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A KIND OF TIME-INCONSISTENT CORPORATE INTERNATIONAL INVESTMENT PROBLEM WITH DISCONTINUOUS CASH FLOW*

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Abstract. In this paper, we study a kind of time-inconsistent corporate international investment problem with discontinuous cash flow in consideration of the exchange risk, information costs and taxes. The time-inconsistency arises from the presence of investment risk in the cost functional. We first define the time-consistent equilibrium strategy for this problem and establish a sufficient condition for it through a flow of forward-backward stochastic differential equations with random jumps. When all market parameters are deterministic, an equilibrium strategy is given explicitly by solutions of several ordinary differential equations. Moreover, we present some numerical examples to discuss the influence of market parameters on the equilibrium strategy. It is confirmed that the information costs provide a useful explanation for home bias puzzle in international finance by our time-inconsistent model.

Keywords. Corporate international investment; Information costs; Time-inconsistency; Equilibrium strategy; Forward-backward stochastic differential equations; Ordinary differential equations.

AMS subject classifications. 91G80; 93E20.

1. Introduction

The study of international diversification problem is very popular and important in finance. In the existing literature, there are several factors that argue for the international diversification at the corporate level, mainly the partial segmentation of international capital market, the agency costs, the uncertainty of operational cash flows and the exchange risk. Reasons for the segmentation of international capital market include restrictions imposed by government policies and regulations, transaction costs, lack of information, and unfamiliarity with foreign markets [21]. Moreover, the degree of segmentation may be greater in product markets than in financial markets [23]. This leaves room for profitable corporate international investments in real assets. With the agency costs, there is some room for corporate activity independent of investor diversification [17]. The corporate decisions can result from optimizing the manager's objective rather than that of shareholders although two objectives may overlap. In an international investment, the gains from the uncertainty of operational cash flows are even greater because of the partial segmentation of national economics and markets [19,20]. Besides, Aliber [1,2] show that the exchange risk can create a difference in the cost of capital of firms located in different currency zones, thereby affecting the international investment.

Choi [11] examines the effect of last two factors, i.e., uncertainty of operational cash flows and exchange risk, on corporate international investment decisions in a two-country dynamic optimization model, taking the first two factors as environmental fac-

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tors. It is found that the exchange rate as well as uncertainty of operational cash flows affect the corporate international investment in a significant way.

The recent literatures of international investment give an important role to the information costs in financial markets [4–7, 22]. Because of the barriers to the international investment, a manager who invests in a foreign asset needs to buy some necessary information for the analysis and valuation of this asset, including a database, the elaboration of models, the costs of gathering and treating this information, etc. All these costs may have an effect on the distribution of returns of this asset and thus the investment decisions. From Merton [22], it appears that taking into account the effect of information costs on the price of an asset is similar to applying an additional discount rate to the asset's future cash flow. Besides, the economic literatures show that the product quantity is a decreasing function of the price and the cash flow of a firm's activity is taxed at a certain rate [9].

In consideration of the factors mentioned above, Bellalah and Wu [6] extend the model of Choi [11] with more economic and behaviour variables. By the dynamic programming principle method, the authors derive the general form of optimal proportion of foreign investment in the firm's total capital budget. We can see that the information costs play a central role and provide a useful explanation for "home bias puzzle" in the international finance. That is to say, a manager accepts to invest abroad only if he/she has some information about the foreign markets. It is noted that the asset prices and exchange rate are all continuous processes in [6] and [11].

Due to some sudden and unpredictable events, rumours and governmental policies, the asset prices and exchange rate are often assumed to have random jumps in financial literatures [3, 10, 14, 18]. So we propose a kind of corporate international investment problem where the processes of asset prices and exchange rate are driven by Brownian motions and Poisson processes jointly and the exchange risk, information costs and taxes are also taken into consideration. Instead of maximizing the manager's utility in [6] and [11], the objective of a firm is direct to maximize the expectation of future wealth and minimize the risk of this investment simultaneously. Thus our problem is time-inconsistent, meaning that an optimality viewed from today will not remain optimal afterwards. There are lots of works coping with the time-inconsistent optimal control problems within a game theoretic framework [8, 12, 13, 26, 27]. The basic idea is that an action made by the controller at each time point is regarded as a game against all actions made by the future incarnations of himself. An equilibrium control is then characterized by the feature that any deviation from it will be worse off. Hu, Jin and Zhou [15, 16] formulate a general time-inconsistent stochastic linear-quadratic (LQ) control problem. By virtue of the so-called local spike variation, the authors introduce the concept of equilibrium via open-loop controls, and derive a necessary and sufficient condition for the equilibrium. Their results are generalized by Sun and Guo [25] in a jump-diffusion setting.

In fact, our corporate international investment problem falls into the formulation of general time-inconsistent stochastic LQ control problems with random jumps proposed by [25]. We define the equilibrium strategy of our problem accordingly, which is time-consistent and optimal infinitesimally at any time point. Thanks to the results obtained by [25], we give a sufficient condition for equilibrium strategy through a flow of forward-backward stochastic differential equations (FBSDEs) with random jumps and an additional constraint. However, the general solvability of these FBSDEs remains an outstanding open problem since they are not standard. When the market parameters are all deterministic, this flow of FBSDEs can be reduced into two coupled nonlinear ordinary differential equations (ODEs) and one linear ODE. We prove the existence and uniqueness of solutions to these ODEs, and thus get an equilibrium strategy explicitly in the form of linear feedback. Then several numerical examples are presented to illustrate the theoretical results obtained above and show the influence of market parameters on the equilibrium strategy. In particular, the information costs can still explain the "home bias puzzle" by our time-inconsistent model.

The rest of this paper is organized as follows. In the next section, we formulate a corporate international investment problem and establish a sufficient condition for equilibrium strategy via a flow of FBSDEs with random jumps. Based on this general result, a linear feedback equilibrium strategy is given explicitly by solutions of several ODEs for the deterministic parameters case in Section 3. We provide some numerical examples in Section 4 to discuss the influence of market parameters on the equilibrium strategy. The last section is devoted to concluding the novelty of this paper and proposing some future work.

