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A FULLY DISCRETE LOW-REGULARITY INTEGRATOR FOR THE KORTEWEG-DE VRIES EQUATION*

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Abstract. In this paper we propose a fully discrete low-regularity integrator for the Korteweg-de Vries equation on the torus. This is an explicit scheme and can be computed with a complexity of $\mathcal{O}(N\log N)$ operations by fast Fourier transform, where N is the degrees of freedom in the spatial discretization. We prove that the scheme is first-order convergent in both time and space variables in H^γ-norm for $H^{\gamma+1}$ initial data under Courant-Friedrichs-Lewy condition $N\geq 1/\tau$, where τ denotes the temporal step size. We also carry out numerical experiments that illustrate the convergence behavior.

Keywords. The KdV equation; low regularity; fully discrete; fast Fourier transform.

AMS subject classifications. 65M12; 65M15; 35Q55.

1. Introduction

In this paper, we are concerned with the numerical solution of the KdV equation

$$\begin{cases}\partial_t u(t,x)+\partial_x^3 u(t,x)=\dfrac{1}{2}\partial_x(u(t,x))^2, & t>0,\ x\in\mathbb{T},\\ u(0,x)=u^0(x), & x\in\mathbb{T},\end{cases}\tag{1.1}$$

on the one-dimensional torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ with initial data $u^0\in H^{s_0}(\mathbb{T})$, where $u(t,x)$ is a real-valued function and $0\leq s_0<\infty$ indicates the regularity of the initial data.

The Korteweg-de Vries (KdV) equation is a model describing the evolution of weakly nonlinear long waves. It arises from the shallow water waves propagating in a channel [14]. This equation has many important properties and applications, and thus has attracted a lot of attention in the field of mathematics and physics. There is a large amount of work devoted to the wellposedness of the KdV equation, in particular, in function spaces of low regularity. For example, for the global wellposedness of the KdV equation for initial data in H^s with $s\geq -1$, we refer readers to, e.g. [1, 11, 13]. It is also very popular to study the numerical solution of the KdV equation, especially the issues about the error estimates, stability and the convergence rate of the numerical solution. So far, various numerical discretization methods have been established, such as finite difference methods [2, 3, 7, 22], operator splitting [8–10, 21], and exponential integrators [4, 5].

Finite-difference formats are easy to construct, but usually require the solutions to be sufficiently smooth. For finite difference methods on the KdV equation, schemes proposed in [22] are conditionally stable and require the stability condition on the time step τ and the mesh size h, $\tau=\mathcal{O}(h^3)$. Thereafter, in order to extend the stability domain, an unconditionally stable implicit scheme was derived in [3]. In the case where the initial value has sufficiently high regularity $u^0\in H^3$, a fully discrete finite difference scheme was proved to be convergent under Courant-Friedrichs-Lewy (CFL) condition $\tau=\mathcal{O}(h^{\frac{3}{2}})$ in [7]. However, no convergence rate was obtained there. By considering the

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explicit Rusanov scheme for the hyperbolic flux term and a 4-point θ-scheme for the dispersive term, the first-order convergence rate was obtained for solution in $H^6(\mathbb{R})$ under a hyperbolic CFL condition $\tau = \mathcal{O}(h)$ when $\theta \geq \frac{1}{2}$ and under an 'Airy' CFL condition $\tau = \mathcal{O}(h^3)$ when $\theta < \frac{1}{2}$ in [2]. The operator splitting methods also require the solutions to have a certain smoothness. For operator splitting methods on the KdV equation, the Strang splitting methods in [8,9] converge with the expected rates if the initial data are sufficiently regular. The schemes have first- and second-order convergence rates in H^γ for rough initial data in $H^{\gamma+3}$ and $H^{\gamma+5}$, respectively.

