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THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE LANDAU-LIFSHITZ EQUATION WITH GILBERT DAMPING TERM*

QI GUO† AND YAMIN XIAO‡

Abstract. In this paper, we establish the existence of global smooth solutions for the Landau-Lifshitz type system on a finite interval $[0,L]$. The proof is based on the technique of finite difference-differential and a priori estimates. Our result matches the known result on periodic boundary condition in [Guo and Huang, Discrete Contin. Dyn. Syst., 5(4):729–740, 1999].

Keywords. Landau-Lifshitz system; initial-boundary value problem; smooth solutions.

AMS subject classifications. 35B65; 35Q60.

1. Introduction

In this paper, we consider the initial-boundary value problem (IVP) on a finite interval $[0,L]$ for the Landau-Lifshitz type system

$$\begin{cases} \vec{Z}_t = \vec{Z}\times\vec{Z}_{xx} + \vec{Z}\times f(\vec{Z}), & (x,t)\in[0,L]\times\mathbb{R}^+ \\ \vec{Z}(0,t)=\vec{g_0}(t), \vec{Z}(L,t)=\vec{g_1}(t), & t\in\mathbb{R}^+ \\ \vec{Z}(x,0)=\vec{\varphi}(x), & x\in[0,L] \end{cases} \tag{1.1}$$

and the system with the Gilbert damping term

$$\begin{cases} \vec{Z}_t = -\varepsilon\vec{Z}\times(\vec{Z}\times\vec{Z}_{xx}) + \vec{Z}\times\vec{Z}_{xx} + \vec{Z}\times f(\vec{Z}), & (x,t)\in[0,L]\times\mathbb{R}^+ \\ \vec{Z}(0,t)=\vec{g_0}(t), \vec{Z}(L,t)=\vec{g_1}(t), & t\in\mathbb{R}^+ \\ \vec{Z}(x,0)=\vec{\varphi}(x), & x\in[0,L] \end{cases} \tag{1.2}$$

where $\vec{Z}(x,t)=(z_1(x,t),z_2(x,t),z_3(x,t)):[0,L]\times\mathbb{R}^+\to\mathbb{S}^2\subset\mathbb{R}^3$ is an unknown vector-valued function with normalized length, $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, $\vec{f}=(f_1,f_2,f_3)$ is a given three-dimensional vector function, the $\vec{\varphi}(x)$, $\vec{g_0}(t)$ and $\vec{g_1}(t)$ are three-dimensional initial and boundary vector functions, respectively. $\varepsilon>0$ is the Gilbert damping parameter, "$\times$" is the cross-product operator of two 3-dimensional vectors. And the system (1.1) and (1.2) satisfies the additional compatibility condition $\vec{\varphi}(0)=\vec{g_0}(0)$. Without loss of generality, we take $L=1$.

The Landau-Lifshitz system was introduced by Landau and Lifshitz [12], which describes the evolution of spin fields in continuum ferromagnetism. The Cauchy problem for the Landau-Lifshitz system for ferromagnets

$$\vec{Z}_t = \vec{Z}\times\vec{Z}_{xx} + \vec{Z}\times J\vec{Z} \tag{1.3}$$

has been proved by the inverse transform method in [16], where $J=\mathrm{diag}(J_1,J_2,J_3), J_1\le J_2\le J_3$ is a diagonal matrix. The Landau-Lifshitz system after neglecting the Gilbert

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†The Graduate School of China Academy of Engineering Physics, Beijing, 100088, P.R. China (guoqi20@gscaep.ac.cn).

‡Corresponding author. The Graduate School of China Academy of Engineering Physics, Beijing, 100088, P.R. China (xiaoyamin20@gscaep.ac.cn).

damping term is known as the Heisenberg system [18]

$$\vec{Z}_t = \vec{Z} \times \vec{Z}_{xx}. \tag{1.4}$$

For system (1.4), the existence and uniqueness of global smooth solutions have been established in [21]. Guo and Han [5] proved the multidimensional case for system (1.4) under the condition that the gradient of the solutions is bounded in space $L^2(0,T;L^\infty(\mathbb{R}^n))$. Zhou and Guo [20] considered the system (1.4) with several variables

$$\vec{Z}_t = \vec{Z} \times \vec{Z}_{xx} + f(x,t,\vec{Z}), \tag{1.5}$$

and they proved the global existence of weak solutions with the nonlinear boundary conditions $\vec{Z}_x(0,t) = \mathrm{grad}\psi_0(t,\vec{Z}(0,t)), -\vec{Z}_x(L,t) = \mathrm{grad}\psi_1(t,\vec{Z}(L,t))$, and the initial condition $\vec{Z}(x,0) = \vec{\varphi}(x)$, where $f(x,t,\vec{Z})$ is a known three-dimensional vector function with variables x, t, $\vec{Z}$, $\psi_0(t,\vec{Z})$ and $\psi_1(t,\vec{Z})$ are the scalar functions, "grad" denotes the gradient operator with respect to $\vec{Z}$. Guo and Hong [6] studied the existence of global smooth solutions for the 2D system (1.6)

$$\vec{Z}_t = -\varepsilon\vec{Z} \times (\vec{Z} \times \vec{Z}_{xx}) + \vec{Z} \times \vec{Z}_{xx}, \tag{1.6}$$

by using the properties of Harmonic mapping. Recently, Guo and Huang [9] considered the global well-posedness for the $n-$dimensional system (1.6) in a critical Besov space with $n \geq 3$. There are several interesting results for the system (1.6) (see, e.g., [1, 4, 8, 10, 13, 14, 17]).

In this paper, we mainly consider the initial-boundary problem for system (1.1) by the spatial difference method and a priori estimates. On studying the solution of the Landau-Lifshitz type systems in a difference scheme, we can refer to [2, 3, 7, 15]. In particular, Guo and Huang [7] constructed the smooth solutions of the initial value problem with periodic boundary conditions for system (1.1). Here, we are first concerned with the initial-boundary problem for the following diffusion system

$$\begin{cases} \vec{Z}_t = \varepsilon\vec{Z}_{xx} + \varepsilon|\vec{Z}_x|^2\vec{Z} + \vec{Z} \times \vec{Z}_{xx} + \vec{Z} \times f(\vec{Z}), \\ \vec{Z}(0,t) = \vec{g_0}(t), \vec{Z}(1,t) = \vec{g_1}(t), \\ \vec{Z}(x,0) = \vec{\varphi}(x). \end{cases} \tag{1.7}$$

By the spatial difference method, we establish the existence of the local smooth solution for (1.7). At the same time, we shall show that initial-boundary value problem (1.7) with $\varepsilon > 0$ is equivalent in the classical sense to the system (1.2).

THEOREM 1.1. *For any $\varepsilon > 0$. Suppose $\vec{g_0}(t)$, $\vec{g_1}(t) \in C^{2m+1}(\mathbb{R}^+)$, $\vec{\varphi}(x) \in H^{2m+1}([0,1])$, $f(\vec{Z}) \in C^{2m+1}(\mathbb{R}^3)$, $m \geq 1$. Then the initial-boundary value problem (1.2) and (1.7) admits at least one local smooth solution $\vec{Z}_\varepsilon(x,t)$ satisfying*

$$\vec{Z}_\varepsilon(x,t) \in \Big(\bigcap_{s=0}^{m} W^s_\infty(0,T_0;H^{2(m-s)+1}([0,1]))\Big) \cap \Big(\bigcap_{s=0}^{m+1} H^s(0,T_0;H^{2(m-s)+2}([0,1]))\Big),$$

where $T_0 > 0$ is independent of m,s, and m,s are non-negative integers with $m - s \geq 0$.

Next, we prove the uniform estimates for system (1.2) independent of ε and obtain the existence of global smooth solutions for system (1.2) or (1.7).

