COMMUN. MATH. SCI.
Vol. 21, No. 7, pp. 1815–1863

HIGH FRICTION LIMITS OF EULER–NAVIER–STOKES–KORTEWEG EQUATIONS FOR MULTICOMPONENT MODELS*

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Abstract. In this paper we analyze the high friction regime for the Navier–Stokes–Korteweg equations for multicomponent systems. According to the shape of the mixing and friction terms, we shall perform two limits: the high friction limit toward an equilibrium system for the limit densities and the *barycentric* velocity, and, after an appropriate time scaling, the diffusive relaxation toward parabolic, gradient flow equations for the limit densities. The rigorous justification of these limits is done by means of relative entropy techniques in the framework of weak, finite energy solutions of the relaxation models, rewritten in the enlarged formulation in terms of the *drift velocity*, toward smooth solutions of the corresponding equilibrium dynamics. Finally, since our estimates are uniform for small viscosity, the results are also valid for the Euler–Korteweg multicomponent models, and the corresponding estimates can be obtained by sending the viscosity to zero.

Keywords. High friction limit; diffusive relaxation; Euler–Navier–Stokes–Korteweg equations; Stefan–Maxwell systems; relative entropy method.

AMS subject classifications. 35L65; 35B25.

1. Introduction

In this paper we study the high friction limit for the Navier-Stokes-Korteweg multicomponent systems [19, 24, 26], that is:

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0 \\ \partial_t(\rho_i u_i) + \operatorname{div}(\rho_i u_i \otimes u_i) - 2\nu \operatorname{div}(\mu_L(\rho_i) D(u_i)) - \nu \nabla(\lambda_L(\rho_i) \operatorname{div} u_i) + \nabla \rho_i^\gamma \\ = \rho_i \nabla \left(k(\rho_i) \Delta \rho_i + \frac{1}{2} k'(\rho_i) |\nabla \rho_i|^2 \right) - \frac{1}{\epsilon} \sum_{j=1}^n b_{i,j} \rho_i \rho_j (u_i - u_j) - \frac{M_i}{\epsilon} \rho_i u_i, \end{cases} \tag{1.1}$$

where $i = 1, \cdots, n$, $t > 0$, $x \in \mathbb{T}^3$, the n-dimensional torus, ρ_i are the particles' density, u_i their velocities (and, accordingly, $m_i = \rho_i u_i$ their momenta), $\nu \geq 0$ is the viscosity coefficient, $M_i \geq 0$, and the singular coefficient $1/\epsilon$ in front of the terms

$$-\sum_{j=1}^n b_{i,j} \rho_i \rho_j (u_i - u_j) - M_i \rho_i u_i$$

is responsible for the high friction regime $\epsilon \to 0$. The term $k(\rho_i) \geq 0$ stands for the capillarity coefficient and usually has the form of the power law, while $D(u_i)$ is the symmetric part of the gradient ∇u_i, and the Lamé coefficients $\mu_L(\rho_i)$ and $\lambda_L(\rho_i)$ satisfy, for every particles' density,

$$\mu_L(\rho_i) \geq 0; \qquad \frac{2}{n} \mu_L(\rho_i) + \lambda_L(\rho_i) \geq 0. \tag{1.2}$$

*Received: March 30, 2022; Accepted (in revised form): January 13, 2023. Communicated by François Bouchut.

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We consider, for simplicity, γ-law pressure, that is $p(\rho_i)=\rho_i^\gamma$ with $\gamma>1$ and the corresponding internal energy $h(\rho_i)$ is given by:

$$h(\rho_i)=\frac{\rho_i^\gamma}{\gamma-1};$$

more general hypotheses for the monotone function $p(\rho)$ can be considered. Moreover, as usual for this kind of models, we rewrite the pressure and the Korteweg terms by introducing the stress tensors T_i in the following way:

$$-\nabla\rho_i^\gamma+\rho_i\nabla\left(k(\rho_i)\Delta\rho_i+\frac{1}{2}k'(\rho_i)|\nabla\rho_i|^2\right)=\operatorname{div}T_i.$$

The novelty of the multicomponent model presented here is the presence of many different particles, and as a consequence, the presence of the interaction term

$$\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j),$$

which stands for the momentum production rate due to diffusive mixing. Here, the nonnegative matrix $b_{i,j}$ models the interaction of the i-th and j-th components, with a strength weighted by ϵ. In addition, as already proposed in [24], we also consider a diagonal term $M_i\rho_i u_i$, $M_i\geq 0$, accounting for a (classical) friction term. The term modeling the mixing shall agree with the conservation of the total momentum, while the latter friction term comes as a body force, in accordance with the single component case [24]. Hence, we shall require

$$\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)=0, \tag{1.3}$$

which is true provided the matrix $\{b_{i,j}\}_{i,j=1}^n$ is symmetric. Moreover, this condition and assumptions $b_{i,j}\geq 0$ and $M_i\geq 0$ guarantee the dissipative structure of the model. Indeed, introducing the vectors $\hat\rho=(\rho_1,\cdots,\rho_n)$ and $\hat m=(m_1,\cdots,m_n)$, the total mechanical energy associated to (1.1) is given by

$$\eta_{tot}(\hat\rho,\hat m,\nabla\hat\rho):=\sum_{i=1}^n\eta(\rho_i,m_i,\nabla\rho_i):=\sum_{i=1}^n\left(\frac{1}{2}\rho_i|u_i|^2+\frac{\rho_i^\gamma}{\gamma-1}+\frac{1}{2}k(\rho_i)|\nabla\rho_i|^2\right) \tag{1.4}$$

and it verifies the following relation:

$$\frac{d}{dt}\sum_{i=1}^n\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_i|u_i|^2+\frac{\rho_i^\gamma}{\gamma-1}+\frac{1}{2}k(\rho_i)|\nabla\rho_i|^2\right)dx+\frac{1}{2\epsilon}\sum_{i,j=1}^n\int_{\mathbb{T}^3}\frac{1}{2}b_{i,j}\rho_i\rho_j|u_i-u_j|^2dx$$
$$+\frac{1}{\epsilon}\sum_{i=1}^n\int_{\mathbb{T}^3}M_i\rho_i|u_i|^2\,dx=0,$$

being

$$\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\cdot u_i=\frac{1}{2}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2\geq 0. \tag{1.5}$$

Concerning the aforementioned mixing term, we collect here below all required assumptions, referred to as *Stefan-Maxwell Ansatz* [24]. We introduce the $\mathbb{R}^n \times \mathbb{R}^n$ matrix B defined as

$$B_{i,j} := b_{i,j}\rho_i\rho_j \tag{1.6}$$

and we assume

$$b_{i,j} = b_{j,i} \geq 0 \text{ for any } i,j = 1,\cdots,n; \quad b_{i,i} = -\sum_{j=1, j\neq i} b_{i,j}. \tag{1.7}$$

We observe that, thanks to (1.7), we readily obtain

$$\sum_{j=1}^{n} b_{i,j}\rho_i\rho_j(u_i - u_j) = \sum_{j=1}^{n} B_{i,j}(u_i - u_j) = -\sum_{j=1}^{n} B_{i,j}u_j,$$

because, for any $j = 1,\ldots,n$,

$$\sum_{j=1}^{n} B_{i,j}u_i = 0.$$

For later convenience, we thus introduce the matrix $\tau := -B$ and u solution of (1.3) is equivalent to $u \in Ker(\tau)$. Moreover, since $(1,\cdots,1) \in Ker(\tau)$, then $\dim Ker(\tau) \geq 1$. The study of this kernel will be crucial in the forthcoming discussions and we shall consider appropriate conditions for it in the sequel.

Following [19,24], the present investigation is confined in the analysis of the behavior of weak, finite energy solutions of such systems in the high friction regime, and not on their existence; for the latter, for single component cases, see [3–6] and the references therein. Hence, in the present paper we are dealing with the rigorous justification of relaxation limits [8, 10, 12, 27], in particular using relative entropy approach [25]. As we shall point out later, when the diagonal term $M_i\rho_i u_i$ is present in the model, we shall obtain a nontrivial equilibrium dynamic after a time scaling, leading to a diffusive relaxation limit. These kind of limits have been addressed in different frameworks and with using many tools. In particular, we refer to [15] and the references therein for the results concerning weak solutions and compactness arguments; in this context, see also [2] for a recent study concerning the relaxation limit for weak, finite energy solutions to the Quantum Navier–Stokes–Poisson system toward weak solutions of Quantum drift–diffusion equation. Still in the context of diffusive relaxations, many other (diffusive) limits have been addressed using relative entropy tools; among others, see [7,11,14,16]. Finally, referring in particular to multicomponent models, we shall also mention here the relevant examples of relaxation limits for (bipolar) Euler-Poisson equations describing electrons and positively charged ions in plasmas or semiconductors [1,20,21].

Our analysis takes advantage of relative entropy techniques in the framework of finite energy weak solutions of the enlarged formulation of (1.1), namely we rewrite the latter in terms of the *drift velocity* v_i [9]:

$$v_i = \frac{\nabla\mu(\rho_i)}{\rho_i},$$

where $\mu(\rho_i)$ satisfies $\mu'(\rho_i) = \sqrt{\rho_i k(\rho_i)}$. In this way it is possible to obtain the following

augmented formulation of (1.1):

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0 \\ \partial_t(\rho_i u_i) + \operatorname{div}(\rho_i u_i \otimes u_i) - 2\nu \operatorname{div}(\mu_L(\rho_i) D(u_i)) - \nu \nabla(\lambda_L(\rho_i) \operatorname{div} u_i) + \nabla \rho_i^\gamma \\ = \operatorname{div}(\mu(\rho_i)\nabla v_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} v_i) - \frac{1}{\epsilon}\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i - u_j) - \frac{M_i}{\epsilon}\rho_i u_i \\ \partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes u_i) + \operatorname{div}(\mu(\rho_i)^t \nabla u_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} u_i) = 0, \end{cases} \tag{1.8}$$

where $\lambda(\rho_i) = 2(\mu'(\rho_i)\rho_i - \mu(\rho_i))$. Using the relation between $\mu(\rho_i)$ and $\lambda(\rho_i)$ we define:

$$\begin{aligned} \operatorname{div} S_i &= \operatorname{div}(\mu(\rho_i)\nabla v_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} v_i), \\ \operatorname{div} K_i &= \operatorname{div}(\mu(\rho_i)^t \nabla u_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} u_i), \end{aligned}$$

and we readily obtain

$$\begin{aligned} \int_{\mathbb{T}^3} \operatorname{div} S_i \cdot u_i \, dx &= -\int_{\mathbb{T}^3} \mu(\rho_i)\nabla v_i : \nabla u_i \, dx - \frac{1}{2}\int_{\mathbb{T}^3} \lambda(\rho_i)\operatorname{div} v_i \operatorname{div} u_i \, dx \\ &= -\int_{\mathbb{T}^3} \mu(\rho_i)^t \nabla u_i : \nabla v_i \, dx - \frac{1}{2}\int_{\mathbb{T}^3} \lambda(\rho_i)\operatorname{div} u_i \operatorname{div} v_i \, dx \\ &= \int_{\mathbb{T}^3} \operatorname{div} K_i \cdot v_i \, dx. \end{aligned} \tag{1.9}$$

The above identity comes from the symmetry of ∇v_i, which implies ${}^t\nabla u_i : \nabla v_i = \nabla v_i : \nabla u_i$.

Hence, in contrast with the one used in [19,24], as already done in [13], the strategy we shall use here is to estimate the following relative entropy, expressed in terms of the momenta $m_i = \rho_i u_i$ and the *drift momenta* $J_i = \rho_i v_i = \nabla \mu(\rho_i)$. To this end, using again the notation $\hat{\rho} = (\rho_1, \cdots, \rho_n)$, $\hat{m} = (m_1, \cdots, m_n)$, $\hat{J}(x,t) = (J_1, \cdots, J_n)$, and the relative entropy is then given by

$$\begin{aligned} \eta_{tot}(\hat{\rho}, \hat{m}, \hat{J} | \bar{\hat{\rho}}, \bar{\hat{m}}, \bar{\hat{J}})(t) &:= \sum_{i=1}^n \eta(\rho_i, u_i, J_i | \bar{\rho}_i, \bar{u}_i, \bar{J}_i) \\ := \sum_{i=1}^n \Big(\eta(\rho_i, u_i, J_i) - \eta(\bar{\rho}_i, \bar{u}_i, \bar{J}_i) - \frac{\partial \eta}{\partial \rho_i}(\bar{\rho}_i, \bar{u}_i, \bar{J}_i)(\rho_i - \bar{\rho}_i) \\ - \frac{\partial \eta}{\partial m_i}(\bar{\rho}_i, \bar{u}_i, \bar{J}_i)(m_i - \bar{m}_i) - \frac{\partial \eta}{\partial J_i}(\bar{\rho}_i, \bar{u}_i, \bar{J}_i)(J_i - \bar{J}_i) \Big), \end{aligned}$$

where we recall $\eta_{tot}(\hat{\rho}, \hat{m}, \hat{J})$ is defined as

$$\eta_{tot}(\hat{\rho}, \hat{m}, \hat{J}) = \sum_{i=1}^n \eta(\rho_i, m_i, J_i) = \sum_{i=1}^n \left(\frac{1}{2}\frac{|m_i|^2}{\rho_i} + \frac{1}{2}\frac{|J_i|^2}{\rho_i} + h(\rho_i) \right), \tag{1.10}$$

that is, (1.4) after we have introduced the variables J_i. As a consequence, the quadratic

expression we shall estimate here becomes the integral of η_{tot} in dx, namely:

$$\Sigma_{tot}(\hat{\rho},\hat{m},\hat{J}|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})(t) := \int_{\mathbb{T}^3} \sum_{i=1}^{n} \eta(\rho_i,u_i,J_i|\bar{\rho}_i,\bar{u}_i,\bar{J}_i)\,dx$$
$$= \int_{\mathbb{T}^3} \sum_{i=1}^{n} \left(\frac{1}{2}\rho_i \left| \frac{m_i}{\rho_i} - \frac{\bar{m}_i}{\bar{\rho}_i} \right|^2 + \frac{1}{2}\rho_i \left| \frac{J_i}{\rho_i} - \frac{\bar{J}_i}{\bar{\rho}_i} \right|^2 + h(\rho_i|\bar{\rho}_i) \right) dx. \tag{1.11}$$

The behavior of the system under investigation here in the high friction regime $\epsilon \to 0$ strongly relies on the presence or absence of the diagonal term $M_i\rho_i u_i \geq 0$. Indeed, when this term is present, the high friction regime, after an appropriate time scaling, is given by the following parabolic equation:

$$\partial_t \rho_i = \operatorname{div}_x \left(\rho_i \nabla_x \left(h'(\rho_i) + k(\rho_i)\Delta\rho_i + \frac{1}{2}k'(\rho_i)|\nabla\rho_i|^2 \right) \right), \text{ for any } i = 1,\cdots,n, \tag{1.12}$$

both for Euler–Korteweg and Navier–Stokes–Korteweg systems, as one can check by performing the classical Hilbert expansion; see Section 2 for details. Concerning the Lamé coefficients, besides the natural condition (1.2) needed to guarantee the dissipative nature of the viscosity terms, in this case we shall assume only appropriate uniform integrability conditions, without a precise connection with the capillarity coefficient $k(\rho)$, as it is usually needed in the analysis of these models, as these terms will come as higher order errors in the limit. In other words, in the case $M_i > 0$ we are dealing with a diffusive relaxation, as the hyperbolic system converges toward parabolic equilibrium systems (1.12) in the diffusive scaling. Starting from [22], where the authors discussed the case of the diffusive relaxation of the Euler system with friction toward the porous media equation in the single component case, a general framework for the relative entropy calculation and the analysis of the diffusive limits have been presented in [17,23]; see also [18] for the non–monotone pressure cases. The models under investigation here, without viscosity, mixing, and friction terms, are included in the framework of abstract Euler flows generated by the first variation of an energy functional $\mathcal{E}(\rho)$ introduced in these papers:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho\partial_t u + \rho u \cdot \nabla u = -\rho\nabla \dfrac{\delta \mathcal{E}}{\delta \rho}. \end{cases}$$

Specifically, system (1.1) is obtained for the following particular choice for $\mathcal{E}(\rho)$:

$$\mathcal{E}(\rho) = \int \sum_{i=1}^{n} \left(h(\rho_i) + \frac{1}{2}k(\rho_i)|\nabla\rho_i|^2 \right) dx. \tag{1.13}$$

These results have been improved following the enlarged formulation of [9], and in particular the diffusive relaxation limit for the single component case is treated in [13]. As already mentioned above, in the present work we shall then adopt the same strategy of [13] for the multicomponent system, thus improving the results contained in [24], as we include viscosity and capillarity effects in our model. It is worth observing that the stability result we obtain in the Navier–Stokes–Korteweg case is uniform for $0 < \nu \ll 1$, and thus we obtain in particular the corresponding estimate also for the Euler–Korteweg model, by taking the limit $\nu \to 0^+$ in the former inequality.

The case $M_i = 0$, treated in [19], is significantly different, as we do not perform any diffusive time scaling, and the equilibrium system in the case $\nu = 0$ is not parabolic, namely, there is no emergence of diffusive behavior in the high friction regime. Indeed, following the argument of this paper, after the Hilbert expansion of (1.8) in Section 2, we recover the following hyperbolic system at equilibrium, where we introduce the notation $\bar{\cdot}$ to indicate the equilibrium dynamics for the system:

$$\begin{cases} \partial_t \bar{\rho}_i + \operatorname{div}(\bar{\rho}_i \bar{u}) = 0 \\ \partial_t(\bar{\rho}\bar{u}) + \operatorname{div}(\bar{\rho}\bar{u}\otimes\bar{u}) + \displaystyle\sum_{i=1}^n \nabla p(\bar{\rho}_i) - 2\nu \sum_{i=1}^n \operatorname{div}(\mu_L(\bar{\rho}_i) D(\bar{u})) - \nu \sum_{i=1}^n \nabla(\lambda_L(\bar{\rho}_i)\operatorname{div}(\bar{u})) \\ = \displaystyle\sum_{i=1}^n \operatorname{div}(\mu(\bar{\rho}_i)\nabla \bar{v}_i) + \sum_{i=1}^n \frac{1}{2}\nabla(\lambda(\bar{\rho}_i)\operatorname{div}\bar{v}_i) \\ \partial_t(\bar{\rho}_i \bar{v}_i) + \operatorname{div}(\bar{\rho}_i \bar{v}_i \otimes \bar{u}) + \operatorname{div}(\mu(\bar{\rho}_i)^t \nabla(\bar{u})) + \frac{1}{2}\nabla(\lambda(\bar{\rho}_i)\operatorname{div}(\bar{u})) = 0. \end{cases} \tag{1.14}$$

We recall here that $\mu_L(\rho_i), \lambda_L(\rho_i) \geq 0$ stand for the Lamé coefficients, while $\mu(\rho_i), \lambda(\rho_i)$ are the capillarity ones used in the augmented formulation. Finally, $\bar{u}$ stands for the barycentric velocity defined by the following relations:

$$\bar{\rho} = \sum_{i=1}^n \bar{\rho}_i, \ \ \bar{\rho}\bar{u} = \sum_{i=1}^n \bar{\rho}_i \bar{u}_i.$$

As it is manifested, contrary to the previous analysis, the equilibrium dynamics is described by the group velocity $\bar{u}$ for each particle density $\bar{\rho}_i$. This is a consequence of the absence of the diagonal term $M_i \rho_i u_i$ in the momentum equation and therefore the interaction mixing term

$$\sum_{j=1}^n b_{i,j} \rho_i \rho_j |u_i - u_j|$$

alone plays the role of an alignment term for the velocities u_i.

Another important difference with the previous case concerns the presence of the viscosity term also in the equilibrium system (1.14), which, in this scaling, are not higher order errors (see again Section 2 for details), and we are thus forced to manage them differently in the relative entropy calculation. For this, as in [9], we shall make the following particular choices for the Lamé coefficients: $\mu_L(\rho_i) = \mu(\rho_i)$ and $\lambda_L(\rho_i) = \lambda(\rho_i)$, which will be crucial to estimate the viscosity terms by means of the relative entropy and thus obtain the desired stability for the limit under investigation. Besides the presence of these extra terms in our model, thanks to the enlarged formulation adopted here, we are also able to consider more general capillarity coefficients, and thus slightly generalize the results proved in [19] also under this point of view. Indeed, as already pointed out for the previous case, our stability estimate is uniform for $0 < \nu \ll 1$ and it is consistent with the one obtained in the Euler–Korteweg case in [19] in the limit $\nu \to 0^+$.

The remaining part of this paper is organized as follows. In Section 2 we present the Hilbert expansion of system (1.8) in both cases $M_i = 0$ and $M_i > 0$, underlining in particular the differences due to the presence/absence of the diagonal term $M_i \rho_i u_i$. In Section 3 we investigate the case of (1.8) when $M_i > 0$. As already noted before, the equilibrium is given by the parabolic equation for each particle density ρ_i, namely the gradient flow (1.12) written in terms of the drift velocity v_i, both for Euler–Korteweg and Navier–Stokes–Korteweg, consistent with [13] for the single component case. We

prove a stability estimate between weak, entropic solutions of (1.1) and strong solutions of the equilibrium system, exploiting the multicomponent version of the relative entropy inequality. The last section is devoted to the complementary case $M_i = 0$: the structure is the same as the one in Section 3, however this time no diffusive scaling is present and the equilibrium is given by the system satisfied by the densities and the barycentric velocity.

2. Hilbert expansion

In this section we shall perform a Hilbert expansion for solution of (1.8) as a first step in the analysis of our high friction limit. To this end, we have to study the solvability properties of the linear system depending on $M_i \geq 0$ given by:

$$-\sum_{j=1}^{n} b_{i,j}\rho_i\rho_j(u_i - u_j) - M_i\rho_i u_i = d_i, \quad d_i \in \mathbb{R}^3, \qquad i = 1, \cdots, n, \tag{2.1}$$

and the associated homogeneous system

$$\sum_{j=1}^{n} b_{i,j}\rho_i\rho_j(u_i - u_j) + M_i\rho_i u_i = 0, \qquad i = 1, \cdots, n. \tag{2.2}$$

The authors in [19] make the following hypothesis for the homogeneous system with $M_i = 0$:

$$\sum_{j=1}^{n} B_{i,j}(u_i - u_j) = 0, \qquad i = 1, \cdots, n, \tag{2.3}$$

where $B_{i,j}$ is defined in (1.6).