Exponential integrators are effective in relaxing the regularity restriction. They have been widely used for solving partial differential equations including hyperbolic and parabolic problems, whose computational benefits have been exhibited by some typical applications (see e.g. [4,5]). For the KdV equation, there has been some literature devoted to the approximation of the numerical solution by exponential integrator methods. For example, an exponential-type integrator was proposed to obtain the first-order convergence in H^1 with initial data $u^0 \in H^3$ by twisting technique [6]. Based on exponential integrator, the so-called low-regularity integrators are proposed to further reduce regularity requirements. The first- and second-order convergence in H^γ were investigated under $H^{\gamma+1}$-data and $H^{\gamma+3}$-data by introducing the embedded exponential-type low-regularity integrators [24, 25]. Very recently, the convergence in L^2 was achieved under H^γ-data in [16]. The scheme preserving the underlying geometric structure of the continuous problem was introduced in [17]. For the modified Korteweg-de Vries equation, the first-order accuracy was obtained by requiring the boundedness of one additional spatial derivative of the solution [18].

All the schemes considered above are semi-discrete, for fully discrete schemes of KdV equation, a Lawson-type exponential integrator was introduced to get the first-order convergence rate in both time and space under the CFL condition $\tau = \mathcal{O}(h)$ for initial data in H^3 [19]. A fully discrete symmetric exponential-type low-regularity integrator for nonlinear Klein-Gordon equation was proposed in [23]. Fully discrete schemes for the nonlinear Schrödinger equation were designed in [15, 20].

Motivated by the semi-discretized (in time) scheme of the KdV equation proposed in [24], our purpose in the present paper is to construct a fully discrete low-regularity integrator and obtain the first-order convergence in both time and space in H^γ-norm for initial data in $H^{\gamma+1}$. The main difficulty comes from the fact that the symmetry is broken, making it necessary to do a more detailed design of the temporal discretization. For spatial discretization, we construct the spatial low-regularity exponential integrator based on frequency truncation, which has been used to derive the fully discrete integrator for nonlinear Schrödinger equation in [15]. One main difficulty of full discretization is the proof of stability. After frequency truncation, the terms containing high frequencies are not easy to control because there is a derivative in the nonlinear term of the KdV equation. Therefore, we have to use the CFL condition to get the desired stability estimate.

The rest of this paper is arranged as follows. First, in Section 2 we give some preliminaries, including the notations and some foundational lemmas. We also present our fully discrete low-regularity integrator and state the main convergence result of this paper. In Section 3, we present the construction of the numerical scheme. In Section 4, we prove the main convergence theorem of this paper, including error and stability estimates. Finally, we provide numerical experiments to support our theoretical analysis in Section 5.

2. Preliminaries and main result

In this section, we give some notations and some foundational lemmas. We also state the main convergence result of this paper.

2.1. Some notations. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{T})$, i.e.,

$$\langle f,g\rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}\,dx, \qquad \|f\|_{L^2(\mathbb{T})} = \sqrt{\langle f,f\rangle}.$$

For $s\geq 0$, we adopt the standard Sobolev space $H^s(\mathbb{T})$ with the following norm

$$\|f\|^2_{H^s(\mathbb{T})} = \|J^s f\|^2_{L^2(\mathbb{T})} = 2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^s|\hat{f}_k|^2,$$

where the operator $J^s=(1-\partial_x^2)^{\frac{s}{2}}$ is defined by $J^s f=\sum_{k\in\mathbb{Z}}(1+k^2)^{\frac{s}{2}}\hat{f}_k e^{ikx}$ and $\hat{f}_k$ are the Fourier coefficients of function f,

$$\hat{f}_k = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ixk} f(x)\,dx.$$

Then, the Fourier inverse transform is correspondingly denoted as

$$\big(\mathcal{F}^{-1}\hat{f}_k\big)(x) = \sum_{k\in\mathbb{Z}} e^{ixk}\hat{f}_k. \tag{2.1}$$

For convenience, we employ some additional notations, which will be used frequently in the following. At first, we define the operator ∂_x^{-1} for function f on $\mathbb{T}$ as

$$\widehat{(\partial_x^{-1}f)}_k = \begin{cases} (ik)^{-1}\hat{f}_k, & \text{when } k\neq 0,\\ 0, & \text{when } k=0. \end{cases} \tag{2.2}$$

Furthermore, let P_N and $P_{>N}$ be projection operators, which keep the frequencies in the domains $|k|\leq N$ and $|k|>N$, respectively. More specifically, for an arbitrary positive integer N, P_N and $P_{>N}$ can be written as

$$P_N f = \sum_{k=-N}^{N} e^{ixk}\hat{f}_k, \qquad P_{>N}f = f - P_N f. \tag{2.3}$$

In particular, for the zero-mode operator, we have

$$P_0 f = \hat{f}_0 = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,dx.$$

Let $\mathbb{P}$ be the orthogonal projection onto the space of mean zero functions, i.e.

$$\mathbb{P}f(x) = f(x) - P_0 f. \tag{2.4}$$

In view of spatial discretization in the numerical scheme, for every product of two $(2N+1)$-term Fourier series, we use a $(4N+1)$-point fast Fourier transform (FFT) and truncate it to obtain the $(2N+1)$-term Fourier series again. More details can be seen as follows.