THEOREM 1.2. *For any $\varepsilon>0$. Suppose that $\vec{g_0}(t)$, $\vec{g_1}(t)\in\mathcal{C}^m(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^m([0,1])$, $f(\vec{Z})\in\mathcal{C}^m(\mathbb{R}^3)$ with $m\geq 2$. Then the initial-boundary value problem (1.2) and (1.7) admits a global smooth solution $\vec{Z}_\varepsilon(x,t)$ satisfying $\vec{Z}_\varepsilon(x,t)\in S^2$ and*

$$\vec{Z}_\varepsilon(x,t)\in\Big(\bigcap_{s=0}^{[m/2]}W^s_\infty(0,T;H^{m-2s}([0,1]))\Big)\cap\Big(\bigcap_{s=0}^{[m+1/2]}H^s(0,T;H^{m-2s+1}([0,1]))\Big)$$

for any $T>0$, where m,s are non-negative integers with $m-2s\geq 0$.

Finally, we achieve the existence of a unique global smooth solution for (1.1) by passing to the limit as $\varepsilon\to 0$.

THEOREM 1.3. *Let $\vec{g_0}(t)$, $\vec{g_1}(t)\in C^m(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^m([0,1])$, $f(\vec{Z})\in C^m(\mathbb{R}^3)$ with $m\geq 2$. Then the initial-boundary value system (1.1) admits global smooth solutions $\vec{Z}(x,t)$ satisfying $\vec{Z}(x,t)\in S^2$ and*

$$\vec{Z}(x,t)\in\bigcap_{s=0}^{[m/2]}W^s_\infty(0,T;H^{m-2s}([0,1])).$$

Especially, when $m\geq 3$, the solution $\vec{Z}$ is unique.

Notations.

(i) Setting $x_j=jh(j=0,1,...J)$, where $h=1/J$, J is a positive integer. Then we define the discrete functions $u_j=u(x_j,t)$, and $\Delta_+u_j=u_{j+1}-u_j$, $\Delta_-u_j=u_j-u_{j-1}$. Denote the discrete function spaces

$$\|\delta^k u_h\|_p=\Big(\sum_{j=0}^{J-k}|\frac{\Delta_+^k u_j}{h^k}|^p h\Big)^{1/p},\quad \|\delta^k u_h\|_\infty=\max_{0\leq j\leq J-k}|\frac{\Delta_+^k u_j}{h^k}|,$$

and

$$\|u_h\|_{\tilde{H}^p}=\Big(\sum_{k=0}^{p}\|\delta^k u_h\|_2^2\Big)^{1/2},$$

where $u_h=\{u_j|j=0,1,...,J\},1\leq p<\infty,0\leq k<J$.

(ii) Denote the Sobolev space $W^k_p(\mathbb{R})=\{u\in L^p(\mathbb{R})$ and $\|u\|_{W^k_p}=\sum_{j=0}^k\|\frac{\partial^j u}{\partial x_j}\|_p\leq\infty\}$, where $1\leq p\leq\infty$. In particular, $W^k_2(\mathbb{R})=H^k(\mathbb{R})$.

(iii) Throughout the paper, C stands for a generic positive constant, which may be different from line to line. We will use the notation $A\lesssim B$ to denote the relation $A\leq CB$, $\|\cdot\|_p$ to denote $\|\cdot\|_{L^p}$ for conciseness.

This paper is organized as follows. In Section 2, we present several important lemmas, which will be frequently used throughout the rest of this paper. In Section 3, we construct the finite difference-differential system (3.1) and prove Theorem 1.1. In Section 4, we give some independent estimates about corresponding parameter ε. And the Theorem 1.3 will be proved in Section 5.

2. Preliminaries

In this preliminaries section, we present some lemmas for the discrete functions $u_j=u(x_j,t)$ which play an important role in our proofs.

LEMMA 2.1. *For the dispersed $\{u_j\}$ and $\{v_j\}$, we have*

$$\sum_{j=1}^{J} u_j\Delta_- v_j = -\sum_{j=0}^{J-1} v_j\Delta_+ u_j - u_0v_0 + u_Jv_J,$$
$$\sum_{j=1}^{J-1} u_j\Delta_+\Delta_- v_j = -\sum_{j=0}^{J-1}(\Delta_+ u_j)(\Delta_+ v_j) - u_0\Delta_+ v_0 + u_J\Delta_- v_J.$$

In what follows, let's recall the Gagliardo-Nirenberg inequalities for discrete functions (see [19]).

LEMMA 2.2. *Let p be a real number and k, n be integers such that $2\le p\le\infty$, $0\le k<n$. Then we have*

$$\|\delta^k u_h\|_p \le C\|u_h\|_2^{1-r}(\|\delta^n u_h\|_2 + \|u_h\|_2)^r,$$

for $r=1/n(k+1/2-1/p)$.

LEMMA 2.3 ([11]). *Let $L_h^2=\{u_h|(\sum_{j=0}^{J-1}|u_j|^2h)^{1/2}<\infty\}$ be the space of discrete three-dimensional vector-valued functions. For each $h>0$ there is operator: $I_h: L_h^2\to L^2$ such that if $u=I_hu_h$, then $u(x_j)=u_j, j=1,2,\cdots,J$, and u is entire analytic. The mapping I_h can commute with shift and difference operations. Moreover*

$$C\|\frac{\partial^k u}{\partial x^k}\|_2 \le \|\delta^k u_h\|_2 \le \|\frac{\partial^k u}{\partial x^k}\|_2,$$

where $C>0$ depends on k.

3. Proof of Theorem 1.1

In this section, we prove that system (1.7) admits at least one local smooth solution by using difference in the spatial direction. For simplicity, we let $\varepsilon=1$ and establish the corresponding finite difference-differential system by (i) of Notations

$$\begin{cases} \frac{\mathrm{d}\vec{Z}_j}{\mathrm{d}t} = \frac{\Delta_+\Delta_-\vec{Z}_j}{h^2} + \vec{Z}_j\times\frac{\Delta_+\Delta_-\vec{Z}_j}{h^2} + \vec{Z}_j\times f(\vec{Z}_j) + |\frac{\Delta_+\vec{Z}_j}{h}|^2\vec{Z}_j, \quad j=1,2,\dots J-1,\\ \vec{Z}(0,t)=\vec{Z}_0=\vec{g_0}(t), \vec{Z}(1,t)=\vec{Z}_J=\vec{g_1}(t),\\ \vec{Z}_j|_{t=0}=\vec{\varphi}(x_j)=\vec{\varphi}_j, \quad j=0,1,\dots J, \end{cases} \tag{3.1}$$

where $\vec{Z}_j=\vec{Z}(x_j,t)$, $x_j=jh$, $j=0,1,\dots J$, $h=1/J$, $J>0$.

Note that the local existence of smooth solutions of the Equation (3.1) can be proved by the general theory of the ordinary differential equations. So we only need to derive some a priori estimates of h independently for such solutions, which allows the local existence of system (1.7) to be solved smoothly by sending $h\to 0$.

LEMMA 3.1. *Let $\vec{g_0}(t),\vec{g_1}(t)\in\mathcal{C}^1(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^1([0,1])$, $f(\vec{Z})\in\mathcal{C}^1(\mathbb{R}^3)$. Suppose $\vec{Z}_j(t)$ is the smooth solution of the differential system (3.1), then there exists $C>0$ independent of h such that*

$$\sup_{0\le t\le T_0}\|\vec{Z}_h(\cdot,t)\|_2\le C, \qquad \sup_{0\le t\le T_0}\|\delta\vec{Z}_h(\cdot,t)\|_2\le C$$

for all $0\le t\le T_0$.

