0)}_{p_{\pm,m}}(x,t)\neq 0$ $(m=1,2)$.

Lemma 2.8. *Under the hypothesis of Lemma 2.5, one has*

$$\Phi_{+,1}(x,t;k)=\frac{\zeta^{(0)}_{p_{+,1}}(x,t)}{(k-p_{+,1})^{\frac{1}{4}}}+\frac{\zeta^{(1)}_{p_{+,1}}(x,t)}{(k-p_{+,1})^{-\frac{1}{4}}}+O\left((k-p_{+,1})^{\frac{1}{4}}\right), \ k\to p_{+,1}, \tag{2.49a}$$

$$\Phi_{+,2}(x,t;k)=\frac{\zeta^{(0)}_{p_{+,2}}(x,t)}{(k-p_{+,2})^{\frac{1}{4}}}+\frac{\zeta^{(1)}_{p_{+,2}}(x,t)}{(k-p_{+,2})^{-\frac{1}{4}}}+O\left((k-p_{+,2})^{\frac{1}{4}}\right), \ k\to p_{+,2}, \tag{2.49b}$$

$$\Phi_{-,2}(x,t;k)=\frac{\zeta^{(0)}_{p_{-,1}}(x,t)}{(k-p_{-,1})^{\frac{1}{4}}}+\frac{\zeta^{(1)}_{p_{-,1}}(x,t)}{(k-p_{-,1})^{-\frac{1}{4}}}+O\left((k-p_{-,1})^{\frac{1}{4}}\right), \ k\to p_{-,1}, \tag{2.49c}$$

$$\Phi_{-,1}(x,t;k)=\frac{\zeta^{(0)}_{p_{-,2}}(x,t)}{(k-p_{-,2})^{\frac{1}{4}}}+\frac{\zeta^{(1)}_{p_{-,2}}(x,t)}{(k-p_{-,2})^{-\frac{1}{4}}}+O\left((k-p_{-,2})^{\frac{1}{4}}\right), \ k\to p_{-,2}, \tag{2.49d}$$

for some vectors $\zeta^{(0)}_{p_{\pm,m}}(x,t)\neq 0$ and $\zeta^{(1)}_{p_{\pm,m}}(x,t)\neq 0$ $(m=1,2)$.

The asymptotic behaviours of the scattering coefficients at the branch points $p_{\pm,m}$ $(m=1,2)$ are also derived from the determinant expression, Lemma 2.7 and Lemma 2.8.

Lemma 2.9. *Under the hypothesis of Lemma 2.4 with $m=1,2$, one has*

$$s_{11}(x,t;k)=\zeta^{(0)}_{\pm,11}(k-p_{\pm,2})^{-\frac{1}{4}}+O(k-p_{\pm,2})^{-\frac{1}{4}}, \ k\to p_{\pm,2}, \tag{2.50a}$$

$$s_{12}(x,t;k)=\zeta^{(0)}_{+,12}(k-p_{+,1})^{-\frac{1}{4}}+O(k-p_{+,1})^{-\frac{1}{4}}, \quad k\to p_{+,1}, \tag{2.50b}$$

$$s_{12}(x,t;k)=\zeta^{(0)}_{-,12}(k-p_{-,2})^{-\frac{1}{4}}+O(k-p_{-,2})^{-\frac{1}{4}}, \quad k\to p_{-,2}, \tag{2.50c}$$

$$s_{21}(x,t;k)=\zeta^{(0)}_{+,21}(k-p_{+,2})^{-\frac{1}{4}}+O(k-p_{+,2})^{-\frac{1}{4}}, \quad k\to p_{+,2}, \tag{2.50d}$$

$$s_{21}(x,t,k)=\zeta^{(0)}_{-,21}(k-p_{-,1})^{-\frac{1}{4}}+O(k-p_{-,1})^{-\frac{1}{4}}, \quad k\to p_{-,1}, \tag{2.50e}$$

$$s_{22}(x,t,k)=\zeta^{(0)}_{\pm,22}(k-p_{\pm,1})^{-\frac{1}{4}}+O(k-p_{\pm,1})^{-\frac{1}{4}}, \quad k\to p_{\pm,1}, \tag{2.50f}$$

for some constants $\zeta^{(0)}_{\pm,11}$, $\zeta^{(0)}_{\pm,12}$, $\zeta^{(0)}_{\pm,21}$ and $b^{(0)}_{\pm,22}$.

Clearly, we observe that $\mu_{+,2}(x,t;k)$ and $\mu_{-,1}(x,t;k)$ are linearly independent at the branch points $p_{-,m}$ $(m=1,2)$. Meanwhile, $\mu_{+,1}(x,t;k)$ and $\mu_{-,2}(x,t;k)$ are linearly independent at the branch points $p_{+,m}$. Finally, $\zeta^{(0)}_{\pm,11},\zeta^{(0)}_{\pm,12},\zeta^{(0)}_{\pm,21}$ and $\zeta^{(0)}_{\pm,22}$ are either all zero or all nonzero depending on the linear dependence of $\mu_{+,1}(x,t;k)$ and $\mu_{-,2}(x,t;k)$ at $p_{+,m}$. The generic case is defined when $\mu_{+,1}(x,t;k)$ and $\mu_{-,2}(x,t;k)$ are linearly independent at branch points such that $\zeta^{(0)}_{\pm,11},\zeta^{(0)}_{\pm,12},\zeta^{(0)}_{\pm,21}$ and $\zeta^{(0)}_{\pm,22}$ are all zero.

LEMMA 2.10. *Under the hypothesis of Lemma 2.5 with $m=1,2$, one has*

$$\begin{aligned}
s_{11}(x,t;k)&=\zeta^{(0)}_{\pm,11}(k-p_{\pm,2})^{-\frac14}+\zeta^{(1)}_{\pm,11}(k-p_{\pm,2})^{\frac14}+O(k-p_{\pm,2})^{\frac14},\ k\to p_{\pm,2},\\
s_{12}(x,t;k)&=\zeta^{(0)}_{+,12}(k-p_{+,1})^{-\frac14}+\zeta^{(1)}_{+,12}(k-p_{+,1})^{\frac14}+O(k-p_{+,1})^{\frac14},\ k\to p_{+,1},\\
s_{12}(x,t;k)&=\zeta^{(0)}_{-,12}(k-p_{-,2})^{-\frac14}+\zeta^{(1)}_{+,12}(k-p_{-,2})^{\frac14}+O(k-p_{-,2})^{\frac14},\ k\to p_{-,2},\\
s_{21}(x,t;k)&=\zeta^{(0)}_{+,21}(k-p_{+,2})^{-\frac14}+\zeta^{(1)}_{+,21}(k-p_{+,2})^{\frac14}+O(k-p_{+,2})^{\frac14},\ k\to p_{+,2},\\
s_{21}(x,t;k)&=\zeta^{(0)}_{-,21}(k-p_{-,1})^{-\frac14}+\zeta^{(1)}_{-,21}(k-p_{-,1})^{\frac14}+O(k-p_{-,1})^{\frac14},\ k\to p_{-,1},\\
s_{22}(x,t;k)&=\zeta^{(0)}_{\pm,22}(k-p_{\pm,1})^{-\frac14}+\zeta^{(1)}_{\pm,22}(k-p_{\pm,1})^{\frac12}+O(k-p_{\pm,1})^{\frac14},\ k\to p_{\pm,1},
\end{aligned}$$

for some constants $\zeta^{(0)}_{\pm,11}$, $\zeta^{(0)}_{\pm,12}$, $\zeta^{(0)}_{\pm,21}$, $\zeta^{(0)}_{\pm,22}$, $\zeta^{(1)}_{\pm,11}$, $\zeta^{(1)}_{\pm,12}$, $\zeta^{(1)}_{\pm,21}$ and $\zeta^{(1)}_{\pm,22}$.