(A1) Let $\{b_{i,j}\}_{i,j=1}^{n}$ be a symmetric matrix such that $b_{i,j} \geq 0$ for $i \neq j$. For any $\rho_i > 0$, $i = 1, \cdots n$, system (2.3) has the one dimensional null space span$\{\mathbf{1}\}$, where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$.

Let us emphasize that, if $b_{i,j} > 0$ for any i, j, the above hypothesis is automatically satisfied, as it manifests from (1.5).

The following result is proved in [19, Lemma 1] and it will be used in the Hilbert expansion to provide a semi–explicit solution to system (2.1) in the case $M_i = 0$ for any $i = 1, \ldots, n$. It is worth observing that such system is independent of the specific component of the vectors $u_i \in \mathbb{R}^3$ and therefore its resolution and the subsequent computations of this section are done component-wise (when not otherwise specified).

LEMMA 2.1 (Lemma 1 in [19]). *Let $d_1, \cdots, d_n \in \mathbb{R}^3$ satisfy $\sum_{i=1}^{n} d_i = 0$, $\rho_i > 0$ for $i = 1, \cdots, n$ and assume condition (A1) above. Then the system*

$$-\sum_{j=1}^{n} B_{i,j}(u_i - u_j) = d_i \quad i = 1, \cdots n, \quad \textit{subject to} \ \sum_{i=1}^{n} \rho_i u_i = 0$$

has the unique solution

$$\rho_i u_i = -\sum_{j,k=1}^{n} \left(\delta_{i,j}\rho_i - \frac{\rho_i\rho_j}{\rho}\right)(\tau)^{-1}_{j,k} d_k, \quad \rho_n u_n = -\sum_{j=1}^{n} \rho_j u_j,$$

where $i=1,\cdots,n$ $\rho=\sum_{i=1}^{n}\rho_i>0$ *and* $(\tau)^{-1}_{i,j}\in\mathbb{R}^{(n-1)\times(n-1)}$ *is the inverse of a regular submatrix, obtained from reordering the matrix* $\tau_{i,j}\in\mathbb{R}^{n\times n}$ *of rank* $n-1$ *with coefficients*

$$\tau_{i,j}=\delta_{i,j}\sum_{k=1}^{n}B_{i,k}-B_{i,j},\quad i,j=1,\cdots,n. \tag{2.4}$$

It is worth observing that, in view of the assumption (1.7), the definition (2.4) of τ in Lemma 2.1 coincides with the matrix $\tau=-B$, for B defined in (1.6).

Now, we pass to the study of the solvability of (2.1), when the coefficients M_i do not all vanish. Therefore, let us identify the null-space of system (2.2) and, to this end, we prove the following result concerning the matrix

$$\tau^M:=\tau+M;\quad M=\mathrm{diag}(M_1\rho_1,\dots,M_n\rho_n). \tag{2.5}$$

PROPOSITION 2.1. *Let* B *be any positive semi-definite matrix as in* (1.6) *and let* $\tau=-B$. *Moreover, let* M *be any diagonal, positive semi-definite matrix* M *defined as in* (2.5). *Then, the following property holds:*

$$Ker(M)\cap Ker(\tau)=Ker(\tau^M). \tag{2.6}$$

Proof. ($\subseteq$)
Let us suppose $\mathbf{u}\in Ker(M)\cap Ker(\tau)$, then

$$\tau^M\cdot\mathbf{u}=(\tau+M)\cdot\mathbf{u}=\tau\cdot\mathbf{u}+M\cdot\mathbf{u}=\mathbf{0},$$

therefore $\mathbf{u}\in Ker(\tau^M)$.

($\supseteq$) Let $\mathbf{u}\in Ker(\tau^M)$ so that $\tau\cdot\mathbf{u}+M\cdot\mathbf{u}=\mathbf{0}$. Multiplying this equality by $\mathbf{u}$, using the definition of $\tau_{i,j}$ and the property in (1.7), we can then write

$$\sum_{i,j=1}^{n}b_{i,j}\rho_i\rho_j(u_i-u_j)\cdot u_i+\sum_{i=1}^{n}M_i\rho_i|u_i|^2=0.$$

Since $b_{i,j}=b_{j,i}$, this is equivalent to requiring

$$\frac{1}{2}\sum_{i,j=1}^{n}b_{i,j}\rho_i\rho_j|u_i-u_j|^2+\sum_{i=1}^{n}M_i\rho_i|u_i|^2=0,$$

and, being in addition $b_{i,j}$ and M_i non-negative for any i and j, we further deduce

$$\sum_{i=1}^{n}M_i\rho_i|u_i|^2=0.$$

Hence, for any index for which $M_i\rho_i>0$ we obtain $u_i=0$, that is, since M is diagonal, $M\cdot\mathbf{u}=\mathbf{0}$. This implies $\mathbf{u}\in Ker(M)$ and consequently $\mathbf{u}\in Ker(\tau)$, being $-\tau\cdot\mathbf{u}=M\cdot\mathbf{u}=0$, and the proof is complete. ∎

This simple observation helps us to analyze the kernel of the matrix τ^M if $M\neq 0$. However, even if the $\dim(Ker(M)\cap Ker(\tau))=1=Ker(\tau^M)$ and thus $Rk(\tau^M)=n-1$, we can not follow the proof of Lemma 2.1 to find the unique solution of the non-homogeneous system (2.1). Due to the presence of the drug force in our model, the

total momentum is not conserved, and we do not have, in general, additional information about its solutions. On the other side, when $Ker(\tau)\cap Ker(M)=\{\mathbf{0}\}$ the matrix τ^M is invertible and system (2.1) has a unique solution given by

$$u=-(\tau^M)^{-1}d,$$

as in the case of our framework (A2) below. Further, general conditions which guarantee the invertibility of τ^M could be considered as well, but one needs specific information about the matrices τ and M. For this, we analyze here below some simple examples to show that it is not possible, in general, to get information on the increase of the rank of $\tau+M$ by looking only to the rank of τ and the rank of M, but one needs to know information on the structure of their null spaces, and not only on their dimensions.

EXAMPLE 2.1. *Let us consider a matrix $\tau\in\mathbb{R}^{4\times4}$ defined as in (2.4) such that $Rk(\tau)=2$ and $\rho_i=\rho_j=1$, for every $i,j=1,\cdots,4$; for instance:*

$$\tau=\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}.$$

Then

$$Ker(\tau)=span\{(1,0,1,0),(0,1,0,1)\}.$$

Now, for M defined as follows

$$M=\begin{pmatrix}m_1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

with $m_1>0$, clearly we have

$$Ker(M)=span\{(0,1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

In this case the resulting matrix τ^M is such that $Rk(\tau^M)=3>Rk(\tau)=2$. Indeed, the solution of the homogeneous system $(\tau^M)\cdot\boldsymbol{u}=0$ is

$$\{(0,t,0,t);t\in\mathbb{R}\}=Ker(\tau)\cap Ker(M)=Ker(\tau^M).$$

Using the result of Proposition 2.1, we readily obtain this relation, being

$$\{(0,t,0,t);t\in\mathbb{R}\}=Ker(\tau)\cap Ker(M)=Ker(\tau^M).$$

Let us now consider two 4×4 matrices M_1,M_2 with rank 2 in two possible configurations, that is

$$M_1=\begin{pmatrix}m_1&0&0&0\\0&m_2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad M_2=\begin{pmatrix}m_1&0&0&0\\0&0&0&0\\0&0&m_3&0\\0&0&0&0\end{pmatrix},$$

such that $m_1, m_2, m_3 > 0$. The solutions of the corresponding systems $(\tau + M_i)\boldsymbol{u} = 0$ are discussed here below. In the first case,

$$Ker(M_1) = span\{(0,0,1,0),(0,0,0,1)\}.$$

From a direct inspection of the system $(\tau + M_1)\boldsymbol{u} = 0$ we obtain only the trivial solution, that is $\tau + M_1$ is invertible.

Again, we remark that $Ker(\tau) \cap Ker(M_1) = \{\boldsymbol{0}\}$, and τ^M is invertible from Proposition 2.1.

In the second case,

$$Ker(M_2) = span\{(0,1,0,0),(0,0,0,1)\}$$

the null space of $\tau + M_2$ is given by $span\{(0,1,0,1)\}$, i.e., $\tau + M_2$ has rank 3. As before,

$$\{(0,t,0,t); t \in \mathbb{R}\} = Ker(\tau) \cap Ker(M_2) = Ker(\tau^M).$$

As already mentioned above, we present a possible framework for which τ^M becomes invertible. Analogous to condition (A1) above, we make the following hypothesis; however, see also Remark 2.2.

(A2) Let $\{b_{i,j}\}_{i,j=1}^n$ be such that hypothesis (A1) holds and assume there exists at least one index $\bar{i} \in \{1, \cdots, n\}$ such that $M_{\bar{i}}\rho_{\bar{i}} > 0$.

COROLLARY 2.1. *If (A2) holds, then τ^M is invertible.*

Proof. If $\mathbf{u} \in Ker(\tau^M)$ then, in view of Proposition 2.1, $\mathbf{u} \in Ker(\tau) \cap Ker(M)$ and, in particular, following the proof of this result, we conclude

$$\sum_{i=1}^n M_i \rho_i |u_i|^2 = 0.$$

From (A2) we know that there exists at least one index $\bar{i}$ such that $M_{\bar{i}}\rho_{\bar{i}} > 0$ and, correspondingly, from the above equality we conclude $u_{\bar{i}} = 0$. Finally, since $Ker(\tau)$ reduces to span$\{\mathbf{1}\}$ in view of condition (A1), we conclude $\mathbf{u} = (0, \cdots, 0)$ is the unique solution of (2.2) and the proof is complete. ∎

REMARK 2.1. In the case $M_i > 0$ for every $i = 1, \cdots, n$ we do not need to impose any condition for the null space of system (2.3) as in (A1). Indeed, following the previous analysis we get the same energy equality:

$$\frac{1}{2} \sum_{i,j=1}^n b_{i,j} \rho_i \rho_j |u_i - u_j|^2 + \sum_{i=1}^n M_i \rho_i |u_i|^2 = 0 \tag{2.7}$$

and, as before,

$$\sum_{i=1}^n M_i \rho_i |u_i|^2 = 0.$$

Then, since $M_i > 0$ for every i, then the only solution is the trivial one, namely $u_i = 0$ for $i = 1, \cdots, n$ Summarizing, with $\tau_{i,j}$ given as before, system (2.2) rewrites as follows:

$$\sum_{j=1}^n \tau_{i,j} u_j + M_i \rho_i u_i = 0, \qquad i = 1, \cdots, n, \tag{2.8}$$

and therefore, under the assumption (A2), the matrix τ^M is invertible and we can find the solution u_i for every $i \in \{1,\cdots,n\}$ from (2.1): $u = -(\tau^M_{i,j})^{-1} d$. This is the result corresponding to Lemma 2.1 when a diagonal friction term is present. In the sequel, we shall then refer to the aforementioned frameworks: (A1) if $M=0$ and (A2) if $M \neq 0$. The unique solution of the corresponding non-homogeneous linear system will then be used in the forthcoming Hilbert expansion, leading to the hyperbolic equilibrium dynamics satisfied by the barycentric velocity in the framework (A1), and leading to the (easier) case of a parabolic equilibrium dynamics, in the framework (A2).

We are now ready to perform the Hilbert expansion of system (1.8) in the high friction regime, namely for small $\epsilon > 0$. For this, let us introduce the following quantities:

$$\begin{aligned} \rho_i^\epsilon &= \rho_i^0 + \epsilon \rho_i^1 + O(\epsilon^2); \\ u_i^\epsilon &= u_i^0 + \epsilon u_i^1 + O(\epsilon^2); \\ v_i^\epsilon &= v_i^0 + \epsilon v_i^1 + O(\epsilon^2). \end{aligned} \tag{2.9}$$

We study separately the cases $M > 0$ and $M = 0$.

2.1. M > 0; framework (A2). Let us start by inserting the expansions (2.9) into the Equations (1.8). Collecting the terms of the same order we get:

- $O(1/\epsilon)$:

$$\sum_{j=1}^n b_{i,j} \rho_i^0 \rho_j^0 (u_i^0 - u_j^0) + M_i \rho_i^0 u_i^0 = \sum_{j=1}^n \tau_{i,j}^{M,0} \cdot u_j^0 = 0; \quad \tau_{i,j}^{M,0} = -b_{i,j} \rho_i^0 \rho_j^0 + M_i \rho_i^0;$$

- $O(1)$:

$$\begin{aligned} &\partial_t \rho_i^0 + \operatorname{div}(\rho_i^0 u_i^0) = 0; \\ &\partial_t (\rho_i^0 u_i^0) + \operatorname{div}(\rho_i^0 u_i^0 \otimes u_i^0) - 2\nu \operatorname{div}(\mu_L(\rho_i^0) D(u_i^0)) - \nu \nabla(\lambda_L(\rho_i^0) \operatorname{div} u_i^0) + \nabla p(\rho_i^0) \\ &= \operatorname{div}(\mu(\rho_i^0) \nabla v_i^0) + \frac{1}{2} \nabla(\lambda(\rho_i^0) \operatorname{div} v_i^0 - \sum_{j=1}^n b_{i,j} \rho_i^1 \rho_j^0 (u_i^0 - u_j^0) \\ &- \sum_{j=1}^n b_{i,j} \rho_i^0 \rho_j^1 (u_i^0 - u_j^0) - M_i \rho_i^1 u_i^0 - \sum_{j=1}^n \tau_{i,j}^{M,0} u_j^1; \\ &\partial_t (\rho_i^0 v_i^0) + \operatorname{div}(\rho_i^0 v_i^0 \otimes u_i^0) + \operatorname{div}(\mu(\rho_i^0)^t \nabla u_i^0) + \frac{1}{2} \nabla(\lambda(\rho_i^0) \operatorname{div} u_i^0) = 0. \end{aligned}$$

Under the framework (A2), we know that τ^M is invertible and therefore the only solution of the homogeneous system obtained at order $1/\epsilon$ is $u_i^0 = 0$ for every $i = 1,\cdots,n$. Hence, the momentum equation in $O(1)$ reduces to

$$\sum_{j=1}^n \tau_{i,j}^{M,0} u_j^1 = -\nabla p(\rho_i^0) + \operatorname{div}(\mu(\rho_i^0) \nabla v_i^0) + \frac{1}{2} \nabla(\lambda(\rho_i^0) \operatorname{div} v_i^0 =: d_i^0, \tag{2.10}$$

and, since $\tau^{M,0}$ is invertible, we can find u_i^1 as function of ρ_i^0:

$$u_i^1 = \sum_{k=1}^n (\tau^{M,0})^{-1}_{i,k} d_k^0. \tag{2.11}$$

Moreover, the continuity equation and the "drift" equation reduce to the trivial relations $\partial_t \rho_i^0 = 0$ and $\partial_t(\rho_i^0 v_i^0) = 0$, being $\rho_i^0 v_i^0 = \nabla \mu(\rho_i^0)$.

At this point it remains to express each component in $\mathbb{R}^3$ of the n-velocities u_i. Following the argument in [24] we compute in $\mathbb{T}^3 \times (0,T)$ the Kronecker product:

$$\bar{\tau}^{M,0} = \tau^{M,0} \otimes I_3 \in \mathbb{R}^{3n \times 3n}. \tag{2.12}$$

The matrix $\bar{\tau}^{M,0}$ is invertible since it is defined as the Kronecker product of two invertible matrices, and therefore we can compute the first non-trivial term in the expansion for the momentum, namely $\rho_i^0 u_i^1$ for every i $\in \{1, \cdots, n\}$.

Now, using (2.11) in the $O(\epsilon)$ part of the continuity equation

$$\partial_t \rho_i^1 + \operatorname{div}(\rho_i^0 u_i^1) = 0, \tag{2.13}$$

we readily obtain

$$\partial_t \rho_i^1 + \operatorname{div}\left(\rho_i^0 \, (\bar{\tau}^{M,0})^{-1} \cdot \left(\nabla p(\rho_i^0) - \operatorname{div}(\mu(\rho_i^0)\nabla v_i^0) - \frac{1}{2}\nabla(\lambda(\rho_i^0)\operatorname{div} v_i^0)\right)\right) = 0. \tag{2.14}$$

Therefore, the continuity equation for ρ_i^ϵ and u_i^ϵ in (2.9) up to order ϵ^2 is given by summing the trivial relation $\partial_t \rho_i^0 = 0$ to ϵ times (2.13):

$$\partial_t(\rho_i^0 + \epsilon \rho_i^1) + \epsilon \operatorname{div}(\rho_i^0 u_i^1) = 0,$$

that is, using instead (2.14):

$$\partial_t \rho_i^\epsilon + \epsilon \operatorname{div}\left(\rho_i^\epsilon \, (\bar{\tau}^{M,\epsilon})^{-1} \cdot \left(\nabla p(\rho_i^\epsilon) - \operatorname{div}(\mu(\rho_i^\epsilon)\nabla v_i^\epsilon) - \frac{1}{2}\nabla(\lambda(\rho_i^\epsilon)\operatorname{div} v_i^\epsilon)\right)\right) = 0,$$

where

$$\bar{\tau}^{M,\epsilon} = \tau^{M,\epsilon} \otimes I_3; \quad \tau_{i,j}^{M,\epsilon} = -b_{i,j}\rho_i^\epsilon \rho_j^\epsilon + M_i \rho_i^\epsilon.$$

Hence, in order to recover a non-trivial behaviour in the limit $\epsilon \to 0$, as customary in diffusive limits, one has to rescale the time variable $\partial_t \to \epsilon \partial_t$ in the original model, as done in [24]. In this paper, the authors consider the following system:

$$\begin{aligned} &\partial_t \rho_i + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i) = 0 \\ &\partial_t(\rho_i u_i) + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i \otimes u_i) + \frac{1}{\epsilon}\nabla p(\rho_i) = -\frac{1}{\epsilon^2}\sum_{j=1}^{n} b_{i,j}\rho_i \rho_j (u_i - u_j) - \frac{1}{\epsilon^2} M_i \rho_i u_i \end{aligned} \tag{2.15}$$

and show that as $\epsilon \to 0$ the hyperbolic system converges to the parabolic system:

$$\partial_t \bar{\rho}_i + \operatorname{div}\left(\tilde{C}(\bar{\rho})^{-1}\nabla p(\bar{\rho}_i)\right) = 0 \quad \text{for } i = 1, \cdots, n,$$

where $\tilde{C}(\bar{\rho}) = \bar{C}(\bar{\rho}) \otimes I_3$ as defined in Remark 2.2 below. In the present paper we extend the results of [24] by adding capillarity and viscosity effects in the system, thus studying the diffusive relaxation limit of the model (3.2), obtained after the aforementioned scaling of time in the augmented formulation (1.8); see Section 3 for details.

Specifically, the system we shall investigate becomes

$$\begin{aligned}
&\partial_t \rho_i + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i) = 0\\
&\partial_t(\rho_i u_i) + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i \otimes u_i) - \frac{2}{\epsilon}\nu\operatorname{div}(\mu_L(\rho_i)D(u_i)) - \frac{\nu}{\epsilon}\nabla(\lambda_L(\rho_i)\operatorname{div} u_i) + \frac{1}{\epsilon}\nabla\rho_i^\gamma\\
&= \frac{1}{\epsilon}\operatorname{div}(\mu(\rho_i)\nabla v_i) + \frac{1}{2\epsilon}\nabla(\lambda(\rho_i)\operatorname{div} v_i) - \frac{1}{\epsilon^2}\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i - u_j) - \frac{M_i}{\epsilon^2}\rho_i u_i\\
&\partial_t(\rho_i v_i) + \frac{1}{\epsilon}\operatorname{div}(\rho_i v_i \otimes u_i) + \frac{1}{\epsilon}\operatorname{div}(\mu(\rho_i)^t\nabla u_i) + \frac{1}{2\epsilon}\nabla(\lambda(\rho_i)\operatorname{div} u_i) = 0,
\end{aligned} \tag{2.16}$$

Therefore, as noticed above, using the Hilbert expansion, from the $O(1/\epsilon^2)$ term

$$\sum_{j=1}^n b_{i,j}\rho_i^0\rho_j^0(u_i^0 - u_j^0) + M_i\rho_i^0 u_i^0 = \sum_{j=1}^n \tau_{i,j}^{M,0}\cdot u_j^0 = 0,$$

we readily obtain $u_i^0 = 0$ for any $i = 1,\dots,n$. Hence, the continuity equation starts from order $O(1)$, that is:

$$\partial_t\rho_i^0 + \operatorname{div}(\rho_i^0 u_i^1) = 0,$$

and the momentum equation at order $O(1/\epsilon)$ gives (2.10) and, as a consequence, (2.11). Finally, for $\bar\tau^{M,0}$ defined in (2.12), we obtain the following parabolic system at equilibrium:

$$\partial_t\rho_i^1 + \operatorname{div}\left(\rho_i^0\,(\bar\tau^{M,0})^{-1}\cdot\left(\nabla p(\rho_i^0) - \operatorname{div}(\mu(\rho_i^0)\nabla v_i^0) - \frac{1}{2}\nabla(\lambda(\rho_i^0)\operatorname{div} v_i^0)\right)\right) = 0.$$

REMARK 2.2. Let us remark that the authors in [24] make the following assumption on the friction coefficients, motivated by experimental evidence: $M_i >> b_{i,j}$, ensuring that the matrix $\bar C = \operatorname{diag}(M_i) - \operatorname{diag}(\rho_i)B$, where $B = \{b_{i,j}\}_{i,j=1}^n$ is in fact diagonally dominant and positive definite, thus invertible using the Kronecker product as above. Thanks to this condition, each component in the limiting dynamics evolves according to its velocity, while, without the diagonal term M, one needs to introduce a *barycentric velocity* as done in next section. We stress that the invertibility condition, referred to the matrix $\bar C$ associated to the non-homogeneous system (2.10) has the purpose to solve for $\rho_i^0 u_i^1$ while, from the discussion above, we deduced the invertibility of the matrix $-\tau_{i,j} + \operatorname{diag}(M_i\rho_i)$, which allows us to find directly u_i^1 in function of ρ_i^0; see (2.11).