Proof. Taking the scalar product of $(3.1)_1$ with $\vec{Z}_j h$ and $\frac{\Delta_+\Delta_-\vec{Z}_j}{h}$, respectively, summing from $j=1$ to $J-1$, and by Hölder inequality we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{Z}_h\|_2^2+\|\delta\vec{Z}_h\|_2^2\lesssim\|\vec{Z}_h\|_\infty^2\|\delta\vec{Z}_h\|_2^2+(|\vec{g_0}|+|\vec{g_1}|)\|\delta\vec{Z}_h\|_\infty, \tag{3.2}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\vec{Z}_h\|_2^2+\|\delta^2\vec{Z}_h\|_2^2\\ &\lesssim\|\vec{Z}_h\|_2\|\delta^2\vec{Z}_h\|_2+\|\vec{Z}_h\|_\infty\|\delta\vec{Z}_h\|_4^2\|\delta^2\vec{Z}_h\|_2+(\|\vec{g_0}'\|_\infty+\|\vec{g_1}'\|_\infty)\|\delta\vec{Z}_h\|_\infty.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and applying the following interpolation inequalities by Lemma 2.2

$$\begin{aligned}\|\vec{Z}_h\|_\infty&\lesssim\|\vec{Z}_h\|_2^{1/2}(\|\delta\vec{Z}_h\|_2+\|\vec{Z}_h\|_2)^{1/2},\\ \|\delta\vec{Z}_h\|_\infty&\lesssim\|\delta\vec{Z}_h\|_2^{1/2}(\|\delta^2\vec{Z}_h\|_2+\|\delta\vec{Z}_h\|_2)^{1/2},\\ \|\delta\vec{Z}_h\|_4&\lesssim\|\delta\vec{Z}_h\|_2^{3/4}(\|\delta\vec{Z}_h\|_2+\|\delta^2\vec{Z}_h\|_2)^{1/4}.\end{aligned}$$

It is derived that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\vec{Z}_h\|_2^2+\|\delta\vec{Z}_h\|_2^2)+\|\delta^2\vec{Z}_h\|_2^2\lesssim1+(\|\vec{Z}_h\|_2^2+\|\delta\vec{Z}_h\|_2^2)^5. \tag{3.4}$$

Thus, there exists $C>0$ independent of h such that

$$\|\vec{Z}_h(\cdot,t)\|_2^2+\|\delta\vec{Z}_h(\cdot,t)\|_2^2+\int_0^{T_0}\|\delta^2\vec{Z}_h(\cdot,\tau)\|_2^2\mathrm{d}\tau\le C \tag{3.5}$$

for all $0\le t\le T_0$ and we complete the proof of Lemma 3.1. ∎

LEMMA 3.2. *Let $\vec{g_0}(t)$, $\vec{g_1}(t)\in\mathcal{C}^3(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^3([0,1])$, $f(\vec{Z})\in\mathcal{C}^3(\mathbb{R}^3)$, if $\vec{Z}_j(t)$ is the smooth solution of the differential system (3.1), then there exist constants $T_0>0$, $C>0$ independent of h such that*

$$\|\vec{Z}_{ht}(\cdot,t)\|_2^2+\|\delta\vec{Z}_{ht}(\cdot,t)\|_2^2+\int_0^{T_0}\|\delta^2\vec{Z}_{ht}(\cdot,\tau)\|_2^2\,d\tau\le C.$$

Proof. Differentiating $(3.1)_1$ with respect to t and one gets

$$\begin{aligned}\vec{Z}_{jtt}=&\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}+\vec{Z}_{jt}\times\frac{\Delta_+\Delta_-\vec{Z}_j}{h^2}+\vec{Z}_j\times\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}+\vec{Z}_{jt}\times f(\vec{Z}_j)\\ &+\vec{Z}_j\times f'(\vec{Z}_j)\vec{Z}_{jt}+(|\frac{\Delta_+\vec{Z}_j}{h}|^2\vec{Z}_j)_t.\end{aligned} \tag{3.6}$$

Taking the scalar product of (3.6) with $\vec{Z}_{jt}h$ and summing from $j=1$ to $J-1$ we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{Z}_{ht}\|_2^2+\|\delta\vec{Z}_{ht}\|_2^2\\ &=\sum_{j=1}^{J-1}\vec{Z}_j\times\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}\cdot\vec{Z}_{jt}h+\sum_{j=1}^{J-1}\vec{Z}_j\times f'(\vec{Z}_j)\vec{Z}_{jt}\cdot\vec{Z}_{jt}h+\sum_{j=1}^{J-1}(|\frac{\Delta_+\vec{Z}_j}{h}|^2\vec{Z}_j)_t\cdot\vec{Z}_{jt}h\end{aligned}$$

$$+(\vec{g_0}'\cdot\vec{g_0}''+\vec{g_1}'\cdot\vec{g_1}'')h+\frac{\Delta_+\vec{Z}_{0t}}{h}\cdot\vec{Z}_{0t}-\frac{\Delta_-\vec{Z}_{Jt}}{h}\cdot\vec{Z}_{Jt}. \tag{3.7}$$

Using the Hölder, Young and Gagliardo-Nirenberg inequalities and the fact $\|\delta^2\vec{Z}_h\|_2\lesssim\|\vec{Z}_{ht}\|_2+1$, the terms on the right-hand side of (3.7) can be bounded by

$$\begin{aligned}
&\sum_{j=1}^{J-1}\vec{Z}_j\times\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}\cdot\vec{Z}_{jt}h\\
&=-\sum_{j=1}^{J-1}(\vec{Z}_j\times\vec{Z}_{jt})\cdot\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h\\
&=\sum_{j=0}^{J-1}(\Delta_+\vec{Z}_j\times\vec{Z}_{jt})\cdot\frac{\Delta_+\vec{Z}_{jt}}{h}+(\vec{Z}_0\times\vec{Z}_{0t})\cdot\frac{\Delta_+\vec{Z}_{0t}}{h}-(\vec{Z}_J\times\vec{Z}_{Jt})\cdot\frac{\Delta_-\vec{Z}_{Jt}}{h}\\
&\lesssim\frac{1}{4}\|\delta\vec{Z}_{ht}\|_2^2+\|\vec{Z}_{ht}\|_2^2+\|\delta\vec{Z}_{ht}\|_\infty,
\end{aligned}\tag{3.8}$$

$$\sum_{j=1}^{J-1}\vec{Z}_j\times f'(\vec{Z}_j)\vec{Z}_{jt}\cdot\vec{Z}_{jt}h\le\|\vec{Z}_h\|_\infty\|f'(\vec{Z})\|_\infty\|\vec{Z}_{ht}\|_2^2, \tag{3.9}$$

$$\begin{aligned}
\sum_{j=1}^{J-1}(|\frac{\Delta_+\vec{Z}_j}{h}|^2\vec{Z}_j)_t\cdot\vec{Z}_{jt}h&\lesssim\|\delta\vec{Z}_h\|_\infty\|\vec{Z}_h\|_\infty\|\delta\vec{Z}_{ht}\|_2\|\vec{Z}_{ht}\|_2+\|\delta\vec{Z}_h\|_\infty\|\vec{Z}_{ht}\|_2^2\\
&\lesssim\frac{1}{4}\|\delta\vec{Z}_{ht}\|_2^2+\|\vec{Z}_{ht}\|_2^2+\|\vec{Z}_{ht}\|_2^3,
\end{aligned}\tag{3.10}$$

and

$$(\vec{g_0}'\cdot\vec{g_0}''+\vec{g_1}'\cdot\vec{g_1}'')h+\frac{\Delta_+\vec{Z}_{0t}}{h}\cdot\vec{Z}_{0t}-\frac{\Delta_-\vec{Z}_{Jt}}{h}\cdot\vec{Z}_{Jt}\lesssim\|\delta\vec{Z}_{ht}\|_\infty. \tag{3.11}$$

Inserting the estimates (3.8)-(3.11) into (3.7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{Z}_{ht}\|_2^2+\|\delta\vec{Z}_{ht}\|_2^2\lesssim\|\vec{Z}_{ht}\|_2^3+\|\delta\vec{Z}_{ht}\|_\infty. \tag{3.12}$$

Now, let's estimate $\|\delta\vec{Z}_{ht}\|_2$. Making scalar product of (3.6) with $\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h$ and summing from $j=1$ to $J-1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\vec{Z}_{ht}\|_2^2+\|\delta^2\vec{Z}_{ht}\|_2^2\\
&=\vec{g_1}''\cdot\frac{\Delta_-\vec{Z}_{Jt}}{h}-\vec{g_0}''\cdot\frac{\Delta_+\vec{Z}_{0t}}{h}+\sum_{j=1}^{J-1}\frac{\Delta_+\Delta_-\vec{Z}_j}{h^2}\times\vec{Z}_{jt}\cdot\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h\\
&\quad+\sum_{j=1}^{J-1}f(\vec{Z}_j)\times\vec{Z}_{jt}\cdot\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h+\sum_{j=1}^{J-1}f'(\vec{Z}_j)\vec{Z}_{jt}\times\vec{Z}_j\cdot\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h\\
&\quad-\sum_{j=1}^{J-1}(|\frac{\Delta_+\vec{Z}_j}{h}|^2\vec{Z}_j)_t\cdot\frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2}h.
\end{aligned}\tag{3.13}$$