The asymptotic behavior of the reflection coefficients at the branch points are derived via utilizing the same method, which we omit here.

3. Inverse problem

In order to solve the focusing Hirota Equation (1.2) with the initial conditions (2.1), we need to introduce a generalized RH problem based on the analyticity, symmetries and asymptotic behaviours of the Jost solution $\Phi_\pm(x,t;k)$ and scattering data $s_{ij}(z)$ discussed above. By solving the RH problem, we can obtain the solution of the equation. In Section 1, the RH problem as we have constructed it has been accounted for, and this section focuses on finding the conditions that underpin the RH problem separately.

3.1. Residue conditions.

As in most cases, the matrix $M(x,t;k)$ will have singularities at the eigenvalues that form a non-empty discrete spectrum. This motivates us to complete the formulation of the RH problem via taking these singularities into account when discussing the RH problem. Immediately afterwards, we come to a discussion of the residue conditions for $M(x,t;k)$.

LEMMA 3.1. *If $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$ and for all $t\in\mathbb{R}$, providing that $s_{22}(k)$ contains finite simple zeros, $Z=k_1,\ldots,k_n\subset\mathbb{C}^+\setminus\Sigma$, the matrix $M(x,t;k)$ is analytic in $\mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z})$, which has simple poles at $k_n\in Z$ and $k_n^*\in\bar{Z}$, then there exists norming constants $\gamma_1,\ldots,\gamma_n\in\mathbb{C}$ so that*

$$\operatorname*{Res}_{k=k_n} M(x,t;k)=\left(0\ \gamma_n M_1(x,t;k_n)e^{2i\theta_0(x,t,k_n)}\right),\quad n=1,\ldots,N, \tag{3.1a}$$

$$\operatorname*{Res}_{k=k_n^*} M(x,t;k)=\left(-\gamma_n^* M_2(x,t;k_n^*)e^{-2i\theta_0(x,t,k_n^*)}\ 0\right),\quad n=1,\ldots,N. \tag{3.1b}$$

We now break the symmetry of $x\to -x$ in $M(x,t;k)$ by starting from the scattering relation from the right. Indeed,

$$\widehat{M}(x,t;k)=\begin{cases}\left(\phi_{-,2}(x,t;k)e^{i\theta_0(x,t;k)\sigma_3}\quad \dfrac{\phi_{+,1}(x,t;k)e^{i\theta_0(x,t;k)\sigma_3}}{r_{11}}\right), & k\in\mathbb{C}^+\setminus\Sigma,\\ \left(\dfrac{\phi_{+,2}(x,t;k)e^{i\theta_0(x,t;k)\sigma_3}}{r_{22}}\quad \phi_{-,1}(x,t;k)e^{i\theta_0(x,t;k)\sigma_3}\right), & k\in\mathbb{C}^-\setminus\Sigma.\end{cases} \tag{3.2}$$

The jump condition: $\widehat{M}^{+}(x,t;k)=\widehat{M}^{-}(x,t;k)\hat{J}(x,t;k)$, where

$$\hat{J}(x,t;k)=J(x,t;k)[(\rho,r,\theta_0,\Sigma_+,\Sigma_-)\mapsto(\hat{\rho},\hat{r},-\theta_0,\Sigma_-,\Sigma_+)],$$

with $\hat{\rho}(k)=\frac{r_{21}(k)}{r_{11}(k)}$ and $\hat{r}(k)=\frac{1}{r_{11}(k)s_{12}(k)}$.

3.2. Growth conditions. In addition to discussing the normalization condition, jump condition and residue conditions of the matrix $M(x,t,k)$ as before, the appropriate growth conditions near the branch points should be considerd. In accordance with the behaviors of Jost solutions near the branch points, the following lemma is obtained.

LEMMA 3.2. *Providing that $u-u_\pm\in L_x^{1,1}(\mathbb{R}^\pm)$, for all $t\in\mathbb{R}$, and $W\neq0$ in the generic case, one has*

$$M(x,t;k)=\begin{cases}\left(H^{(0)}_{p_{+,1}}(x,t)+O(1)\right)(k-p_{+,1})^{-\frac{\sigma_3}{4}}, & k\to p_{+,1},\\ \left(H^{(0)}_{p_{+,2}}(x,t)+O(1)\right)(k-p_{+,2})^{\frac{\sigma_3}{4}}, & k\to p_{+,2},\\ H^{(0)}_{p_{-,m}}(x,t)+O(1), & k\to p_{-,m},\end{cases}\tag{3.3}$$

for some invertible matrices $H^{(0)}_{p_\pm,m}(x,t)$ $(m=1,2)$.

Especially, Lemma 3.2 implies that the limit of the following expression exists

$$\lim_{k\to p_{+,1}}M(x,t;k)(k-p_{+,1})^{\frac{\sigma_3}{4}},\qquad \lim_{k\to p_{-,1}}M(x,t;k),\tag{3.4a}$$

$$\lim_{k\to p_{+,2}}M(x,t;k)(k-p_{+,2})^{-\frac{\sigma_3}{4}},\qquad \lim_{k\to p_{-,2}}M(x,t;k).\tag{3.4b}$$

The conditions for the existence of the limits will be used as the growth conditions for the RH problem in the general case to ensure that the solution exists uniquely. In the exceptional case of linear correlation of $\Phi_{+,1}(x,t;k)$ and $\Phi_{-,2}(x,t;k)$ at branch points $p_\pm$, $\bar{p}_\pm$, the asymptote changes. Assuming the case where $\zeta^{(0)}_{\pm,11}$ and $\zeta^{(0)}_{\pm,22}$ are zero and $\zeta^{(1)}_{\pm,11}$ and $\zeta^{(1)}_{\pm,22}$ are not zero, where $\zeta^{(0)}_{\pm,11}$, $\zeta^{(0)}_{\pm,22}$, $\zeta^{(1)}_{\pm,11}$ and $\zeta^{(1)}_{\pm,22}$ are given by Lemma 2.10, the following result is given.