2. Formulation and equilibrium strategy

Let $T>0$ be a time horizon and $(\Omega,\mathcal{F},\mathbb{F},\mathbf{P})$ be a filtered complete probability space. Suppose that $\{W_1(s)\}_{0\le s\le T},\{W_2(s)\}_{0\le s\le T},\{W_3(s)\}_{0\le s\le T}$ are three 1-dimensional Brownian motions with constant correlation coefficients $\rho_{12},\rho_{13},\rho_{23}$ and $\{N_1(s)\}_{0\le s\le T},\{N_2(s)\}_{0\le s\le T},\{N_3(s)\}_{0\le s\le T}$ are three independent Poisson processes with constant intensities ν_1,ν_2,ν_3. Let the filtration $\mathbb{F}=\{\mathcal{F}_s\}_{0\le s\le T}$ be generated by (W_1,W_2,W_3) and (N_1,N_2,N_3) augmented by all $\mathbf{P}$-null sets of $\mathcal{F}$. The following notations are also used throughout the paper:

$$L^q_{\mathcal{F}}(\Omega;\mathbb{R}^m)=\Big\{\xi \text{ is a } \mathbb{R}^m\text{-valued, } \mathcal{F}\text{-measurable random variable} \quad \text{s.t. } \mathbb{E}^{\mathbf{P}}\Big[|\xi|^q\Big]<+\infty\Big\};$$

$$\mathcal{S}^q_{\mathbb{F}}(t,T;\mathbb{R}^m)=\Big\{\{\phi(s),t\le s\le T\} \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{F}\text{-adapted process} \quad \text{s.t. } \mathbb{E}^{\mathbf{P}}\Big[\sup_{t\le s\le T}|\phi(s)|^q\Big]<+\infty\Big\};$$

$$\mathcal{L}^q_{\mathbb{F},p}(t,T;\mathbb{R}^m)=\Big\{\{\phi(s),t\le s\le T\} \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{F}\text{-predictable process} \quad \text{s.t. } \mathbb{E}^{\mathbf{P}}\Big[\int_t^T|\phi(s)|^q ds\Big]<+\infty\Big\};$$

$$\mathcal{L}^\infty_{\mathbb{F},p}(t,T;\mathbb{R}^m)=\Big\{\{\phi(s),t\le s\le T\} \text{ is a } \mathbb{R}^m\text{-valued, } \mathbb{F}\text{-predictable, } \mathbf{P}\text{-essentially uniformly bounded process}\Big\}.$$

We begin with a two-country firm whose static cash flows $R(s)$ and $\widetilde{R}(s)$ at time s from its domestic and foreign operations respectively as in [6]:

$$R(s)=[P(s)-C(s)]Q(s)-\tau[P(s)-C(s)]Q(s),\ Q(s)=P^\alpha(s), \tag{2.1}$$

$$\widetilde{R}(s)=e(s)[\widetilde{P}(s)-\widetilde{C}(s)]\widetilde{Q}(s)-\widetilde{\tau}e(s)[\widetilde{P}(s)-\widetilde{C}(s)]\widetilde{Q}(s),\ \widetilde{Q}(s)=\widetilde{P}^\beta(s). \tag{2.2}$$

Here, $P(s),C(s),Q(s)$ and $\widetilde{P}(s),\widetilde{C}(s),\widetilde{Q}(s)$ are the output price, input price, output quantity of the domestic and foreign production respectively, and $e(s)$ is the exchange rate at time s. τ and $\widetilde{\tau}$ are the fixed tax rates in home and foreign country, and α and β are two given negative constants reflecting the degree of competition in the domestic and foreign market respectively.

Considering the presence of information costs, we adopt the model of Merton [22] to modify the dynamic of assets, which has been applied in [5, 6]. This adjustment leads to the result that the cash flow of an asset should be discounted to the present by the riskless rate plus the information costs rate under the risk-neutral probability. Motivated by the results on information costs, we suppose that the output prices, input prices and exchange rate satisfy the following SDEs with random jumps:

$$\begin{aligned} dP(s) &= r_P(s)P(s)ds + \sigma(s)P(s)dW_1(s) + \delta(s)P(s-)dN_1(s), \\ dC(s) &= r_C(s)C(s)ds + \sigma(s)C(s)dW_1(s) + \delta(s)C(s-)dN_1(s), \\ d\widetilde{P}(s) &= r_{\widetilde{P}}(s)\widetilde{P}(s)ds + \widetilde{\sigma}(s)\widetilde{P}(s)dW_2(s) + \widetilde{\delta}(s)\widetilde{P}(s-)dN_2(s), \\ d\widetilde{C}(s) &= r_{\widetilde{C}}(s)\widetilde{C}(s)ds + \widetilde{\sigma}(s)\widetilde{C}(s)dW_2(s) + \widetilde{\delta}(s)\widetilde{C}(s-)dN_2(s), \\ de(s) &= [r_e(s) + \lambda_e(s)]e(s)ds + \sigma_e(s)e(s)dW_3(s) + \delta_e(s)e(s-)dN_3(s), \end{aligned}$$

with initial values $P_0, C_0, \widetilde{P}_0, \widetilde{C}_0$ and e_0 respectively. Here, $r_P, r_C, r_{\widetilde{P}}, r_{\widetilde{C}}$ and r_e are the expected rates of corresponding variables. $\sigma, \widetilde{\sigma}$ and σ_e are the volatility rate, and $\delta, \widetilde{\delta}$ and δ_e are the relative jump size of the domestic variables, the foreign variables and the exchange rate respectively. λ_e reflects the information costs carried by investors so as to get some information about the exchange market. $(W_1, N_1), (W_2, N_2)$ and (W_3, N_3) represent the continuous and jump sources of uncertainties affecting the domestic market, the foreign market and the exchange market.

REMARK 2.1. Since the jump sources of uncertainties are sudden, unpredictable and usually local, it is reasonable to suppose that N_1, N_2, N_3 are mutually independent, while the continuous sources W_1, W_2, W_3 can be regarded as the sum of many micro factors influencing the markets and thus assumed to be correlative.

By the technique of solving linear SDEs with random jumps, we can get

$$\begin{aligned} P(s) &= P_0 e^{\int_0^s [r_P(r) - \frac{1}{2}\sigma^2(r)]dr} e^{\int_0^s \sigma(r)dW_1(r)} \prod_{0<r\le s} [1 + \delta(r)\Delta N_1(r)], \\ C(s) &= C_0 e^{\int_0^s [r_C(r) - \frac{1}{2}\sigma^2(r)]dr} e^{\int_0^s \sigma(r)dW_1(r)} \prod_{0<r\le s} [1 + \delta(r)\Delta N_1(r)], \end{aligned}$$

where $\Delta N_1(s) = N_1(s) - N_1(s-)$. Then the domestic cash flow $R(\cdot)$ can be written as

$$\begin{aligned} R(s) &= (1-\tau)P^\alpha(s)[P(s) - C(s)] \\ &= K e^{(1+\alpha)\int_0^s \sigma(r)dW_1(r)} \prod_{0<r\le s} [1 + \delta(r)\Delta N_1(r)]^{1+\alpha} F(s) \end{aligned}$$

with $K = (1-\tau)P_0^\alpha$ and

$$F(s) = e^{\alpha \int_0^s [r_P(r) - \frac{1}{2}\sigma^2(r)]dr} \left(P_0 e^{\int_0^s [r_P(r) - \frac{1}{2}\sigma^2(r)]dr} - C_0 e^{\int_0^s [r_C(r) - \frac{1}{2}\sigma^2(r)]dr}\right).$$