Similar to the estimate of (3.7), we obtain

$$\vec{g_1}'' \cdot \frac{\Delta_-\vec{Z}_{Jt}}{h} - \vec{g_0}'' \cdot \frac{\Delta_+\vec{Z}_{0t}}{h} \lesssim \|\delta\vec{Z}_{ht}\|_\infty \lesssim \frac{1}{10}\|\delta^2\vec{Z}_{ht}\|_2^2 + \|\delta\vec{Z}_{ht}\|_2^2 + 1, \tag{3.14}$$

$$\sum_{j=1}^{J-1} \frac{\Delta_+\Delta_-\vec{Z}_j}{h^2} \times \vec{Z}_{jt} \cdot \frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2} h \lesssim \|\vec{Z}_{ht}\|_\infty \|\delta^2\vec{Z}_h\|_2 \|\delta^2\vec{Z}_{ht}\|_2 \lesssim \frac{1}{10}\|\delta^2\vec{Z}_{ht}\|_2^2 + \|\vec{Z}_{ht}\|_2^4 + \|\delta\vec{Z}_{ht}\|_2^4, \tag{3.15}$$

$$\sum_{j=1}^{J-1} f(\vec{Z}_j) \times \vec{Z}_{jt} \cdot \frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2} h \lesssim \|\vec{Z}_{ht}\|_2 \|\delta^2\vec{Z}_{ht}\|_2 \lesssim \frac{1}{10}\|\delta^2\vec{Z}_{ht}\|_2^2 + \|\vec{Z}_{ht}\|_2^2, \tag{3.16}$$

$$\sum_{j=1}^{J-1} f'(\vec{Z}_j)\vec{Z}_{jt} \times \vec{Z}_j \cdot \frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2} h \lesssim \|\vec{Z}_{ht}\|_2 \|\delta^2\vec{Z}_{ht}\|_2 \lesssim \frac{1}{10}\|\delta^2\vec{Z}_{ht}\|_2^2 + \|\vec{Z}_{ht}\|_2^2, \tag{3.17}$$

and

$$-\sum_{j=1}^{J-1} \left(\left|\frac{\Delta_+\vec{Z}_j}{h}\right|^2 \vec{Z}_j\right)_t \cdot \frac{\Delta_+\Delta_-\vec{Z}_{jt}}{h^2} h \lesssim \|\vec{Z}_{ht}\|_2 \|\delta^2\vec{Z}_{ht}\|_2 + \|\delta\vec{Z}_{ht}\|_2 \|\delta^2\vec{Z}_{ht}\|_2 \lesssim \frac{1}{10}\|\delta^2\vec{Z}_{ht}\|_2^2 + \|\vec{Z}_{ht}\|_2^2 + \|\delta\vec{Z}_{ht}\|_2^2. \tag{3.18}$$

Inserting the above estimates into (3.13), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\vec{Z}_{ht}\|_2^2 + \|\delta^2\vec{Z}_{ht}\|_2^2 \lesssim (\|\vec{Z}_{ht}\|_2^2 + \|\delta\vec{Z}_{ht}\|_2^2)^2 + 1. \tag{3.19}$$

Combining (3.12) and (3.19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\vec{Z}_{ht}\|_2^2 + \|\delta\vec{Z}_{ht}\|_2^2) + \|\delta^2\vec{Z}_{ht}\|_2^2 \lesssim (\|\vec{Z}_{ht}\|_2^2 + \|\delta\vec{Z}_{ht}\|_2^2)^2 + 1. \tag{3.20}$$

Thus there exists $C > 0$ independent of h such that

$$\|\vec{Z}_{ht}(\cdot,t)\|_2^2 + \|\delta\vec{Z}_{ht}(\cdot,t)\|_2^2 \le C, \quad \int_0^{T_0} \|\delta^2\vec{Z}_{ht}(\cdot,\tau)\|_2^2 \mathrm{d}\tau \le C \tag{3.21}$$

for all $0 \le t \le T_0$. Then we finish the proof of Lemma 3.2. ∎

Similar to the proof of Lemmas 3.1 and 3.2, assuming that $\vec{g_0}(t), \vec{g_1}(t) \in C^{2m+1}(\mathbb{R}^+)$, $\vec{\varphi}(x) \in H^{2m+1}([0,1])$, $f(\vec{Z}) \in C^{2m+1}(\mathbb{R}^3)$, $m \ge 0$, we obtain the following lemma by the induction argument.

LEMMA 3.3. *Assume that $\vec{g_0}(t)$, $\vec{g_1}(t) \in \mathcal{C}^{2m+1}(\mathbb{R}^+)$, $\vec{\varphi}(x) \in H^{2m+1}([0,1])$, $f(\vec{Z}) \in \mathcal{C}^{2m+1}(\mathbb{R}^3)$, $m \ge 1$, then there exists $C > 0$ independent of h such that, for all $0 \le t \le T_0$,*

$$\|\vec{Z}_{ht^m}(\cdot,t)\|_2 + \|\delta\vec{Z}_{ht^m}(\cdot,t)\|_2 + \int_0^{T_0} \|\delta^2\vec{Z}_{ht^m}(\cdot,\tau)\|_2^2 \, d\tau \le C.$$

COROLLARY 3.1. *Under the conditions in Lemma 3.3, we have, for some* $C>0$ *independent of* h,

$$\|\vec{Z}_h(\cdot,t)\|^2_{\tilde{H}^{2m+1}}+\int_0^{T_0}\|\vec{Z}_h(\cdot,\tau)\|^2_{\tilde{H}^{2m+2}}d\tau\le C$$

for all $0\le t\le T_0$.

Therefore, it follows from Corollary 3.1 that, the discrete solutions $\vec{Z}_h(t)$ of ordinary difference system (3.1) are uniformly bounded concerning the step $h=1/J$ in $W^s_\infty(0,T_0;\tilde{H}^{2(m-s)+1}([0,1]))$, $0\le s\le m$. In addition, applying the Lemma 2.3, let $u_h(x,t)\in L^2([0,1])$ for $t>0$ as the image of $\vec{Z}_h(t)$ under the map $I_h:L^2_h\to L^2$, namely $u_h(x_j,t)=\vec{Z}_j(t)$, the set $\{u_h(x,t)\}$ is bounded in

$$G(T_0)=\Big(\bigcap_{s=0}^{m}W^s_\infty(0,T_0;H^{2(m-s)+1}([0,1]))\Big)\cap\Big(\bigcap_{s=0}^{m+1}H^s(0,T_0;H^{2(m-s)+2}([0,1]))\Big).$$

As demonstrated in [7], making a similar argument we obtain the existence of local smooth solutions to (1.7) with $\varepsilon=1$. And for any $\varepsilon>0$, the proof procedure can be obtained by the same way. Hence, we have the result as follows.

THEOREM 3.1. *For any* $\varepsilon>0$, $\vec{g_0}(t)$, $\vec{g_1}(t)\in\mathcal{C}^{2m+1}(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^{2m+1}([0,1])$, $f(\vec{Z})\in\mathcal{C}^{2m+1}(\mathbb{R}^3)$, $m\ge1$. *Then the initial-boundary value problem* (1.7) *admits at least one local smooth solution* $\vec{Z}(x,t)$ *satisfying* $\vec{Z}(x,t)\in G(T_0)$, *where* $T_0>0$ *is independent of* m *and* s, *and* m *and* s *are non-negative integers with* $m-s\ge0$.

Now, we prove that the diffusion system (1.7) is equivalent to the system (1.2) with $\varepsilon>0$ in the classical sense.

THEOREM 3.2. *Under the conditions in Theorem 3.1, we assume that*

$$|\vec{\varphi}(x)|=1, \tag{3.22}$$

for $x\in[0,1]$. *Then in the classical sense problem* (1.7) *is equivalent to the system* (1.2).