LEMMA 3.3. *Let $W\neq0$, providing that $u-u_\pm\in L_x^{1,2}(\mathbb{R}^\pm)$ and for all $t\in\mathbb{R}$, the modified eigenfunctions $\mu_\pm(x,t;k)$ are continuous at the branch points $p_{\pm,m}$ $(m=1,2)$, in the exceptional case we have*

$$M=\begin{cases}\left(H^{(0)}_{p_{+,1}}+H^{(1)}_{p_{+,1}}(k-p_{+,1})^{\frac12}+O(k-p_{+,1})^{\frac12}\right)(k-p_{+,1})^{-\frac14}, & k\to p_{+,1},\\ \left(H^{(0)}_{p_{+,2}}+H^{(1)}_{p_{+,2}}(k-p_{+,2})^{\frac12}+O(k-p_{+,2})^{\frac12}\right)(k-p_{+,2})^{\frac14}, & k\to p_{+,2},\\ \left(H^{(0)}_{p_{-,1}}+H^{(1)}_{p_{-,1}}(k-p_{-,1})^{\frac12}+O(k-p_{-,1})^{\frac12}\right)(k-p_{-,1})^{-\frac14+\frac{\sigma_3}{4}}, & k\to p_{-,1},\\ \left(H^{(0)}_{p_{-,2}}+H^{(1)}_{p_{-,2}}(k-p_{-,2})^{\frac12}+O(k-p_{+,2})^{\frac12}\right)(k-p_{-,2})^{-\frac14-\frac{\sigma_3}{4}}, & k\to p_{-,2},\end{cases}\tag{3.5}$$

where $M=M(x,t;k)$, $H=H(x,t)$, and some matrices $H^{(0)}_{p_\pm,m}(x,t)$ and $H^{(1)}_{p_\pm,m}(x,t)$ with $\det H^{(0)}_{p_\pm,m}=0$ $(m=1,2)$.

We can obviously notice that the growth conditions at $p_{+,1}$ and $p_{+,2}$ are not of the same form as those at $p_{-,1}$ and $p_{-,2}$. This asymmetry is due to the fact that we have chosen to obtain the relationship of $\Phi_-(x,t;k)$ from the right-hand side by obtaining the analytic scattering coefficients, which further leads to the definition of $M(x,t;k)$. If we choose the definition of $\hat{M}(x,t;k)$ in (3.2), then the growth conditions would match exactly to the above, except that the difference in $p_{\pm,1}$ needs to interchange. In Section 1, we show the RH problem given in RH Problem 1.1.

REMARK 3.1. For $W\neq 0$ and $t\in\mathbb{R}$, if $u-u_\pm\in L_x^{1,1}(\mathbb{R}^\pm)$ and $(u-u_\pm)_x\in L_x^{1,1}(\mathbb{R}^\pm)$ with $\mu_{+,1}(x,t;k)$ and $\mu_{-,2}(x,t;k)$ are linearly independent at the branch points $p_{\pm,m}$ $(m=1,2)$, then the matrix $M(x,t;k)$ still admits RH Problem 1.1.

3.3. Linear algebraic-integral equation and reconstruction formula.

We have attempted to convert the solution of the above RH problem into a suitable set of the linear algebraic-integral equations. For the sake of brevity, we will suppress the dependence of M, J and θ_0 in time t and space x, but this does not create the ambiguity.

LEMMA 3.4. *Let $M(x,t;k)$ be any solution of the RH problem, then $M^\dagger(x,t;k)^{-1}$ also solves the RH problem. Furthermore, $\det M(x,t;k)=1$ for $k\in\mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z})$.*

THEOREM 3.1. *The solution of the RH problem for $k\in\mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z})$ with reflectionless is given as*

$$M(x,t;k)=I+\sum_{n=1}^{N}\frac{\operatorname*{Res}_{k=k_n^*}M_-}{k-k_n^*}+\sum_{n=1}^{N}\frac{\operatorname*{Res}_{k=k_n}M_+}{k-k_n}+\frac{1}{2\pi i}\int_\Sigma\frac{M_-(\xi)(J(\xi)-I)}{\xi-k}d\xi$$

$$=I+\sum_{n=1}^{N}\left(-\frac{\gamma_n^*e^{-2i\theta_0(\bar{k}_n)}M_2(\bar{k}_n)}{k-\bar{k}_n},\frac{\gamma_ne^{2i\theta_0(k_n)}M_1(k_n)}{k-k_n}\right).$$

In what follows, we will reconstruct the solution of the focusing Hirota equation.

LEMMA 3.5. *The matrix $M(x,t;k)$ is the solution of the RH problem, then $M(x,t;k)$ admits the modified Lax pair in Lemma 1.1 with*

$$Q(x,t):=-i\lim_{k\to\infty}k[\sigma_3,M(x,t;k)]. \tag{3.6}$$

COROLLARY 3.1. *The matrix $M(x,t;k)$ is the solution of the RH problem, then the corresponding solution of the focusing Hirota equation is derived by*

$$u(x,t)=-2i\sum_{n=1}^{N}\gamma_ne^{2i\theta_0(k_n)}M_{11}(k_n)-\frac{1}{\pi}\int_\Sigma[M_-(\xi)(J(\xi)-I)]_{12}d\xi. \tag{3.7}$$

4. Soliton solution

In what follows, the soliton solution without reflection potential is considered.

Considering the element 1,2 of $M_-(x,t,k)$ at $k=k_n^*$ and the element 1,1 of $M_+(x,t,k)$ at $k=k_n$, we find

$$M_{12}(x,t,k_n^*)=\sum_{j=1}^{N}\frac{M_{11}(x,t,k_j)e^{2i\theta_0(k_j)}\gamma_n(k_j)}{k_n^*-k_j}, \tag{4.1a}$$

$$M_{11}(x,t,k_n)=1+\sum_{j=1}^{N}\frac{M_{12}(x,t,k_j^*)e^{-2i\theta_0(k_j^*)}\gamma_n^*(k_j^*)}{k_n-k_j^*}. \tag{4.1b}$$

For later convenience and use, we introduce

$$C_j^*(k_\ell^*) = -\frac{\gamma_n^*(k_j^*)}{k_\ell - k_j^*} e^{-2i\theta_0(k_j^*)}, \quad C_j(k_n) = \frac{\gamma_n(k_j)}{k_n^* - k_j} e^{2i\theta_0(k_j)}. \tag{4.2}$$

Substituting (4.2) into the expression about $M_{11}(x,t,k)$, and finally we can obtain

$$M_{11}(x,t,k_n) = 1 + \sum_{j=1}^{N}\sum_{\ell=1}^{N} C_j^*(k_n^*) C_\ell(k_j) M_{11}(k_\ell). \tag{4.3}$$

In order to write the solution into a simplified matrix form, the following notation is further introduced,

$$\mathbb{W} = (W_1, W_2, \cdots, W_N)^T, \; W_n = M_{11}(k_n), \; \mathbb{I} = \mathbb{I}_{N\times 1} = (1,1,\cdots,1)^T, \tag{4.4}$$

$$G = G_{n,\ell} = \sum_{j=1}^{N} C_j^*(k_n^*) C_\ell(k_j) M_{-,11}(k_j), \; n,j = 1,2,\ldots,N, \tag{4.5}$$

then $M_{11}(k_n)$ can be expressed by $(\mathbb{I}+G)\mathbb{W} = \mathbb{I} \triangleq \mathbb{F}\mathbb{W} = \mathbb{I}$, the solution of the Hirota equation can be derived by

$$u(x,t) = -2i\frac{\mathbb{F}^{aug}}{\mathbb{F}}, \tag{4.6}$$

where

$$\mathbb{F}_{aug} = \begin{pmatrix} 0 & -\gamma_1 e^{-2i\theta_0(k_1^*)} & \cdots & -\gamma_N e^{-2i\theta_0(k_N^*)} \\ 1 & 1+G_{11} & \cdots & G_{1N} \\ 1 & \vdots & \ddots & \vdots \\ 1 & G_{N1} & \cdots & 1+G_{NN} \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} 1+G_{11} & \cdots & G_{1N} \\ \vdots & \ddots & \vdots \\ G_{N1} & \cdots & 1+G_{NN} \end{pmatrix}.$$

5. The modified Riemann-Hilbert problem

5.1. Alternative solutions of the Lax pair. This section of work prepares for the construction of the modified RH problem, for which a solution is found for the Lax pair with the initial value conditions.