Denote by I_{2N} the $(4N+1)$-point trigonometric interpolation operator

$$I_{2N}f(x)=\sum_{k=-2N}^{2N}\mathrm{e}^{ikx}\tilde{f}_k, \tag{2.5}$$

where

$$\tilde{f}_k=\frac{1}{4N+1}\sum_{n=-2N}^{2N}\mathrm{e}^{-ikx_n}f(x_n),\quad x_n=\frac{2\pi n}{4N+1},\quad n=-2N,\cdots,2N,$$

can be computed with complexity $\mathcal{O}(N\log N)$ by FFT. From the above definition, we observe that if the Fourier coefficients $\hat{f}_k$ of the function f satisfy $\hat{f}_k=0$ for $|k|>2N$, then $I_{2N}f=f$ and $\tilde{f}_k=\hat{f}_k$.

Let S_N be the space that consists of all functions $f\in L^2(\mathbb{T})$ satisfying $\hat{f}_k=0$ for $|k|>N$. If f and $g\in S_N$, we store $\hat{f}_k=\hat{g}_k=0$ for $N<|k|\le 2N$ in the computer, then $f=I_{2N}f$ and $g=I_{2N}g$ can be computed by (2.5) and FFT with cost $\mathcal{O}(N\log N)$. For the product of f and g, we have the Fourier coefficients $\widehat{(fg)}_k=0$ for $|k|>2N$, which implies that $fg=I_{2N}(fg)$. Hence, computing the Fourier coefficients of $P_NI_{2N}(fg)\in S_N$ is of $\mathcal{O}(N\log N)$ complexity. So, we conclude that for f and $g\in S_N$, computing the Fourier coefficients of $P_N(fg)\in S_N$ is also of $\mathcal{O}(N\log N)$ complexity.

In the sequal, we use $A\lesssim B$ or $B\gtrsim A$ to denote the statement that $A\le CB$ for some absolute constant $C>0$ which may vary from line to line but independent of τ, N and n, and we denote $A\sim B$ for $A\lesssim B\lesssim A$.

2.2. Some foundational lemmas. We first state the following estimates which will be used frequently below. The first estimate is the Kato-Ponce inequality which was originally proved in [12] and other estimates were given in [24].

LEMMA 2.1. *For $\gamma>\frac{1}{2}$, the following inequalities hold:*

(1) *For any f, $g\in H^\gamma$, we have*

$$\left\|fg\right\|_{H^\gamma}\lesssim\|f\|_{H^\gamma}\|g\|_{H^\gamma}. \tag{2.6}$$

(2) *For any $f\in H^\gamma$, $g\in H^{\gamma+1}$, we have*

$$\left\langle J^\gamma\partial_x(fg),J^\gamma f\right\rangle\lesssim\|f\|_{H^\gamma}^2\|g\|_{H^{\gamma+1}}. \tag{2.7}$$

Now we present a lemma for the nonlinear term on $[t_n,t_{n+1}]$, which will be frequently used later. The proof can be seen in [24].

LEMMA 2.2. *For $\tau>0$, we have*

(1) *For any $t\in[0,\tau)$, $f(t,\cdot)$, $g(t,\cdot)\in L^2$, and $\hat{f}(t,0)=\hat{g}(t,0)=0$. Then for any $t_n\ge 0$, we have*