Proof. Let $\vec{Z}(x,t)$ be a classical solution of the system (1.2) with $\varepsilon>0$, we shall prove that $\vec{Z}(x,t)$ is also a solution of system (1.7). Indeed, due to $\vec{Z}(x,t)$ being a classical solution of system (1.2), it is easy to verify that $|\vec{Z}(x,t)|=1$ for $(x,t)\in[0,1]\times[0,T]$. Thus we have

$$-\varepsilon\vec{Z}\times(\vec{Z}\times\vec{Z}_{xx})=\alpha|\vec{Z}|^2\vec{Z}_{xx}-\varepsilon(\vec{Z}\cdot\vec{Z}_{xx})\vec{Z}=\varepsilon\vec{Z}_{xx}+\varepsilon|\vec{Z}_x|^2\vec{Z}, \tag{3.23}$$

where we have used the fact that $\vec{Z}\cdot\vec{Z}_x=0$, $|\vec{Z}_x|^2+\vec{Z}\cdot\vec{Z}_{xx}=0$, which implies that $\vec{Z}(x,t)$ is a classical solution of the problem (1.7).

On the other hand, let $\vec{Z}(x,t)$ be a classical solution of the system (1.7), we need to show that $\vec{Z}(x,t)$ satisfies the identity (1.2) for any $(x,t)\in[0,1]\times(0,T)$. In fact, we suppose that $u(x,t)$ satisfies Equation (1.7). Set $u(x,t)=|\vec{Z}(x,t)|^2$. By calculation, the Equation (1.7) becomes

$$\begin{cases} u_t=\varepsilon u_{xx}+2\varepsilon|\vec{Z}_x|^2(u-1),\\ u(x,0)=1,\\ u(0,t)=u(1,t)=1.\end{cases} \tag{3.24}$$

It is obvious that $\bar{u}=1$ is the classical solution of system (3.24). Denote that $w=u-\bar{u}=|\vec{Z}(x,t)|^2-1$, we obtain that

$$\begin{cases} w_t=\varepsilon w_{xx}+2\varepsilon|\vec{Z}_x|^2 w, \\ w(x,0)=0, \\ w(0,t)=w(1,t)=0. \end{cases} \tag{3.25}$$

Taking the scalar product of (3.25) with w and integrating it with respect to x, one gets

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |w|^2\mathrm{d}x+\varepsilon\int_0^1 |w_x|^2\mathrm{d}x &= 2\varepsilon\int_0^1 |\vec{Z}_x|^2|w|^2\mathrm{d}x \\ &\le 2\varepsilon\max_{x,t}|\vec{Z}_x|^2\int_0^1 |w|^2\mathrm{d}x. \end{aligned} \tag{3.26}$$

Thus it follows by the Grönwall inequality and $w(x,0)=0$ that $w(x,t)=0$ for any $(x,t)\in[0,1]\times(0,T)$, namely, $|\vec{Z}(x,t)|^2=1$ and we complete the proof of Theorem 3.2. □

4. Proof of Theorem 1.2

In this section, we give some a priori estimates which are independent of ε for the solution of system (1.7). First, we give two lemmas which will be used to prove those estimates.

LEMMA 4.1. *For any given positive number T, assume that $f(t)$ is a nonnegative function which makes the following inequality hold for any $t\ge 0$*

$$f(t)\le A+B\int_0^t f(\tau)d\tau+\alpha\int_0^t G(f(\tau))d\tau, \tag{4.1}$$

where A,B are normal numbers, $\alpha>0$ is a parameter, $G(\cdot)$ is a smooth function which holds $\lim_{y\to\infty}\frac{G(y)}{y^r}=0$ for some $r>1$. Then there is a normal number $C=C(A,B,T)$ such that $f(t)\le C$ for any $t\in[0,T]$ and $\alpha\in(0,\alpha_0]$, where the positive number $\alpha_0<\frac{BA^{1-r}}{e^{B(r-1)T}-1}$.

Proof. Let $F(t)=e^{-Bt}(A+B\int_0^t f(\tau)\mathrm{d}\tau+\alpha\int_0^t G(f(\tau))\mathrm{d}\tau)$. Then one has $F'(t)\le \alpha e^{B(r-1)t}F^r(t)$, where $F(0)=A$, and the proof of Lemma 4.1 is thus completed. □

LEMMA 4.2. *Under the conditions of Theorem 3.1, let $\vec{Z}(x,t)$ be a smooth solution of (1.7), and denote*

$$\mathcal{A}_i(t)=\vec{Z}_{xx}(i,t),\mathcal{B}_i(t)=\vec{Z}(i,t)\times\vec{Z}_{xx}(i,t),\mathcal{C}_i(t)=\vec{Z}(i,t)\cdot(\vec{Z}_x(i,t)\times\vec{Z}_{xx}(i,t)),$$

$$\mathcal{D}_i(t)=\vec{Z}_x(i,t)\cdot\vec{Z}_{xx}(i,t),\mathcal{E}_i(t)=\vec{Z}_x(i,t)\cdot\vec{Z}_{xxt}(i,t),$$

where $i=0,1$. Then we have

$$\begin{aligned} \mathcal{A}_i(t)=&-|\vec{Z}_x(i,t)|^2\vec{g_i}+\frac{\varepsilon}{\varepsilon^2+1}\vec{g_i}'-\frac{1}{\varepsilon^2+1}\vec{g_i}\times\vec{g_i}' \\ &-\frac{\varepsilon}{\varepsilon^2+1}\vec{g_i}\times f(\vec{g_i})+\frac{1}{\varepsilon^2+1}\vec{g_i}\times(\vec{g_i}\times f(\vec{g_i})), \\ \mathcal{C}_i(t)=&\frac{1}{\varepsilon^2+1}(\vec{g_i}\times f(\vec{g_i}))\cdot\vec{Z}_x(i,t)-\frac{1}{\varepsilon^2+1}\vec{g_i}'\cdot\vec{Z}_x(i,t)-\frac{\varepsilon}{\varepsilon^2+1}(\vec{g_i}\times\vec{g_i}')\cdot\vec{Z}_x(i,t) \end{aligned}$$

$$
\begin{aligned}
&-\frac{\varepsilon}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot f(\vec{g_i}),\\
\mathcal{D}_i(t)=&\frac{\varepsilon}{\varepsilon^2+1}\vec{g_i}'\cdot\vec{Z}_x(i,t)-\frac{1}{\varepsilon^2+1}(\vec{g_i}\times\vec{g_i}')\cdot\vec{Z}_x(i,t)-\frac{\varepsilon}{\varepsilon^2+1}(\vec{g_i}\times f(\vec{g_i}))\cdot\vec{Z}_x(i,t)\\
&-\frac{1}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot f(\vec{g_i}),\\
\mathcal{E}_i(t)=&-|\vec{Z}_x(i,t)|^2\vec{Z}_x(i,t)\cdot\vec{g_i}'+\frac{\varepsilon}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot\vec{g_i}''-\frac{1}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot(\vec{g_i}\times\vec{g_i}'')\\
&-\frac{\varepsilon}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot(\vec{g_i}'\times f(\vec{g_i}))+\frac{1}{\varepsilon^2+1}(\vec{g_i}\cdot f(\vec{g_i}))(\vec{g_i}'\cdot\vec{Z}_x(i,t))\\
&-\frac{\varepsilon}{\varepsilon^2+1}\vec{Z}_x(i,t)\cdot[\vec{g_i}\times f'(\vec{g_i})\vec{g_i}'].
\end{aligned}
$$

Proof. It follows from (1.7) that

$$\vec{Z}\times\vec{Z}_t=\varepsilon\vec{Z}\times\vec{Z}_{xx}-|\vec{Z}_x|^2\vec{Z}-\vec{Z}_{xx}+(\vec{Z}\cdot f(\vec{Z}))\vec{Z}-f(\vec{Z}).$$