PROPOSITION 5.1. *Suppose that $u(x,t)$ is a bounded classical solution of the Equation (1.2) defined for $(x,t) \in \mathbb{R}\times[0,\infty)$. For any $k \in \mathbb{C}$, there is a unique simultaneous fundamental solution $\psi(x,t;k)$ for both parts of Lax pair (1.4) together with initial conditions $\psi(0,0;k) = I$. Furthermore, $\psi(x,t;k)$ is an entire function with respect to k. Moreover, $\det\psi\ (x,t;k) = 1$.*

PROPOSITION 5.2. *Under the hypotheses of Theorem 2.1 and the Lemma 5.1, one has*

$$\psi(x,t;k) = \Psi(x,t;k) C^{-1}(k), \quad k \in \mathbb{C}\setminus(\Sigma\cup Z\cup\bar{Z}), \tag{5.1}$$

where $C(k) = \Psi(0,0;k)$.

This definition is given above as the definition of $\psi(x,t;k)$, and it is not difficult to find that $\psi(x,t;k)$ is still an entire function. The symmetry is then passed to $\psi(x,t;k)$, although it follows directly from (2.29) and Lemma 5.1 that $\psi(x,t;k)$ satisfies the following symmetry property. Under the hypotheses of Lemma 5.1 with $k \in \mathbb{C}$, one has $\psi^*(x,t;k^*) = -\sigma\psi(x,t;k)\sigma$.

5.2. Existence and uniqueness of solutions of the RH problem. Owing to the singular behavior of $M(x,t;k)$ at the branch points, the existence and uniqueness of the solution to the RH problem is nontrivial. We can define a substitutable matrix and the RH problem that are also regular at the branch points, and we develop this result below. First we choose a sufficiently large $R>0$ such that the ball B_R, centre at the origin of the complex k-plane and with R as its radius, contains the branch cuts $\Sigma_\pm$ and the zeros of analytic scattering coefficients $s_{11}(k)$ and $s_{22}(k)$. The modified piecewise analytic matrix function will be introduced as follows.

$$M^{(1)}(x,t;k)=\begin{cases}\Psi(x,t;k)e^{-i\theta_0(x,t;k)\sigma_3}, & k\in\mathbb{C}\setminus(-\infty,-R]\cup B_R\cup[R,\infty),\\ \psi(x,t;k)e^{-i\theta_0(x,t;k)\sigma_3}, & k\in B_R,\end{cases} \tag{5.2}$$

where $\Psi(x,t;k)$ and $\psi(x,t;k)$ are defined in (2.28) and Proposition 5.2, respectively. In this way, we put all the points with spectral singularities in a large circle, so that the constructed $M^{(1)}$ is a completely analytic function (detail in Section 1, the modified RH problem). The modified RH problem is constructed, which places the zeros of $s_{11}(k)$ and $s_{22}(k)$ inside B_R, $\psi(x,t;k)$ is analytic, so $M^{(1)}$ is not only piecewise meromorphic, but piecewise analytic. It also indicates that the modified RH problem does not need the residue conditions, and it is obvious that $M^{(1)}$ is analytic on the branch cuts $\Sigma_\pm$ and also on the branch points $p_{\pm,m}$ $(m=1,2)$. In Section 1, we show the RH problem given in (5.2).

THEOREM 5.1. *For all $t\in\mathbb{R}$, if $u-u_\pm\in L^1_x(\mathbb{R}^\pm)$ and $(u-u_\pm)_x\in L^1_\infty(\mathbb{R}^\pm)$, then the matrix $M^{(1)}(x,t;k)$ is defined in Section 5.2 still admits the modified RH problem.*

Compared with RH Problem 1.1, the advantage of modified RH problem 1.2 is that the singularity of branch secant line and branch points $p_{\pm,m}$ $(m=1,2)$ are eliminated by introducing a large enough circle B_R. It is worth noting that the existence and uniqueness of the solution of modified RH problem can be guaranteed by vanishing lemma [48]. For simplicity, we state the vanishing inducements explicitly.

LEMMA 5.1 (The vanishing lemma shown in [48]). *For the simple smooth closed curves $\Sigma^{(1)}$, an oriented contour, with a finite number of self-intersections, find a matrix $M^{(1)}(x,t;k)$ satisfies the RH problem as follows:*

(a) *$M^{(1)}(x,t;k)$ can be analytical in $k\in\mathbb{C}\setminus\Sigma^{(1)}$.*

(b) *$M^{(1)}_+(x,t;k)=M^{(1)}_-(x,t;k)G^{(1)}(x,t;k), k\in\Sigma^{(1)}$.*

(c) *$M^{(1)}(x,t;k)=I+O(\frac{1}{k}), k\to\infty$.*

Suppose the contour $\Sigma^{(1)}$ is Schwarz symmetric and the jump matrix $G^{(1)}(x,t;k)$ such that the conditions:

(i) *$G^{(1)}(x,t;k)$ is $C^1(\Sigma^{(1)})$,*

(ii) *$G^{(1)}(x,t;k)=(G^{(1)})^\dagger(x,t;k), k\in\Sigma^{(1)}\setminus\mathbb{R}$, where $\dagger$ denotes the Schwarz conjugate-transpose,*

(iii) *$\operatorname{Res}G^{(1)}(x,t;k)$ is positive definite for $k\in\Sigma^{(1)}\cap\mathbb{R}$. Then the RH problem admits a unique solution.*

Because the contour of RH problem is not closed, Lemma 5.1 is not applicable to RH Problem 1.1, but it does satisfy the requirements of the modified RH Problem 1.2.

THEOREM 5.2. *Providing that $\rho(k)\in C^1(\mathbb{R}\setminus(-R,R))$ and $C(k)\in C^1(\partial B_R\cap\mathbb{C}^\pm)$ with $C(k)C^\dagger(k)=I$. The modified RH problem admits a unique solution.*

Finally, we construct an appropriate mapping between the two RH problems (RH problem and modified RH problem) to find the correlation, so as to further establish the uniqueness of the solution of the first RH problem. In detail, if $M_c(x,t;k)$ is any solution to the first RH problem, then let $C_0(k)=M_c(0,0;k)$. From Lemma 3.4, we know that $C_0(k)$ must have the same determinant. After fixing $C_0(k)$, we now define the following mapping F_c that operates on the matrix-valued function $\nu(x,t;k)$.

$$F_c(m)(x,t;k)=\begin{cases}\nu(x,t;k), & k\in\mathbb{C}\setminus(\Sigma\cup B_R),\\ \nu(x,t;k)e^{i\theta_0(x,t;k)\sigma_3}C_0^{-1}(k)e^{-i\theta_0(x,t;k)}, & k\in B_R\setminus\Sigma.\end{cases} \tag{5.3}$$

THEOREM 5.3. *For $x\in\mathbb{R}$ and $t\in\mathbb{R}^+$, the RH problem has a unique solution under the condition that $\rho(k)\in C^1(\mathbb{R}\setminus(-R,R))$ and the RH problem processes a solution $C_0(k)$ such that $C_0(k)C_0^\dagger(k)=I$.*

6. Solution for special case

In this section, we discuss the soliton solution and simple breather solution of the focusing Hirota equation with special initial value conditions, that is, the initial value is a nonzero constant at infinity, i.e., for $t=0$ and the constants $A_-=A_+>0$, where considering the initial condition $u(x,0)=A$, $x\to\pm\infty$ with $\epsilon=h_\pm=0$.