2.2. M = 0; framework (A1). In the case under investigation, namely for $M_i = 0$ and $\nu \geq 0$, we follow the approach of [19]. We introduce the *barycentric velocity* u, the *relative velocities* $w_i = u_i - u$, and their expansions, namely:

$$\begin{aligned}
w_i^\epsilon &= w_i^0 + \epsilon w_i^1 + O(\epsilon^2);\\
u^\epsilon &= u^0 + \epsilon u^1 + O(\epsilon^2).
\end{aligned} \tag{2.17}$$

Moreover, we have to consider also the total density and the moments:

$$\rho = \sum_{i=1}^n \rho_i, \quad \rho u = \sum_{i=1}^n \rho_i u_i, \quad \rho v = \sum_{i=1}^n \rho_i v_i. \tag{2.18}$$

Using these extra quantities, the original system (1.8) becomes:

$$\begin{aligned}
&\partial_t\rho_i+\operatorname{div}(\rho_i w_i+\rho_i u)=0\\
&\partial_t(\rho_i w_i+\rho_i u)+\operatorname{div}(\rho_i(w_i+u)\otimes(w_i+u))+\nabla\rho_i^\gamma-2\nu\operatorname{div}(\mu_L(\rho_i)D(w_i+u))\\
&-\nu\nabla(\lambda_L(\rho_i)\operatorname{div}(w_i+u))=\operatorname{div}(\mu(\rho_i)\nabla v_i)+\frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div}v_i)-\frac{1}{\epsilon}\sum_{i,j=1}^{n}b_{i,j}\rho_i\rho_j(w_i-w_j)\\
&\partial_t(\rho_i v_i)+\operatorname{div}(\rho_i v_i\otimes(w_i+u))+\operatorname{div}(\mu(\rho_i)^t\nabla(w_i+u))+\frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div}(w_i+u))=0,
\end{aligned}\tag{2.19}$$

subject to the condition:

$$\sum_{i=1}^{n}\rho_i w_i=\sum_{i=1}^{n}\rho_i(u_i-u)=\sum_{i=1}^{n}\rho_i u_i-\rho u=0.\tag{2.20}$$

Using (2.17) in (2.20) the latter becomes:

$$0=\sum_{i=1}^{n}\rho_i w_i=\sum_{i=1}^{n}\rho_i^0 w_i^0+\epsilon\sum_{i=1}^{n}(\rho_i^0 w_i^1+\rho_i^1 w_i^0)+O(\epsilon^2),$$

that is, looking at order $O(1)$ and at order $O(\epsilon)$:

$$\sum_{i=1}^{n}\rho_i^0 w_i^0=0;\qquad\sum_{i=1}^{n}(\rho_i^0 w_i^1+\rho_i^1 w_i^0)=0.\tag{2.21}$$

Finally, we use $(2.9)_1$ to deduce the following expansion of the total mass ρ:

$$\rho=\rho^0+\epsilon\rho^1+O(\epsilon^2),$$

where

$$\rho^0:=\sum_{i=1}^{n}\rho_i^0;\quad\rho^1:=\sum_{i=1}^{n}\rho_i^1.$$

Plugging the above expansions in (2.19) and collecting the terms of the same order in ϵ we obtain:

- $O(1/\epsilon)$:

$$\sum_{j=1}^{n}b_{i,j}\rho_i\rho_j(w_i^0-w_j^0)=0;\tag{2.22}$$

- $O(1)$:

$$\begin{aligned}
&\partial_t\rho_i^0+\operatorname{div}(\rho_i^0 w_i^0+\rho_i^0 u^0)=0;\\
&\partial_t(\rho_i^0 w_i^0+\rho_i^0 u^0)+\operatorname{div}(\rho_i^0(w_i^0+u^0)\otimes(w_i^0+u^0))+\nabla p(\rho_i^0)\\
&\quad-2\nu\operatorname{div}(\mu_L(\rho_i)D(w_i^0+u^0))-\nu\nabla(\lambda_L(\rho_i^0)\operatorname{div}(w_i^0+u^0))\\
&=\operatorname{div}(\mu(\rho_i^0)\nabla v_i^0)+\frac{1}{2}\nabla(\lambda(\rho_i^0)\operatorname{div}v_i^0)-\sum_{j=1}^{n}b_{i,j}\rho_i^0\rho_j^0(w_i^1-w_j^1)
\end{aligned}$$

$$-\sum_{j=1}^{n} b_{i,j}(\rho_i^1\rho_j^0+\rho_i^0\rho_j^1)(w_i^0-w_j^0); \tag{2.23}$$

$$\partial_t(\rho_i^0 v_i^0)+\operatorname{div}(\rho_i^0 v_i^0\otimes(w_i^0+u^0))+\operatorname{div}(\mu(\rho_i^0)^t\nabla(w_i^0+u^0))$$
$$+\frac{1}{2}\nabla(\lambda(\rho_i^0)\operatorname{div}(w_i^0+u^0))=0.$$

We note that, being all densities strictly positive, condition $(2.21)_1$ implies $w_i^0=0$ for $i=1,\cdots,n$, simplifying (2.23) considerably. Then we sum from $i=1,\cdots,n$ the momentum Equation $(2.23)_2$ obtaining in this way the equation satisfied by the barycentric velocity. Specifically:

$$\partial_t\left(\sum_{i=1}^{n}\rho_i^0u^0\right)+\operatorname{div}\left(\sum_{i=1}^{n}\rho_i^0u^0\otimes u^0\right)+\sum_{i=1}^{n}\nabla p(\rho_i^0)-2\nu\sum_{i=1}^{n}\operatorname{div}(\mu_L(\rho_i^0)D(u^0))$$
$$-\nu\sum_{i=1}^{n}\nabla(\lambda_L(\rho_i^0)\operatorname{div}(u^0))=\sum_{i=1}^{n}\operatorname{div}(\mu(\rho_i^0)\nabla v_i^0)+\sum_{i=1}^{n}\frac{1}{2}\nabla(\lambda(\rho_i^0)\operatorname{div}v_i^0)=0, \tag{2.24}$$

where we used the equality

$$\sum_{i,j=1}^{n} b_{i,j}\rho_i^0\rho_j^0(w_i^1-w_j^1)=0$$

due to symmentry. Hence, denoting $\rho_i^0=\bar\rho_i$, $\sum_{i=1}^n\rho_i^0=\bar\rho$, $v_i^0=\bar v_i$ and $u^0=\bar u$, the leading term in the Hilbert expansion of (2.19) is given by the following system:

$$\partial_t\bar\rho_i+\operatorname{div}(\bar\rho_i\bar u)=0;$$

$$\partial_t(\bar\rho\bar u)+\operatorname{div}(\bar\rho\bar u\otimes\bar u)+\sum_{i=1}^{n}\nabla p(\bar\rho_i)-2\nu\sum_{i=1}^{n}\operatorname{div}(\mu_L(\bar\rho_i)D(\bar u))-\nu\sum_{i=1}^{n}\nabla(\lambda_L(\bar\rho_i)\operatorname{div}(\bar u))$$
$$=\sum_{i=1}^{n}\operatorname{div}(\mu(\bar\rho_i)\nabla\bar v_i)+\sum_{i=1}^{n}\frac{1}{2}\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar v_i);$$

$$\partial_t(\bar\rho_i\bar v_i)+\operatorname{div}(\bar\rho_i\bar v_i\otimes\bar u)+\operatorname{div}(\mu(\bar\rho_i)^t\nabla(\bar u))+\frac{1}{2}\nabla(\lambda(\bar\rho_i)\operatorname{div}(\bar u))=0.$$

The latter can be rewritten in the following way:

$$\begin{cases}\partial_t\bar\rho_i+\operatorname{div}(\bar\rho_i\bar u)=0\\ \partial_t(\bar\rho_i\bar u)+\operatorname{div}(\bar\rho_i\bar u\otimes\bar u)-2\nu\operatorname{div}(\mu_L(\bar\rho_i)D(\bar u))-\nu\nabla(\lambda_L(\bar\rho_i)\operatorname{div}\bar u)+\nabla p(\bar\rho_i)\\ =\operatorname{div}(\mu(\bar\rho_i)\nabla\bar v_i)+\frac{1}{2}\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar v_i)+\bar R_i\\ \partial_t(\bar\rho_i\bar v_i)+\operatorname{div}(\bar\rho_i\bar v_i\otimes\bar u)+\operatorname{div}(\mu(\bar\rho_i)^t\nabla(\bar u))+\frac{1}{2}\nabla(\lambda(\bar\rho_i)\operatorname{div}(\bar u))=0.\end{cases} \tag{2.25}$$

The term $\bar R_i$ is defined as:

$$\bar R_i:=\frac{\rho_i}{\bar\rho}\sum_{j=1}^{n}\left[\operatorname{div}\bar S_j-\nabla p(\bar\rho_j)\right]-\operatorname{div}\bar S_i$$

$$
\begin{aligned}
&+\nabla p(\bar\rho_i)+\frac{\rho_i}{\bar\rho}\sum_{j=1}^n[2\nu\operatorname{div}\mu_L(\bar\rho_j)D(\bar u)+\nu\nabla(\lambda_L(\bar\rho_j)\operatorname{div}\bar u)]\\
&-2\nu\operatorname{div}\mu_L(\bar\rho_i)D(\bar u)-\nu\nabla(\lambda_L(\bar\rho_i)\operatorname{div}\bar u)\\
&=\frac{\rho_i}{\bar\rho}\operatorname{div}\bar T-\operatorname{div}\bar T_i+\nu\operatorname{div}\bar D\frac{\bar\rho_i}{\bar\rho}-\nu\operatorname{div}\bar D_i,
\end{aligned}
\tag{2.26}
$$

where:

$$
\operatorname{div}\bar T_i=\operatorname{div}\bar S_i-\nabla p(\bar\rho_i);\qquad \operatorname{div}\bar T=\sum_{j=1}^n\operatorname{div}\bar T_j;
$$

$$
\operatorname{div}\bar D_i=2\operatorname{div}(\mu_L(\bar\rho_i)D(\bar u))+\nabla(\lambda_L(\bar\rho_i)\operatorname{div}\bar u);\qquad \operatorname{div}\bar D=\sum_{j=1}^n\operatorname{div}\bar D_j.
$$

In Section 4 we compute the relative entropy between weak solutions of (2.19) and strong solutions of (2.25), thus justifying in this framework the formal analysis above.

REMARK 2.3. It is worth observing that in the proposed scaling, contrarily to what happens under the diffusive scaling used in the previous case $M>0$, the viscosity terms are not higher order, and they persist in the limit (2.25), as it is also manifest looking at Equation (2.23). For this reason, as proposed in [9], in Section 4 we shall make the particular choice of Lamé coefficients $\mu_L(\rho_i)=\mu(\rho_i)$ and $\lambda_L(\rho_i)=\lambda(\rho_i)$, which allows us to control the viscosity terms by means of the relative entropy. On the other hand, being $\bar u_i,\operatorname{div}\bar u_i,D(\bar u_i)$ of order ϵ in the diffusive scaling, these terms can be treated as errors, and therefore this particular choice for Lamé coefficients is not required; concerning this, see also [13].

REMARK 2.4. Let us underline the relations between Equation (2.24) and the solvability for w_i^1 of $(2.23)_2$ when $w_i^0=0$. With the notation

$$
\begin{aligned}
d_i^0:=&\partial_t(\rho_i^0u^0)+\operatorname{div}(\rho_i^0u^0\otimes u^0)+\operatorname{div}S[\rho_i^0,v_i^0]-2\nu\operatorname{div}(\mu_L(\rho_i^0)D(u^0))\\
&-\nu\nabla(\lambda_L(\rho_i^0)\operatorname{div}(u^0)),
\end{aligned}
$$

Equation $(2.23)_2$ rewrites as

$$
-\sum_{j=1}^n b_{i,j}\rho_i^0\rho_j^0(w_i^1-w_j^1)=d_i^0,
\tag{2.27}
$$

hence (2.24) is equivalent to the condition $\sum_{i=1}^n d_i^0=0$, which ensures there exists a unique solution $(w_i^1,\dots,w_n^1)$ to (2.27) in view of Lemma 2.1.

3. High friction limit with diffusive scaling for $M_i>0$

The system we are going to study in this framework generalizes the one considered in [24], where no capillarity and viscosity effects are taken into account and, for reader's convenience, let us rewrite it here below:

$$
\begin{aligned}
&\partial_t\rho_i+\operatorname{div}(\rho_iu_i)=0\\
&\partial_t(\rho_iu_i)+\operatorname{div}(\rho_iu_i\otimes u_i)-2\nu\operatorname{div}(\mu_L(\rho_i)D(u_i))-\nu\nabla(\lambda_L(\rho_i)\operatorname{div}u_i)+\nabla\rho_i^\gamma\\
&=\operatorname{div}S_i-\frac{1}{\epsilon}\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)-\frac{M_i}{\epsilon}\rho_iu_i\\
&\partial_t(\rho_iv_i)+\operatorname{div}(\rho_iu_i\otimes v_i)+\operatorname{div}K_i=0,
\end{aligned}
\tag{3.1}
$$

where

$$\operatorname{div} S_i = \operatorname{div}(\mu(\rho_i)\nabla v_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} v_i) = \operatorname{div} T_i + \nabla\rho_i^\gamma,$$
$$\operatorname{div} K_i = \operatorname{div}(\mu(\rho_i)^t\nabla u_i) + \frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div} u_i).$$

As already noticed in Section 2, in order to obtain a non-trivial limiting dynamics as $\epsilon \to 0$, we rescale the time variable $\partial_t \to \epsilon\partial_t$ to end up to a diffusive limit:

$$\begin{aligned}
&\partial_t\rho_i + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i) = 0\\
&\partial_t(\rho_i u_i) + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i\otimes u_i) - \frac{2}{\epsilon}\nu\operatorname{div}(\mu_L(\rho_i)D(u_i)) - \frac{\nu}{\epsilon}\nabla(\lambda_L(\rho_i)\operatorname{div} u_i) + \frac{1}{\epsilon}\nabla\rho_i^\gamma\\
&= \frac{1}{\epsilon}\operatorname{div} S_i - \frac{1}{\epsilon^2}\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j) - \frac{M_i}{\epsilon^2}\rho_i u_i\\
&\partial_t(\rho_i v_i) + \frac{1}{\epsilon}\operatorname{div}(\rho_i u_i\otimes v_i) + \frac{1}{\epsilon}\operatorname{div} K_i = 0.
\end{aligned} \tag{3.2}$$

Then, the leading terms of the Hilbert expansion are the following equilibrium relations for each component of the mixture:

$$\begin{aligned}
&u_i^0 = 0,\\
&u_i^1 = (\bar\tau^{M,0})^{-1}(-\nabla p(\rho_i^0) + \operatorname{div} S_i(\rho_i^0, v_i^0)),\ \textit{the Darcy's law} \text{ in our context},\\
&v_i^0 = \frac{\nabla\mu(\rho_i^0)}{\rho_i^0},
\end{aligned}$$

and each ρ_i^0 solves the gradient flow equation

$$\partial_t\rho_i^0 + \operatorname{div}(\rho_i^0(\bar\tau^{M,0})^{-1}(-\nabla p(\rho_i^0) + \operatorname{div} S_i(\rho_i^0, v_i^0))) = 0. \tag{3.3}$$

In order to compare weak solution of (3.2) and strong solution of the parabolic equilibrium (3.3) we rewrite the latter as Euler-Korteweg system with high friction and an error term of order ϵ, using the same strategy of [13,22,23]. Indeed (3.3) is equivalent to:

$$\begin{aligned}
&\partial_t\bar\rho_i + \frac{1}{\epsilon}\operatorname{div}(\bar\rho_i\bar u_i) = 0\\
&\partial_t(\bar\rho_i\bar u_i) + \frac{1}{\epsilon}\operatorname{div}(\bar\rho_i\bar u_i\otimes\bar u_i) + \frac{1}{\epsilon}\nabla\bar\rho_i^\gamma = \frac{1}{\epsilon}\operatorname{div}(\bar\rho_i\nabla\bar v_i) - \frac{1}{\epsilon^2}\sum_{j=1}^n b_{i,j}\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j) - \frac{M_i}{\epsilon^2}\bar\rho_i\bar u_i + \bar e_i\\
&\partial_t(\bar\rho_i\bar v_i) + \frac{1}{\epsilon}\operatorname{div}(\bar\rho_i\bar u_i\otimes\bar v_i) + \frac{1}{\epsilon}\operatorname{div}(\bar\rho_i{}^t\nabla\bar u_i) = 0,
\end{aligned} \tag{3.4}$$

where we have denoted

$$\begin{aligned}
&\rho_i^0 = \bar\rho_i;\\
&\bar u_i = \epsilon(\bar\tau^M)^{-1}(-\nabla p(\bar\rho_i) + \operatorname{div} S_i(\bar\rho_i, \bar v_i));\\
&\bar v_i = \frac{\nabla\mu(\bar\rho_i)}{\bar\rho_i};\\
&\bar\tau^M = \tau^M(\bar\rho_i)\otimes I_3
\end{aligned}$$

$$\begin{aligned}\bar{e}_i &= \partial_t(\bar{\rho}_i\bar{u}_i) + \frac{1}{\epsilon}\operatorname{div}(\bar{\rho}_i\bar{u}_i\otimes\bar{u}_i)\\ &= \epsilon\partial_t(\bar{\rho}_i(\bar{\tau}^M)^{-1}(-\nabla p(\bar{\rho}_i)+\operatorname{div} S_i(\bar{\rho}_i,\bar{v}_i)))\\ &\quad + \epsilon\operatorname{div}(\bar{\rho}_i(\bar{\tau}^M)^{-1}(-\nabla p(\bar{\rho}_i)+\operatorname{div} S_i(\bar{\rho}_i,\bar{v}_i))\otimes(\bar{\tau}^M)^{-1}(-\nabla p(\bar{\rho}_i)+\operatorname{div} S_i(\bar{\rho}_i,\bar{v}_i)))\\ &= O(\epsilon).\end{aligned}$$

The aim of this part is to validate the large friction limit using relative entropy techniques. For this, we start by recalling the definition of the total mechanical energy of our system $\eta_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon)$:

$$\eta_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon) = \sum_{i=1}^n \eta(\rho_i^\epsilon,m_i^\epsilon,J_i^\epsilon) = \sum_{i=1}^n\left(\frac{1}{2}\frac{|m_i^\epsilon|^2}{\rho_i^\epsilon}+\frac{1}{2}\frac{|J_i^\epsilon|^2}{\rho_i^\epsilon}+h(\rho_i^\epsilon)\right),$$

and its space–integrated version

$$\Sigma_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon)(t) = \int_{\mathbb{T}^3}\sum_{i=1}^n \eta(\rho_i^\epsilon,m_i^\epsilon,J_i^\epsilon)(t)\,dx.$$

Hence, the relative entropy between solutions of (3.2) and (3.4) reads

$$\begin{aligned}&\eta_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})(t) := \sum_{i=1}^n \eta(\rho_i^\epsilon,u_i^\epsilon,J_i^\epsilon|\bar{\rho}_i,\bar{u}_i,\bar{J}_i)\\ &= \sum_{i=1}^n \eta(\rho_i^\epsilon,m_i^\epsilon,J_i^\epsilon) - \sum_{i=1}^n \eta(\bar{\rho}_i,\bar{m}_i,\bar{J}_i) - \sum_{i=1}^n \eta_{\rho_i}(\bar{\rho},\bar{m},\bar{J})(\rho_i^\epsilon-\bar{\rho}_i)\\ &\quad - \sum_{i=1}^n \eta_{m_i}(\bar{\rho},\bar{m},\bar{J})(m_i^\epsilon-\bar{m}_i) - \sum_{i=1}^n \eta_{J_i}(\bar{\rho},\bar{m},\bar{J})(J_i^\epsilon-\bar{J}_i),\end{aligned} \tag{3.5}$$

and

$$\Sigma_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})(t) = \int_{\mathbb{T}^3}\eta_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})(t)\,dx.$$

Let us now clarify the notion of periodic *weak* and *dissipative weak* solutions of the relaxation system (3.2) we shall consider in the sequel.