$$\begin{aligned}
&\int_0^\tau \mathrm{e}^{(t_n+t)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+t)\partial_x^3}f(t)\cdot\mathrm{e}^{-(t_n+t)\partial_x^3}g(t)\right)dt\\
=&\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f(\tau)\cdot\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}g(\tau)\right)\\
&-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}f(0)\cdot\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}g(0)\right)\\
&-\frac{1}{3}\int_0^\tau \mathrm{e}^{(t_n+t)\partial_x^3}\Big(\mathrm{e}^{-(t_n+t)\partial_x^3}\partial_x^{-1}\partial_tf(t)\cdot\mathrm{e}^{-(t_n+t)\partial_x^3}\partial_x^{-1}g(t)\\
&\qquad+\mathrm{e}^{-(t_n+t)\partial_x^3}\partial_x^{-1}f(t)\cdot\mathrm{e}^{-(t_n+t)\partial_x^3}\partial_x^{-1}\partial_tg(t)\Big)\,dt.
\end{aligned}$$

(2) *Let* $t_n \geq 0$, $\gamma > \frac{1}{2}$, f, $g \in L^\infty((0,T);H^\gamma)$, *and* $\hat{f}(t_n,0) = \hat{g}(t_n,0) = 0$. *Then the following inequality holds*

$$\Big\| \int_0^\tau \mathrm{e}^{(t_n+t)\partial_x^3} \partial_x \big(\mathrm{e}^{-(t_n+t)\partial_x^3} f(t_n) \cdot \mathrm{e}^{-(t_n+t)\partial_x^3} g(t_n) \big)\, dt \Big\|_{H^\gamma} \lesssim \sqrt{\tau} \|f\|_{L^\infty((0,T);H^\gamma)} \|g\|_{L^\infty((0,T);H^\gamma)}.$$

2.3. The main result of the fully discrete integrator. For fixed time T and a positive integer L, we denote by τ the temporal step size given by $\tau = T/L$. Correspondingly, $t_n = n\tau$, $n = 1, \cdots, L$ are discretization points in the time interval $[0,T]$. Then, we construct the fully discrete low-regularity integrator for Equation (1.1) in the following way:

$$u_{\tau,N}^{n+1} = \Phi_{\tau,N}^n (u_{\tau,N}^n), \tag{2.8}$$

where $u_{\tau,N}^0 = P_N I_{2N} u^0$ and the operator $\Phi_{\tau,N}^n$ is defined by

$$\begin{aligned} \Phi_{\tau,N}^n(f) = {} & \mathrm{e}^{-\tau\partial_x^3} f + \frac{1}{6} P_N \left(\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \right)^2 - \frac{1}{6} \mathrm{e}^{-\tau\partial_x^3} P_N \left(\partial_x^{-1} f \right)^2 \\ & + \frac{1}{18} \mathbb{P} \Big[P_N \Big(\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \cdot \partial_x^{-1} P_N \big(\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \big)^2 \Big) \Big] \\ & - \frac{1}{18} \mathbb{P} \Big[P_N \Big(\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \cdot \mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} P_N \big(\partial_x^{-1} f \big)^2 \Big) \Big] \\ & - \frac{1}{54} \partial_x^{-1} P_N \Big[\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \cdot P_N \big(\mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \big)^2 \Big] \\ & + \frac{1}{54} \mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} P_N \Big[\partial_x^{-1} f \cdot P_N \big(\partial_x^{-1} f \big)^2 \Big] \\ & - \frac{\tau}{18} \mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} P_N \Big[f \cdot P_N \big(f \big)^2 \Big] + \frac{\tau}{6} \mathrm{e}^{-\tau\partial_x^3} \partial_x^{-1} f \cdot P_0(f)^2, \qquad f \in S_N. \end{aligned} \tag{2.9}$$

Then, we observe that the numerical flow $\Phi_{\tau,N}^n$ is a mapping from f to $\Phi_{\tau,N}^n(f) \in S_N$.

The scheme is explicit and can be computed with cost $\mathcal{O}(N \log N)$ by FFT. Indeed, for the initial data $u_{\tau,N}^0 = P_N I_{2N} u^0$, we deduce first that it can be computed by FFT from the above analysis. Then, we note that the scheme consists only of the forms $\mathrm{e}^{i\tau\partial_x^2} f$ and $P_N(fg)$, and these terms can be computed with computational cost $\mathcal{O}(N \log N)$ by FFT.