In this case, the fundamental solution of Lax pair (1.4) can be written as

$$\Phi_A(x,t,k)=E_\pm^{(1)}(k)e^{i\theta_A\sigma_3} \tag{6.1}$$

with

$$E_\pm^{(1)}(k)=\begin{pmatrix}1 & \frac{i(\lambda-k)}{A}\\ \frac{i(\lambda-k)}{A} & 1\end{pmatrix},\quad \theta_A(x,t,k)=\lambda(k)\left[x+4\beta k^2t-2\alpha kt-2\beta A^2t\right]. \tag{6.2}$$

We then define the Jost solutions of Lax pairs under boundary conditions as

$$\Phi_\pm(x,t,k)=\upsilon_\pm(x,t,k)e^{i\theta_A\sigma_3},\quad x\to\pm\infty, \tag{6.3}$$

which satisfy the following Volterra integral form as

$$\upsilon_-(x,t,k)=E_+^{(1)}(k)+\int_{-\infty}^{x}E_-^{(1)}(k)e^{i\lambda(x-y)\hat{\sigma_3}}(E_-^{(1)}(k))^{-1}\Delta Q_-(y,t)\upsilon_-(y,t,k)dy, \tag{6.4a}$$

$$\upsilon_+(x,t,k)=E_+^{(1)}(k)-\int_{x}^{+\infty}E_+^{(1)}(k)e^{i\lambda(x-y)\hat{\sigma_3}}(E_+^{(1)}(k))^{-1}\Delta Q_+(y,t)\upsilon_+(y,t,k)dy, \tag{6.4b}$$

where $\Delta Q_\pm=Q-Q_\pm$.

PROPOSITION 6.1. *Assume that $u-u_\pm\in\mathbb{L}^1(\mathbb{R}^\pm)$, the matrices $\upsilon_\pm$ have the following properties:*

(1) *The columns $\upsilon_{+,1}$ and $\upsilon_{-,2}$ are analytic in $k\in\mathbb{C}^+\setminus\Sigma^+$;*

(2) *The columns $\upsilon_{-,1}$ and $\upsilon_{+,2}$ are analytic in $k\in\mathbb{C}^-\setminus\Sigma^-$;*

(3) *The solution $\upsilon_\pm(x,t,k)$ of (6.4) are existent and unique,*

where we denote by $\upsilon_{\pm,j}$ $(j=1,2)$ the j-th column of $\upsilon_\pm(x,t,k)$, $\mathbb{C}^\pm=\{k\in\mathbb{C}: Imk\gtrless 0\}$, $\Sigma^\pm=\Sigma\cap\mathbb{C}^\pm$ and $\Sigma=i[-q_0,q_0]$. There is no doubt that $\upsilon_\pm$ and $\Phi_\pm$ have the same analytical properties.

An application of Abel's theorem yields

$$\det\Phi_{\pm}=\lim_{x\to\pm\infty}\det\Phi_{\pm}(x,t,k)=\frac{2\lambda}{\lambda+k}\triangleq d^{(1)}(k). \tag{6.5}$$

Not only that, we can obtain the symmetry property as follows:

$$Q^*(x,t)=-\sigma_*Q(x,t)\sigma_*;\quad \Phi^*(x,t,k^*)=-\sigma_*\Phi(x,t,k)\sigma_*, \tag{6.6}$$

with $\sigma_*=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. The scattering matrix $S(k)$ is defined by $\Phi_-(x,t,k)=\Phi_+(x,t,k)S(k)$ with $k\in\Sigma$. We can further rewrite the scattering matrix into the following form via using the symmetry condition

$$S(k)=\begin{pmatrix}s_{11}(k) & -s_{21}^*(k^*)\\ s_{21}(k) & s_{11}^*(k^*)\end{pmatrix}. \tag{6.7}$$

Note that $s_{11}(k)$ has Schwarz conjugate $s_{11}^*(k)$, which means $s_{11}(k)=s_{11}^*(k)$. Further, the elements are represented by Wronskian determinant, and it can be deduced that $s_{11}(k)$ is analytic in $\mathbb{C}^-\setminus\Sigma^-$ and $s_{11}^*(k^*)$ is analytic in $\mathbb{C}^+\setminus\Sigma^+$. The jump discontinuity of $\lambda(k)$ across $i[-A,A]$ induces a corresponding jump for the eigenfunctions and scattering data. Taking Σ to be oriented upwards, we have

$$\upsilon_{+,1}^+(x,t,k)=\frac{\lambda+k}{iu_+}\upsilon_{+,2}(x,t,k),\quad \upsilon_{+,2}^+(x,t,k)=\frac{\lambda+k}{iu_+^*}\upsilon_{+,1}(x,t,k), \tag{6.8a}$$

$$\upsilon_{-,1}^+(x,t,k)=\frac{\lambda+k}{iu_-}\upsilon_{-,2}(x,t,k),\quad \upsilon_{-,2}^+(x,t,k)=\frac{\lambda+k}{iu_-^*}\upsilon_{-,1}(x,t,k), \tag{6.8b}$$

$$(s_{11}^*)^+(k^*)=\frac{u_-}{u_+}s_{11}(k). \tag{6.8c}$$

Using the scattering relation, the analytic properties about $\epsilon_{\pm,j}$ $j=1,2$ are further assigned, and the meromorphic function matrix $\check{M}(x,t,k)$ is defined by

$$\check{M}(x,t,k)=\begin{cases}\left(\dfrac{\upsilon_{+,1}(x,t;k)}{s_{11}^*(k)d^{(1)}(k)},\upsilon_{-,2}(x,t;k)\right), & k\in\mathbb{C}^+\setminus\Sigma^+,\\ \left(\upsilon_{-,1}(x,t;k)\ \ \dfrac{\upsilon_{+,2}(x,t;k)}{s_{11}d^{(1)}(k)}\right), & k\in\mathbb{C}^-\setminus\Sigma^-,\end{cases} \tag{6.9}$$

and

$$r_1(k)=-\frac{s_{21}(k)}{s_{11}^*(k^*)},\quad \check{M}_{\pm}(x,t,k)=\lim_{\varepsilon\to 0_+}\check{M}(x,t,k+i\varepsilon). \tag{6.10}$$