It follows from Itô's formula that

$$\begin{aligned} dR(s) &= f(s)R(s)ds + (1+\alpha)\sigma(s)R(s)dW_1(s) \\ &\quad + [(1+\delta(s))^{1+\alpha} - 1]R(s-)dN_1(s), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} f(s) &= \frac{1}{2}(1+\alpha)^2\sigma^2(s) + \frac{F'(s)}{F(s)} \\ &= \frac{1}{2}(1+\alpha)^2\sigma^2(s) + (1+\alpha)[r_P(s) - \frac{1}{2}\sigma^2(s)] + \frac{(r_P(s) - r_C(s))C_0 e^{\int_0^s r_C(r)dr}}{P_0 e^{\int_0^s r_P(r)dr} - C_0 e^{\int_0^s r_C(r)dr}} \end{aligned}$$

is the expected return rate of domestic cash flow, and $(1+\alpha)\sigma(s),(1+\delta(s))^{1+\alpha}-1$ are the volatility rate and the relative jump size respectively.

However, the investors require an additional return to get compensated for their costs engaged in the search of information. So we further modify (2.3) by adding an incremental rate of expected return to the drift term. Thus the real domestic cash flow $R(\cdot)$ satisfies the following SDE with jumps:

$$\begin{cases} dR(s)=[f(s)+\lambda_R(s)]R(s)ds+(1+\alpha)\sigma(s)R(s)dW_1(s)\\ \qquad\qquad +[(1+\delta(s))^{1+\alpha}-1]R(s-)dN_1(s),\\ R_0=(1-\tau)P_0^{\alpha}(P_0-C_0),\end{cases} \tag{2.4}$$

where λ_R is called the information costs rate of the domestic market.

For the foreign cash flow, set

$$\overline{R}(s)=(1-\widetilde{\tau})\widetilde{P}^{\beta}(s)[\widetilde{P}(s)-\widetilde{C}(s)].$$

Similar to the derivation of (2.3), we can get

$$\begin{aligned} d\overline{R}(s)=&\overline{f}(s)\overline{R}(s)ds+(1+\beta)\widetilde{\sigma}(s)\overline{R}(s)dW_2(s)\\ &+[(1+\widetilde{\delta}(s))^{1+\beta}-1]\overline{R}(s-)dN_2(s),\end{aligned}$$

with

$$\overline{f}(s)=\frac{1}{2}(1+\beta)^2\widetilde{\sigma}^2(s)+(1+\beta)[r_{\widetilde{P}}(s)-\frac{1}{2}\widetilde{\sigma}^2(s)]+\frac{(r_{\widetilde{P}}(s)-r_{\widetilde{C}}(s))\widetilde{C}_0e^{\int_0^s r_{\widetilde{C}}(r)dr}}{\widetilde{P}_0e^{\int_0^s r_{\widetilde{P}}(r)dr}-\widetilde{C}_0e^{\int_0^s r_{\widetilde{C}}(r)dr}}.$$

Then applying Itô's formula to $\widetilde{R}(s)=e(s)\overline{R}(s)$, we obtain

$$\begin{aligned} d\widetilde{R}(s)=&\widetilde{f}(s)\widetilde{R}(s)ds+(1+\beta)\widetilde{\sigma}(s)\widetilde{R}(s)dW_2(s)+\sigma_e(s)\widetilde{R}(s)dW_3(s)\\ &+[(1+\widetilde{\delta}(s))^{1+\beta}-1]\widetilde{R}(s-)dN_2(s)+\delta_e(s)\widetilde{R}(s-)dN_3(s),\end{aligned} \tag{2.5}$$

where

$$\widetilde{f}(s)=\overline{f}(s)+r_e(s)+\lambda_e(s)+(1+\beta)\widetilde{\sigma}(s)\sigma_e(s)\rho_{23}$$

is the expected return rate of foreign cash flow converted into the domestic currency, and $(1+\beta)\widetilde{\sigma}(s),\sigma_e(s),(1+\widetilde{\delta}(s))^{1+\beta}-1,\delta_e(s)$ are the volatility rates and the relative jump sizes respectively.

In the presence of information costs, the dynamic of foreign cash flow $\widetilde{R}(\cdot)$ should also be modified by adding an incremental rate of expected return to the drift term and actually given by

$$\begin{cases} d\widetilde{R}(s)=[\widetilde{f}(s)+\lambda_{\widetilde{R}}(s)]\widetilde{R}(s)ds+(1+\beta)\widetilde{\sigma}(s)\widetilde{R}(s)dW_2(s)+\sigma_e(s)\widetilde{R}(s)dW_3(s)\\ \qquad\qquad +[(1+\widetilde{\delta}(s))^{1+\beta}-1]\widetilde{R}(s-)dN_2(s)+\delta_e(s)\widetilde{R}(s-)dN_3(s),\\ \widetilde{R}_0=e_0(1-\widetilde{\tau})\widetilde{P}_0^{\beta}(\widetilde{P}_0-\widetilde{C}_0),\end{cases} \tag{2.6}$$

where $\lambda_{\widetilde{R}}$ is called the information costs rate of the foreign market. We further assume that

Assumption 2.1. *All market parameters* $r_P,r_C,r_{\widetilde{P}},r_{\widetilde{C}},r_e,\sigma,\widetilde{\sigma},\sigma_e,\delta,\widetilde{\delta},\delta_e,\lambda_R,\lambda_{\widetilde{R}},\lambda_e$ *belong to* $\mathcal{L}^{\infty}_{\mathbb{F},p}(0,T;\mathbb{R})$. *Moreover,* $\delta,\widetilde{\delta},\delta_e\geq -1$ *on* $[0,T]$.

ASSUMPTION 2.2. *The covariance matrix Σ of (W_1, W_2, W_3), i.e.,*

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$$

is definitely positive.