Definition 3.1 (weak and dissipative weak solutions). *A triple* $(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon)$ *(where* $\hat{\rho}^\epsilon=(\rho_1^\epsilon,\cdots,\rho_n^\epsilon)$, $\hat{m}^\epsilon=(m_1^\epsilon,\cdots,m_n^\epsilon)$, $\hat{J}^\epsilon=(J_1^\epsilon,\cdots,J_n^\epsilon)$*) such that*

$$\begin{aligned}&0\le\rho_i^\epsilon\in C^0([0,\infty);L^1(\mathbb{T}^3)),\\ &(\rho_i^\epsilon u_i^\epsilon,\rho_i^\epsilon v_i^\epsilon)\in C^0([0,\infty);L^1(\mathbb{T}^3)^3\times L^1(\mathbb{T}^3)^3)\end{aligned}$$

is called a periodic weak solution *of* (3.2) *if for any* $i=1,\cdots,n$*:*

$$\begin{aligned}&\sqrt{\rho_i^\epsilon}u_i^\epsilon\in L^\infty((0,T);L^2(\mathbb{T}^3)^3)\\ &\sqrt{\rho_i^\epsilon}v_i^\epsilon\in L^\infty((0,T);L^2(\mathbb{T}^3)^3)\\ &\rho_i^\epsilon\in C^0([0,\infty);L^\gamma(\mathbb{T}^3))\\ &\mu_L(\rho_i^\epsilon)D(u_i^\epsilon)\in L^1((0,T);L^1(\mathbb{T}^3)^{3\times3})\\ &\lambda_L(\rho_i^\epsilon)\operatorname{div}u_i^\epsilon\in L^1((0,T);L^1(\mathbb{T}^3))\end{aligned}$$

$$(\rho_i^\epsilon)^2 u_i^\epsilon \in L^\infty((0,T);L^1(\mathbb{T}^3)^3)$$

and $(\rho_i^\epsilon, u_i^\epsilon, v_i^\epsilon)$ *satisfy for all* $\psi_i \in C_0^1([0,\infty);C^1(\mathbb{R}^3))$ *and* $\phi_i, \varphi_i \in C_0^1([0,\infty);C^1(\mathbb{T}^3)^3)$*:*

$$-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon \partial_t \psi_i + \frac{1}{\epsilon}\rho_i^\epsilon u_i^\epsilon \cdot \nabla \psi_i)\, dxdt = \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon(x,0)\psi_i(x,0)\, dx,$$

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon u_i^\epsilon \partial_t \phi_i + \frac{1}{\epsilon}\rho_i^\epsilon u_i^\epsilon \otimes u_i^\epsilon : \nabla \psi_i + \frac{1}{\epsilon}\rho_i^{\epsilon\gamma} \operatorname{div} \psi_i + \frac{1}{\epsilon}\mu(\rho_i^\epsilon) v_i \cdot \nabla \operatorname{div} \phi_i \\
&\quad + \frac{1}{\epsilon}\nabla \mu(\rho_i^\epsilon) \cdot (\nabla \phi_i v_i) dxdt - \int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{1}{\epsilon}\left(\frac{1}{2}\nabla \lambda(\rho_i^\epsilon) \cdot v_i^\epsilon \operatorname{div} \phi_i + \frac{1}{2}\lambda(\rho_i^\epsilon) v_i^\epsilon \cdot \nabla \operatorname{div} \phi_i\right) dxdt \\
&\quad + \frac{2\nu}{\epsilon}\int_0^\infty \int_{\mathbb{T}^3} \mu_L(\rho_i^\epsilon) D(u_i^\epsilon) : D(\phi_i) dxdt + \frac{\nu}{\epsilon}\int_0^\infty \int_{\mathbb{T}^3} \lambda_L(\rho_i^\epsilon) \operatorname{div} u_i^\epsilon \operatorname{div} \phi_i dxdt \\
&\quad + \frac{1}{\epsilon^2}\int_0^\infty \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}\rho_i^\epsilon \rho_j^\epsilon (u_i^\epsilon - u_j^\epsilon) \cdot \phi_i dxdt + \frac{1}{\epsilon^2}\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n M_i \rho_i^\epsilon u_i^\epsilon \cdot \phi\, dxdt \\
=&\int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon u_i^\epsilon)(x,0) \cdot \phi_i(x,0)\, dx, \\
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon v_i^\epsilon \partial_t \varphi_i + \frac{1}{\epsilon}\rho_i^\epsilon u_i^\epsilon \otimes v_i^\epsilon : \nabla \varphi_i + \frac{1}{\epsilon}\mu(\rho_i^\epsilon) v_i \cdot \nabla \operatorname{div} \varphi_i + \frac{1}{\epsilon}\nabla \mu(\rho_i^\epsilon) \cdot (\nabla \varphi_i v_i))\, dxdt \\
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \frac{1}{\epsilon}\left(\frac{1}{2}\nabla \lambda(\rho_i^\epsilon) \cdot v_i^\epsilon \operatorname{div} \varphi_i + \frac{1}{2}\lambda(\rho_i^\epsilon) v_i^\epsilon \cdot \nabla \operatorname{div} \varphi_i\right) dxdt \\
=&\int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon v_i^\epsilon)(x,0) \cdot \varphi_i(x,0)\, dx.
\end{aligned}$$

If in addition $\eta_{tot}(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon) \in C([0,\infty);L^1(\mathbb{T}^3))$ *and the integrated energy inequality*

$$\begin{aligned}
&-\int_0^\infty \Sigma_{tot}(\hat\rho^\epsilon(t), \hat m^\epsilon(t), \hat J^\epsilon(t))\theta'(t)\, dt \\
&\quad + \frac{\nu}{\epsilon}\int_0^\infty \int_{\mathbb{T}^3} \sum_{i,j=1}^n \left(2\mu_L(\rho_i^\epsilon)|D(u_i^\epsilon)|^2 + \lambda_L(\rho_i^\epsilon)|\operatorname{div} u_i^\epsilon|^2\right)\theta(t)\, dxdt \\
&\quad + \frac{1}{2\epsilon^2}\int_0^\infty \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}\rho_i^\epsilon \rho_j^\epsilon |u_i^\epsilon - u_j^\epsilon|^2 \theta(t)\, dxdt + \frac{1}{\epsilon^2}\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n M_i \rho_i^\epsilon |u_i^\epsilon|^2 \theta(t)\, dxdt \\
\le&\Sigma_{tot}(\hat\rho^\epsilon(0), \hat m^\epsilon(0), \hat J^\epsilon(0))\theta(0)
\end{aligned} \tag{3.6}$$

holds for any $\theta \in W^{1,\infty}([0,\infty))$ *compactly supported in* $[0,\infty)$*, then we call* $(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon)$ *a periodic* dissipative weak solution.

We say that a periodic weak (dissipative) solution $(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon)$ *for* (3.2) *has finite total mass and finite total energy if for any* $T>0$ *there exists a constant* $K>0$ *independent of* ϵ *such that:*

$$\begin{aligned}
&\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon\, dx \le K; \\
&\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \sum_{i=1}^n \left(\frac{1}{2}\frac{|m_i^\epsilon|^2}{\rho_i^\epsilon} + \frac{1}{2}\frac{|J_i^\epsilon|^2}{\rho_i^\epsilon} + \frac{\rho_i^{\epsilon\gamma}}{\gamma-1}\right) dx \le K.
\end{aligned} \tag{3.7}$$

The relative entropy inequality is stated in the following proposition; to shorten notations, we shall omit the ϵ dependence in the remaining part of the section.

PROPOSITION 3.1. *Assume condition (A2) holds and let $(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon)$ be a dissipative weak periodic solution in the sense of Definition 3.1 with finite total mass and energy (3.7). Let $\bar w_i=(\bar\rho_i,\bar u_i,\bar v_i)$ be a smooth solution to (3.4) such that:*

$$\bar w_i,\partial_t\bar w_i,\nabla\bar w_i,D^2\bar w_i,D^3\bar\rho_i\in L^\infty([0,T];L^\infty(\mathbb{T}^3))$$

for any $i=1,\dots,n$. Then, the following inequality holds:

$$\begin{aligned}
&\Sigma_{tot}(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon|\bar{\bar\rho},\bar{\bar m},\bar{\bar J})\Big|_{s=0}^t+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|(u_i-u_j)-(\bar u_i-\bar u_j)|^2dxds\\
&+\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i-\bar u_i)|^2dxds+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}(u_i-\bar u_i)|^2dxds\\
&+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2dxds\\
&\le-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar\rho_i)\operatorname{div}\bar u_i dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar u_i:(u_i-\bar u_i)\otimes(u_i-\bar u_i))\,dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\cdot((v-\bar v_i)\operatorname{div}\bar u_i-(u_i-\bar u_i)\operatorname{div}\bar v_i)dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i))((v-\bar v_i)\cdot\nabla\operatorname{div}\bar u_i-(u_i-\bar u_i)\cdot\nabla\operatorname{div}\bar v_i)dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar u_i:(v_i-\bar v_i)\otimes(v_i-\bar v_i))\,dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\rho_j-\bar\rho_j)(\bar u_i-u_i)\cdot(\bar u_i-\bar u_j)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bar e_i\cdot(\bar u_i-u_i)dxds\\
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)D(\bar u_i):D(u_i-\bar u_i)dxds\\
&-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)\operatorname{div}\bar u_i(\operatorname{div}u_i-\operatorname{div}\bar u_i)dxds.
\end{aligned}\tag{3.8}$$

Proof.

Step 1: Energy Inequalities.

As customary in these contexts, we use the following test function in (3.6):

$$\theta(s)=\begin{cases}1, & \text{for } 0\le s<t,\\ \dfrac{t-s}{\delta}+1, & \text{for } t\le s<t+\delta,\\ 0, & \text{for } s>t+\delta,\end{cases}\tag{3.9}$$

and, passing to the limit as $\delta \to 0$ we get:

$$\begin{aligned}
&\Sigma_{tot}(\hat{\rho}^\epsilon, \hat{m}^\epsilon, \hat{J}^\epsilon)(t) + \frac{\nu}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n \left(2\mu_L(\rho_i)|D(u_i)|^2 + \lambda_L(\rho_i)|\operatorname{div} u_i|^2\right) dxds \\
&\quad + \frac{1}{2\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i - u_j|^2\, dxds + \frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n M_i\rho_i|u_i|^2\, dxds \\
&\leq \Sigma_{tot}(\hat{\rho}(0), \hat{m}(0), \hat{J}(0)).
\end{aligned} \tag{3.10}$$

We multiply $(3.4)_2$ by $\bar{u}_i$ and $(3.4)_3$ by $\bar{v}_i$, integrating in time and space we get the energy inequality of system (3.4):

$$\begin{aligned}
&\Sigma_{tot}(\bar{\hat{\rho}}(t), \bar{\hat{m}}(t), \bar{\hat{J}}(t)) + \frac{1}{2\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}\bar{\rho}_i\bar{\rho}_j|\bar{u}_i - \bar{u}_j|^2\, dxds \\
&\quad + \frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n M_i\bar{\rho}_i|\bar{u}_i|^2\, dxds \\
&\leq \Sigma_{tot}(\bar{\hat{\rho}}(0), \bar{\hat{m}}(0), \bar{\hat{J}}(0)) + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{e}_i \cdot \bar{u}_i\, dxds.
\end{aligned} \tag{3.11}$$

Step II: Equations for the difference.

Now we evaluate the linear part of the relative entropy using suitable test functions in the weak formulation according to Definition 3.1. If we take the differences $(\rho_i - \bar{\rho}_i, u_i - \bar{u}_i, v_i - \bar{v}_i)$ they have to satisfy:

$$-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i - \bar{\rho}_i)\partial_s\psi_i + \frac{1}{\epsilon}(\rho_i u_i - \bar{\rho}_i\bar{u}_i)\cdot\nabla\psi_i dxds = \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i(x,0) - \bar{\rho}_i(x,0)\psi_i(x,0))dx,$$

$$\begin{aligned}
&-\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i u_i - \bar{\rho}_i\bar{u}_i)\cdot\partial_s\phi_i + \frac{1}{\epsilon}(\rho_i u_i \otimes u_i - \bar{\rho}_i\bar{u}_i\otimes\bar{u}_i) : \nabla\phi_i + \frac{1}{\epsilon}(\rho_i^\gamma - \bar{\rho}_i^\gamma)\operatorname{div}\phi_i\, dxds \\
&\quad - \frac{1}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n [(\mu(\rho_i)v_i - \mu(\bar{\rho}_i)\bar{v}_i)\nabla\operatorname{div}\phi_i + (\nabla\mu(\rho_i)v_i - \nabla\mu(\bar{\rho}_i)\bar{v}_i) : \nabla\phi_i]\, dxds \\
&\quad - \frac{1}{2\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n [(\nabla\lambda(\rho_i)v_i - \nabla\lambda(\bar{\rho}_i)\bar{v}_i)\operatorname{div}\phi_i + (\lambda(\rho_i)v_i - \lambda(\bar{\rho}_i)\bar{v}_i)\nabla\operatorname{div}\phi_i] \\
&\quad + \frac{\nu}{\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n 2\mu_L(\rho_i)D(u_i) : D(\phi_i) + \lambda_L(\rho_i)\operatorname{div} u_i \operatorname{div}\phi_i\, dxds \\
&= -\frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}(\rho_i\rho_j(u_i - u_j) - \bar{\rho}_i\bar{\rho}_j(\bar{u}_i - \bar{u}_j))\phi_i\, dxds \\
&\quad - \frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n M_i(\rho_i u_i - \bar{\rho}_i\bar{u}_i)\cdot\phi_i\, dxds - \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n \bar{e}_i\phi\, dxds \\
&\quad + \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^n ((\rho_i u_i)(x,0) - (\bar{\rho}_i\bar{u}_i)(x,0))\phi_i(x,0)\, dx,
\end{aligned}$$

and

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i v_i-\bar\rho_i\bar v_i)\cdot\partial_s\varphi_i+\frac{1}{\epsilon}(\rho_i u_i\otimes v_i-\bar\rho_i\bar u_i\otimes\bar v_i):\nabla\varphi_i\,dxds\\
&\quad+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n[(\mu(\rho_i)u_i-\mu(\bar\rho_i)\bar u_i)\nabla\operatorname{div}\varphi_i+(\nabla\mu(\rho_i)u_i-\nabla\mu(\bar\rho_i)\bar u_i):\nabla\varphi_i]\,dxds\\
&\quad+\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n[(\nabla\lambda(\rho_i)u_i-\nabla\lambda(\bar\rho_i)\bar u_i)\operatorname{div}\varphi_i+(\lambda(\rho_i)u_i-\lambda(\bar\rho_i)\bar u_i)\nabla\operatorname{div}\varphi_i]\,dxds\\
&=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n((\rho_i v_i)(x,0)-(\bar\rho_i\bar v_i)(x,0))\varphi_i(x,0)\,dx,
\end{aligned}
$$

where ψ,ϕ,φ are Lipschitz test functions (ϕ,φ vector-valued) compactly supported in $[0,\infty)$ in time and periodic in space. In the above formulation we choose:

$$
\begin{aligned}
\psi_i&=\theta(s)\left(h'(\bar\rho_i)-\frac{1}{2}\frac{|\bar m_i|^2}{\bar\rho_i}-\frac{1}{2}\frac{|\bar J_i|^2}{\bar\rho_i}\right),\\
\phi_i&=\theta(s)\left(\frac{\bar m_i}{\bar\rho_i}\right)\quad \varphi_i=\theta(s)\left(\frac{\bar J_i}{\bar\rho_i}\right),
\end{aligned}
$$

where θ is defined in (3.9). The previous relations become as $\delta\to 0$:

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\sum_{i=1}^n\left(h'(\bar\rho_i)-\frac{1}{2}\frac{|\bar m_i|^2}{\bar\rho_i}-\frac{1}{2}\frac{|\bar J_i|^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)\Big|_{s=0}^t\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(h'(\bar\rho_i)-\frac{1}{2}\frac{|\bar m_i|^2}{\bar\rho_i}-\frac{1}{2}\frac{|\bar J_i|^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)dxds\\
&\quad-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla_x\left(h'(\bar\rho_i)-\frac{1}{2}\frac{\bar m_i{}^2}{\bar\rho_i}-\frac{1}{2}\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u_i)dxds=0,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\bar m_i}{\bar\rho_i}(\rho_i u_i-\bar\rho_i\bar u_i)\Big|_{s=0}^t-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(\frac{\bar m_i}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u_i)dxds\\
&\quad-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\rho_i u_i\otimes u_i-\bar\rho_i\bar u_i\otimes\bar u_i):\nabla\left(\frac{\bar m_i}{\bar\rho_i}\right)+(p(\rho_i)-p(\bar\rho_i))\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)\right)dxds\\
&\quad-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)v_i-\mu(\bar\rho_i)\bar v_i)\nabla\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)\\
&\qquad\qquad+(\nabla\mu(\rho_i)v_i-\nabla\mu(\bar\rho_i)\bar v_i):\nabla\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds\\
&\quad-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\nabla\lambda(\rho_i)v_i-\nabla\lambda(\bar\rho_i)\bar v_i)\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)\\
&\qquad\qquad+(\lambda(\rho_i)v_i-\lambda(\bar\rho_i)\bar v_i)\nabla\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds\\
&\quad+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n2\mu_L(\rho_i)D(u_i):D(\bar u_i)+\lambda_L(\rho_i)\operatorname{div}u_i\operatorname{div}\bar u_i dxds
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}(\rho_i\rho_j(u_i-u_j)-\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j))\bar u_i dxds\\
&\quad -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i(\rho_i u_i-\bar\rho_i\bar u_i)\cdot\bar u_i\,dxds-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \bar e_i\bar u_i dxds, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\bar J_i}{\bar\rho_i}(\rho_i v_i-\bar\rho_i\bar v_i)\Big|_{s=0}^t-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(\frac{\bar J_i}{\bar\rho_i}\right)(\rho_i v_i-\bar\rho_i\bar v_i)\,dxds\\
&\quad -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\rho_i u_i\otimes v_i-\bar\rho_i\bar u\otimes\bar v_i):\nabla\left(\frac{\bar J_i}{\bar\rho_i}\right)\right)dxds\\
&\quad +\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)u_i-\mu(\bar\rho_i)\bar u_i)\nabla\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)\\
&\qquad\qquad +(\nabla\mu(\rho_i)u_i-\nabla\mu(\bar\rho_i)\bar u_i):\nabla\left(\frac{\bar J_i}{\bar\rho_i}\right)dxds\\
&\quad +\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\nabla\lambda(\rho_i)u_i-\nabla\lambda(\bar\rho_i)\bar u_i)\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)\\
&\qquad\qquad +(\lambda(\rho_i)u_i-\lambda(\bar\rho_i)\bar u_i)\nabla\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)dxds\\
&=0. \qquad (3.14)
\end{aligned}
$$

Recalling the definition of the relative entropy (3.5), and using (3.10),(3.11), (3.12), (3.13), (3.14) we obtain:

$$
\begin{aligned}
&\Sigma_{tot}(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon|\bar{\hat\rho},\bar{\hat m},\bar{\hat J})\Big|_{s=0}^t\\
&\le-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(h'(\bar\rho_i)-\frac{1}{2}\frac{|\bar m_i|^2}{\bar\rho_i}-\frac{1}{2}\frac{|\bar J_i|^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)\\
&\qquad\qquad +\partial_s\left(\frac{\bar m_i}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u_i)+\partial_s\left(\frac{\bar J_i}{\bar\rho_i}\right)(\rho_i v_i-\bar\rho_i\bar v_i)dxds\\
&\quad -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla_x\left(h'(\bar\rho_i)-\frac{1}{2}\frac{\bar m_i{}^2}{\bar\rho_i}-\frac{1}{2}\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u_i)\\
&\qquad\qquad +(p(\rho_i)-p(\bar\rho_i))\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds\\
&\quad -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i u_i\otimes u_j-\bar\rho_i\bar u_i\otimes\bar u_i):\nabla\bar u_i+(\rho_i u_i\otimes v_j-\bar\rho_i\bar u_i\otimes\bar v_j):\nabla\bar v_i dxds\\
&\quad -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)[(v_i-\bar v_i)\nabla\operatorname{div}\bar u_i-(u_i-\bar u_i)\nabla\operatorname{div}\bar v_i]\\
&\qquad\qquad +\nabla\mu(\rho_i)[\nabla\bar u_i(v_i-\bar v_i)-\nabla\bar v_i(u_i-\bar u_i)]dxds\\
&\quad -\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\lambda(\rho_i)[(v_i-\bar v_i)\operatorname{div}\bar u_i-(u_i-\bar u_i)\operatorname{div}\bar v_i]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \lambda(\rho_i)[(v_i-\bar v_i)\nabla \operatorname{div}\bar u_i - (u_i-\bar u_i)\nabla \operatorname{div}\bar v_i]dxds\\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)-\mu(\bar\rho_i))(\bar u_i\nabla \operatorname{div}\bar v_i - \bar v_i\nabla\operatorname{div}\bar u_i)\\
&\qquad + (\nabla\mu(\rho_i)-\nabla\mu(\bar\rho_i))[\nabla\bar v_i\bar u - \nabla\bar u_i\bar v_i]dxds\\
&+\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\lambda(\rho_i)-\lambda(\bar\rho_i))[\bar u_i\nabla\operatorname{div}\bar v_i - \bar v_i\nabla\operatorname{div}\bar u_i]\\
&\qquad + (\nabla\lambda(\rho_i)-\nabla\lambda(\bar\rho_i))(\bar u_i\operatorname{div}\bar v_i - \bar v_i\operatorname{div}\bar u_i)dxds\\
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i)|^2dxds - \frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}u_i|^2dxds\\
&+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n 2\mu_L(\rho_i)D(u_i):D(\bar u_i)+\lambda_L(\rho_i)\operatorname{div}u_i\operatorname{div}\bar u_i dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}(\rho_i\rho_j(u_i-u_j)-\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j))\bar u_i dxds\\
&-\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar\rho_i\bar\rho_j|\bar u_i-\bar u_j|^2dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i u_i^2dxds+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\bar\rho_i\bar u_i^2dxds\\
&\qquad +\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i(\rho_i u_i-\bar\rho_i\bar u_i)\bar u_i dxds\\
=:&\sum_{i=1}^{12} I_i.
\end{aligned}
$$

Let us observe that the I_6+I_7 term is equal to zero thanks to the property (1.9) of the stress tensors S_i, K_i. Using the relation $h''(\bar\rho_i)=p'(\bar\rho_i)/\bar\rho_i$ we find that:

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s h'(\bar\rho_i)(\rho_i-\bar\rho_i)+\frac{1}{\epsilon}\nabla_x h'(\bar\rho_i)(\rho_i u_i-\bar\rho_i\bar u_i)dxds\\
=&\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p'(\bar\rho_i)(\rho_i-\bar\rho_i)\operatorname{div}\bar u_i+\frac{\rho_i}{\bar\rho_i}\nabla p(\bar\rho_i)(\bar u_i-u_i)dxds.
\end{aligned}
$$

Then we notice that the following quantity:

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\partial_s\left(\frac12|\bar u_i|^2\right)(\rho_i-\bar\rho_i)+\frac{1}{\epsilon}\nabla_x\left(\frac12|\bar u_i|^2\right)(\rho_i u_i-\bar\rho_i\bar u_i)\right)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(-\partial_s\bar u(\rho_i u_i-\bar\rho_i\bar u_i)-\frac{1}{\epsilon}\nabla_x\bar u_i:(\rho_i u_i\otimes u_i-\bar\rho_i\bar u_i\otimes\bar u_i)\right)dxds
\end{aligned}
$$

can be rearranged by multiplying the momentum equations of the strong solution $\bar u_i,\bar v_i$, by $\rho_i(\bar u_i-u_i)$ and $\rho_i(\bar v_i-v_i)$ respectively. We get:

$$
\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\partial_s\left(\frac12|\bar u_i|^2\right)(\rho_i-\bar\rho_i)+\frac{1}{\epsilon}\nabla_x\left(\frac12|\bar u_i|^2\right)(\rho_i u_i-\bar\rho_i\bar u_i)-\partial_s\bar u_i(\rho_i u_i-\bar\rho_i\bar u_i)\right)dxds
$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{1}{\epsilon}\nabla_x\bar{u}_i:(\rho_i u_i\otimes u_i-\bar{\rho}_i\bar{u}_i\otimes\bar{u}_i)dxds\\
=&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{1}{\epsilon}\rho_i\nabla_x\bar{u}_i:(u_i-\bar{u}_i)\otimes(u_i-\bar{u}_i)+\frac{1}{\epsilon}\frac{\rho_i}{\bar{\rho}_i}\nabla p(\bar{\rho}_i)(\bar{u}_i-u_i)\\
&-\frac{1}{\epsilon}\frac{\rho_i}{\bar{\rho}_i}\operatorname{div}\bar{S}_i(\bar{u}_i-u_i)-\frac{\rho_i}{\bar{\rho}_i}\bar{e}(\bar{u}_i-u_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{\epsilon^2}M_i\rho_i(\bar{u}_i-u_i)+\frac{1}{\epsilon^2}b_{i,j}\rho_i\bar{\rho}_j(\bar{u}_i-u_i)(\bar{u}_i-\bar{u}_j)\right)dxds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\partial_s\left(\frac{1}{2}|\bar{v}_i|^2\right)(\rho_i-\bar{\rho}_i)+\frac{1}{\epsilon}\nabla_x\left(\frac{1}{2}|\bar{v}_i|^2\right)(\rho_i u_i-\bar{\rho}_i\bar{u}_i)\right)dxds\\
&\quad+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(-\partial_s\bar{v}_i(\rho_i u_i-\bar{\rho}_i\bar{u}_i)-\frac{1}{\epsilon}\nabla_x\bar{v}_i:(\rho_i u_i\otimes v_i-\bar{\rho}_i\bar{u}_i\otimes\bar{v}_i)\right)dxds\\
=&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{\epsilon}\rho_i\nabla\bar{v}_i:(u_i-\bar{u}_i)\otimes(v_i-\bar{v}_i)+\frac{1}{\epsilon}\frac{\rho_i}{\bar{\rho}_i}\operatorname{div}\bar{K}_i(\bar{v}_i-v_i)\right)dxds.
\end{aligned}
$$

From the previous calculations the relative entropy becomes:

$$
\begin{aligned}
&\Sigma_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})\Big|_{s=0}^t\\
\le&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar{\rho}_i)\operatorname{div}\bar{u}_i dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar{u}_i:(u_i-\bar{u}_i)\otimes(u_i-\bar{u}_i)+\rho_i\nabla\bar{v}_i:(u_i-\bar{u}_i)\otimes(v_i-\bar{v}_i))dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)[(v_i-\bar{v}_i)\nabla\operatorname{div}\bar{u}_i-(u_i-\bar{u}_i)\nabla\operatorname{div}\bar{v}_i]\\
&\qquad\qquad+\nabla\mu(\rho_i)[\nabla\bar{u}_i(v_i-\bar{v}_i)-\nabla\bar{v}_i(u_i-\bar{u}_i)]dxds\\
&-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\lambda(\rho_i)[(v_i-\bar{v}_i)\operatorname{div}\bar{u}_i-(u_i-\bar{u}_i)\operatorname{div}\bar{v}_i]\\
&\qquad\qquad+\lambda(\rho_i)[(v_i-\bar{v}_i)\nabla\operatorname{div}\bar{u}_i-(u_i-\bar{u}_i)\nabla\operatorname{div}\bar{v}_i]dxds\\
&+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar{\rho}_i}\Big(\mu(\bar{\rho}_i)\operatorname{div}\nabla\bar{v}_i+{}^t\nabla\mu(\bar{\rho}_i)^t\nabla\bar{v}_i\\
&\qquad\qquad+\frac{1}{2}\nabla\lambda(\bar{\rho}_i)\operatorname{div}\bar{v}_i+\frac{1}{2}\lambda(\bar{\rho}_i)\nabla\operatorname{div}\bar{v}_i\Big)(\bar{u}_i-u_i)dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar{\rho}_i}\Big(\mu(\bar{\rho}_i)\operatorname{div}{}^t\nabla\bar{u}_i+{}^t\nabla\mu(\bar{\rho}_i)\nabla\bar{u}_i\\
&\qquad\qquad+\frac{1}{2}\nabla\lambda(\bar{\rho}_i)\operatorname{div}\bar{u}_i+\frac{1}{2}\lambda(\bar{\rho}_i)\nabla\operatorname{div}\bar{u}_i\Big)(\bar{v}_i-v_i)dxds\\
&-\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar{\rho}_i\bar{\rho}_j|\bar{u}_i-\bar{u}_j|^2dxds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}(\rho_i\rho_j(u_i-u_j)-\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j))\bar u_i dxds\\
&\qquad\qquad -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\bar\rho_j(\bar u_i-\bar u_j)(\bar u_i-u_i)dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i u_i^2 dxds+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\bar\rho_i\bar u_i^2 dxds\\
&\qquad\qquad +\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i(\rho_i u_i-\bar\rho_i\bar u_i)\bar u_i dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i\bar u_i(\bar u_i-u_i)dxds+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bar e(\bar u_i-u_i)dxds\\
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i)|^2dxds-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}u_i|^2dxds\\
&+\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)Du_i:D\bar u_i dxds+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)\operatorname{div}u_i\operatorname{div}\bar u_i dxds\\
=:&\sum_{i=1}^{13} I_i,
\end{aligned}
$$

where, with a slight abuse of notation, we redefine the right-hand side of the above relation using the same symbols. Let us mention that $\operatorname{div}{}^t\nabla\bar u=\nabla\operatorname{div}\bar u$ while $\operatorname{div}\nabla\bar v_i=\nabla\operatorname{div}\bar v_i$ since $\nabla\bar v_i$ is symmetric as $\bar v_i$ is a gradient. We define $\tilde I_2$:

$$
\begin{aligned}
\tilde I_2:=&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\mu(\rho_i)-\frac{\rho_i}{\bar\rho_i}\mu(\bar\rho_i)\right)(\nabla\operatorname{div}\bar u_i(v_i-\bar v_i)-\nabla\operatorname{div}\bar v_i(u_i-\bar u_i))\,dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\nabla\mu(\rho_i)-\frac{\rho_i}{\bar\rho_i}\nabla\mu(\bar\rho_i)\right)\cdot(\nabla\bar u_i(v_i-\bar v_i)-\nabla\bar v_i(u_i-\bar u_i))\,dxds,
\end{aligned}
$$

since $v_i=\nabla\mu(\rho_i)/\rho_i$ then $\tilde I_2$ reads as:

$$
\begin{aligned}
\tilde I_2=&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left(\frac{\mu(\rho_i)}{\rho_i}-\frac{\mu(\bar\rho_i)}{\bar\rho_i}\right)(\nabla\operatorname{div}\bar u_i(v_i-\bar v_i)-\nabla\operatorname{div}\bar v_i(u_i-\bar u_i))\,dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(v_i-\bar v_i)\cdot(\nabla\bar u_i(v_i-\bar v_i)-\nabla\bar v_i(u_i-\bar u_i))\,dxds.
\end{aligned}
$$

We define

$$
\begin{aligned}
\tilde I_3:=&-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\lambda(\rho_i)-\frac{\rho_i}{\bar\rho_i}\lambda(\bar\rho_i)\right)((v_i-\bar v_i)\nabla\operatorname{div}\bar u_i-(u-\bar u_i)\nabla\operatorname{div}\bar v)\,dxds\\
&-\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\nabla\lambda(\rho_i)-\frac{\rho_i}{\bar\rho_i}\nabla\lambda(\bar\rho_i)\right)((v-\bar v_i)\operatorname{div}\bar u_i-(u_i-\bar u)\operatorname{div}\bar v_i)dxds,
\end{aligned}
$$

since $\lambda(\rho_i) = 2(\rho_i\mu'(\rho_i) - \mu(\rho_i))$ one has:

$$\tilde{I_3} = -\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\left(\frac{\lambda(\rho_i)}{\rho_i} - \frac{\lambda(\bar{\rho}_i)}{\bar{\rho}_i}\right)((v_i-\bar{v}_i)\nabla \operatorname{div}\bar{u}_i - (u_i-\bar{u}_i)\nabla\operatorname{div}\bar{v}_i)\,dxds$$
$$-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i - \mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)((v_i-\bar{v}_i)\operatorname{div}\bar{u}_i - (u_i-\bar{u}_i)\operatorname{div}\bar{v}_i)dxds.$$

Therefore $I_3 + I_4 + I_5 + I_6 = \tilde{I_2} + \tilde{I_3}$ becomes:

$$\tilde{I_2} + \tilde{I_3} = -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i - \mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)((v_i-\bar{v}_i)\operatorname{div}\bar{u}_i + (\bar{u}_i-u_i)\operatorname{div}\bar{v}_i)dxds$$
$$-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu'(\rho_i) - \mu'(\bar{\rho}_i))((v_i-\bar{v}_i)\nabla\operatorname{div}\bar{u}_i + (\bar{u}_i-u_i)\nabla\operatorname{div}\bar{v}_i)dxds$$
$$-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\left[(v_i-\bar{v}_i)\nabla\bar{u}_i(v_i-\bar{v}_i) - (v_i-\bar{v}_i)\nabla\bar{v}_i(u_i-\bar{u}_i)\right]dxds.$$

Let us manage the remaining terms, starting with the interaction terms $I_7 + I_8$. Following the computations in [19] we get:

$$I_8 = \frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}(\rho_i\rho_j(u_i-u_j) - \bar{\rho}_i\bar{\rho}_j(\bar{u}_i-\bar{u}_j))\bar{u}_i dxds$$
$$-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\bar{\rho}_j(\bar{u}_i-\bar{u}_j)(\bar{u}_i-u_i)dxds$$
$$= -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)(u_i-\bar{u}_i)dxds$$
$$+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)u_i dxds - \frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar{\rho}_i\bar{\rho}_j(\bar{u}_i-\bar{u}_j)\bar{u}_i dxds$$
$$-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\bar{\rho}_j(\bar{u}_i-\bar{u}_j)(\bar{u}_i-u_i)dxds$$
$$= -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)(u_i-\bar{u}_i)dxds$$
$$+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2 dxds - \frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar{\rho}_i\bar{\rho}_j|\bar{u}_i-\bar{u}_j|^2 dxds$$
$$-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\rho_j+\bar{\rho}_j-\rho_j)(\bar{u}_i-\bar{u}_j)(\bar{u}_i-u_i)dxds$$
$$= \frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2 dxds - \frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar{\rho}_i\bar{\rho}_j|\bar{u}_i-\bar{u}_j|^2 dxds$$
$$-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\bar{\rho}_j-\rho_j)(u_i-\bar{u}_i)(\bar{u}_i-\bar{u}_j)dxds$$

$$
\begin{aligned}
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-\bar u_i)[(u_i-u_j)-(\bar u_i-\bar u_j)]dxds\\
=&\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds-\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\bar\rho_i\bar\rho_j|\bar u_i-\bar u_j|^2dxds\\
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\bar\rho_j-\rho_j)(\bar u_i-u_i)(\bar u_i-\bar u_j)dxds\\
&-\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|(u_i-u_j)-(\bar u_i-\bar u_j)|^2dxds.
\end{aligned}
$$

Therefore I_7+I_8 reduces to:

$$
\begin{aligned}
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\rho_j-\bar\rho_j)(\bar u_i-u_i)(\bar u_i-\bar u_j)\,dxds\\
&-\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|(u_i-u_j)-(\bar u_i-\bar u_j)|^2\,dxds.
\end{aligned}
$$

The friction terms and the error part $I_9+I_{10}+I_{11}$ becomes:

$$
I_9+I_{10}+I_{11}=-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2\,dxds+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bar e(\bar u_i-u_i)\,dxds.
$$

Finally the viscosity parts $I_{12}+I_{13}$ rewrite as:

$$
\begin{aligned}
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i)|^2dxds-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}u_i|^2dxds\\
&+\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)Du_i:D\bar u_i dxds+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)\operatorname{div}u_i\operatorname{div}\bar u_i dxds\\
=&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i-\bar u_i)|^2dxds-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}(u_i-\bar u_i)|^2dxds\\
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)D(\bar u_i):D(u_i-\bar u_i)dxds\\
&-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)\operatorname{div}\bar u_i(\operatorname{div}u_i-\operatorname{div}\bar u_i)dxds
\end{aligned}
$$

and we end up with (3.8):

$$
\begin{aligned}
&\Sigma_{tot}(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon|\bar{\hat\rho},\bar{\hat m},\bar{\hat J})\Big|_{s=0}^t+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|(u_i-u_j)-(\bar u_i-\bar u_j)|^2dxds\\
&+\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2dxds+\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i-\bar u_i)|^2dxds\\
&+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}(u_i-\bar u_i)|^2dxds
\end{aligned}
$$

$$
\begin{aligned}
\leq & -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar{\rho}_i)\operatorname{div}\bar{u}_i dxds \\
& -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n (\rho_i\nabla\bar{u}_i : (u_i-\bar{u}_i)\otimes(u_i-\bar{u}_i))\,dxds \\
& -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(v_i-\bar{v}_i)\nabla\bar{u}_i(v_i-\bar{v}_i)dxds \\
& -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)((v-\bar{v}_i)\operatorname{div}\bar{u}_i-(u_i-\bar{u}_i)\operatorname{div}\bar{v}_i)dxds \\
& -\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu'(\rho_i)-\mu'(\bar{\rho}_i))((v-\bar{v}_i)\nabla\operatorname{div}\bar{u}_i-(u_i-\bar{u}_i)\nabla\operatorname{div}\bar{v}_i)dxds \\
& -\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\bar{\rho}_j-\rho_j)(\bar{u}_i-u_i)(\bar{u}_i-\bar{u}_j)dxds \\
& +\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar{\rho}_i}\bar{e}(\bar{u}_i-u_i)dxds \\
& -\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu_L(\rho_i)D(\bar{u}_i):D(u_i-\bar{u}_i)dxds \\
& -\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda_L(\rho_i)\operatorname{div}\bar{u}_i(\operatorname{div}u_i-\operatorname{div}\bar{u}_i)dxds.
\end{aligned}
$$

□

3.1. Stability result and convergence of the diffusive limit. In this part we prove the stability result using the estimate contained in Proposition 3.1. Let us present the "distance" used to measure the difference between a weak dissipative solution of (3.2) and a strong solution of (3.4)

$$
\Psi(t) := \int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar{m}_i}{\bar{\rho}_i}\right|^2+\frac{1}{2}\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar{J}_i}{\bar{\rho}_i}\right|^2+h(\rho_i|\bar{\rho}_i)\right)dx. \tag{3.15}
$$

The proof presented here follows the ones contained in [13,24], in particular generalizing the results of the latter by including the capillarity and viscosity terms in the model under investigation. In addition, we shall also take advantage the enlarged formulation (1.8) in terms of the drift velocity [9] to be able to handle more general Korteweg tensors with respect to the ones considered in [19]. We stress that, under the hypothesis of γ-law pressure, with $\gamma>1$, we have

$$
h(\rho_i)=\frac{\rho_i^\gamma}{\gamma-1},
$$

and therefore

$$
p(\rho_i|\bar{\rho}_i)=(\gamma-1)h(\rho_i|\bar{\rho}_i),
$$

this means that the term in (3.8) involving the pressure can be controlled in terms of the relative entropy. To control the lines involving the Korteweg tensor in (4.6) we make use of the results contained in [9], we state below for the sake of completeness.

LEMMA 3.1 ([9, Lemma 35]). *Let us assume* $\mu(\rho)=\rho^{\frac{s+3}{2}}$ *with* $\gamma\geq s+2$ *and* $s\geq -1$. *We have*

$$\rho|\mu'(\rho)-\mu'(\bar\rho)|^2\leq C(\bar\rho)h(\rho|\bar\rho),$$

with $C(\bar\rho)$ *uniformly bounded for* $\bar\rho$ *belonging to compact sets in* $\mathbb{R}^+\times\mathbb{T}^n$.

Finally, to take full advantage of the diagonal relaxation term in (3.8), as already done in [24], we strengthen assumption (A2) as follows:

(A2b) Let $\{b_{i,j}\}_{i,j=1}^n$ be such that hypothesis (A1) holds and assume that for any index $i\in\{1,\cdots,n\}$ we have $M_i\rho_i>0$.

In view of (A2b), in the proof of the main result below, we shall use $\widehat{M}:=\min_{i\in\{1,\dots,n\}}M_i>0$.

THEOREM 3.1. *Let* $T>0$ *be fixed and let* $(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon)$ *be as in Definition 3.1 and* $(\bar\rho,\bar m,\bar J)$ *be a smooth solution of (3.4). Assume the pressure* $p(\rho_i)$ *is given by the* γ*–law* ρ_i^γ *such that* $\gamma>1$. *Let* $\mu(\rho_i)=\rho^{\frac{s+3}{2}}$ *with* $\gamma\geq s+2$ *and* $s\geq -1$, *and assume*

$$||\lambda_L(\rho_i)||_{L^\infty((0,t);L^1(\mathbb{T}^3))},||\mu_L(\rho_i)||_{L^\infty((0,t);L^1(\mathbb{T}^3))}\leq\tilde E$$

for a positive constant $\tilde E$ *independent on* ϵ. *Let us also assume that* $\rho_i^\epsilon\in L^\infty([0,T];L^\infty(\mathbb{T}^3))$ *and there exist two constants* k,N *such that:*

$$0<k\leq\rho_i^\epsilon\leq N\ \text{ in }\mathbb{R}^3,0<t<T. \tag{3.16}$$

Finally, assume condition (A2b) holds. Then, for any $t\in[0,T]$, *the stability estimate*

$$\Psi(t)\leq(\Psi(0)+C\nu\epsilon+C\epsilon^4)\exp^{\hat Ct}, \tag{3.17}$$

holds true, where $\hat C>0$ *depends on* $\widehat{M}$ *and* C *is a positive constant depending on* T, K, k, N, $\widehat{M}$, *the* L^1 *bound ((3.7)) for* ρ_i^ϵ, *assumed to be uniform in* ϵ, $\bar\rho_i$ *and its derivatives. Moreover, if* $\Psi(0)\to 0$ *as* $\epsilon\to 0$, *then as* $\epsilon\to 0$

$$\sup_{t\in[0,T]}\Psi(t)\to 0.$$

Proof. In the following, we shall estimate the right–hand–side of (3.8)

$$\begin{aligned}\sum_{l=1}^8\mathcal{I}_l:=&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar\rho_i)\operatorname{div}\bar u_i dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar u_i:(u_i-\bar u_i)\otimes(u_i-\bar u_i))dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar u_i:(v_i-\bar v_i)\otimes(v_i-\bar v_i))dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\cdot((v-\bar v_i)\operatorname{div}\bar u_i-(u_i-\bar u_i)\operatorname{div}\bar v_i)dxds\\
&-\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i))((v-\bar v_i)\cdot\nabla\operatorname{div}\bar u_i-(u_i-\bar u_i)\cdot\nabla\operatorname{div}\bar v_i)dxds\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i(\rho_j-\bar\rho_j)(\bar u_i-u_i)\cdot(\bar u_i-\bar u_j)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bar e_i\cdot(\bar u_i-u_i)dxds\\
&-\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)D(\bar u_i):D(u_i-\bar u_i)dxds\\
&-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)\operatorname{div}\bar u_i(\operatorname{div}u_i-\operatorname{div}\bar u_i)dxds
\end{aligned}
$$

term by term, using the strict positive terms at the left–hand–side and in terms of the quantity (3.15). Following standard arguments [13, 22, 23], and using in particular that $\bar u_i$ and their derivatives are $O(\epsilon)$, we readily obtain

$$
\begin{aligned}
&|\mathcal{I}_1|\le\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|p(\rho_i|\bar\rho_i)\operatorname{div}\bar u_i|dxds\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n h(\rho_i|\bar\rho_i)dxds;\\
&|\mathcal{I}_2|\le\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i\nabla\bar u_i:(u_i-\bar u_i)\otimes(u_i-\bar u_i)|dxds\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar m_i}{\bar\rho_i}\right|^2dxds;\\
&|\mathcal{I}_3|\le\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i\nabla\bar u_i:(v_i-\bar v_i)\otimes(v_i-\bar v_i)|dxds\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar J_i}{\bar\rho_i}\right|^2dxds.
\end{aligned}
$$

Then we bound $\mathcal{I}_4$ by means of Young's inequality as follows:

$$
\begin{aligned}
|\mathcal{I}_4|&\le\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i||(v-\bar v_i)\operatorname{div}\bar u_i-(u_i-\bar u_i)\operatorname{div}\bar v_i)|dxds\\
&\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i||v_i-\bar v_i|dxds\\
&\quad+\frac{C}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i||u_i-\bar u_i|dxds\\
&\le\frac{C}{\widehat M}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar J_i}{\bar\rho_i}\right|^2dxds+\frac{1}{12\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar m_i}{\bar\rho_i}\right|^2dxds,
\end{aligned}
$$

being $\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i=\frac{s+1}{2}(v_i-\bar v_i)$. For $\mathcal{I}_5$ we use Lemma 3.1 to obtain:

$$
\begin{aligned}
|\mathcal{I}_5|&\le\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i)||(v_i-\bar v_i)\cdot\nabla\operatorname{div}\bar u_i-(u_i-\bar u_i)\cdot\nabla\operatorname{div}\bar v_i|dxds\\
&\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i)||v_i-\bar v_i|dxds\\
&\quad+\frac{C}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i)||u_i-\bar u_i|dxds\\
&\le\frac{C}{\widehat M}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(h(\rho_i|\bar\rho_i)+\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar J_i}{\bar\rho_i}\right|^2\right)dxds
\end{aligned}
$$

$$+\frac{1}{12\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar m_i}{\bar\rho_i}\right|^2 dxds.$$