It follows that the limit value $\check{M}_{\pm}(x,t,k)$ satisfies

$$\check{M}_+(x,t,k)=\check{M}_-(x,t,k)J(x,t,k), \tag{6.11}$$

where

$$J(k)=\begin{cases}\begin{pmatrix}\frac{1+r(k)r^*(k)}{d^{(1)}(k)} & r^*(k)e^{2i\theta_A}\\ r(k)e^{-2i\theta_A} & d^{(1)}(k)\end{pmatrix}, & k\in\mathbb{R}\setminus\{0\},\\ \begin{pmatrix}-\frac{\lambda-k}{iu_-}r^*(k)e^{2i\theta_A} & \frac{2\lambda}{iu_-^*}\\ \frac{u_-^*}{2i\lambda}[1+r(k)r^*(k)] & -\frac{\lambda+k}{iu_-^*}r(k)e^{-2i\theta_A}\end{pmatrix}, & k\in\Sigma_+,\\ \begin{pmatrix}\frac{\lambda+k}{iu_-}r^*(k)e^{2i\theta_A} & \frac{u_-}{2i\lambda}[1+r(k)r^*(k)]\\ \frac{2\lambda}{iu_-} & \frac{\lambda-k}{iu_-^*}r(k)e^{-2i\theta_A}\end{pmatrix}, & k\in\Sigma_-.\end{cases} \tag{6.12}$$

Note that if $s_{11}(k_\ell)=0$ for some $k_\ell\in\mathbb{C}^-\setminus\Sigma^-$, then $s_{11}^*(k_\ell^*)=0$. Consequently, assume $s_{11}(k_\ell)=0$, then either (a) k_ℓ is a purely imaginary number; (b) $-k_\ell^*$ is also a zero point of $s_{11}(k)$. We assume that $s_{11}(k_\ell)=0$ for simple zeros $k_\ell\in\mathbb{C}^-\setminus\Sigma^-$ $(\ell=1,2,\ldots,n)$ and then obtain the following residue formulae

$$\begin{aligned}\operatorname*{Res}_{k=k_\ell}\check{M}(x,t,k)&=\lim_{k\to k_\ell}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & C_{k_\ell}e^{2i\theta_A}\\ 0 & 0\end{pmatrix},\\ \operatorname*{Res}_{k=k_\ell^*}\check{M}(x,t,k)&=\lim_{k\to k_\ell^*}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & 0\\ C_{k_\ell^*}e^{-2i\theta_A} & 0\end{pmatrix},\end{aligned}\tag{6.13}$$

with $C_{k_\ell}=\frac{c_{k_\ell}}{\dot{s}_{11}(k_\ell)}$, $\dot{s}_{11}(k_\ell)$ is the derivative of $s_{11}(k)$ at $k=k_\ell$, which is a number that is not equal to 0. In addition, c_{k_ℓ} is a nonzero constant. If k_ℓ is not purely imaginary, we also derive the relevant residue conditions as follows

$$\begin{aligned}\operatorname*{Res}_{k=-k_\ell}\check{M}(x,t,k)&=\lim_{k\to -k_\ell}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & 0\\ C_{k_\ell}e^{-2i\theta_A} & 0\end{pmatrix},\\ \operatorname*{Res}_{k=-k_\ell^*}\check{M}(x,t,k)&=\lim_{k\to -k_\ell^*}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & -C_{k_\ell^*}e^{2i\theta_A}\\ 0 & 0\end{pmatrix}.\end{aligned}\tag{6.14}$$

Finally, the main Riemann-Hilbert problem is obtained, which is stated as follows.

RH PROBLEM 6.1. *The matrix $\check{M}(x,t;k)$ satifies the properties as follows:*

(i) *Jump condition:*

$$\check{M}_+(x,t;k)=\check{M}_-(x,t;k)J(x,t;k),$$

where the jump matrix $J(x,t,k)$ is defined in (6.12).

(ii) *Asymptotic behavior:*

$$\check{M}(x,t;k)=\mathbb{I}+O\left(\frac{1}{k}\right).$$

(iii) *Residue conditions: $\check{M}(x,t;k)$ has simple poles at which the residue conditions* (6.13) *and* (6.14) *satisfy.*

Finally, the solution $u(x,t)$ of the focusing Hirota equation can be expressed from the solution of the RH Problem 6.1 as follows:

$$u(x,t)=-2i\lim_{k\to\infty}(k\check{M}(x,t,k))_{12}.\tag{6.15}$$

6.1. One soliton solution. In this section, the RH problem when the reflection coefficient $r(k)=0$ is discussed.

RH PROBLEM 6.2. *According to the definition of piecewise analytic function $\check{M}(x,t,k)$, the RH problem admits*

(i) *Jump condition: The jump conditions are determined by*

$$\check{M}_+(x,t,k)=\check{M}_-(x,t,k)J_1(x,t,k),\quad k\in\mathbb{R}\setminus\{0\},\tag{6.16a}$$

$$\check{M}_+(x,t,k)=\check{M}_-(x,t,k)J_2(x,t,k),\quad k\in\Sigma_+,\tag{6.16b}$$

$$\check{M}_+(x,t,k)=\check{M}_-(x,t,k)J_3(x,t,k),\quad k\in\Sigma_-,\tag{6.16c}$$

where $J_\ell(x,t,k)$ $\ell=1,2,3$ *are*

$$J_1(x,t,k)=\begin{pmatrix} \frac{1+r(k)r^*(k)}{d^{(1)}(k)} & r^*(k)e^{2i\theta_A} \\ r(k)e^{-2i\theta_A} & d^{(1)}(k) \end{pmatrix}, \tag{6.17a}$$

$$J_2(x,t,k)=\begin{pmatrix} -\frac{\lambda-k}{iu_-}r^*(k)e^{2i\theta_A} & \frac{2\lambda}{iu_-^*} \\ \frac{u_-^*}{2i\lambda}[1+r(k)r^*(k)] & -\frac{\lambda+k}{iu_-^*}r(k)e^{-2i\theta_A} \end{pmatrix}, \tag{6.17b}$$

$$J_3(x,t,k)=\begin{pmatrix} \frac{\lambda+k}{iu_-}r^*(k)e^{2i\theta_A} & \frac{u_-}{2i\lambda}[1+r(k)r^*(k)] \\ \frac{2\lambda}{iu_-} & \frac{\lambda-k}{iu_-^*}r(k)e^{-2i\theta_A} \end{pmatrix}. \tag{6.17c}$$

(ii) *Residue condition:*

$$\begin{aligned} \operatorname*{Res}_{k=-ik_1} \check{M}(x,t,k) &= \lim_{k\to -ik_1} \frac{\check{M}(x,t,k)}{d^{(1)}(k)} \begin{pmatrix} 0 & i\eta e^{2i\theta_A} \\ 0 & 0 \end{pmatrix}, \\ \operatorname*{Res}_{k=ik_1} \check{M}(x,t,k) &= \lim_{k\to ik_1} \frac{\check{M}(x,t,k)}{d^{(1)}(k)} \begin{pmatrix} 0 & 0 \\ i\eta e^{-2i\theta_A} & 0 \end{pmatrix}, \end{aligned} \tag{6.18}$$

where $\eta\neq 0$ *is a real constant.*

(iii) *Normalization:* $\check{M}(x,t,k)\to I$, *as* $k\to\infty$.