Let $X(s)$ be the total wealth of this firm and $\pi(s)$ be the amount of money invested into the foreign market at time s. According to (2.4) and (2.6), we know that the wealth process $\{X(s)\}_{0\leq s\leq T}$ satisfies the following SDE with jumps (the argument s is suppressed):

$$\begin{aligned} dX =& \pi\Big\{[\widetilde{f}+\lambda_{\widetilde{R}}]ds+(1+\beta)\widetilde{\sigma}dW_2+\sigma_e dW_3+[(1+\widetilde{\delta})^{1+\beta}-1]dN_2+\delta_e dN_3\Big\} \\ &+(X-\pi)\Big\{[f+\lambda_R]ds+(1+\alpha)\sigma dW_1+[(1+\delta)^{1+\alpha}-1]dN_1\Big\} \\ =&[(f+\lambda_R)X+(\widetilde{f}-f+\lambda_{\widetilde{R}}-\lambda_R)\pi]ds \\ &+(1+\alpha)\sigma(X-\pi)dW_1+(1+\beta)\widetilde{\sigma}\pi dW_2+\sigma_e\pi dW_3 \\ &+[(1+\delta)^{1+\alpha}-1](X-\pi)dN_1+[(1+\widetilde{\delta})^{1+\beta}-1]\pi dN_2+\delta_e\pi dN_3, \end{aligned} \tag{2.7}$$

with the initial wealth $X(0)=x_0$. It follows from [24] that (2.7) admits a unique solution $X\in\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})$ for any admissible strategy $\pi\in\mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R})$.

By Assumption 2.2, there exists an invertible matrix Γ such that

$$\Gamma\Sigma\Gamma^T = I_3. \tag{2.8}$$

It follows that

$$\widetilde{W}(s)=(\widetilde{W}_1(s),\widetilde{W}_2(s),\widetilde{W}_3(s))^T \triangleq \Gamma(W_1(s),W_2(s),W_3(s))^T,\ 0\leq s\leq T,$$

is a 3-dimensional standard Brownian motion. We further denote

$$\begin{aligned} A(s)=&f(s)+\lambda_R(s)+[(1+\delta(s))^{1+\alpha}-1]\nu_1, \\ B(s)=&\widetilde{f}(s)-f(s)+\lambda_{\widetilde{R}}(s)-\lambda_R(s) \\ &-[(1+\delta(s))^{1+\alpha}-1]\nu_1+[(1+\widetilde{\delta}(s))^{1+\beta}-1]\nu_2+\delta_e(s)\nu_3, \\ C(s)=&\Big[[(1+\alpha)\sigma(s),0,0]\Gamma^{-1}\Big]^T, \\ D(s)=&\Big[[-(1+\alpha)\sigma(s),(1+\beta)\widetilde{\sigma}(s),\sigma_e(s)]\Gamma^{-1}\Big]^T, \\ E(s)=&[(1+\delta(s))^{1+\alpha}-1,0,0]^T, \\ F(s)=&[-(1+\delta(s))^{1+\alpha}+1,(1+\widetilde{\delta}(s))^{1+\beta}-1,\delta_e(s)]^T, \end{aligned} \tag{2.9}$$

and define the compensated Poisson process $\widetilde{N}(\cdot)$ as

$$\widetilde{N}(s)=(\widetilde{N}_1(s),\widetilde{N}_2(s),\widetilde{N}_3(s))^T \triangleq (N_1(s)-\nu_1 s,N_2(s)-\nu_2 s,N_3(s)-\nu_3 s)^T,\ 0\leq s\leq T.$$

Thus the dynamic (2.7) of wealth process $X(\cdot)$ can be rewritten as

$$dX(s)=[A(s)X(s)+B(s)\pi(s)]ds+[C(s)X(s-)+D(s)\pi(s)]^T d\widetilde{W}(s)$$

$$+[E(s)X(s-)+F(s)\pi(s)]^T d\widetilde{N}(s),\ 0\le s\le T. \tag{2.10}$$

In view of both profit and risk, this firm's objective is to look for the optimal investment strategy $\pi^*\in\mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R})$ that minimizes

$$\begin{aligned}J(0,x_0;\pi)&=\frac{\gamma}{2}\mathrm{Var}[X(T)]-\mathbb{E}[X(T)]\\&=\frac{\gamma}{2}\mathbb{E}[X^2(T)]-\frac{\gamma}{2}(\mathbb{E}[X(T)])^2-\mathbb{E}[X(T)],\end{aligned} \tag{2.11}$$

where γ is a given positive constant reflecting the degree of risk aversion of the manager.

This is a typical time-inconsistent linear-quadratic (LQ) optimization problem proposed by [25]. To be precise, we consider the dynamic of wealth process $X(\cdot)$ starting from any $(t,x_t)\in[0,T)\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$, i.e.,

$$\begin{cases}dX(s)=[A(s)X(s)+B(s)\pi(s)]ds+[C(s)X(s-)+D(s)\pi(s)]^T d\widetilde{W}(s)\\ \qquad\qquad +[E(s)X(s-)+F(s)\pi(s)]^T d\widetilde{N}(s),\ t\le s\le T,\\ X(t)=x_t,\end{cases} \tag{2.12}$$

where the coefficients A,B,C,D,E,F are defined by (2.9). Our aim is to minimize the cost functional

$$J(t,x_t;\pi)=\frac{\gamma}{2}\mathbb{E}_t[X^2(T)]-\frac{\gamma}{2}(\mathbb{E}_t[X(T)])^2-\mathbb{E}_t[X(T)] \tag{2.13}$$

over $\pi\in\mathcal{L}^2_{\mathbb{F},p}(t,T;\mathbb{R})$, where $\mathbb{E}_t[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_t]$. Due to the appearance of term $\frac{\gamma}{2}(\mathbb{E}_t[X(T)])^2$, an admissible strategy $\bar\pi\in\mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R})$ minimizing $J(0,x_0;\pi)$ is no longer optimal for $J(t,\bar X(t);\pi)$ when the investment is reviewed at any time $t\in(0,T)$. Taking both the time-consistency and the optimality into consideration, we define the equilibrium strategy of our corporate international investment problem inspired by [15, 25]:

Definition 2.1. *Let $\pi^*\in\mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R})$ be an admissible strategy and $X^*\in\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})$ be the wealth process corresponding to π^*. Then π^* is called an equilibrium strategy if for any $t\in[0,T)$ and $v\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$,*

$$\lim_{\varepsilon\downarrow 0}\frac{J(t,X^*(t);\pi^{t,\varepsilon,v})-J(t,X^*(t);\pi^*)}{\varepsilon}\ge 0,$$

where $\pi^{t,\varepsilon,v}$ is defined by

$$\pi^{t,\varepsilon,v}(s)=\pi^*(s)+v\mathbf{1}_{s\in[t,t+\varepsilon)},\quad t\le s\le T.$$

An equilibrium strategy defined above is optimal infinitesimally via spike variation at any time point. Since the cost functional (2.13) varies from time to time, it is reasonable to regard the firm as a player at any time $t\in[0,T)$ against all its incarnations in the future. Thus it can make a decision to optimize the investment only in an infinitesimal time internal $[t,t+\varepsilon)$. Moreover, an equilibrium strategy on $[0,T]$ is still that on subinterval $[t,T]$ for any $t\in[0,T)$.