Moreover, it preserves the mean-value of the solution of the KdV equation (1.1) at the discrete level, i.e.

$$\int_{\mathbb{T}} u_{\tau,N}^n(x) dx \equiv \int_{\mathbb{T}} u^0(x) dx, \quad 0 \leq n\tau \leq T.$$

We now present the convergence theorem of the fully discrete integrator in (2.9), which will be proved in Section 4.

THEOREM 2.1. *Assume that* $u^0 \in H^{\gamma+1}(\mathbb{T})$ *for some* $\gamma > \frac{1}{2}$, *and* $u(t,\cdot)$ *is the solution to* (1.1). *Given a fixed time* $T > 0$, *let* $u_{\tau,N}^n$ *be the numerical solution of the scheme* (2.9) *with* $\hat{u}^0 = 0$ *and* $N \geq 1/\tau$. *Then there exist constants* τ_0, $C > 0$ *such that for any* $0 < \tau \leq \tau_0$ *we have*

$$\|u(t_n,\cdot) - u_{\tau,N}^n\|_{H^\gamma} \leq C(\tau + N^{-1}), \quad 0 \leq n\tau \leq T, \tag{2.10}$$

where the constants τ_0 *and* C *depend only on* T *and* $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

3. The construction of the low-regularity integrator

In this section, we derive the fully discrete low-regularity integrator. The process is based on the frequency truncation and harmonic analysis techniques. In particular, without loss of generality, we shall assume the zero-mode of initial data u^0 to be zero. Then $\hat{u}(t,0)=0$ for any $t\geq 0$ by the conservation law.

Let τ be the time step. First we introduce the twisted function $v=\mathrm{e}^{\partial_x^3 t}u$. Then by the Duhamel's formula initiated from $t_n=n\tau$ we have:

$$\begin{aligned} v(t_n+t) &= v(t_n)+\frac{1}{2}\int_0^t \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\right)^2 ds \\ &= v(t_n)+\frac{1}{2}F_n(v(t_n+t),t), \end{aligned} \tag{3.1}$$

where F_n is defined as

$$F_n(v,t)=\int_0^t \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(s)\right)^2 ds. \tag{3.2}$$

Inserting the Duhamel's formula $v(t_n+s)$ into (3.1) and taking $t=\tau$ leads to the transformed formulation

$$v(t_{n+1})=v(t_n)+\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\left(v(t_n)+\frac{1}{2}F_n(v(t_n+s),s)\right)\right)^2 ds.$$

First of all, for $n\geq 0$, we approximate $F_n\big(v(t_n+t),s\big)$ by $F_n\big(v(t_n),s\big)$ in the above integral to get

$$v(t_{n+1})\approx v(t_n)+\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\left(v(t_n)+\frac{1}{2}F_n(v(t_n),s)\right)\right)^2 ds.$$

Our aim is to get the first-order convergence in time and space. In order to make the analysis simpler, we will get rid of higher-order terms. Let

$$v(t_{n+1})=v(t_n)+\frac{1}{2}\big(\mathrm{I}(t_n)+\mathrm{II}(t_n)+\mathcal{R}_1(v)\big), \tag{3.3}$$

where

$$\begin{aligned} \mathrm{I}(t_n) &:= \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^2 ds, \\ \mathrm{II}(t_n) &:= \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}F_n(v(t_n),s)\right) ds, \end{aligned}$$

and $\mathcal{R}_1(v)$ is the remainder term. Further, to obtain the fully discrete integrator, we will apply the high and low frequency decomposition and consider the terms $\mathrm{I}(t_n)$ and $\mathrm{II}(t_n)$ one by one. For $\mathrm{I}(t_n)$, we divide it into two parts

$$\mathrm{I}(t_n)=\mathrm{I}_1(t_n)+\mathcal{R}_2(v), \tag{3.4}$$

where

$$\mathrm{I}_1(t_n)=\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^2 ds,$$

and the remainder is

$$\mathcal{R}_2(v)=\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_{>N}\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^2 ds, \tag{3.5}$$

P_N and $P_{>N}$ are the operators given in (2.3). Lemma 2.2 (1) implies that the term $\mathrm{I}_1(t_n)$ is integrable and can be written as

$$\mathrm{I}_1(t_n)=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\right)^2-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}P_N\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\right)^2. \tag{3.6}$$