Hence, we get

$$\vec{g_i}'=\varepsilon\mathcal{A}_i(t)+\mathcal{B}_i(t)+\varepsilon|\vec{Z}_x(i,t)|^2\vec{g_i}+\vec{g_i}\times f(\vec{g_i}), \tag{4.2}$$

$$\vec{g_i}\times\vec{g_i}'=\varepsilon\mathcal{B}_i(t)-|\vec{Z}_x(i,t)|^2\vec{g_i}-\mathcal{A}_i(t)+(\vec{g_i}\cdot f(\vec{g_i}))\vec{g_i}-f(\vec{g_i}). \tag{4.3}$$

Combining (4.2) with (4.3), it yields that

$$
\begin{aligned}
\mathcal{A}_i(t)=&-|\vec{Z}_x(i,t)|^2\vec{g_i}+\frac{\varepsilon}{\varepsilon^2+1}\vec{g_i}'-\frac{1}{\varepsilon^2+1}\vec{g_i}\times\vec{g_i}'\\
&-\frac{\varepsilon}{\varepsilon^2+1}\vec{g_i}\times f(\vec{g_i})+\frac{1}{\varepsilon^2+1}\vec{g_i}\times(\vec{g_i}\times f(\vec{g_i})).
\end{aligned} \tag{4.4}
$$

The proofs of $\mathcal{C}_i(t)$, $\mathcal{D}_i(t)$ and $\mathcal{E}_i(t)$ are similar to the proof of $\mathcal{A}_i(t)$, we shall omit the details. □

Now, we give the a priori uniform estimate on ε. Denote the global solution of (1.7) with $\varepsilon>0$ by $\vec{Z}_\varepsilon(x,t)$.

Lemma 4.3. *Assume that $\vec{g_0}(t)$, $\vec{g_1}(t)\in\mathcal{C}^2(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^2([0,1])$, $f(\vec{Z}_\varepsilon)\in\mathcal{C}^2(\mathbb{R}^3)$. Then the solution $\vec{Z}_\varepsilon(x,t)$ obeys the following bounds uniformly, for any $T>0$,*

$$\sup_{0\le t\le T}\|\vec{Z}_{\varepsilon x}(\cdot,t)\|_2\le C,\qquad \sup_{0\le t\le T}\|\vec{Z}_{\varepsilon xx}(\cdot,t)\|_2\le C,$$

where $\varepsilon\in(0,\varepsilon_0]$, C is dependent on T and independent of ε.

Proof. Drop the subscript ε for simplicity. First, taking the scalar product of $(1.7)_1$ with $\vec{Z}_{xx}$ and integrating it with respect to x, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{Z}_x\|_2^2+2\varepsilon\int_0^1|\vec{Z}\times\vec{Z}_{xx}|^2\mathrm{d}x=&\int_0^1(f(\vec{Z})\times\vec{Z})\cdot\vec{Z}_{xx}\,\mathrm{d}x+\vec{g_1}'\cdot\vec{Z}_x(1,t)-\vec{g_0}'\cdot\vec{Z}_x(0,t)\\
&\lesssim\|\vec{Z}_x\|_\infty+\|\vec{Z}_x\|_2^2,
\end{aligned} \tag{4.5}
$$

where we have used the fact $\vec{Z}\cdot\vec{Z}_{xx}=-|\vec{Z}_x|^2$. By the Grönwall inequality, one has

$$\|\vec{Z}_x\|_2^2\lesssim 1+\int_0^t\|\vec{Z}_x\|_\infty \mathrm{d}\tau. \tag{4.6}$$

Second, we estimate $\sup\limits_{0\le t\le T}\|\vec{Z}_{xx}\|_2$. By integrating by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{Z}_{xx}\|_2^2=\int_0^1\vec{Z}_{xx}\cdot\vec{Z}_{xxt}\,\mathrm{d}x=-\int_0^1\vec{Z}_{xxx}\cdot\vec{Z}_{xt}\,\mathrm{d}x+\vec{Z}_{xx}\cdot\vec{Z}_{xt}|_{x=0}^1, \tag{4.7}$$

or

$$\|\vec{Z}_{xx}\|_2^2=\|\vec{\varphi}_{xx}\|_2^2-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot\vec{Z}_{xt}\,\mathrm{d}x\mathrm{d}\tau+2\int_0^t\vec{Z}_{xx}\cdot\vec{Z}_{xt}\,\mathrm{d}\tau|_{x=0}^1. \tag{4.8}$$

Using the Hölder and Young inequalities and Lemma 4.2, the terms on the right-hand side of (4.8) can be bounded by

$$\begin{aligned}2\int_0^t\vec{Z}_{xx}\cdot\vec{Z}_{x\tau}\mathrm{d}\tau|_{x=0}^1=&2\vec{Z}_{xx}\cdot\vec{Z}_x|_{x=0}^1|_{\tau=0}^t-2\int_0^t\vec{Z}_{xx\tau}\cdot\vec{Z}_x\mathrm{d}\tau|_{x=0}^1\\=&2[\mathcal{D}_1(\tau)-\mathcal{D}_0(\tau)]|_{\tau=0}^t-2\int_0^t\mathcal{E}_1(\tau)-\mathcal{E}_0(\tau)\mathrm{d}\tau\\\lesssim&\|\vec{Z}_x(\cdot,t)\|_\infty+\int_0^t\|\vec{Z}_x(\cdot,\tau)\|_\infty^3\mathrm{d}\tau,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot\vec{Z}_{xt}\,\mathrm{d}x\mathrm{d}\tau\\=&-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot\{\varepsilon\vec{Z}\times(\vec{Z}\times\vec{Z}_{xx})+\vec{Z}\times\vec{Z}_{xx}+\vec{Z}\times f(\vec{Z})\}_x\,\mathrm{d}x\mathrm{d}\tau\\\le&-\varepsilon\int_0^t\|\vec{Z}_{xxx}\|_2^2\mathrm{d}\tau-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(f(\vec{Z})\times\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau\\&-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}\times f'(\vec{Z})\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau+2\varepsilon\int_0^t\|\vec{Z}_x\|_6^6\mathrm{d}\tau\\\lesssim&-\varepsilon\int_0^t\|\vec{Z}_{xxx}\|_2^2\mathrm{d}\tau-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau\\&+\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau+\varepsilon\int_0^t\|\vec{Z}_x\|_6^6\mathrm{d}\tau+1,\end{aligned} \tag{4.10}$$

where we have used the following estimates

$$\begin{aligned}&2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(f(\vec{Z})\times\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau\\=&2\int_0^t\vec{Z}_{xx}\cdot(f(\vec{Z})\times\vec{Z}_x)|_{x=0}^1\mathrm{d}\tau-2\int_0^t\int_0^1\vec{Z}_{xx}\cdot(f(\vec{Z})\times\vec{Z}_x)_x\mathrm{d}x\mathrm{d}\tau\\\lesssim&\int_0^t\mathcal{A}_i(\tau)\cdot[f(\vec{g_i})\times\vec{Z}_x(i,\tau)]|_{i=0}^1\mathrm{d}\tau+\int_0^t\|\vec{Z}_{xx}\|_2\|f'(\vec{g_i})\|_\infty\|\vec{Z}_x\|_4^2\mathrm{d}\tau\\\lesssim&\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau,\end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
&-2\int_0^t\int_0^1 \vec{Z}_{xxx}\cdot(\vec{Z}\times f'(\vec{Z})\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau \\
=&-2\int_0^t \vec{Z}_{xx}\cdot(\vec{Z}\times f'(\vec{Z})\vec{Z}_x)|_{x=0}^1\mathrm{d}\tau+2\int_0^t\int_0^1 \vec{Z}_{xx}\cdot(\vec{Z}_x\times f'(\vec{Z})\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau \\
&+2\int_0^t\int_0^1 \vec{Z}_{xx}\cdot(\vec{Z}\times f'(\vec{Z})\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int_0^1 \vec{Z}_{xx}\cdot(\vec{Z}\times f''(\vec{Z})\vec{Z}_x\vec{Z}_x)\mathrm{d}x\mathrm{d}\tau \\
\lesssim&-\int_0^t \mathcal{A}_i(\tau)[\vec{g}_i\times f'(\vec{g}_i)\vec{Z}_x(i,\tau)]|_{i=0}^1\mathrm{d}\tau+\int_0^t\|\vec{Z}_{xx}\|_2\|\vec{Z}_x\|_4^2\mathrm{d}\tau+\int_0^t\|\vec{Z}_{xx}\|_2^2\mathrm{d}\tau \\
\lesssim&\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau+1.
\end{aligned}
\tag{4.12}
$$