In order to solve the RH Problem 6.2, firstly, we make the following transformation

$$\hat{M}(x,t,k)=\begin{cases} \check{M}(x,t,k)\begin{pmatrix} \sqrt{d^{(1)}(k)} & 0 \\ 0 & \frac{1}{\sqrt{d^{(1)}(k)}} \end{pmatrix}, & k\in\mathbb{C}^+\setminus\Sigma^+, \\ \check{M}(x,t,k)\begin{pmatrix} \frac{1}{\sqrt{d^{(1)}(k)}} & 0 \\ 0 & \sqrt{d^{(1)}(k)} \end{pmatrix}, & k\in\mathbb{C}^-\setminus\Sigma^-, \end{cases} \tag{6.19}$$

such that

$$\hat{M}(x,t,k)=\tilde{M}(x,t,k)T(k), \tag{6.20}$$

where $T(k)$ is the solution of the continuous spectrum component problem

$$T_+(k)=T_-(k)\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad k\in i(-A,A), \tag{6.21}$$

$$T(k)\to\mathbb{I}, as \quad k\to\infty, \tag{6.22}$$

with

$$T(k)=\frac{1}{2}\begin{pmatrix} \ell(k)+\ell^{-1}(k) & \ell(k)-\ell^{-1}(k) \\ \ell(k)-\ell^{-1}(k) & \ell(k)+\ell^{-1}(k) \end{pmatrix}=\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{11} \end{pmatrix} \tag{6.23}$$

and $\ell(k)=\left(\frac{k+iA}{k-iA}\right)^{\frac{1}{4}}$. We rearrange (6.20) to $\tilde{M}(x,t,k)=\hat{M}(x,t,k)T^{-1}(k)$, which infers that $\tilde{M}(x,t,k)$ is the discrete spectral component of $\hat{M}(x,t,k)$. In addition, it also signifies analytic anywhere in $k\in\mathbb{C}$ except for the poles ik_1 and $-ik_1$ satisfying the residue condition

$$\operatorname*{Res}_{k=-ik_1} \tilde{M}_1(k)=-W_{12}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1), \tag{6.24a}$$

$$\operatorname*{Res}_{k=-ik_1} \tilde{M}_2(k) = W_{11}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1), \tag{6.24b}$$

$$\operatorname*{Res}_{k=ik_1} \tilde{M}_1(k) = -W_{11}(ik_1)\delta^*(x,t)\hat{M}_1(ik_1), \tag{6.24c}$$

$$\operatorname*{Res}_{k=ik_1} \tilde{M}_2(k) = W_{12}(ik_1)\delta^*(x,t)\hat{M}_1(ik_1). \tag{6.24d}$$

Therefore, applying Liouville's theorem without reflection potential, there appears the relation

$$\tilde{M}(k) = I + \frac{\operatorname*{Res}_{k=-ik_1} \tilde{M}(k)}{k+ik_1} + \frac{\operatorname*{Res}_{k=ik_1} \tilde{M}(k)}{k-ik_1}. \tag{6.25}$$

Considering (6.23), (6.24), and (6.25), we have

$$\begin{aligned}
\hat{M}_1(x,t,k) = T_1(k) + T_{11}(k)&\left(-\frac{W_{12}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1)}{k+ik_1} - \frac{W_{11}(ik_1)\delta^*(x,t)\hat{M}_2(ik_1)}{k-ik_1}\right)\\
+T_{12}(k)&\left(\frac{W_{11}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1)}{k+ik_1} - \frac{W_{12}(ik_1)\delta^*(x,t)\hat{M}_2(ik_1)}{k-ik_1}\right),\\
\hat{M}_2(x,t,k) = T_2(k) + T_{12}(k)&\left(-\frac{W_{12}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1)}{k+ik_1} - \frac{W_{11}(ik_1)\delta^*(x,t)\hat{M}_2(ik_1)}{k-ik_1}\right)\\
+T_{22}(k)&\left(\frac{W_{11}(-ik_1)\delta(x,t)\hat{M}_1(-ik_1)}{k+ik_1} - \frac{W_{12}(ik_1)\delta^*(x,t)\hat{M}_2(ik_1)}{k-ik_1}\right).
\end{aligned}$$

Ulteriorly, one has

$$[I+\delta(x,t)\mathcal{D}]\,\hat{M}_1(-ik_1) = T_1(-ik_1) + \mathcal{F}\delta^*(x,t)\hat{M}_2(ik_1), \tag{6.26a}$$

$$[I-\delta^*(x,t)\mathcal{D}^*]\,\hat{M}_2(ik_1) = T_2(ik_1) + \mathcal{F}\delta(x,t)\hat{M}_1(-ik_1). \tag{6.26b}$$

It is thereby inferred that

$$\hat{M}_1(-ik_1) = \frac{T_1(-ik_1)(I-\delta^*(x,t)\mathcal{D}^*) + T_2(ik_1)\delta^*(x,t)\mathcal{F})}{(I+\delta\mathcal{D})(I-\delta^*(x,t)\mathcal{D}^*) + \mathcal{F}^2|\delta|^2}, \tag{6.27a}$$

$$\hat{M}_2(ik_1) = \frac{T_2(ik_1)(I+\delta(x,t)\mathcal{D}) + T_1(-ik_1)\delta(x,t)\mathcal{F})}{(I+\delta\mathcal{D})(I-\delta^*(x,t)\mathcal{D}^*) + \mathcal{F}^2|\delta|^2}, \tag{6.27b}$$

where

$$\mathcal{D} = -T_{11}(-ik_1)\dot{T}_{12}(-ik_1) + \dot{T}_{11}(-ik_1)T_{12}(-ik_1) = \frac{iA}{2(k_1^2-A^2)}, \tag{6.28a}$$

$$\mathcal{F} = \frac{|T_{11}(-ik_1)|^2 + |T_{12}(-ik_1)|^2}{2ik_1} = -\frac{i}{2}(k_1^2-A^2)^{-\frac{1}{2}}. \tag{6.28b}$$

Based on the above discussion and the expression of the focusing Hirota solution, we give the following theorem to illustrate the specific expressions of the soliton solution of the focusing Hirota equation.

Theorem 6.3. *A pair of single soliton solutions for purely imaginary discrete spectrums $\pm ik_1$ are obtained for the focusing Hirota equation on a constant background A satisfying*

$$u(x,t) = A - 2i\left(\frac{T_{11}^2(-ik_1)\delta(I-\delta^*\mathcal{D}^*) + T_{12}^2(ik_1)\delta^*(I-\delta\mathcal{D} + 2\mathcal{F}|\delta|^2 T_{11}(-ik_1)T_{12}(ik_1))}{(I+\delta\mathcal{D})(I-\delta^*(x,t)\mathcal{D}^*) + \mathcal{F}^2|\delta|^2}\right), \tag{6.29}$$

where $\delta(x,t)=i\eta e^{2i\theta_A(x,t,-ik_1)}$ *and the expressions of* $T_{1\ell}(k)$ $\ell=1,2$ *are defined in* (6.23).

6.2. Simple breather solution. In this section, the simple breathers solution of the focusing Hirota equation with the nonzero constant background is dicussed. We first expound the corresponding RH problem.