Hence, combining (3.4) with (3.6), we have

$$\mathrm{I}(t_n)=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\right)^2-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}P_N\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\right)^2+\mathcal{R}_2(v). \tag{3.7}$$

A similar approach to the above can be applied to handle the term $\mathrm{II}(t_n)$. We decompose $\mathrm{II}(t_n)$ into the low frequency part $\mathrm{II}_1(t_n)$ and the remainder $\mathcal{R}_3(v)$,

$$\mathrm{II}(t_n)=\mathrm{II}_1(t_n)+\mathcal{R}_3(v), \tag{3.8}$$

where

$$\begin{aligned}\mathrm{II}_1(t_n)&=\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\cdot\mathrm{e}^{-(t_n+s)\partial_x^3}F_n(v(t_n),s)\right)ds,\\ \mathcal{R}_3(v)&=\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_{>N}\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\cdot\mathrm{e}^{-(t_n+s)\partial_x^3}F_n(v(t_n),s)\right)ds.\end{aligned} \tag{3.9}$$

We estimate $\mathrm{II}_1(t_n)$ first and postpone the estimation of the remainder term $\mathcal{R}_3(v)$ to Section 4. By applying Lemma 2.2 (1), we express $\mathrm{II}_1(t_n)$ as

$$\mathrm{II}_1(t_n)=\mathrm{II}_{11}+\mathrm{II}_{12}+\mathrm{II}_{13},$$

where

$$\mathrm{II}_{11}=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right); \tag{3.10}$$

$$\mathrm{II}_{12}=-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}P_N\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}F_n\big(v(t_n),0\big)\right); \tag{3.11}$$

$$\mathrm{II}_{13}=-\frac{1}{3}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}\partial_s F_n\big(v(t_n),s\big)\right)ds. \tag{3.12}$$

We truncate the frequency of F_n in (3.10) into low and high frequencies and obtain

$$\mathrm{II}_{11}=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}P_N F_n\big(v(t_n),\tau\big)\right)+\mathcal{R}_4(v),$$

where $\mathcal{R}_4(v)$ is remainder term defined as

$$\mathcal{R}_4(v)=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}P_{>N}F_n\big(v(t_n),\tau\big)\right). \tag{3.13}$$

According to definition, $F_n(v,s)$ is integrable and can be written into the following form by Lemma 2.2

$$F_n(v,s)=\frac{1}{3}\mathrm{e}^{(t_n+s)\partial_x^3}\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v\right)^2-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v\right)^2.$$

Inserting the above formula into (3.10), we rewrite the term II_{11} as

$$\begin{aligned}\mathrm{II}_{11}=&\mathcal{R}_4(v)+\frac{1}{9}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\partial_x^{-1}P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big)\\&-\frac{1}{9}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big),\end{aligned}\tag{3.14}$$

where $\mathbb{P}$ is defined in (3.15) and

$$\begin{aligned}c_n=&\frac{1}{18\pi}\int_{\mathbb{T}}\mathrm{e}^{t_{n+1}\partial_x^3}\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\partial_x^{-1}P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big)\,dx\\&-\frac{1}{18\pi}\int_{\mathbb{T}}\mathrm{e}^{t_{n+1}\partial_x^3}\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big)\,dx.\end{aligned}\tag{3.15}$$

Then, we rewrite (3.14) as

$$\begin{aligned}\mathrm{II}_{11}=&c_n+\mathcal{R}_4(v)+\frac{1}{9}\mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\partial_x^{-1}P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big)\Big]\\&-\frac{1}{9}\mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\Big)\Big].\end{aligned}\tag{3.16}$$

Since $F_n\big(v(t_n),0\big)=0$, we have

$$\mathrm{II}_{12}=0.\tag{3.17}$$

Now we consider the term II_{13} in (3.12). Note that

$$\partial_s F_n(v(t_n),s)=\mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2,$$

then by the definition of operator ∂_x^{-1} in (2.2), we get

$$\begin{aligned}\partial_x^{-1}\partial_s F_n(v(t_n),s)&=\mathbb{P}\Big[\mathrm{e}^{(t_n+s)\partial_x^3}\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2\Big]\\&=\mathrm{e}^{(t_n+s)\partial_x^3}\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2-\frac{1}{2\pi}\int_{\mathbb{T}}\big(v(t_n)\big)^2\,dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathrm{II}_{13}=&\frac{\tau}{6\pi}\partial_x^{-1}P_Nv(t_n)\int_{\mathbb{T}}\big(v(t_n)\big)^2\,dx\\&-\frac{1}{3}\int_0^\tau\mathrm{e}^{(t_n+s)\partial_x^3}P_N\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2\Big)\,ds\\\triangleq&J_1+J_2.\end{aligned}\tag{3.18}$$