We estimate the interaction term $\mathcal{I}_6$ in (3.8) when $b_{i,j}>0$ as in [24]:

$$\begin{aligned}
|\mathcal{I}_6| &\le \frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n |b_{i,j}\rho_i(\bar\rho_j-\rho_j)(u_i-\bar u_i)||\bar u_i-\bar u_j|dxds\\
&\le \frac{C}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n |b_{i,j}\rho_i(\bar\rho_j-\rho_j)(u_i-\bar u_i)|dxds\\
&\le \frac{C}{\widehat{M}}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n |\rho_j-\bar\rho_j|^2 dxds+\frac{1}{12\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2 dxds\\
&\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n h(\rho_i|\bar\rho_i)dxds+\frac{1}{12\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2 dxds.
\end{aligned}$$

In the first line we use again $\bar u_i-\bar u_j=O(\epsilon)$ and in the second (3.16) and Young's inequality, while the last inequality follows from the strict convexity of Ψ, which is again a consequence of (3.16). Indeed, in the framework of solutions with bounded density, $|\rho_i-\bar\rho_i|^2$ can be controlled in terms of $h(\rho_i|\bar\rho_i)$, as shown also in [17, 22]. For $\mathcal{I}_7$, since $\bar e_i=O(\epsilon)$, using Young's inequality we get:

$$\begin{aligned}
|\mathcal{I}_7| &\le \int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar\rho_i}|\bar e_i||u_i-\bar u_i|dxds\\
&\le \frac{C\epsilon^2}{\widehat{M}}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar\rho_i^2}|\bar e|^2 dxds+\frac{1}{4\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2 dxds\\
&\le C\epsilon^4 T+\frac{1}{4\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2 dxds.
\end{aligned}$$

Finally, as in [13], the viscosity terms in $\mathcal{I}_8$ can be controlled thanks to the uniform bounds of $\mu_L(\rho_i)$ and $\lambda_L(\rho_i)$, and again using that $\bar u_i$ is $O(\epsilon)$. Indeed:

$$\begin{aligned}
\mathcal{I}_8 =\mathcal{I}_{8,1}+\mathcal{I}_{8,2} :=& -\frac{2\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu_L(\rho_i)D(\bar u_i):D(u_i-\bar u_i)dxds\\
&-\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda_L(\rho_i)\operatorname{div}\bar u_i(\operatorname{div}u_i-\operatorname{div}\bar u_i)dxds
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{I}_{8,1}| &\le \frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu_L(\rho_i)|D(u_i-\bar u_i)|^2\,dxds+\nu CT\epsilon;\\
|\mathcal{I}_{8,2}| &\le \frac{\nu}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda_L(\rho_i)|\operatorname{div}(u_i-\bar u_i)|^2\,dxds+\nu CT\epsilon.
\end{aligned}$$

Therefore, collecting all estimates above, we obtain:

$$\Psi(t)+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n M_i\rho_i|u_i-\bar u_i|^2 dxds$$

$$+\frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|(u_i-u_j)-(\bar{u}_i-\bar{u}_j)|^2dxds$$

$$+\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i-\bar{u}_i)|^2dxds+\frac{\nu}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}(u_i-\bar{u}_i)|^2dxds$$

$$\leq\Psi(0)+C\epsilon^4T+\nu\bar{C}T\epsilon+\hat{C}\int_0^t\Psi(s)ds,$$

and, since from relation (1.2) we have

$$0\leq\frac{\nu}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\lambda_L(\rho_i)+\frac{2}{n}\mu_L(\rho_i)\right)|\operatorname{div}(u_i-\bar{u}_i)|^2dxds$$

$$\leq\frac{\nu}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu_L(\rho_i)|D(u_i-\bar{u}_i)|^2dxds+\frac{\nu}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda_L(\rho_i)|\operatorname{div}(u_i-\bar{u}_i)|^2dxds,$$

see also [13], the Gronwall lemma concludes the proof. ∎

REMARK 3.1. In the above estimate, we keep the dependence on the viscosity constant ν explicit, showing in this way the final estimate (3.17) is indeed uniform for $0\leq\nu\ll1$. In other words, our results apply also for the Euler-Korteweg case, and the corresponding estimate is given by taking the limit $\nu\to0^+$ in (3.17), that is

$$\Psi(t)\leq(\Psi(0)+C\epsilon^4)\exp^{\hat{C}t}.$$

4. High friction limit in the case $M_i=0$

This section is devoted to the study of the high–friction limit when no diagonal damping is present, and without diffusive scaling. Hence, referring to the discussions of Section 2, the system we are going to study is the following:

$$\begin{cases}\partial_t\rho_i+\operatorname{div}(\rho_iu_i)=0\\ \partial_t(\rho_iu_i)+\operatorname{div}(\rho_iu_i\otimes u_i)-2\nu\operatorname{div}(\mu(\rho_i)D(u_i))-\nu\nabla(\lambda(\rho_i)\operatorname{div}u_i)+\nabla\rho_i^\gamma\\ =\operatorname{div}S_i-\dfrac{1}{\epsilon}\displaystyle\sum_{j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\\ \partial_t(\rho_iv_i)+\operatorname{div}(\rho_iv_i\otimes u_i)+\operatorname{div}K_i=0.\end{cases}\tag{4.1}$$

The argument we shall use is closely related to the ones in [19]; however, in the present work, we consider viscosity effects and a more general class of capillarity coefficients, the latter thanks to the augmented formulation already presented in the previous section. Thus, for readers' convenience, we recall the following relations involving in particular the drift velocities v_i:

$$v_i=\frac{\nabla\mu(\rho_i)}{\rho_i},\ m_i=\rho_iu_i,\ J_i=\rho_iv_i=\nabla\mu(\rho_i);$$

$$\operatorname{div}S_i=\rho_i\nabla\left(k(\rho_i)\Delta\rho_i+\frac{1}{2}k'(\rho_i)|\nabla\rho_i|^2\right)=\operatorname{div}(\mu(\rho_i)\nabla v_i)+\frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div}v_i);$$

$$\operatorname{div}K_i=\operatorname{div}(\mu(\rho_i)^t\nabla u_i)+\frac{1}{2}\nabla(\lambda(\rho_i)\operatorname{div}u_i).$$

As already emphasized in Section 2, in system (4.1) we made the particular choice $\mu_L(\rho_i)=\mu(\rho_i)$ and $\lambda_L(\rho_i)=\lambda(\rho_i)=2(\mu'(\rho_i)\rho_i-\mu(\rho_i))$, as in [9]. This allows us to rewrite the viscosity terms in the forthcoming relative entropy calculation (4.6) in terms of the quadratic quantity already used (3.15); see subsections below. Indeed, we recall that from the Hilbert expansion of (4.1) performed in Section 2, in the present high friction limit without time scaling, which does not lead to a diffusive limit, the viscosity part is not an $O(\epsilon)$ error term, and therefore it can not be managed in the same way as in Section 3.

Hence, let us now compare via a relative entropy inequality weak solutions of (4.1) with strong solutions of system (2.25), that is:

$$\begin{cases} \partial_t\bar\rho_i+\operatorname{div}(\bar\rho_i\bar u)=0\\ \partial_t(\bar\rho_i\bar u)+\operatorname{div}(\bar\rho_i\bar u\otimes\bar u)-2\nu\operatorname{div}(\mu(\bar\rho_i)D(\bar u))-\nu\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u)+\nabla\bar\rho_i^\gamma=\operatorname{div}\bar S_i+\bar R_i\\ \partial_t(\bar\rho_i\bar v_i)+\operatorname{div}(\bar\rho_i\bar v_i\otimes\bar u)+\operatorname{div}\bar K_i=0 \end{cases} \tag{4.2}$$

where $\bar m_i=\bar\rho_i\bar u$, $\bar J_i=\bar\rho_i\bar v_i=\nabla\mu(\bar\rho_i)$, and $\bar R_i$ is defined as in (2.26):

$$\begin{aligned} \bar R_i&=\frac{\rho_i}{\bar\rho}\sum_{j=1}^n\left[\operatorname{div}\bar S_j-\nabla p(\bar\rho_j)\right]-\operatorname{div}\bar S_i+\nabla p(\bar\rho_i)\\ &\quad+\tfrac{\rho_i}{\bar\rho}\textstyle\sum_{j=1}^n[2\nu\operatorname{div}\mu(\bar\rho_j)D(\bar u)+\nu\nabla(\lambda(\bar\rho_j)\operatorname{div}\bar u)]-2\nu\operatorname{div}\mu(\bar\rho_i)D(\bar u)-\nu\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u)\\ &=\frac{\rho_i}{\bar\rho}\operatorname{div}\bar T-\operatorname{div}\bar T_i+\nu\operatorname{div}\bar D\frac{\bar\rho_i}{\bar\rho}-\nu\operatorname{div}\bar D_i;\\ \operatorname{div}\bar T_i&=\operatorname{div}\bar S_i-\nabla p(\bar\rho_i);\\ \operatorname{div}\bar T&=\sum_{i=1}^n\operatorname{div}\bar T_i;\\ \operatorname{div}\bar D_i&=\operatorname{div}(\mu(\bar\rho_i)D(\bar u))+\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u);\\ \operatorname{div}\bar D&=\sum_{i=1}^n\operatorname{div}\bar D_i. \end{aligned}$$

The definition of weak solutions of system (4.1) is the following.

DEFINITION 4.1 (weak and dissipative weak solution). *A triple* $(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon)$ *(where* $\hat\rho^\epsilon=(\rho_1^\epsilon,\cdots,\rho_n^\epsilon)$, $\hat m^\epsilon=(m_1^\epsilon,\cdots,m_n^\epsilon)$, $\hat J^\epsilon=(J_1^\epsilon,\cdots,J_n^\epsilon)$*) such that*

$$\begin{aligned} &0\le\rho_i^\epsilon\in C^0([0,\infty);L^1(\mathbb{T}^3));\\ &(\rho_i^\epsilon u_i^\epsilon,\rho_i^\epsilon v_i^\epsilon)\in C^0([0,\infty);L^1(\mathbb{T}^3;\mathbb{R}^3)) \end{aligned}$$

is called a periodic weak solution *of* (4.1) *if for any* $i=1,\cdots,n$*:*

$$\begin{aligned} &\sqrt{\rho_i^\epsilon}u_i^\epsilon\in L^\infty((0,T);L^2(\mathbb{T}^3)^3)\\ &\sqrt{\rho_i^\epsilon}v_i^\epsilon\in L^\infty((0,T);L^2(\mathbb{T}^3)^3)\\ &\rho_i^\epsilon\in C^0([0,\infty);L^\gamma(\mathbb{T}^3))\\ &\mu(\rho_i^\epsilon)D(u_i^\epsilon)\in L^1((0,T);L^1(\mathbb{T}^3)^{3\times3})\\ &\lambda(\rho_i^\epsilon)\operatorname{div}u_i^\epsilon\in L^1((0,T);L^1(\mathbb{T}^3))\\ &(\rho_i^\epsilon)^2u_i^\epsilon\in L^\infty((0,T);L^1(\mathbb{T}^3)^3) \end{aligned}$$

and $(\rho_i^\epsilon, u_i^\epsilon, v_i^\epsilon)$ *satisfy for all* $\psi_i \in C_0^1([0,\infty); C^1(\mathbb{R}^3))$ *and* $\phi_i, \varphi_i \in C_0^1([0,\infty); C^1(\mathbb{T}^3)^3)$*:*

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon \partial_t \psi_i + \rho_i^\epsilon u_i^\epsilon \cdot \nabla \psi_i)\, dxdt = \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon(x,0)\psi_i(x,0)\, dx,\\
&-\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon u_i^\epsilon \partial_t \phi_i + \rho_i^\epsilon u_i^\epsilon \otimes u_i^\epsilon : \nabla \phi_i + \rho_i^{\epsilon\gamma} \operatorname{div} \psi_i + \mu(\rho_i^\epsilon) v_i \cdot \nabla \operatorname{div} \phi_i + \nabla \mu(\rho_i^\epsilon) \cdot (\nabla \phi_i v_i) dxdt\\
&\quad -\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \left(\frac{1}{2} \nabla \lambda(\rho_i^\epsilon) \cdot v_i^\epsilon \operatorname{div} \phi_i + \frac{1}{2} \lambda(\rho_i^\epsilon) v_i^\epsilon \cdot \nabla \operatorname{div} \phi_i \right) dxdt\\
&\quad +\nu \int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n (2\mu(\rho_i) D(u_i) : \nabla \phi_i + \lambda(\rho_i) \operatorname{div} u_i \operatorname{div} \phi_i)\, dxdt\\
&\quad +\frac{1}{\epsilon} \int_0^\infty \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j} \rho_i^\epsilon \rho_j^\epsilon (u_i^\epsilon - u_j^\epsilon) \cdot \phi_i dxdt\\
&= \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon u_i^\epsilon)(x,0) \cdot \phi_i(x,0) dx,\\
&\quad -\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon v_i^\epsilon \partial_t \varphi_i + \rho_i^\epsilon u_i^\epsilon \otimes v_i^\epsilon : \nabla \varphi_i + \mu(\rho_i^\epsilon) v_i \cdot \nabla \operatorname{div} \varphi_i + \nabla \mu(\rho_i^\epsilon) \cdot (\nabla \varphi_i v_i)) dxdt\\
&\quad -\int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \left(\frac{1}{2} \nabla \lambda(\rho_i^\epsilon) \cdot v_i^\epsilon \operatorname{div} \varphi_i + \frac{1}{2} \lambda(\rho_i^\epsilon) v_i^\epsilon \cdot \nabla \operatorname{div} \varphi_i \right) dxdt\\
&= \int_{\mathbb{T}^3} \sum_{i=1}^n (\rho_i^\epsilon v_i^\epsilon)(x,0) \cdot \varphi_i(x,0) dx.
\end{aligned}
$$

If in addition $\eta_{tot}(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon) \in C([0,\infty); L^1(\mathbb{T}^3))$ *and the integrated energy inequality of* (1.11)

$$
\begin{aligned}
&-\int_0^\infty \Sigma_{tot}(\hat\rho^\epsilon(t), \hat m^\epsilon(t), \hat J^\epsilon(t)) \theta'(t) dt + \frac{1}{2\epsilon} \int_0^\infty \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j} \rho_i^\epsilon \rho_j^\epsilon |u_i^\epsilon - u_j^\epsilon|^2 \theta(t) dxdt\\
&\quad +2\nu \int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \mu(\rho_i^\epsilon) |D(u_i^\epsilon)|^2\, dxdt + \nu \int_0^\infty \int_{\mathbb{T}^3} \sum_{i=1}^n \lambda(\rho_i^\epsilon) |\operatorname{div} u_i^\epsilon|^2 dxdt\\
&\le \Sigma_{tot}(\hat\rho^\epsilon(0), \hat m^\epsilon(0), \hat J^\epsilon(0)) \theta(0) \qquad (4.3)
\end{aligned}
$$

holds for any $\theta \in W^{1,\infty}([0,\infty))$ *compactly supported in* $[0,\infty)$*, then we call* $(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon)$ *a periodic* dissipative weak solution*.*

We say that a weak periodic (dissipative) weak solution $(\hat\rho^\epsilon, \hat m^\epsilon, \hat J^\epsilon)$ *for* (4.1) *has finite total mass and finite total energy if for any* $T > 0$ *there exists a constant* $K > 0$ *independent of* ϵ *such that:*

$$
\begin{aligned}
&\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \sum_{i=1}^n \rho_i^\epsilon dx \le K;\\
&\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \sum_{i=1}^n \left(\frac{1}{2} \frac{m_i^{\epsilon 2}}{\rho_i^\epsilon} + \frac{1}{2} \frac{J_i^{\epsilon 2}}{\rho_i^\epsilon} + \frac{\rho_i^{\epsilon\gamma}}{\gamma - 1} \right) dx \le K. \qquad (4.4)
\end{aligned}
$$

We recall the definition of the relative entropy where the subscripts ϵ have been

omitted for simplicity:

$$\eta_{tot}(\hat{\rho},\hat{m},\hat{J}|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})=\sum_{i=1}^{n}\eta(\rho_i,m_i,J_i)-\sum_{i=1}^{n}\eta(\bar{\rho}_i,\bar{m}_i,\bar{J}_i)-\sum_{i=1}^{n}\eta_{\rho_i}(\bar{\rho}_i,\bar{m}_i,\bar{J}_i)(\rho_i-\bar{\rho}_i)$$
$$-\sum_{i=1}^{n}\eta_{m_i}(\bar{\rho}_i,\bar{m}_i,\bar{J}_i)(m_i-\bar{m}_i)-\sum_{i=1}^{n}\eta_{J_i}(\bar{\rho}_i,\bar{m}_i,\bar{J}_i)(J_i-\bar{J}_i). \quad (4.5)$$

PROPOSITION 4.1. *Let $(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon)$ be a dissipative weak periodic solution to (4.1) as in Definition 4.1 with finite total mass and energy (4.4) such that (A1) holds. Let $\bar{w}_i=(\bar{\rho}_i,\bar{m}_i=\bar{\rho}_i\bar{u},\bar{J}_i=\bar{\rho}_i\bar{v}_i)$ be a smooth solution to (4.2) such that:*

$$\bar{w}_i,\partial_t\bar{w}_i,\nabla\bar{w}_i,D^2\bar{w}_i,D^3\bar{\rho}_i\in L^\infty([0,T];L^\infty(\mathbb{T}^3)),\ \textit{for any } i=1,\cdots,n$$

then:

$$\Sigma_{tot}(\hat{\rho}^\epsilon,\hat{m}^\epsilon,\hat{J}^\epsilon|\bar{\hat{\rho}},\bar{\hat{m}},\bar{\hat{J}})\Big|_{s=0}^{t}+\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^{n}b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds$$
$$+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\mu(\rho_i)|D(u_i-\bar{u})|^2dxds+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\lambda(\rho_i)|\operatorname{div}(u_i-\bar{u})|^2dxds$$
$$\le 2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_iD(\bar{u}):(v_i-\bar{v}_i)\otimes(u_i-\bar{u})dxds$$
$$+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_i\nabla(\operatorname{div}\bar{u})(\mu'(\rho_i)-\mu'(\bar{\rho}_i))(u_i-\bar{u})dxds$$
$$+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)\operatorname{div}\bar{u}(u_i-\bar{u})dxds$$
$$+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\frac{\rho_i}{\bar{\rho}_i}\bar{R}_i(\bar{u}-u_i)dxds-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}p(\rho_i|\bar{\rho}_i)\operatorname{div}\bar{u}dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}(\rho_i\nabla\bar{u}:(u_i-\bar{u})\otimes(u_i-\bar{u}))dxds-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_i\nabla\bar{u}:(v_i-\bar{v}_i)\otimes(v_i-\bar{v}_i)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)\cdot((v-\bar{v}_i)\operatorname{div}\bar{u}-(u_i-\bar{u})\operatorname{div}\bar{v}_i)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^{n}\rho_i(\mu'(\rho_i)-\mu'(\bar{\rho}_i))((v-\bar{v}_i)\cdot\nabla\operatorname{div}\bar{u}-(u_i-\bar{u})\cdot\nabla\operatorname{div}\bar{v}_i)dxds. \quad (4.6)$$

Proof. Using the definition of periodic dissipative weak solution as in Definition 4.1 and the regularity of strong solution of (4.2) we compute the first two terms of (4.5), while the weak formulation allows us to estimate the remaining linear parts.

Step I: The Energy Inequalities.

As in Section 3, we use the following test function in (4.3):

$$\theta(s)=\begin{cases}1 & \text{for} \quad 0\le s<t\\ \dfrac{t-s}{\delta}+1 & \text{for} \quad t\le s<t+\delta\\ 0 & \text{for} \quad s>t+\delta.\end{cases} \quad (4.7)$$

Passing to the limit as $\delta \to 0$ we get:

$$\Sigma_{tot}(\hat{\rho}^\epsilon(t),\hat{m}^\epsilon(t),\hat{J}^\epsilon(t)) + \frac{1}{2\epsilon}\int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2 dxds$$
$$+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu(\rho_i)|D(u_i)|^2 dxds + \nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda(\rho_i)|\operatorname{div} u_i|^2 dxds$$
$$\leq \Sigma_{tot}(\hat{\rho}^\epsilon(0),\hat{m}^\epsilon(0),\hat{J}^\epsilon(0)). \tag{4.8}$$

We compute the energy associated to the strong solution $(\bar{\rho}_i,\bar{m}_i,\bar{J}_i)$ multiplying Equations $(4.2)_2$ and $(4.2)_3$ by $\bar{u}$ and $\bar{v}_i$ respectively. If we sum all the contributions integrating over $(0,T)\times\mathbb{T}^3$ we obtain:

$$\Sigma_{tot}(\bar{\hat{\rho}}(t),\bar{\hat{m}}(t),\bar{\hat{J}}(t)) + 2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu(\bar{\rho}_i)|D(\bar{u})|^2 dxds + \nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda(\bar{\rho}_i)|\operatorname{div}\bar{u}|^2 dxds$$
$$\leq \Sigma_{tot}(\bar{\hat{\rho}}(0),\bar{\hat{m}}(0),\bar{\hat{J}}(0)) + \int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \bar{R}_i\bar{u}\,dxds. \tag{4.9}$$

Step II: Equations for the difference.