Remark 2.2. When the Poisson processes are involved in the model of [6], the optimal foreign investment π^* cannot be represented explicitly by the partial derivatives of solution $u(t,x)$ to the corresponding HJB equation as in [6] because the terms $u(t,(1+\delta)^{1+\alpha}x-((1+\delta)^{1+\alpha}-1)\pi)$, $u(t,x+((1+\widetilde{\delta})^{1+\beta}-1)\pi)$, $u(t,x+\delta_e\pi)$ appear in the HJB equation (see Chapter 9 of [24]). Moreover, there is no literature concerned with the influence of information costs on the corporate international investment in the time-inconsistent situation. So we formulate a time-inconsistent corporate international investment problem with discontinuous cash flow in consideration of information costs and give an equilibrium strategy inspired by the maximum principle.

Thanks to Theorem 3.1 of [25], we can establish a sufficient condition for the equilibrium strategy of our problem as follows:

THEOREM 2.1. *Let Assumption 2.1 and 2.2 hold. An admissible strategy* $\pi^* \in \mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R})$ *is an equilibrium strategy if for any* $t \in [0,T)$,

(i) *The system of FBSDEs with jumps*

$$\begin{cases} dX^*(s) = [A(s)X^*(s) + B(s)\pi^*(s)]ds \\ \qquad\qquad + [C(s)X^*(s-) + D(s)\pi^*(s)]^T d\widetilde{W}(s) \\ \qquad\qquad + [E(s)X^*(s-) + F(s)\pi^*(s)]^T d\widetilde{N}(s), \quad 0 \le s \le T, \\ X(0) = x_0, \\ dY(s;t) = -[A(s)Y(s;t) + C^T(s)Z(s;t) + E^T(s)VK(s;t)]ds \\ \qquad\qquad + Z^T(s;t)d\widetilde{W}(s) + K^T(s;t)d\widetilde{N}(s), \quad t \le s \le T, \\ Y(T;t) = \gamma X^*(T) - \gamma \mathbb{E}_t[X^*(T)] - 1, \end{cases} \tag{2.14}$$

admits a solution $(X^*(\cdot), Y(\cdot;t), Z(\cdot;t), K(\cdot;t)) \in \mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R}) \times \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},p}(t,T;\mathbb{R}^3) \times \mathcal{L}^2_{\mathbb{F},p}(t,T;\mathbb{R}^3)$;

(ii) $\Lambda(s;t) \triangleq B(s)Y(s-;t) + D^T(s)Z(s;t) + F^T(s)VK(s;t)$ *satisfies*

$$\Lambda(t;t) = 0, \quad \mathbf{P}\text{-}a.s., \tag{2.15}$$

where V *is defined by*

$$V \triangleq \mathrm{Diag}(\nu_1, \nu_2, \nu_3). \tag{2.16}$$

The proof is straightforward, so we just omit it. For each $t \in [0,T)$, the BSDE in (2.14) is the adjoint equation and the constraint (2.15) is the maximum condition of our optimization problem. More detailed derivation can be referred to [25].

REMARK 2.3. The authors consider $\bigcup_{q>4} \mathcal{L}^q_{\mathbb{F},p}(0,T;\mathbb{R}^l)$ as the set of all admissible controls in [25]. It is crucial for establishing the necessary condition of an equilibrium control in Theorem 3.1 there. However, the sufficient condition there still holds when the set of all admissible controls is relaxed to $\mathcal{L}^2_{\mathbb{F},p}(0,T;\mathbb{R}^l)$. Interested readers can refer to the proof of Theorem 3.1 and Remark 3.3 in [25] for more details.

In fact, (2.14) form a flow of FBSDEs with random jumps as time t evolves from 0 to T. Thus the equilibrium strategies can be obtained by solving such type of equations. However, the general existence of solutions to the flow of FBSDEs (2.14) with constraint (2.15) remains a challenging open problem.

REMARK 2.4. The FBSDEs in (2.14) are not standard since a flow of unknowns $(Y(\cdot;t), Z(\cdot;t), K(\cdot;t))$ is involved and there is an additional constraint acting on the diagonal (i.e., when $s=t$) of the flow. In fact, the forward SDE and backward SDEs are coupled with $(Y(s;s), Z(s;s), K(s;s))_{0\le s\le T}$. So we cannot simply regard (2.14) as parameterized FBSDEs or discuss its wellposedness by the usual arguments for FBSDEs.

In next section, we will solve (2.14) when the market parameters are all deterministic and give an equilibrium strategy of our corporate international investment problem explicitly for this case.

3. Equilibrium strategy when market parameters are deterministic

Throughout this section, we assume

ASSUMPTION 3.1. *All market parameters* $r_P, r_C, r_{\widetilde{P}}, r_{\widetilde{C}}, r_e, \sigma, \widetilde{\sigma}, \sigma_e, \delta, \widetilde{\delta}, \delta_e, \lambda_R, \lambda_{\widetilde{R}}, \lambda_e$ *are bounded deterministic functions on* $[0,T]$*. Moreover,* $\delta, \widetilde{\delta}, \delta_e \geq -1$ *on* $[0,T]$.

Then the coefficients defined in (2.9) are all deterministic and bounded. We assume further that

ASSUMPTION 3.2. *There exists a constant* $\varepsilon > 0$ *such that* $D^T(t)D(t) + F^T(t)VF^T(t) \geq \varepsilon$ *for any* $t \in [0,T]$.

Motivated by the linear feedback optimality of a classical LQ control problem, we conjecture that for any $t \in [0,T)$,

$$Y(s;t) = M(s)X^*(s) - N(s)\mathbb{E}_t[X^*(s)] + \Phi(s), \quad t \leq s \leq T, \tag{3.1}$$

where $M(\cdot)$, $N(\cdot)$ and $\Phi(\cdot)$ are deterministic differentiable functions to be determined later.

For any fixed t, applying Itô's formula to $Y(\cdot;t)$, we have

$$\begin{aligned} dY(s;t) = &\{M(s)[A(s)X^*(s) + B(s)\pi^*(s)] + \dot{M}(s)X^*(s) \\ &- N(s)\mathbb{E}_t[A(s)X^*(s) + B(s)\pi^*(s)] - \dot{N}(s)\mathbb{E}_t[X^*(s)] + \dot{\Phi}(s)\}dt \\ &+ M(s)[C(s)X^*(s-) + D(s)\pi^*(s)]^T d\widetilde{W}(s) \\ &+ M(s)[E(s)X^*(s-) + F(s)\pi^*(s)]^T d\widetilde{N}(s). \end{aligned} \tag{3.2}$$

Comparing the $d\widetilde{W}(s)$ and $d\widetilde{N}(s)$ terms above with those in (2.14), we get

$$Z(s;t) = M(s)[C(s)X^*(s-) + D(s)\pi^*(s)], \tag{3.3}$$

$$K(s;t) = M(s)[E(s)X^*(s-) + F(s)\pi^*(s)]. \tag{3.4}$$

It can be seen that $Z(s;t)$ and $K(s;t)$ are independent of t.