We need only to consider the second term J_2. First, we calculate the zero Fourier frequency of J_2 and obtain

$$\widehat{(J_2)}_0=-\frac{1}{6\pi}\int_0^\tau\int_{\mathbb{T}}\mathrm{e}^{(t_n+s)\partial_x^3}\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2\Big)\,dxds.$$

We truncate the high and low frequency domain of $\widehat{(J_2)}_0$ as

$$\widehat{(J_2)}_0=-\frac{1}{6\pi}\int_0^\tau\int_{\mathbb{T}}\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot P_N\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2\,dxds$$

$$-\frac{1}{6\pi}\int_0^\tau\int_{\mathbb{T}} \mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot P_{>N}\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\big)^2\,dxds$$
$$=: \widehat{J_3}+\mathcal{R}_5(v).$$

We can directly calculate $\widehat{J_3}$. Integrating $\widehat{J_3}$ with respect to s yields

$$\begin{aligned}\widehat{J_3} &= \frac{1}{18\pi}\int_{\mathbb{T}} \mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-2}v(t_n)\cdot P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\,dx\\ &\quad -\frac{1}{18\pi}\int_{\mathbb{T}} \mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-2}v(t_n)\cdot \mathrm{e}^{-\tau\partial_x^3}P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\big)^2\,dx\\ &= -c_n,\end{aligned} \tag{3.19}$$

where c_n is introduced in (3.15). Thus

$$\widehat{(J_2)}_0 = -c_n+\mathcal{R}_5(v). \tag{3.20}$$

On the other hand, when $k\neq 0$, the k-th Fourier coefficient of J_2 is

$$\widehat{(J_2)}_k = \frac{1}{3}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{1}{k_1}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds,$$

where α_4 is the phase function defined as

$$\alpha_4 = k^3-k_1^3-k_2^3-k_3^3, \tag{3.21}$$

and $\hat{v}_{k_j} := \hat{v}(t_n,k_j), \quad j=1,2,3.$

By symmetry, $\widehat{(J_2)}_k$ is equal to

$$\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\Big(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\Big)\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds.$$

Since the decomposition of the frequency will destroy the symmetry, we used symmetry here first. Under the condition of $k=k_1+k_2+k_3$, from the definition of α_4 in (3.21), we have the following relation

$$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3} = \frac{\alpha_4}{3kk_1k_2k_3}+\frac{1}{k}.$$

Therefore we can split $\widehat{(J_2)}_k$ into two terms

$$\begin{aligned}\widehat{(J_2)}_k &= \frac{1}{27}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{\alpha_4}{kk_1k_2k_3}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds\\ &\quad +\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{1}{k}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds\\ &=: \widehat{(J_4)}_k+\widehat{(J_5)}_k.\end{aligned}$$

We consider the terms $\widehat{(J_4)}_k$ and $\widehat{(J_5)}_k$ separately. First, we truncate $\widehat{(J_4)}_k$ to the frequency domain $|k_2+k_3|\le N$ and denote $\mathcal{R}_6(v)$ as the remaining part whose Fourier coefficients are

$$\hat{\mathcal{R}}_6(v,k)=\frac{1}{27}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|>N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{\alpha_4}{kk_1k_2k_3}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds. \tag{3.22}$$

Then

$$\widehat{(J_4)}_k-\hat{\mathcal{R}}_6(v,k)=\frac{1}{27}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{\alpha_4}{kk_1k_2k_3}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds. \tag{3.23}$$

Calculating the integration in (3.23) with respect to s, we get

$$\widehat{(J_4)}_k-\hat{\mathcal{R}}_6(v,k)=\frac{1}{27}i \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \frac{1}{-ikk_1k_2k_3}\left(\mathrm{e}^{-it_{n+1}\alpha_4}-\mathrm{e}^{-it_n\alpha_4}\right)\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}.$$