Now we evaluate the linear part of the relative entropy using suitable test functions in the weak formulation according to Definition 4.1. If we take the differences $(\rho_i-\bar{\rho}_i,\rho_iu_i-\bar{\rho}_i\bar{u},\rho_iv_i-\bar{\rho}_i\bar{v}_i)$ they have to satisfy:

$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n (\rho_i-\bar{\rho}_i)\partial_s\psi_i + (\rho_iu_i-\bar{\rho}_i\bar{u})\cdot\nabla\psi_i\,dxds$$
$$=\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i(x,0)-\bar{\rho}_i(x,0)\psi_i(x,0))dx,$$

$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_iu_i-\bar{\rho}_i\bar{u})\cdot\partial_s\phi_i + (\rho_iu_i\otimes u_i-\bar{\rho}_i\bar{u}\otimes\bar{u}):\nabla\phi_i + (\rho_i^\gamma-\bar{\rho}_i^\gamma)\operatorname{div}\phi_i\,dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)v_i-\mu(\bar{\rho}_i)\bar{v}_i)\nabla\operatorname{div}\phi_i + (\nabla\mu(\rho_i)v_i-\nabla\mu(\bar{\rho}_i)\bar{v}_i):\nabla\phi_i\,dxds$$
$$-\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\nabla\lambda(\rho_i)v_i-\nabla\lambda(\bar{\rho}_i)\bar{v}_i)\operatorname{div}\phi_i + (\lambda(\rho_i)v_i-\lambda(\bar{\rho}_i)\bar{v}_i)\nabla\operatorname{div}\phi_i\,dxds$$
$$+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(2\mu(\rho_i)D(u_i)-2\mu(\bar{\rho}_i)D(\bar{u})):\nabla\phi_i + (\lambda(\rho_i)\operatorname{div}u_i-\lambda(\bar{\rho}_i)\operatorname{div}\bar{u})\operatorname{div}\phi_i\,dxds$$
$$+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\phi_i\,dxds + \int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\bar{R}_i\phi_i\,dxds$$
$$=\int_{\mathbb{T}^3}\sum_{i=1}^n((\rho_iu_i)(x,0)-(\bar{\rho}_i\bar{u}_i)(x,0)\phi_i(x,0))dx,$$

and

$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_iv_i-\bar{\rho}_i\bar{v}_i)\cdot\partial_s\varphi_i + (\rho_iu_i\otimes v_i-\bar{\rho}_i\bar{u}\otimes\bar{v}_i):\nabla\varphi_i\,dxds$$

$$+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)u_i-\mu(\bar\rho_i)\bar u)\nabla\operatorname{div}\varphi_i+(\nabla\mu(\rho_i)u_i-\nabla\mu(\bar\rho_i)\bar u):\nabla\varphi_i dxds$$
$$+\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\nabla\lambda(\rho_i)u_i-\nabla\lambda(\bar\rho_i)\bar u)\operatorname{div}\varphi_i+(\lambda(\rho_i)u_i-\lambda(\bar\rho_i)\bar u)\nabla\operatorname{div}\varphi_i dxds$$
$$=\int_{\mathbb{T}^3}\sum_{i=1}^n((\rho_iv_i)(x,0)-(\bar\rho_i\bar v_i)(x,0))\phi_i(x,0)dx,$$

where ψ,ϕ,φ are Lipschitz test functions (ϕ,φ vector-valued) compactly supported in $[0,\infty)$ in time and periodic in space. In the above formulation we choose:

$$\psi_i=\theta(s)\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right),$$
$$\phi_i=\theta(s)\left(\frac{\bar m_i}{\bar\rho_i}\right)\quad\varphi_i=\theta(s)\left(\frac{\bar J_i}{\bar\rho_i}\right),$$

where θ is defined in (4.7) and we recall that $\bar m_i/\bar\rho_i=\bar u$ is indeed independent from i. The previous relations become as $\delta\to0$:

$$\int_{\mathbb{T}^3}\sum_{i=1}^n\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)\Big|_{s=0}^t$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)\,dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla_x\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_iu_i-\bar\rho_i\bar u)\,dxds=0,\tag{4.10}$$

$$\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\bar m_i}{\bar\rho_i}(\rho_iu_i-\bar\rho_i\bar u)\Big|_{s=0}^t-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(\frac{\bar m_i}{\bar\rho_i}\right)(\rho_iu_i-\bar\rho_i\bar u)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\rho_iu_i\otimes u_i-\bar\rho_i\bar u\otimes\bar u):\nabla\left(\frac{\bar m_i}{\bar\rho_i}\right)+(p(\rho_i)-p(\bar\rho_i))\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)\right)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\mu(\rho_i)v_i-\mu(\bar\rho_i)\bar v_i)\nabla\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)+(\nabla\mu(\rho_i)v_i-\nabla\mu(\bar\rho_i)\bar v_i):\nabla\left(\frac{\bar m_i}{\bar\rho_i}\right)\right)dxds$$
$$-\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\nabla\lambda(\rho_i)v_i-\nabla\lambda(\bar\rho_i)\bar v_i)\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)+(\lambda(\rho_i)v_i-\lambda(\bar\rho_i)\bar v_i)\nabla\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)\right)dxds$$
$$+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(2\mu(\rho_i)D(u_i)-2\mu(\bar\rho_i)D(\bar u)):\nabla\bar u+(\lambda(\rho_i)\operatorname{div}u_i-\lambda(\bar\rho_i)\operatorname{div}\bar u)\operatorname{div}\bar u dxds$$
$$+\frac1\epsilon\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\bar R_i\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds=0,\tag{4.11}$$

and

$$\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\bar J_i}{\bar\rho_i}(\rho_iv_i-\bar\rho_i\bar v_i)\Big|_{s=0}^t-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(\frac{\bar J_i}{\bar\rho_i}\right)(\rho_iv_i-\bar\rho_i\bar v_i)dxds$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\rho_i u_i\otimes v_i-\bar\rho_i\bar u\otimes\bar v_i):\nabla\left(\frac{\bar J_i}{\bar\rho_i}\right)\right)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left((\mu(\rho_i)u_i-\mu(\bar\rho_i)\bar u)\nabla\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)+(\nabla\mu(\rho_i)u_i-\nabla\mu(\bar\rho_i)\bar u):\nabla\left(\frac{\bar J_i}{\bar\rho_i}\right)\right)dxds\\
&+\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\nabla\lambda(\rho_i)u_i-\nabla\lambda(\bar\rho_i)\bar u)\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)\\
&+\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\lambda(\rho_i)u_i-\lambda(\bar\rho_i)\bar u)\nabla\operatorname{div}\left(\frac{\bar J_i}{\bar\rho_i}\right)dxds=0.
\end{aligned}\tag{4.12}
$$

We notice that

$$
\frac1\epsilon\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\cdot\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds=\frac1\epsilon\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j(u_i-u_j)\cdot\bar u dxds=0,
$$

thanks to the symmetry of the matrix $\{b_{i,j}\}_{i,j=1,\cdots,n}$. Recalling the definition of the relative entropy (4.5), and using (4.8),(4.9), (4.10), (4.11), (4.12) we obtain:

$$
\begin{aligned}
&\Sigma_{tot}(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon|\bar{\hat\rho},\bar{\hat m},\bar{\hat J})\Big|_{s=0}^t dx+\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds\\
&\le-2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)|D(u_i)|^2dxds-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\rho_i)|\operatorname{div}u_i|^2dxds\\
&\quad+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\bar\rho_i)|D(\bar u)|^2dxds+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\bar\rho_i)|\operatorname{div}\bar u|^2dxds\\
&\quad+\nu\int_0^\infty\int_{\mathbb{T}^3}\sum_{i=1}^n(2\mu(\rho_i)D(u_i)-2\mu(\bar\rho_i)D(\bar u)):\nabla\bar u+(\lambda(\rho_i)\operatorname{div}u_i-\lambda(\bar\rho_i)\operatorname{div}\bar u)\operatorname{div}\bar u dxds\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i-\bar\rho_i)+\partial_s\left(\frac{\bar m_i}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u)\\
&\qquad\qquad+\partial_s\left(\frac{\bar J_i}{\bar\rho_i}\right)(\rho_i v_i-\bar\rho_i\bar v_i)dxds\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla_x\left(h'(\bar\rho_i)-\frac12\frac{\bar m_i{}^2}{\bar\rho_i}-\frac12\frac{\bar J_i{}^2}{\bar\rho_i}\right)(\rho_i u_i-\bar\rho_i\bar u)dxds\\
&\qquad\qquad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(p(\rho_i)-p(\bar\rho_i))\operatorname{div}\left(\frac{\bar m_i}{\bar\rho_i}\right)dxds\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i u_i\otimes u_j-\bar\rho_i\bar u\otimes\bar u):\nabla\bar u+(\rho_i u_i\otimes v_j-\bar\rho_i\bar u\otimes\bar v_j):\nabla\bar v_i dxds\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)[(v_i-\bar v_i)\nabla\operatorname{div}\bar u-(u_i-\bar u)\nabla\operatorname{div}\bar v_i]\\
&\qquad\qquad+\nabla\mu(\rho_i)[\nabla\bar u(v_i-\bar v_i)-\nabla\bar v_i(u_i-\bar u)]dxds\\
&\quad-\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\lambda(\rho_i)[(v_i-\bar v_i)\operatorname{div}\bar u-(u_i-\bar u)\operatorname{div}\bar v_i]\\
&\qquad\qquad+\lambda(\rho_i)[(v_i-\bar v_i)\nabla\operatorname{div}\bar u-(u_i-\bar u)\nabla\operatorname{div}\bar v_i]dxds\\
&\quad+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\mu(\rho_i)-\mu(\bar\rho_i))(\bar u\nabla\operatorname{div}\bar v_i-\bar v_i\nabla\operatorname{div}\bar u)+(\nabla\mu(\rho_i)-\nabla\mu(\bar\rho_i))[\nabla\bar v_i\bar u-\nabla\bar u\bar v_i]dxds
\end{aligned}
$$

$$+\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\lambda(\rho_i)-\lambda(\bar\rho_i))[\bar u\nabla\operatorname{div}\bar v_i-\bar v_i\nabla\operatorname{div}\bar u]$$
$$+(\nabla\lambda(\rho_i)-\nabla\lambda(\bar\rho_i))(\bar u\operatorname{div}\bar v_i-\bar v_i\operatorname{div}\bar u)dxds.$$

As in Proposition 3.1 we observe that the last two lines are equal to zero thanks to the property (1.9) of the stress tensors S_i, K_i. Using the relation $h''(\bar\rho_i)=p'(\bar\rho_i)/\bar\rho_i$ we find that:

$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s h'(\bar\rho_i)(\rho_i-\bar\rho_i)+\nabla_x h'(\bar\rho_i)(\rho_iu_i-\bar\rho_i\bar u)dxds$$
$$=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p'(\bar\rho_i)(\rho_i-\bar\rho_i)\operatorname{div}\bar u+\frac{\rho_i}{\bar\rho_i}\nabla p(\bar\rho_i)(\bar u-u_i)dsdx.$$

Then, the following quantity

$$\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\left(\frac{1}{2}|\bar u|^2\right)(\rho_i-\bar\rho_i)+\nabla_x\left(\frac{1}{2}|\bar u|^2\right)(\rho_iu_i-\bar\rho_i\bar u)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\partial_s\bar u(\rho_iu_i-\bar\rho_i\bar u)+\nabla\bar u:(\rho_iu_i\otimes u_i-\bar\rho_i\bar u_i\otimes\bar u_i)dxds$$

can be rearranged by multiplying the momentum equations of the strong solution $\bar u,\bar v_i$, by $\rho_i(\bar u-u_i)$ and $\rho_i(\bar v_i-v_i)$ respectively. We get:

$$\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\partial_s\left(\frac{1}{2}|\bar u|^2\right)(\rho_i-\bar\rho_i)+\nabla_x\left(\frac{1}{2}|\bar u|^2\right)(\rho_iu_i-\bar\rho_i\bar u)-\partial_s\bar u(\rho_iu_i-\bar\rho_i\bar u)\right)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\bar u:(\rho_iu_i\otimes u_i-\bar\rho_i\bar u_i\otimes\bar u_i)dxds$$
$$=-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\nabla\bar u:(u_i-\bar u)\otimes(u_i-\bar u)+\frac{\rho_i}{\bar\rho_i}\nabla p(\bar\rho_i)(\bar u-u_i)$$
$$-\frac{\rho_i}{\bar\rho_i}\operatorname{div}\bar S_i(\bar u-u_i)-\frac{\rho_i}{\bar\rho_i}\bar R_i(\bar u-u_i)dxds+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\operatorname{div}(\mu(\bar\rho_i)D\bar u))\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds$$
$$+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u)\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds$$

and

$$\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\partial_s\left(\frac{1}{2}|\bar v_i|^2\right)(\rho_i-\bar\rho_i)+\nabla_x\left(\frac{1}{2}|\bar v_i|^2\right)(\rho_iu_i-\bar\rho_i\bar u)-\partial_s\bar v_i(\rho_iu_i-\bar\rho_i\bar u)\right)dxds$$
$$-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\bar v_i:(\rho_iu_i\otimes v_i-\bar\rho_i\bar u_i\otimes\bar v_i)dxds$$
$$=-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\rho_i\nabla\bar v_i:(u_i-\bar u)\otimes(v_i-\bar v_i)+\frac{\rho_i}{\bar\rho_i}\operatorname{div}\bar K_i(\bar v_i-v_i)\right)dxds.$$

From the previous calculations the relative entropy verifies:

$$\Sigma_{tot}(\hat\rho^\epsilon,\hat m^\epsilon,\hat J^\epsilon|\bar{\hat\rho},\bar{\hat m},\bar{\hat J})\Big|_{s=0}^t+\frac{1}{\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds$$

$$
\begin{aligned}
\leq & -2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)|D(u_i)|^2dxdt-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\rho_i)|\operatorname{div}u_i|^2dxds\\
&+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(2\mu(\rho_i)D(u_i):D(\bar u)+\lambda(\rho_i)\operatorname{div}u_i\operatorname{div}\bar u)dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\operatorname{div}(\mu(\bar\rho_i)D\bar u))\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds\\
&\qquad+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u)\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar\rho_i)\operatorname{div}\bar u dxds+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bar R_i(\bar u-u_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n(\rho_i\nabla\bar u:(u_i-\bar u)\otimes(u_i-\bar u)+\rho_i\nabla\bar v_i:(u_i-\bar u)\otimes(v_i-\bar v_i))dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)[(v_i-\bar v_i)\nabla\operatorname{div}\bar u-(u_i-\bar u)\nabla\operatorname{div}\bar v_i]\\
&\qquad+\nabla\mu(\rho_i)[\nabla\bar u(v_i-\bar v_i)-\nabla\bar v_i(u_i-\bar u)]dxds\\
&-\frac12\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\nabla\lambda(\rho_i)[(v_i-\bar v_i)\operatorname{div}\bar u-(u_i-\bar u)\operatorname{div}\bar v_i]\\
&\qquad+\lambda(\rho_i)[(v_i-\bar v_i)\nabla\operatorname{div}\bar u-(u_i-\bar u)\nabla\operatorname{div}\bar v_i]dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bigg(\mu(\bar\rho_i)\operatorname{div}\nabla\bar v_i+{}^t\nabla\mu(\bar\rho_i)^t\nabla\bar v_i\\
&\qquad+\frac12\nabla\lambda(\bar\rho_i)\operatorname{div}\bar v_i+\frac12\lambda(\bar\rho_i)\nabla\operatorname{div}\bar v_i\bigg)(\bar u-u_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\rho_i}{\bar\rho_i}\bigg(\mu(\bar\rho_i)\operatorname{div}{}^t\nabla\bar u+{}^t\nabla\mu(\bar\rho_i)\nabla\bar u\\
&\qquad+\frac12\nabla\lambda(\bar\rho_i)\operatorname{div}\bar u+\frac12\lambda(\bar\rho_i)\nabla\operatorname{div}\bar u\bigg)(\bar v_i-v_i)dxds\\
=:&\sum_{i=1}^9 I_i.
\end{aligned}
$$

Concerning the viscous terms $I_1+I_2+I_3$, as already said above, here we should take advantage of the particular choice for Lamé constants to rearrange them properly and then control them using the relative entropy. To this end we compute

$$
\begin{aligned}
I_1+I_2+I_3=&-2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)|D(u_i-\bar u)|^2dxds-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\rho_i)|\operatorname{div}(u_i-\bar u)|^2dxds\\
&-2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)D(\bar u):D(u_i-\bar u)dxds\\
&-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\rho_i)\operatorname{div}\bar u(\operatorname{div}u_i-\operatorname{div}\bar u)dxds
\end{aligned}
$$

$$
\begin{aligned}
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \operatorname{div}(\mu(\bar\rho_i)D\bar u))\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds\\
&+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \nabla(\lambda(\bar\rho_i)\operatorname{div}\bar u)\frac{\rho_i}{\bar\rho_i}(\bar u-u_i)dxds\\
&=:\hat I_1+\hat I_2+\hat I_3.
\end{aligned}
$$

Applying the divergence theorem in $\hat I_2$ we get

$$
\begin{aligned}
\hat I_2=&2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i D(\bar u):\frac{\nabla\mu(\rho_i)}{\rho_i}\otimes(u_i-\bar u)+\frac{\mu(\rho_i)}{\rho_i}\operatorname{div}(D(\bar u))\rho_i(u_i-\bar u)dxds\\
&+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\nabla\lambda(\rho_i)}{\rho_i}\operatorname{div}\bar u\,\rho_i(u_i-\bar u)+\frac{\lambda(\rho_i)}{\rho_i}\nabla(\operatorname{div}\bar u)\,\rho_i(u_i-\bar u)dxds,
\end{aligned}
$$

while $\hat I_3$ is equal to:

$$
\begin{aligned}
\hat I_3=&-2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i D(\bar u):\frac{\nabla\mu(\bar\rho_i)}{\bar\rho_i}\otimes(u_i-\bar u)+\frac{\mu(\bar\rho_i)}{\bar\rho_i}\operatorname{div}(D(\bar u))\rho_i(u_i-\bar u)dxds\\
&-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\nabla\lambda(\bar\rho_i)}{\bar\rho_i}\operatorname{div}\bar u\rho_i(u_i-\bar u)+\frac{\lambda(\bar\rho_i)}{\bar\rho_i}\nabla\operatorname{div}\bar u\,\rho_i(u_i-\bar u)dxds.
\end{aligned}
$$

Using the definition of v_i, $\lambda(\rho_i)=2(\mu'(\rho_i)\rho_i-\mu(\rho_i))$ and the property $\operatorname{div}D(\bar u)=\nabla\operatorname{div}\bar u$ being a symmetric matrix, we have:

$$
\begin{aligned}
\hat I_2+\hat I_3=&2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i D(\bar u):(v_i-\bar{v_i})\otimes(u_i-\bar u)dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\nabla(\operatorname{div}\bar u)(\mu'(\rho_i)-\mu'(\bar\rho_i))(u_i-\bar u)dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\operatorname{div}\bar u(u_i-\bar u)dxds.
\end{aligned}
$$

Hence, we conclude that the viscous part $I_1+I_2+I_3$ rewrites as follows:

$$
\begin{aligned}
I_1+I_2+I_3=&-2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \mu_L(\rho_i)|D(u_i-\bar u)|^2dxds\\
&-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \lambda_L(\rho_i)|\operatorname{div}(u_i-\bar u)|^2dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i D(\bar u):(v_i-\bar{v_i})\otimes(u_i-\bar u)\,dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\nabla(\operatorname{div}\bar u)(\mu'(\rho_i)-\mu'(\bar\rho_i))(u_i-\bar u)dxds\\
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\operatorname{div}\bar u(u_i-\bar u)dxds.
\end{aligned}
$$

Let us recall that $\operatorname{div}{}^t\nabla\bar{u}=\nabla\operatorname{div}\bar{u}$ and $\nabla\operatorname{div}\bar{v}_i=\operatorname{div}\nabla\bar{v}_i$ since $\nabla\bar{v}_i$ is symmetric, being $\bar{v}_i$ a gradient. We define $\tilde{I}_3$ as follows:

$$\begin{aligned}\tilde{I}_3:=&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\mu(\rho_i)-\frac{\rho_i}{\bar{\rho}_i}\mu(\bar{\rho}_i)\right)(\nabla\operatorname{div}\bar{u}(v_i-\bar{v}_i)-\nabla\operatorname{div}\bar{v}_i(u_i-\bar{u}))dxds\\&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\nabla\mu(\rho_i)-\frac{\rho_i}{\bar{\rho}_i}\nabla\mu(\bar{\rho}_i)\right)\cdot(\nabla\bar{u}(v_i-\bar{v}_i)-\nabla\bar{v}_i(u_i-\bar{u}))dxds.\end{aligned}$$

Since $v_i=\nabla\mu(\rho_i)/\rho_i$ then $\tilde{I}_3$ reads as:

$$\begin{aligned}\tilde{I}_3=&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left(\frac{\mu(\rho_i)}{\rho_i}-\frac{\mu(\bar{\rho}_i)}{\bar{\rho}_i}\right)(\nabla\operatorname{div}\bar{u}(v_i-\bar{v}_i)-\nabla\operatorname{div}\bar{v}_i(u_i-\bar{u}))dxds\\&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(v_i-\bar{v}_i)\cdot(\nabla\bar{u}(v_i-\bar{v}_i)-\nabla\bar{v}_i(u_i-\bar{u}))dxds.\end{aligned}$$

Moreover, we define

$$\begin{aligned}\tilde{I}_4:=&-\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\lambda(\rho_i)-\frac{\rho_i}{\bar{\rho}_i}\lambda(\bar{\rho}_i)\right)((v_i-\bar{v}_i)\nabla\operatorname{div}\bar{u}-(u-\bar{u})\nabla\operatorname{div}\bar{v})dxds\\&-\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\nabla\lambda(\rho_i)-\frac{\rho_i}{\bar{\rho}_i}\nabla\lambda(\bar{\rho}_i)\right)((v-\bar{v}_i)\operatorname{div}\bar{u}-(u_i-\bar{u})\operatorname{div}\bar{v}_i)dxds.\end{aligned}$$

Since $\lambda(\rho_i)=2(\rho_i\mu'(\rho_i)-\mu(\rho_i))$ one has:

$$\begin{aligned}\tilde{I}_4=&-\frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left(\frac{\lambda(\rho_i)}{\rho_i}-\frac{\lambda(\bar{\rho}_i)}{\bar{\rho}_i}\right)((v_i-\bar{v}_i)\nabla\operatorname{div}\bar{u}-(u-\bar{u})\nabla\operatorname{div}\bar{v})dxds\\&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)((v-\bar{v}_i)\operatorname{div}\bar{u}-(u_i-\bar{u})\operatorname{div}\bar{v}_i)dxds.\end{aligned}$$

Therefore:

$$\begin{aligned}&I_6+I_7+I_8+I_9=\tilde{I}_3+\tilde{I}_4\\=&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar{\rho}_i)\nabla\bar{\rho}_i)((v-\bar{v}_i)\operatorname{div}\bar{u}-(u_i-\bar{u})\operatorname{div}\bar{v}_i)dxds\\&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(\mu'(\rho_i)-\mu'(\bar{\rho}_i))((v-\bar{v}_i)\nabla\operatorname{div}\bar{u}-(u_i-\bar{u})\nabla\operatorname{div}\bar{v}_i)dxds\\&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i\left[(v_i-\bar{v}_i)\nabla\bar{u}(v_i-\bar{v}_i)-(v_i-\bar{v}_i)\nabla\bar{v}_i(u_i-\bar{u})\right]dxds.\end{aligned}$$

Finally, the relative entropy verifies (4.6) and the proof is complete. ∎

4.1. Stability result and convergence of the limit. As in the previous limit, with the relative entropy estimate (4.6) of Proposition 4.1 at hand, we are able to control the relaxation limit using the quadratic quantity (3.15):

$$\Psi(t)=\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{2}\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar{m}_i}{\bar{\rho}_i}\right|^2+\frac{1}{2}\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar{J}_i}{\bar{\rho}_i}\right|^2+h(\rho_i|\bar{\rho}_i)\right)dx$$

$$= \int_{\mathbb{T}^3} \sum_{i=1}^{n} \left(\frac{1}{2} \rho_i |u_i - \bar{u}|^2 + \frac{1}{2} \rho_i |v_i - \bar{v}_i|^2 + h(\rho_i | \bar{\rho}_i) \right) dx.$$

The proof of our convergence follows the blueprint of [19], again generalizing the results of the latter by including viscosity terms and considering a more general class of capillarity coefficients. As already mentioned above, this last generalization can be done thanks to the enlarged formulation (4.1) in terms of the drift velocity; see [9] and Lemma 3.1 in Section 3. The crucial estimate in the proof done in [19] under the framework (A1) is the following control of the kinetic energy in terms of the interaction energy; see [19, Theorem 9 and Remark 12],

$$\frac{1}{2} \sum_{i,j=1}^{n} b_{i,j} \rho_i \rho_j |u_i - u_j|^2 \geq \delta \sum_{i=1}^{n} \rho_i^2 |u_i - u|^2 \tag{4.13}$$

for a constant $\delta > 0$.