Plugging (3.1), (3.3) and (3.4) into the constraint (2.15), we have

$$\begin{aligned} B(s)[(M(s) - N(s))X^*(s-) + \Phi(s)] + D^T(s)[C(s)X^*(s-) + D(s)\pi^*(s)]M(s) \\ + F^T(s)V[E(s)X^*(s-) + F(s)\pi^*(s)]M(s) = 0, \end{aligned}$$

from which we can formally deduce

$$\pi^*(s) = \psi(s)X^*(s-) + \varphi(s), \tag{3.5}$$

where

$$\begin{aligned} \psi(s) = &-[D^T(s)D(s) + F^T(s)VF(s)]^{-1}M^{-1}(s) \\ &\times [(M(s) - N(s))B(s) + D^T(s)C(s)M(s) + F^T(s)VE(s)M(s)], \\ \varphi(s) = &-[D^T(s)D(s) + F^T(s)VF(s)]^{-1}M^{-1}(s)\Phi(s)B(s). \end{aligned}$$

Next, comparing the ds term in (3.2) with that in (2.14), we obtain (the argument s is suppressed)

$$\begin{aligned}0=&\dot{M}X^*+M([AX^*+B\pi^*)-\dot{N}\mathbb{E}_t[X^*]-N(A\mathbb{E}_t[X^*]+B\mathbb{E}_t[\pi^*])+\dot{\Phi}\\&+A(MX^*-N\mathbb{E}_t[X^*]+\Phi)+C^T(CX^*+D\pi^*)M+E^TV(EX^*+F\pi^*)M\\=&[\dot{M}+2AM+C^TCM+E^TVEM+(BM+C^TDM+E^TVFM)\psi]X^*\\&-[\dot{N}+2AN+BN\psi]\mathbb{E}_t[X^*]\\&+[\dot{\Phi}+A\Phi+(C^TD+E^TVF)M\varphi+B(M-N)\varphi].\end{aligned}$$

This leads to the following ODEs for $M(\cdot)$, $N(\cdot)$ and $\Phi(\cdot)$ (again the argument s is suppressed):

$$\begin{cases}0=\dot{M}+(2A+C^TC+E^TVE)M-(B+C^TD+E^TVF)\\\qquad\times(D^TD+F^TVF)^{-1}[(M-N)B+C^TDM+E^TVFM],\\M(T)=\gamma;\end{cases}\tag{3.6}$$

$$\begin{cases}0=\dot{N}+2AN-BNM^{-1}\\\qquad\times(D^TD+F^TVF)^{-1}[(M-N)B+C^TDM+E^TVFM],\\N(T)=\gamma;\end{cases}\tag{3.7}$$

$$\begin{cases}0=\dot{\Phi}+\{A-[(M-N)B+C^TDM+E^TVFM]\\\qquad\times(D^TD+F^TVF)^{-1}M^{-1}B\}\Phi,\\\Phi(T)=-1.\end{cases}\tag{3.8}$$

In fact, (3.6) and (3.7) form a system of coupled nonlinear ODEs for (M,N). Once we get their solutions, (3.8) is simply a linear ODE. So it is crucial to solve (3.6) and (3.7). Fortunately, we have

PROPOSITION 3.1. *Let Assumptions 3.1 and 3.2 hold. Then (3.6) and (3.7) admit a unique positive bounded solution pair (M,N).*

Proof. Set $J=\frac{M}{N}$. We can formally derive

$$\begin{cases}\dot{J}=-[C^TC+E^TVE\\\qquad-(C^TD+E^TVF)(D^TD+F^TVF)^{-1}(B+C^TD+E^TVF)]J\\\qquad-B(D^TD+F^TVF)^{-1}(C^TD+E^TVF),\\J(T)=1,\end{cases}\tag{3.9}$$

and

$$\begin{cases}\dot{M}=-[2A+C^TC+E^TVE\\\qquad-(B+C^TD+E^TVF)(D^TD+F^TVF)^{-1}(B+C^TD+E^TVF)\\\qquad+B(D^TD+F^TVF)^{-1}(B+C^TD+E^TVF)\dfrac{1}{J}]M,\\M(T)=\gamma.\end{cases}\tag{3.10}$$

(3.9) is simply a linear ODE with bounded coefficients. Hence it admits a unique bounded solution $J(\cdot)$.

Define $\tilde{J}=J-1$. Then we have

$$\dot{\tilde{J}}=-\theta\tilde{J}-\phi,$$

where

$$\theta=C^TC+E^TVE-(C^TD+E^TVF)(D^TD+F^TVF)^{-1}(B+C^TD+E^TVF)$$

is bounded, and

$$\begin{aligned}\phi=&\theta+B(D^TD+F^TVF)^{-1}(C^TD+E^TVF)\\=&C^TC+E^TVE-(C^TD+E^TVF)(D^TD+F^TVF)^{-1}(C^TD+E^TVF)\\\geq&0,\end{aligned}$$

because

$$\begin{aligned}&(C^TC+E^TVE)(D^TD+F^TVF)-(C^TD+E^TVF)^2\\=&|C|^2|D|^2+|EV^{\frac{1}{2}}|^2|FV^{\frac{1}{2}}|^2+|C|^2|FV^{\frac{1}{2}}|^2+|D|^2|EV^{\frac{1}{2}}|^2\\&-|C^TD|^2-|E^TVF|^2-2(C^TD)(E^TVF)\\\geq&|C|^2|FV^{\frac{1}{2}}|^2+|D|^2|EV^{\frac{1}{2}}|^2-2|C||D||EV^{\frac{1}{2}}||FV^{\frac{1}{2}}|\\\geq&0.\end{aligned}$$

So $J\geq 1$. Thus (3.10) also becomes a linear ODE with bounded coefficients, and it admits a unique positive bounded solution $M(\cdot)$. Then the desired result is obtained. ☐

Now, we can present the main result of this section.

THEOREM 3.1. *Let Assumptions 2.2, 3.1 and 3.2 hold. Then π^* given by (3.5) is an equilibrium strategy of our corporate international investment problem, where $M(\cdot)$, $N(\cdot)$, $\Phi(\cdot)$ are solutions of (3.6), (3.7), (3.8) respectively, and X^* is the wealth process corresponding to π^*.*

Proof. By Proposition 3.1, (3.6) and (3.7) admit a unique positive bounded solution pair (M,N). Thus there exists a unique bounded solution Φ for (3.8).