Putting (4.9)-(4.10) into (4.8), we have

$$
\begin{aligned}
\|\vec{Z}_{xx}\|_2^2\lesssim&\|\vec{\varphi}_{xx}\|_2^2+|\vec{Z}_x\|_\infty+\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau-\varepsilon\int_0^t\|\vec{Z}_{xxx}\|_2^2\mathrm{d}\tau+\varepsilon\int_0^t\|\vec{Z}_x\|_6^6\mathrm{d}\tau \\
&-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau+1.
\end{aligned}
\tag{4.13}
$$

Now, we estimate $-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau$ as follows. Owing to $|\vec{Z}|^2=1$, $\vec{Z}\cdot\vec{Z}_x=0$, then it is obvious that $\vec{Z}$, $\vec{Z}_x$, $\vec{Z}\times\vec{Z}_x$ form an orthogonal basis in $\mathbb{R}^3$ for $|\vec{Z}_x|\neq 0$. Supposing $\vec{Z}_{xx}=\alpha\vec{Z}+\beta\vec{Z}_x+\gamma\vec{Z}\times\vec{Z}_x$, then by the direct computation, one gets

$$
\alpha=-|\vec{Z}_x|^2,\quad \beta=\frac{\vec{Z}_x\cdot\vec{Z}_{xx}}{|\vec{Z}_x|^2},\quad \gamma=\frac{(\vec{Z}\times\vec{Z}_x)\cdot\vec{Z}_{xx}}{|\vec{Z}_x|^2}.
\tag{4.14}
$$

Therefore, by (4.14) and Lemma 4.2, it is derived that

$$
\begin{aligned}
&-2\int_0^t\int_0^1\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z}_{xx})\mathrm{d}x\mathrm{d}\tau \\
=&2\int_0^t\int_0^1|\vec{Z}_x|^2\vec{Z}_{xxx}\cdot(\vec{Z}_x\times\vec{Z})\mathrm{d}x\mathrm{d}\tau+3\int_0^t\int_0^1(\vec{Z}\times\vec{Z}_x)\cdot\vec{Z}_{xx}(|\vec{Z}_x|^2)_x\mathrm{d}x\mathrm{d}\tau \\
\leq&5\int_0^t\int_0^1|\vec{Z}_x|^2\vec{Z}_x\cdot(\vec{Z}_t-\varepsilon\vec{Z}_{xx}-\varepsilon|\vec{Z}_x|^2\vec{Z}-\vec{Z}\times f(\vec{Z}))_x\mathrm{d}x\mathrm{d}\tau+C\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau \\
\lesssim&\|\vec{Z}_x\|_4^4+\varepsilon\int_0^t\|\vec{Z}_{xxx}\|_2^2\mathrm{d}\tau+\varepsilon\int_0^t\|\vec{Z}_x\|_6^6\mathrm{d}\tau+\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau,
\end{aligned}
\tag{4.15}
$$

where we have used the fact $\vec{Z}\cdot\vec{Z}_{xxx}=-\frac{3}{2}(|\vec{Z}_x|^2)_x$. Combining the estimates (4.8), (4.13) and (4.15), we conclude that

$$
\|\vec{Z}_{xx}\|_2^2+\|\vec{Z}_x\|_2^2\lesssim\|\vec{Z}_x\|_4^4+\|\vec{Z}_x\|_\infty+\int_0^t\|\vec{Z}_x\|_\infty^3\mathrm{d}\tau+\varepsilon\int_0^t\|\vec{Z}_x\|_6^6\mathrm{d}\tau+1.
\tag{4.16}
$$

Set $F(t)=\|\vec{Z}_{xx}\|_2^2+\delta(\|\vec{Z}_x\|_2^2+\|\vec{Z}_x\|_2^6)$, where δ is an undetermined positive number. It follows from (4.16) that

$$
F(t)\lesssim\frac{1}{2}\|\vec{Z}_{xx}\|_2^2+C(\|\vec{Z}_x\|_2^2+\|\vec{Z}_x\|_2^6)+\int_0^t F(\tau)\mathrm{d}\tau+\varepsilon\int_0^t F^{\frac{5}{3}}(\tau)\mathrm{d}\tau+1.
\tag{4.17}
$$

Taking $\delta > 2C$, we have

$$F(t) \lesssim 1 + \int_0^t F(\tau)\mathrm{d}\tau + \varepsilon \int_0^t F^{\frac{5}{3}}(\tau)\mathrm{d}\tau.$$

Applying the Lemma 4.1, we know that $\forall t \in [0,T], \varepsilon \in (0,\varepsilon_0]$, there holds

$$\|\vec{Z}_x(\cdot,t)\|_2^2 + \|\vec{Z}_{xx}(\cdot,t)\|_2^2 \le C,$$

where C, ε_0 are associated with $T, \|f(\vec{Z})\|_\infty$, $\|f'(\vec{Z})\|_\infty$, $\|f''(\vec{Z})\|_\infty$, $\|\vec{\varphi}\|_{H^2}$, $\|\vec{g_i}'\|_\infty$, $\|\vec{g_i}''\|_\infty$ and independent of ε. And the proof of Lemma 4.3 is thus finished. □

LEMMA 4.4. *Assume that $\vec{g_0}(t)$, $\vec{g_1}(t) \in \mathcal{C}^3(\mathbb{R}^+)$, $\vec{\varphi}(x) \in H^3([0,1])$, $f(\vec{Z}_\varepsilon) \in \mathcal{C}^3(\mathbb{R}^3)$. Then the solution $\vec{Z}_\varepsilon(x,t)$ obeys the following bounds uniformly, for any $T > 0$, $t \in [0,T]$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\sup_{0\le t\le T} \|\vec{Z}_{\varepsilon xt}(\cdot,t)\|_2 \le C, \qquad \sup_{0\le t\le T} \|\vec{Z}_{\varepsilon xxx}(\cdot,t)\|_2 \le C,$$

where C and ε_0 are dependent on T and independent of ε.

Proof. Drop the subscript ε for simplicity. Applying integration by parts gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |\vec{Z}_{xt}|^2 \mathrm{d}x &= 2\int_0^1 \vec{Z}_{xt}\cdot\vec{Z}_{xtt}\,\mathrm{d}x \\ &= 2\vec{Z}_{xt}\cdot\vec{Z}_{tt}|_{x=0}^1 - 2\int_0^1 \vec{Z}_{xxt}\cdot\vec{Z}_{tt}\,\mathrm{d}x,\end{aligned}$$

and integrating it in t, we have

$$\begin{aligned}\|\vec{Z}_{xt}(\cdot,t)\|_2^2 &= \|\vec{Z}_{xt}(\cdot,0)\|_2^2 + 2\int_0^t .\vec{Z}_{xt}\cdot\vec{Z}_{tt}|_{x=0}^1 \mathrm{d}\tau - 2\int_0^t\int_0^1 \vec{Z}_{xxt}\cdot\vec{Z}_{tt}\,\mathrm{d}x\mathrm{d}\tau \\ &\lesssim -\int_0^t\int_0^1 \vec{Z}_{xxt}\cdot\vec{Z}_{tt}\,\mathrm{d}x\mathrm{d}\tau + 1.\end{aligned} \tag{4.18}$$

Differentiating Equation (1.7) with respect to t, and taking the scalar product of it with $\vec{Z}_{xxt}$, then integrating in x and t, we have

$$\begin{aligned}&-\int_0^t\int_0^1 \vec{Z}_{xxt}\cdot\vec{Z}_{tt}\,\mathrm{d}x\mathrm{d}\tau \\ &\lesssim -\varepsilon\int_0^t \|\vec{Z}_{xxt}\|_2^2\mathrm{d}\tau + \int_0^t \|\vec{Z}_{xt}\|_2^2\mathrm{d}\tau - \int_0^t\int_0^1 (\vec{Z}_t\times\vec{Z}_{xx})\cdot\vec{Z}_{xxt}\,\mathrm{d}x\mathrm{d}\tau + 1.\end{aligned} \tag{4.19}$$