THEOREM 4.1. *Let $T > 0$ be fixed and let $(\hat{\rho}^\epsilon, \hat{m}^\epsilon, \hat{J}^\epsilon)$ be as in Definition 4.1 and $(\bar{\rho}, \bar{\hat{m}}, \bar{\hat{J}})$ be a smooth solution of (4.2). Assume the pressure $p(\rho_i)$ is given by the γ–law ρ_i^γ, $\gamma > 1$. Let $\mu(\rho_i) = \rho_i^{\frac{s+3}{2}}$ with $\gamma \geq s+2$ and $s \geq -1$, and assume that $\rho_i^\epsilon \in L^\infty([0,T]; L^\infty(\mathbb{T}^3))$, namely there exist $0 < k, N$ such that*

$$0 < k \leq \rho_i^\epsilon \leq N \text{ in } \mathbb{R}^3, 0 < t < T. \tag{4.14}$$

Finally, assume condition (A1) holds. Then, for any $t \in [0,T]$, the stability estimate

$$\Psi(t) \leq (\Psi(0) + \epsilon C(\delta)(\nu^2 + 1)) \exp^{C(\nu^2 + \nu + 1)t} \tag{4.15}$$

holds true, where C is a positive constant depending on T, K the L^1 bound (4.4) for ρ_i^ϵ, assumed to be uniform in ϵ, $\bar{\rho}_i$, and its derivatives. Moreover, if $\Psi(0) \to 0$ as $\epsilon \to 0$, then as $\epsilon \to 0$

$$\sup_{t \in [0,T]} \Psi(t) \to 0.$$

Proof. Starting from the relative entropy calculation (4.6) of Proposition 4.1, and the definition of $\Psi(t)$ we have

$$\begin{aligned}
&\Psi(t) + \frac{1}{2\epsilon} \int_0^t \int_{\mathbb{T}^3} \sum_{i,j=1}^{n} b_{i,j} \rho_i \rho_j |u_i - u_j|^2 dx ds + 2\nu \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \mu(\rho_i) |D(u_i - \bar{u})|^2 dx ds \\
&\quad + \nu \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \lambda(\rho_i) |\operatorname{div}(u_i - \bar{u})|^2 dx ds \\
&\leq \Psi(0) + \sum_{i=1}^{9} \mathcal{J}_i,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{l=1}^{9} \mathcal{J}_l :=& 2\nu \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \rho_i D(\bar{u}) : (v_i - \bar{v}_i) \otimes (u_i - \bar{u}) dx ds \\
&+ 2\nu \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^{n} \rho_i \nabla(\operatorname{div} \bar{u})(\mu'(\rho_i) - \mu'(\bar{\rho}_i))(u_i - \bar{u}) dx ds
\end{aligned}$$

$$
\begin{aligned}
&+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\operatorname{div}\bar u(u_i-\bar u)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar\rho_i}\bar R_i\cdot(\bar u-u_i)dxds-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n p(\rho_i|\bar\rho_i)\operatorname{div}\bar u dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\nabla\bar u:(u_i-\bar u)\otimes(u_i-\bar u)dxds-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i\nabla\bar u:(v_i-\bar v_i)\otimes(v_i-\bar v_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)\cdot((v-\bar v_i)\operatorname{div}\bar u-(u_i-\bar u)\operatorname{div}\bar v_i)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i))((v_i-\bar v_i)\cdot\nabla\operatorname{div}\bar u-(u_i-\bar u)\cdot\nabla\operatorname{div}\bar v_i)dxds.
\end{aligned}
$$

Similarly to what we did for Theorem 3.1, we shall estimate this remainder term by term; the main difference is related with the different scaling we are using here, which in particular forces us to rewrite the first three terms as shown above, and treat them as quadratic terms controlled by Ψ, thanks to the particular choice $\mu_L(\rho_i)=\mu(\rho_i)$ and $\lambda_L(\rho_i)=\lambda(\rho_i)$. Specifically, applying Young's inequality and using Lemma 3.1, and $\mu''(\rho_i)\nabla\rho_i=(s+1)/v_i$, we get:

$$
\begin{aligned}
|\mathcal{J}_1|&\le C\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i|u_i-\bar u|^2dxds+C\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i|v_i-\bar v_i|^2dxds,\\
|\mathcal{J}_2|&\le C\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n h(\rho_i|\bar\rho_i)dxds+C\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i|u_i-\bar u|^2dxds,\\
|\mathcal{J}_3|&\le C(s)\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i|v_i-\bar v_i|^2dxds+C\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i|u_i-\bar u|^2dxds,
\end{aligned}
$$

that is

$$
|\mathcal{J}_1|+|\mathcal{J}_2|+|\mathcal{J}_3|\le C\nu\int_0^t\Psi(s)ds.
$$

We then split $\mathcal{J}_4$ in four terms as follows:

$$
\begin{aligned}
\mathcal{J}_4&=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar\rho_i}\bar R_i(\bar u-u_i)\,dxds\\
&=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \frac{\rho_i}{\bar\rho_i}\bar R_i(\bar u-u_i+u-u)\,dxds\\
&=-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(u_i-u)\cdot\left(\frac{\operatorname{div}\bar T}{\bar\rho}-\frac{\operatorname{div}\bar T_i}{\bar\rho_i}\right)dxds\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(u-\bar u)\cdot\left(\frac{\operatorname{div}\bar T}{\bar\rho}-\frac{\operatorname{div}\bar T_i}{\bar\rho_i}\right)dxds\\
&\quad-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(u_i-u)\cdot\left(\frac{\operatorname{div}\bar D}{\bar\rho}-\frac{\operatorname{div}\bar D_i}{\bar\rho_i}\right)dxds\\
&\quad-\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n \rho_i(u-\bar u)\cdot\left(\frac{\operatorname{div}\bar D}{\bar\rho}-\frac{\operatorname{div}\bar D_i}{\bar\rho_i}\right)dxds
\end{aligned}
$$

$$=:\mathcal{J}_{4,1}+\mathcal{J}_{4,2}+\mathcal{J}_{4,3}+\mathcal{J}_{4,4}.$$

Thanks to Young's inequality and using (2.18) we obtain

$$\begin{aligned}\mathcal{J}_{4,1}&=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(u_i-u)\frac{\operatorname{div}\bar{T}_i}{\bar{\rho}_i}dxds\\&\le\frac{\delta}{4\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i^2|u_i-u|^2dxds+\frac{C\epsilon}{\delta}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left|\frac{\operatorname{div}\bar{T}_i}{\bar{\rho}_i}\right|^2dxds,\end{aligned}$$

and therefore (4.13) implies

$$|\mathcal{J}_{4,1}|\le\frac{1}{8\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2\,dxds+\epsilon CT.$$

For what concerns $\mathcal{J}_{4,2}$ we have:

$$\begin{aligned}\mathcal{J}_{4,2}&=-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(u-\bar{u})\cdot\left(\frac{\operatorname{div}\bar{T}}{\bar{\rho}}-\frac{\operatorname{div}\bar{T}_i}{\bar{\rho}_i}\right)dxds\\&=-\int_0^t\int_{\mathbb{T}^3}(u-\bar{u})\cdot\left(\frac{\rho}{\bar{\rho}}\sum_{i=1}^n\operatorname{div}\bar{T}_i-\sum_{i=1}^n\frac{\rho_i}{\bar{\rho}_i}\operatorname{div}\bar{T}_i\right)dxds\\&=-\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{\bar{\rho}}-\frac{\rho_i}{\bar{\rho}_i\rho}\right)\rho(u-\bar{u})\cdot\operatorname{div}\bar{T}_i\,dxds\\&\le\int_0^t\int_{\mathbb{T}^3}\rho|u-\bar{u}|^2\,dxds+C\int_0^t\int_{\mathbb{T}^3}\rho\sum_{i=1}^n\left|\frac{1}{\bar{\rho}}-\frac{\rho_i}{\bar{\rho}_i\rho}\right|^2dxds.\end{aligned}$$

Then we use the uniform bounds (4.14) for ρ_i to conclude

$$\rho|u-\bar{u}|^2=\frac{1}{\rho}\left|\sum_{i=1}^n\rho_i(u_i-\bar{u})\right|^2\le\frac{n}{\rho}\sum_{i=1}^n\rho_i^2|u_i-\bar{u}|^2\le\frac{nN}{k}\sum_{i=1}^n\rho_i|u_i-\bar{u}|^2,$$

and

$$\sum_{i=1}^n\left|\frac{1}{\bar{\rho}}-\frac{\rho_i}{\bar{\rho}_i\rho}\right|^2=\sum_{i=1}^n\left|\frac{\rho-\bar{\rho}}{\rho\bar{\rho}}+\frac{\bar{\rho}_i-\rho_i}{\bar{\rho}_i\rho}\right|^2\le C\sum_{i=1}^n|\rho_i-\bar{\rho}_i|^2\le C\sum_{i=1}^n h(\rho_i|\bar{\rho}_i),$$

where the last inequality follows again from the uniform bound (4.14); see also [17,22]. Therefore, there exists $C:=C(n,k,N)$ such that:

$$\mathcal{J}_{4,2}\le C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\rho_i|u_i-\bar{u}|^2+|\rho_i-\bar{\rho}_i|^2\right)dxds\le C\int_0^t\Psi(s)ds.$$

The terms $\mathcal{J}_{4,3}$ and $\mathcal{J}_{4,4}$ can be controlled similarly. Indeed, for $\mathcal{J}_{4,3}$ we obtain

$$\begin{aligned}\mathcal{J}_{4,3}&=\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\frac{\sqrt{\delta}}{\sqrt{8\epsilon}}\rho_i(u_i-u)\frac{\nu\sqrt{8\epsilon}}{\sqrt{\delta}}\frac{\operatorname{div}\bar{D}_i}{\bar{\rho}_i}dxds\\&\le\frac{\delta}{8\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i^2|u_i-u|^2dxds+\frac{C\epsilon}{\delta}\nu^2\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left|\frac{\operatorname{div}\bar{D}_i}{\bar{\rho}_i}\right|^2dxds\end{aligned}$$

$$\leq \frac{1}{8\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2\,dxds+C\epsilon\nu^2T,$$

thanks to (2.18), (4.13), and Young's inequality. Concerning $\mathcal{J}_{4,4}$ we get

$$\begin{aligned}\mathcal{J}_{4,4}&=\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i(u-\bar u)\cdot\left(\frac{\operatorname{div}\bar D}{\bar\rho}-\frac{\operatorname{div}\bar D_i}{\bar\rho_i}\right)dxds\\&=\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{\bar\rho}-\frac{\rho}{\bar\rho_i\bar\rho}\right)\rho(u-\bar u)\operatorname{div}\bar D_i dxds\\&\leq C\int_0^t\int_{\mathbb{T}^3}\rho^2|u-\bar u|^2dxds+C\nu^2\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(\frac{1}{\bar\rho}-\frac{\rho_i}{\bar\rho_i\rho}\right)^2dxds\\&\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|u_i-\bar u|^2dxds+C\nu^2\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n h(\rho_i|\bar\rho_i)dxds\\&\leq C(\nu^2+1)\int_0^t\Psi(s)ds.\end{aligned}$$

Summarizing, the term $\mathcal{J}_4$ satisfies

$$|\mathcal{J}_4|\leq\frac{1}{4\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds+C(\nu^2+1)\int_0^t\psi(s)ds+C\epsilon(\nu^2+1)T.$$

The quadratic terms $\mathcal{J}_5$ $\mathcal{J}_6$, and $\mathcal{J}_7$ are treated in a standard way, as in Theorem 3.1:

$$\begin{aligned}|\mathcal{J}_5|&\leq\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|p(\rho_i|\bar\rho_i)\operatorname{div}\bar u|\,dxds\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n h(\rho_i|\bar\rho_i)dxds;\\|\mathcal{J}_6|&\leq\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i\nabla\bar u:(u_i-\bar u)\otimes(u_i-\bar u)|\,dxds\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|u_i-\bar u|^2dxds;\\|\mathcal{J}_7|&\leq\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i\nabla\bar u:(v_i-\bar v_i)\otimes(v_i-\bar v_i)|\,dxds\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|v_i-\bar v_i|^2dxds.\end{aligned}$$

Moreover, concerning $\mathcal{J}_8$, recalling the relation $\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i=\frac{s+1}{2}(v_i-\bar v_i)$, we have

$$\begin{aligned}|\mathcal{J}_8|&\leq\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu''(\rho_i)\nabla\rho_i-\mu''(\bar\rho_i)\nabla\bar\rho_i)||((v-\bar v_i)\operatorname{div}\bar u-(u_i-\bar u)\operatorname{div}\bar v_i)|dxds\\&\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|v_i-\bar v_i|^2dxds+C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\rho_i|u_i-\bar u|^2dxds,\end{aligned}$$

using again Young's inequality. Finally, for $\mathcal{J}_9$, we use Lemma 3.1 as in Section 3 to conclude

$$\begin{aligned}|\mathcal{J}_9|&\leq\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu'(\rho_i)-\mu'(\bar\rho_i))((v-\bar v_i)\cdot\nabla\operatorname{div}\bar u|dxds\\&\quad+\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n|\rho_i(\mu'(\rho_i)+\mu'(\bar\rho_i))(u_i-\bar u)\cdot\nabla\operatorname{div}\bar v_i|dxds\end{aligned}$$

$$\leq C\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\left(h(\rho_i|\bar{\rho}_i)+\rho_i\left|\frac{m_i}{\rho_i}-\frac{\bar{m}_i}{\bar{\rho}_i}\right|^2+\rho_i\left|\frac{J_i}{\rho_i}-\frac{\bar{J}_i}{\bar{\rho}_i}\right|^2\right)dxds.$$

Collecting all estimates above, the relative entropy inequality becomes:

$$\begin{aligned}&\Psi(t)+\frac{1}{4\epsilon}\int_0^t\int_{\mathbb{T}^3}\sum_{i,j=1}^n b_{i,j}\rho_i\rho_j|u_i-u_j|^2dxds+2\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\mu(\rho_i)|D(u_i-\bar{u})|^2dxds\\&\quad+\nu\int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^n\lambda(\rho_i)|\operatorname{div}(u_i-\bar{u})|^2dxds\\&\leq\Psi(0)+\epsilon C(\delta)(\nu^2+1)+C(\nu^2+\nu+1)\int_0^t\Psi(s)ds,\end{aligned}$$

where $C:=C(s,k,\delta,n,N,T)$, and the Gronwall lemma concludes the proof. ∎

Remark 4.1. As already pointed out in Section 3, see in particular Remark 3.1, it is worth observing that that the stability estimate (4.15) is consistent with the one obtained in [19] for the inviscid case. Indeed as $\nu\to 0^+$ we obtain:

$$\Psi(t)\leq(\Psi(0)+\epsilon C(\delta))\exp^{Ct}$$

which is exactly the one obtained by the authors in [19] for the Euler-Korteweg case.

REFERENCES

[1] N.J. Alves and A.E. Tzavaras, *The relaxation limit of bipolar fluid models*, Discrete Contin. Dyn. Syst., 42(1):211–237, 2022. 1

[2] P. Antonelli, G. Cianfarani Carnevale, C. Lattanzio, and S. Spirito, *Relaxation limit from the quantum Navier–Stokes equations to the quantum drift–diffusion equation*, J. Nonlinear Sci., 31:71, 2021. 1

[3] P. Antonelli and P. Marcati, *On the finite energy weak solutions to a system in quantum fluid dynamics*, Commun. Math. Phys., 287:657–686, 2009. 1

[4] P. Antonelli and P. Marcati, *The quantum hydrodynamics system in two space dimensions*, Arch. Ration. Mech. Anal., 203:499–527, 2012. 1

[5] P. Antonelli and S. Spirito, *Global existence of finite energy weak solutions of quantum Navier–Stokes equations*, Arch. Ration. Mech. Anal., 225:1161–1199, 2017. 1

[6] P. Antonelli and S. Spirito, *Global existence of weak solutions to the Navier–Stokes–Korteweg equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 39(1):171–200, 2022. 1

[7] R. Bianchini, *Strong convergence of a vector–BGK model to the incompressible Navier–Stokes equations via the relative entropy method*, J. Math. Pures Appl., 132(9):280–307, 2019. 1

[8] F. Bouchut, *A reduced stability condition for nonlinear relaxation to conservation laws*, J. Hyperbolic Differ. Equ., 1:149–170, 2004. 1

[9] D. Bresch, M. Gisclon, and I. Lacroix–Violet, *On Navier-Stokes-Korteweg and Euler-Korteweg systems: application to quantum fluids models*, Arch. Ration. Mech. Anal., 233(3):975–1025, 2019. 1, 1, 1, 2.3, 3.1, 3.1, 4, 4.1

[10] R. Caflisch and G. Papanicolau, *The fluid-dynamical limit of a nonlinear model Boltzmann equation*, Commun. Pure Appl. Math., 32:589–616, 1979. 1

[11] J.A. Carrillo, Y. Peng, and A. Wróblewska-Kamińska, *Relative entropy method for the relaxation limit of hydrodynamic models*, Netw. Heterog. Media, 15(3):369–387, 2020. 1

[12] G.Q. Chen, C.D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Commun. Pure Appl. Math., 47:787–830, 1994. 1

[13] G. Cianfarani Carnevale and C. Lattanzio, *High friction limit for Euler-Korteweg and Navier-Stokes-Korteweg models via relative entropy approach*, J. Differ. Equ., 269:10495–10526, 2020. 1, 1, 1, 2.3, 3, 3.1, 3.1

[14] J-.F. Coulombel and T. Goudon, *The strong relaxation limit of the multidimensional isothermal Euler equations*, Trans. Amer. Math. Soc., 359:637–648, 2007. 1

[15] D. Donatelli and P. Marcati, *Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems*, Trans. Amer. Math. Soc., 356:2093–2121, 2004. 1

[16] E. Feireisl and T. Tang, *On a singular limit for the stratified compressible Euler system*, Asymptot. Anal., 114(1-2):59–72, 2019. 1

[17] J. Giesselmann, C. Lattanzio, and A.E. Tzavaras, *Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics*, Arch. Ration. Mech. Anal., 223(3):1427–1484, 2017. 1, 3.1, 4.1

[18] J. Giesselmann and A.E. Tzavaras, *Stability properties of the Euler-Korteweg system with non-monotone pressures*, Appl. Anal., 96:1528–1546, 2017. 1

[19] X. Huo, A. Jüngel, and A.E. Tzavaras, *High-friction limits of Euler flows for multicomponent systems*, Nonlinearity, 32(8):2875–2913, 2019. 1, 1, 1, 1, 1, 2, 2, 2.1, 2.2, 3, 3.1, 4, 4.1, 4.1

[20] A. Jüngel and Y.-J. Peng, *A hierarchy of hydrodynamics models for plasmas: zero-relaxation-time limits*, Commun. Partial Differ. Equ., 24:1007–1033, 1999. 1

[21] C. Lattanzio, *On the 3-D bipolar isentropic Euler-Poisson model for semiconductors and the drift-diffusion limit*, Math. Models Meth. Appl. Sci., 10(3):351–360, 2000. 1

[22] C. Lattanzio and A.E. Tzavaras, *Relative entropy in diffusive relaxation*, SIAM J. Math. Anal., 45(3):1563–1584, 2013. 1, 3, 3.1, 4.1

[23] C. Lattanzio and A.E. Tzavaras, *From gas dynamics with large friction to gradient flows describing diffusion theories*, Commun. Partial Differ. Equ., 42(2):261–290, 2017. 1, 3, 3.1

[24] L. Ostrowski and C. Rohde, *Compressible multicomponent flow in porous media with Maxwell–Stefan diffusion*, Math. Meth. Appl. Sci., 43(7):4200–4221, 2020. 1, 1, 1, 1, 1, 1, 2.1, 2.1, 2.1, 2.2, 3, 3.1, 3.1, 3.1

[25] A. Tzavaras, *Relative entropy in hyperbolic relaxation*, Commun. Math. Sci., 3:119–132, 2005. 1

[26] J. Wesslingh and R. Krishna, *Mass Transfer in Multicomponent Mixtures*, Delft University Press, Delft, Netherlands, 2000. 1

[27] W.-A. Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Ration. Mech. Anal., 172:247–266, 2004. 1