Substituting (3.5) into the forward SDE of (2.14), we can see that it admits a unique solution $X^*\in\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})$. So π^* is admissible.

For any $t\in(0,T]$, define $Y(\cdot;t)$, $Z(\cdot;t)$ and $K(\cdot;t)$ by (3.1), (3.3) and (3.4) respectively. It is straightforward to check that $(Y(\cdot;t),Z(\cdot;t),K(\cdot;t))\in\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R})\times\mathcal{L}^2_{\mathbb{F},p}(t,T;\mathbb{R}^3)\times\mathcal{L}^2_{\mathbb{F},p}(t,T;\mathbb{R}^3)$ satisfies the backward SDE of (2.14) and the constraint (2.15).

By Theorem 2.1, π^* is an equilibrium strategy. ☐

Though proving the existence and uniqueness of solutions to (3.6), (3.7) and (3.8), we are unable to express $M(\cdot)$, $N(\cdot)$ and $\Phi(\cdot)$ by the market parameters analytically or discuss their effect on the equilibrium strategy directly as in [6]. Fortunately, $M(\cdot)$, $N(\cdot)$ and $\Phi(\cdot)$ can be computed numerically by the finite difference method. We shall present some numerical examples to illustrate our results obtained above and show the influence of market parameters on the equilibrium strategy in the next section.

REMARK 3.1. Comparing to the optimal proportion of foreign investment in the firm's total wealth in [6], we give the equilibrium proportion of that by $\frac{\pi^*}{X^*}$. It is adopted to discuss the influence of market parameters in harmony with the discussion in [6].

4. Numerical examples and influence of market parameters

In practice, the market parameters can be estimated by some statistical methods from the historical data while the coefficient in the cost functional can be determined by the manager's preference. We first apply Euler's method to solve (3.6)-(3.8), and substitute (3.5) into (2.10). Then the equilibrium wealth process and the equilibrium strategy can be approximately computed by Euler's method for SDEs according to the processes of prices and exchange rate. Now we give some simulating examples to show the computation results and the influence of some market parameters.

For the domestic and foreign production, we suppose $r_P=0.06, r_C=0.02, r_{\widetilde{P}}=0.08, r_{\widetilde{C}}=0.03, r_e=0.02, \sigma=0.2, \widetilde{\sigma}=0.15, \sigma_e=0.05, \delta=0, \widetilde{\delta}=0, \lambda_e=0.01$ and $\alpha=\beta=-2$. Moreover, we assume $T=1$, $x_0=1$ and $\gamma=1$ for the manager and $\rho_{12}=0.8, \rho_{13}=0.6, \rho_{23}=0.4, \nu_1=\nu_2=\nu_3=3$ in the market.

Let us begin with the effect of information costs rate of the foreign market.

EXAMPLE 4.1. *Set $\delta_e=0$, $\lambda_R=0.005$ and $\lambda_{\widetilde{R}}=0.008, 0.010, 0.012$ respectively. By simulating a 3-dimensional standard Brownian motion, we can get three paths of equilibrium wealth process and thus those of equilibrium proportion $\frac{\pi^*}{X^*}$ corresponding to three different values of $\lambda_{\widetilde{R}}$. They are plotted in Figure 4.1.*

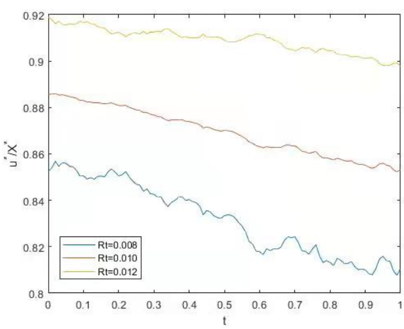

FIG. 4.1. $\frac{\pi^*}{X^*}$ *corresponding to different* $\lambda_{\widetilde{R}}$.

Figure 4.1 shows that the equilibrium proportion of foreign investment in the firm's total wealth increases as the information costs rate of the foreign market increases. The higher information costs rate of the foreign market means that the manager engages more costs in the search of information and thus knows more about the foreign market. So he/she prefers to invest in the foreign market. Most wealth of the firm is invested in the foreign market in Example 4.1 where the three different values of $\lambda_{\widetilde{R}}$ are relatively high, i.e., the manager is fairly familiar with the foreign market.

Then we discuss the influence of information costs rate of the domestic market.

EXAMPLE 4.2. *Set $\delta_e=0$, $\lambda_{\widetilde{R}}=0.001$ and $\lambda_R=0.010, 0.011, 0.012$ respectively. By simulating a 3-dimensional standard Brownian motion again, we can also get three paths of equilibrium wealth process and thus those of equilibrium proportion $\frac{\pi^*}{X^*}$ corresponding to three different values of λ_R. They are plotted in Figure 4.2.*

It can be seen from Figure 4.2 that the higher the information costs rate of the domestic market is, the lower the equilibrium proportion of foreign investment becomes. As the information costs rate of the domestic market increases, the manager knows more and more information of the domestic market and thus is more familiar with that than

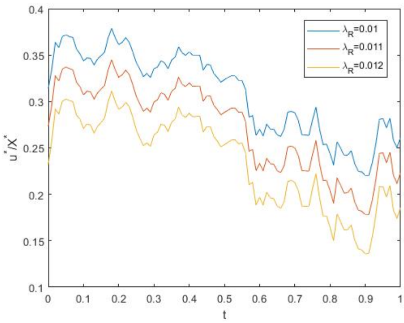

FIG. 4.2. $\frac{\pi^*}{X^*}$ *corresponding to different* λ_R.

the foreign market. So he/she prefers to invest in the domestic market. In Example 4.2 where the value of $\lambda_{\widetilde{R}}$ is very low, i.e., the manager knows little about the foreign market, most wealth of the firm is invested in the domestic market. This is exactly the "home bias puzzle" in international finance.

Besides, the discontinuity of exchange rate also has an effect on the equilibrium strategy.