RH Problem 6.4. *The piecewise analytic function matrix* $\check{M}(x,t,k)$ *with simple poles at* $k_1=a+ib$ *and* $k_1^*=a-ib$, *where* $a>0,b<0$, *satisfies the RH problem*

- (I) *Jump condition: The limit value* $M_\pm(x,t,k)$ *satisfies the jump relation* (6.11) *with the jump matrices derived in* (6.12)*;*
- (II) *Residue condition:*

$$\begin{aligned}
\operatorname*{Res}_{k=k_1}\check{M}(x,t,k)&=\lim_{k\to k_1}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & \varpi e^{2i\theta_A}\\ 0 & 0\end{pmatrix},\\
\operatorname*{Res}_{k=k_1^*}\check{M}(x,t,k)&=\lim_{k\to k_1^*}\frac{\check{M}(x,t,k)}{d^{(1)}(k)}\begin{pmatrix}0 & 0\\ -\varpi^* e^{-2i\theta_A} & 0\end{pmatrix},
\end{aligned}\tag{6.30}$$

  where $\eta\neq 0$ *is a complex constant.*
- (III) *Normalization:* $\check{M}(x,t,k)\to\mathbb{I}$, *as* $k\to\infty$.

As in the previous section, we do the same transformation $\check{M}(x,t,k)\to\hat{M}(x,t,k)$. We know that $\hat{M}(x,t,k)$ satisfies the same jump condition (6.21), and the residue condition at the pole is restated as

$$\operatorname*{Res}_{k=k_1}\hat{M}_1(k)=\lim_{k\to k_1}\hat{M}_1(k_1)\begin{pmatrix}0 & \varpi e^{2i\theta_A(x,t,k)}\\ 0 & 0\end{pmatrix},\tag{6.31a}$$

$$\operatorname*{Res}_{k=k_1^*}\hat{M}_1(k)=\lim_{k\to k_1}\hat{M}_1(k_1)\begin{pmatrix}0 & 0\\ -\varpi^* e^{-2i\theta_A(x,t,k)} & 0\end{pmatrix}.\tag{6.31b}$$

The function $\hat{M}(x,t,k)$ can be solved in the form $\hat{M}(x,t,k)=\tilde{M}(x,t,k)T(k)$, $T(k)$ is defined as in the previous section and $\tilde{M}(x,t,k)$ reads

$$\tilde{M}(x,t,k)=\begin{pmatrix}1+\frac{F}{k-k_1}-\frac{H^*}{k-k_1^*} & -\frac{K^*}{k-k_1^*}+\frac{L}{k-k_1}\\ \frac{K}{k-k_1}-\frac{L^*}{k-k_1^*} & 1+\frac{F^*}{k-k_1^*}-\frac{H}{k-k_1}\end{pmatrix},\tag{6.32}$$

where $F=F(x,t)$, $H=H(x,t)$, $K=K(x,t)$ and $L=L(x,t)$ are unknown functions to be determined. Accordingly, the residue conditions of $\hat{M}(x,t,k)$ at $k=k_1$ is such that

$$\begin{aligned}
&F(x,t)T_{11}(k_1)+L(x,t)T_{12}(k_1)=0,\\
&K(x,t)T_{11}(k_1)-H(x,t)T_{12}(k_1)=0,\\
&F(x,t)T_{12}(k_1)+L(x,t)T_{11}(k_1)=\eta\left[T_{11}(k_1)\left(1-\frac{F^*(x,t)}{k_1+k_1^*}\right)+F(x,t)\dot{T}_{11}(k)\right.\\
&\qquad\left.+T_{12}(k_1)\left(-\frac{K^*(x,t)}{k_1-k_1^*}\right)+L(x,t)\dot{T}_{12}(k_1)\right],\\
&K(x,t)T_{12}(k_1)-H(x,t)T_{11}(k_1)=\eta\left[T_{11}(k_1)\left(-\frac{L(x,t)^*}{k_1-k_1^*}\right)+K(x,t)\dot{T}_{11}(k)\right.\\
&\qquad\left.+T_{12}(k_1)\left(1+\frac{F(x,t)^*}{k_1-k_1^*}\right)-H(x,t)\dot{T}_{12}(k_1)\right],
\end{aligned}\tag{6.33}$$

with $\eta=\varpi e^{2i\theta_A}$. We divide the last two equations in Equation (6.33) by $T_{12}(k)$ at the same time. Letting $\mathcal{A}=F+iK$ and $\mathcal{B}=F-iK$, one has

$$\begin{aligned} c\mathcal{A}+(if)\mathcal{B}^* &= \Upsilon+i, \\ (if)\mathcal{A}+(c^*)\mathcal{B}^* &= \Upsilon^*+i, \end{aligned} \tag{6.34}$$

where $\Upsilon(k)=\frac{T_{11}(k_1)}{T_{12}(k_1)}$ and

$$c=-\frac{\eta^{-1}}{T_{12}^2(k_1)}-\dot{\Upsilon}(k_1), \quad f=\frac{1+|R|^2}{k_1-k_1^*}.$$

Utilizing Cramer's rule, we have

$$\mathcal{A}=\frac{\mathcal{X}+i\mathcal{Y}}{\mathcal{U}}, \tag{6.35}$$

where

$$\mathcal{X}\triangleq\mathcal{X}_1+i\mathcal{X}_2=\Upsilon c^*-f, \quad \mathcal{Y}\triangleq\mathcal{Y}_1+i\mathcal{Y}_2=c^*+f\Upsilon^*, \;\; \mathcal{U}=|c|^2-h^2. \tag{6.36}$$

It is worth noting that e and f are purely imaginary, and $\mathcal{U}$ and $\mathcal{V}$ are real. Based on the above discussion and the expression of the focusing Hirota solution, we give the following theorem to illustrate the specific expression of the soliton solution of the focusing Hirota equation.

THEOREM 6.5. *The solution of the focusing Hirota equation obtained from the solution of RH problem 6.4 with discrete spectrum k_1 and complex parameter ϖ on a constant background A satisfying*

$$\begin{aligned} u(x,t) &= A+4Im(\Upsilon F-K) \\ &= A-\frac{4\mathcal{U}(\Upsilon_1\mathcal{X}_2+\Upsilon_2\mathcal{X}_1-\mathcal{Y}_2)}{\mathcal{U}^2}, \end{aligned} \tag{6.37}$$

where $\Upsilon(k)\triangleq\Upsilon_1(k)+i\Upsilon_2(k)=-\frac{i\sqrt{k_1^2+A^2}+ik_1}{A}$.

7. Conclusions and discussions

In general, we extended the IST rigorously to study the focusing Hirota equation at infinity with a class of initial value conditions. Spectral characteristics of scattering problem describes that there are four branch points due to $W\neq 0$ with counterpropagating flows. Therefore, we should not only discuss the properties of the eigenfunctions and scattering coefficients in the appropriate region, including analytical properties, symmetries and asymptotic behaviors, but also discuss the asymptotic behaviors of the eigenfunctions and scattering coefficients at the branch points. In this work, we can not introduce a uniformization variable to map the multivalued eigenvalues to the complex plane, which makes it a single-valued function. Then, the inverse problem is transformed into a matrix RH problem jumping along the real axis and branch cuts, and the reconstruction formula is successfully obtained. Finally, using the strict framework given by Zhou [48], we transform the residue condition of the existing spectral singularity (the possible zero of the analytical scattering coefficient on the continuous spectrum) into a jump on the circumference, which is solved by introducing a modified RH problem.

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