By (4.19), (4.18) can be written as

$$\|\vec{Z}_{xt}\|_2^2 + \varepsilon\int_0^t \|\vec{Z}_{xxt}\|_2^2\mathrm{d}\tau \lesssim \int_0^t \|\vec{Z}_{xt}\|_2^2\mathrm{d}\tau - \int_0^t\int_0^1 (\vec{Z}_t\times\vec{Z}_{xx})\cdot\vec{Z}_{xxt}\,\mathrm{d}x\mathrm{d}\tau + 1. \tag{4.20}$$

Thanks to $|\vec{Z}|^2 = 1$, $\vec{Z}\cdot\vec{Z}_t = 0$, we assume that $\vec{Z}_{xxt} = \mu\vec{Z} + \nu\vec{Z}_t + \xi\vec{Z}\times\vec{Z}_t$, where

$$\mu = -2\vec{Z}_x\cdot\vec{Z}_{xt} - \vec{Z}_{xx}\cdot\vec{Z}_t, \quad \nu = \frac{\vec{Z}_t\cdot\vec{Z}_{xxt}}{|\vec{Z}_t|^2}, \quad \xi = \frac{(\vec{Z}\times\vec{Z}_t)\cdot\vec{Z}_{xxt}}{|\vec{Z}_t|^2}.$$

Then

$$\begin{aligned}
&-\int_0^t\int_0^1(\vec{Z}_t\times\vec{Z}_{xx})\cdot\vec{Z}_{xxt}\,\mathrm{d}x\mathrm{d}\tau\\
=&2\int_0^t\int_0^1(\vec{Z}_x\cdot\vec{Z}_{xt})(\vec{Z}\times\vec{Z}_t)\cdot\vec{Z}_{xx}\,\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_0^1(\vec{Z}_{xx}\cdot\vec{Z}_t)(\vec{Z}\times\vec{Z}_t)\cdot\vec{Z}_{xx}\,\mathrm{d}x\mathrm{d}\tau\\
&+\int_0^t\int_0^1\frac{(\vec{Z}\times\vec{Z}_t)\cdot\vec{Z}_{xxt}}{|\vec{Z}_t|^2}|\vec{Z}_t|^2\vec{Z}\cdot\vec{Z}_{xx}\,\mathrm{d}x\mathrm{d}\tau\\
\lesssim&\int_0^t\|\vec{Z}_{xt}\|_2^2\mathrm{d}\tau+\frac{\varepsilon}{2}\int_0^t\|\vec{Z}_{xxt}\|_2^2\mathrm{d}\tau+1. \qquad (4.21)
\end{aligned}$$

Inserting (4.21) into (4.20) leads to

$$\|\vec{Z}_{xt}\|_2^2+\frac{\varepsilon}{2}\int_0^t\|\vec{Z}_{xxt}\|_2^2\mathrm{d}\tau\lesssim\int_0^t\|\vec{Z}_{xt}\|_2^2\mathrm{d}\tau+1.$$

Combining the Grönwall inequality, we derive the desired result. In addition, using system $(1.7)_1$, we also have the estimate for $\|\vec{Z}_{xxx}\|_2$ and we complete the proof of Lemma 4.4. □

By induction we can obtain that the smooth solution of the problem (1.7) or (1.2) has the following uniform estimation.

LEMMA 4.5. *Suppose that $\vec{g_0}(t)$, $\vec{g_1}(t)\in\mathcal{C}^m(\mathbb{R}^+)$, $\vec{\varphi}(x)\in H^m([0,1])$, $f(\vec{Z}_\varepsilon)\in\mathcal{C}^m(\mathbb{R}^3)$ with $m\geq 2$. Let $\vec{Z}_\varepsilon(x,t)$ is a classical solution of the problem (1.7) or (1.2), then for $T>0$, $\varepsilon\in(0,\varepsilon_0]$, there holds*

$$\|\vec{Z}_{\varepsilon x^r t^s}(\cdot,t)\|_2+\int_0^T\|\vec{Z}_{\varepsilon x^{r+1}t^s}(x,t)\|_2^2dt\leq C,$$

where $r+2s\leq m$, r,s,m are nonnegative numbers, C and ε_0 are dependent on T and independent of ε.

Thanks to Lemmas 4.3-4.5, it completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

In this section, we prove the Theorem 1.3. From the global prior estimate and the independent uniformly-bounded estimate of ε and t established in the previous section, as well as the standard compactness argument, we can obtain the global existence of smooth solutions for the initial-boundary value problem (1.1) by taking the limit $\varepsilon\to 0$ for the solution set $\{\vec{Z}_\varepsilon(x,t)\}$ of (1.7). Now, we only need to prove the uniqueness part for system (1.1).

Proof. **(Proof of Theorem 1.3.)** Suppose that $\tilde{Z}(x,t)$ is also a smooth solution to the system (1.1) and let $\vec{u}=\vec{Z}-\tilde{Z}$, we have

$$\vec{u}_t=\vec{u}\times\vec{Z}_{xx}+\tilde{Z}\times\vec{u}_{xx}+\vec{u}\times f(\vec{Z})+\tilde{Z}\times[f(\vec{Z})-f(\tilde{Z})]. \qquad (5.1)$$

Firstly, taking the L^2 product of (5.1) with $\vec{u}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}\|_2^2=\int_0^1\tilde{Z}_x\times\vec{u}\cdot\vec{u}_x\,\mathrm{d}x+\int_0^1\vec{u}\cdot\tilde{Z}\times[f(\vec{Z})-f(\tilde{Z})]\,\mathrm{d}x$$

$$\lesssim \|\vec{u}\|_2\|\vec{u}_x\|_2\|\tilde{Z}_x\|_\infty + \|\vec{u}\|_2^2\|\tilde{Z}\|_2$$
$$\lesssim \|\vec{u}\|_2^2(1+\|\tilde{Z}\|_2) + \|\vec{u}_x\|_2^2\|\tilde{Z}_{xx}\|_2. \tag{5.2}$$

Secondly, taking the inner product of (5.1) with $\vec{u}_{xx}$ and by integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}_x\|_2^2 =& \int_0^1 (\vec{u}\times\vec{Z}_{xxx})\cdot\vec{u}_x\,\mathrm{d}x + \int_0^1 (\vec{u}\times f'(\vec{Z})\vec{Z}_x)\cdot\vec{u}_x\,\mathrm{d}x \\ &+ \int_0^1 \vec{u}_x\cdot\Big(\tilde{Z}\times(f(\vec{Z})-f(\tilde{Z}))\Big)_x\,\mathrm{d}x \\ \lesssim& \|\vec{u}\|_\infty\|\vec{u}_x\|_2\|\vec{Z}_{xxx}\|_2 + \|\vec{u}\|_\infty\|\vec{u}_x\|_2\|\vec{Z}_x\|_2 + \|\vec{u}_x\|_2\|\tilde{Z}_x\|_2\|\vec{u}\|_\infty + \|\vec{u}_x\|_2^2\|\tilde{Z}\|_\infty \\ \lesssim& \|\vec{u}_x\|_2^2(\|\vec{Z}_{xxx}\|_2 + \|\vec{Z}_x\|_2 + \|\tilde{Z}_x\|_2).\end{aligned} \tag{5.3}$$

Combining (5.2) with (5.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\vec{u}\|_2^2 + \|\vec{u}_x\|_2^2) \le (\|\vec{u}\|_2^2 + \|\vec{u}_x\|_2^2)(\|\vec{Z}_{xxx}\|_2 + \|\vec{Z}_x\|_2 + \|\tilde{Z}_{xx}\|_2 + \|\tilde{Z}_x\|_2 + 1).$$

Thus, by the Grönwall inequality we obtain $\vec{u}=0$ when $\|\vec{Z}_{xxx}\|_2$, $\|\tilde{Z}_{xx}\|_2 \in L^1((0,T))$. This completes the proof of Theorem 1.3. ∎

Data availability statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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