EXAMPLE 4.3. *Set* $\lambda_R = 0.005$, $\lambda_{\widetilde{R}} = 0.008$ *and* $\delta_e = -0.01, 0, 0.01$ *respectively. By simulating a 3-dimensional standard Brownian motion and a Poisson process with intensity* $\nu_3 = 3$, *we can get three paths of equilibrium wealth process and thus those of equilibrium proportion* $\frac{\pi^*}{X^*}$ *corresponding to three different values of* δ_e. *They are also plotted in Figure 4.3.*

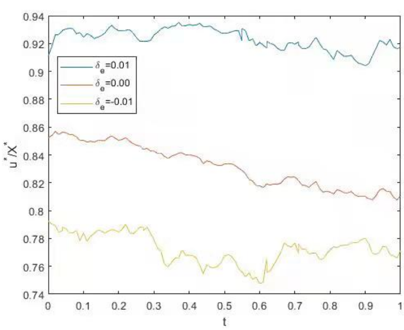

FIG. 4.3. $\frac{\pi^*}{X^*}$ *corresponding to different* δ_e

Figure 4.3 tells us that the change of relative jump size of the exchange rate and that of equilibrium proportion are in the same direction in Example 4.3 where the degree of risk aversion of the manager reflected by γ is relatively low. It is because that the increase of relative jump size of the exchange rate raises the rate of expected return for the foreign market. However, the risk of investing in the foreign market also becomes bigger as the absolute value of relative jump size of the exchange rate increases. So it may have an opposite influence on the equilibrium proportion when the degree of risk aversion of the manager is fairly high.

REMARK 4.1. Since $\delta=\widetilde{\delta}=\delta_e=0$ in Examples 4.1 and 4.2, there is no necessity for us to simulate Poisson processes. We only need to simulate one Poisson process for the dynamic of exchange rate in Example 4.3 where δ and $\widetilde{\delta}$ still equal to 0.

5. Conclusion

To the authors' best knowledge, it is the first time to study a time-inconsistent corporate international investment problem with discontinuous cash flow. We define the equilibrium strategy of our problem and establish a sufficient condition for it via a flow of FBSDEs with jumps. When all market parameters are deterministic, this flow of FBSDEs is further reduced into two coupled highly nonlinear ODEs and one linear ODE, of which the solvability is an open problem in existing literature. We dedicate to prove the existence and uniqueness of solutions to these ODEs by the method of variable transformation. Thus an equilibrium strategy is given explicitly by solving these ODEs numerically. Moreover, we present several simulating examples to show the computation results and the influence of some market parameters on the equilibrium strategy. It is worth pointing out that the information costs also play an important role and explain the "home bias puzzle" in our time-inconsistent international investment problem.

However, the general wellposedness of the flow of FBSDEs above is still an outstanding problem. We hope to address this problem in the future work.

REFERENCES

[1] R.Z. Aliber, *A theory of direct foreign investment*, in C.P. Kindleberger (ed.), The International Corporation, Cambridge: MIT Press, 1970. 1

[2] R.Z. Aliber, *Money, multinationals and sovereigns*, in C.P. Kindleberger and D.B. Audretsch (eds.), The Multinational Corporation in the 1980s, Cambridge: MIT Press, 1983. 1

[3] D.S. Bates, *Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options*, Rev. Financ. Stud., 9:69–107, 1996. 1

[4] M. Bellalah, M. Bradford, and D. Zhang, *A general theory of corporate international investment under incomplete information, short sales and taxes*, Econ. Model., 58:615–626, 2016. 1

[5] M. Bellalah and Z. Wu, *A model for market closure and international portfolio management within incomplete information*, Int. J. Theor. Appl. Finance, 5:479–495, 2002. 1, 2

[6] M. Bellalah and Z. Wu, *A simple model of corporate international investment under incomplete information and taxes*, Ann. Oper. Res., 165:123–143, 2009. 1, 2, 2, 2.2, 3, 3.1

[7] M. Bellalah and D. Zhang, *A model for international capital markets closure in an economy with incomplete information markets and short sales*, Econ. Model., 67:316–324, 2017. 1

[8] T. Bjork, A. Murgoci, and X.Y. Zhou, *Mean-variance portfolio optimization with state dependent risk aversion*, Math. Finance, 24:1–24, 2014. 1

[9] F. Black, *International capital market equilibrium with investment barriers*, J. Financ. Econ., 1:337–352, 1974. 1

[10] A. Chockalingam and K. Muthuraman, *Pricing American options when asset prices jump*, Oper. Res. Lett., 38:82–86, 2010. 1

[11] J.J. Choi, *Diversification, exchange risks and corporate international investment*, J. Int. Bus. Stud., 20:145–155, 1989. 1

[12] I. Ekeland and A. Lazrak, *The golden rule when preferences are time inconsistent*, Math. Finan. Econ., 4:29–55, 2010. 1

[13] I. Ekeland and T.A. Pirvu, *Investment and consumption without commitment*, Math. Finan. Econ., 2:57–86, 2008. 1

[14] H. Geman, *Pure jump Levy processes for asset price modelling*, J. Bank. Financ., 26:1297–1316, 2002. 1

[15] Y. Hu, H. Jin, and X. Zhou, *Time-inconsistent stochastic linear-quadratic control*, SIAM J. Control Optim., 50:1548–1572, 2012. 1, 2

[16] Y. Hu, H. Jin, and X. Zhou, *Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium*, SIAM J. Control Optim., 55:1261–1279, 2017. 1

[17] M.C. Jensen and W.H. Meckling, *Theories of the firm: Managerial behavior, agency costs, and ownership structure*, J. Financ. Econ., 3:305–360, 1976. 1

[18] S. Kindermann and P.A. Mayer, *On the calibration of local jump-diffusion asset price models*, Finance Stoch., 15:685–724, 2011. 1

[19] H. Levy and M. Sarnat, *Diversification, portfolio analysis and the uneasy case for conglomerate mergers*, J. Finance, 25:795–802, 1970. 1

[20] W.G. Lewellen, *A pure financial rationale for the conglomerate merger*, J. Finance, 26:521–537, 1971. 1

[21] I. Mathur and K. Hanagan, *Are multinational corporations superior investment vehicles for achieving international diversification?* J. Int. Bus. Stud., 14:135–146, 1983. 1

[22] R.C. Merton, *An equilibrium market model with incomplete information*, J. Finance, 42:483–511, 1987. 1, 2

[23] A.M. Rugman, *Risk, direct investment and international investment*, Rev. World Econ., 113:487–500, 1977. 1

[24] R. Situ, *Theory of Stochastic Differential Equations with Jumps and Applications*, Springer, 2005. 2, 2.2

[25] Z. Sun and X. Guo, *Equilibrium for a time-inconsistent stochastic linear-quadratic control system with jumps and its application to the mean-variance problem*, J. Optim. Theory Appl., 181:383–410, 2019. 1, 2, 2, 2, 2, 2.3

[26] J.M. Yong, *A deterministic linear quadratic time-inconsistent optimal control problem*, Math. Control Relat. Fields, 1:83–118, 2011. 1

[27] J.M. Yong, *Time-inconsistent optimal control problems and the equilibrium HJB equation*, Math. Control Relat. Fields, 2:271–329, 2012. 1