Taking the inverse Fourier transform, we have

$$\begin{aligned} J_4=&-\frac{1}{27}\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}P_N\left[\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\right)^2\right]\\ &+\frac{1}{27}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}P_N\left[\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\cdot P_N\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\right)^2\right]+\mathcal{R}_6(v). \end{aligned} \tag{3.24}$$

For J_5, we adopt the same strategy as J_4. We truncate J_5 to the frequency domain $|k_2+k_3|\le N$ and denote $\mathcal{R}_7(v)$ as the remainder whose Fourier coefficients are

$$\hat{\mathcal{R}}_7(v,k)=\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|>N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{1}{k}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds. \tag{3.25}$$

Then

$$\widehat{(J_5)}_k=\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{1}{k}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds+\hat{\mathcal{R}}_7(v,k).$$

In order to get a point-wise scheme in physical space, we take an approximation for the exponential $\mathrm{e}^{-i(t_n+s)\alpha_4}\approx\mathrm{e}^{-it_n\alpha_4}$, and denote the error as $\mathcal{R}_8(v)$, then

$$\widehat{(J_5)}_k=\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \mathrm{e}^{-it_n\alpha_4}\frac{1}{k}\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds+\hat{\mathcal{R}}_7(v,k)+\hat{\mathcal{R}}_8(v,k),$$

where $\hat{\mathcal{R}}_8(v,k)$ are the Fourier coefficients of $\mathcal{R}_8(v)$,

$$\hat{\mathcal{R}}_8(v,k)=\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \frac{1}{k}\left(\mathrm{e}^{-i(t_n+s)\alpha_4}-\mathrm{e}^{-it_n\alpha_4}\right)\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}ds.$$

Therefore, we take the inverse Fourier transform to obtain

$$J_5=-\frac{\tau}{9}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}P_N\left[\mathrm{e}^{-t_n\partial_x^3}v(t_n)\cdot P_N\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2\right]+\mathcal{R}_7(v)+\mathcal{R}_8(v). \tag{3.26}$$

Putting together (3.7), (3.8), (3.16), (3.17), (3.18), (3.20), (3.24) and (3.26) yields

$$v(t_{n+1}) = \Phi^n_{\tau,N}\big(v(t_n)\big) + \frac{1}{2}\sum_{j=1}^{8}\mathcal{R}_j(v), \tag{3.27}$$

where $\Phi^n_{\tau,N}$ is given by

$$\begin{aligned}\Phi^n_{\tau,N}(f) = {} & f + \frac{1}{6}\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\Big)^2 - \frac{1}{6}\mathrm{e}^{t_n\partial_x^3}P_N\Big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}f\Big)^2 \\ & + \frac{1}{18}\mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\cdot\partial_x^{-1}P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\big)^2\Big)\Big] \\ & - \frac{1}{18}\mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3}P_N\Big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}f\big)^2\Big)\Big] \\ & - \frac{1}{54}\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}P_N\Big[\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\cdot P_N\big(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\big)^2\Big] \\ & + \frac{1}{54}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}P_N\Big[\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}f\cdot P_N\big(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}f\big)^2\Big] \\ & - \frac{\tau}{18}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}P_N\Big[\mathrm{e}^{-t_n\partial_x^3}f\cdot P_N\big(\mathrm{e}^{-t_n\partial_x^3}f\big)^2\Big] + \frac{\tau}{6}\partial_x^{-1}P_Nf\cdot P_0(f)^2. \end{aligned}\tag{3.28}$$

For given $v^n_{\tau,N}\in S_N$, we define $v^{n+1}_{\tau,N}\in S_N$ by

$$v^{n+1}_{\tau,N} = \Phi^n_{\tau,N}\big(v^n_{\tau,N}\big), \qquad n\ge 0;\quad v^0_{\tau,N} = u^0_{\tau,N}. \tag{3.29}$$

Up to now we finish the construction of the numerical scheme.

REMARK 3.1. We approximate $\widehat{(J_5)}_k$ by

$$\frac{1}{9}i\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \frac{1}{k}\mathrm{e}^{-it_n\alpha_4}\Big(1-\frac{i}{2}\tau\alpha_4\mathrm{e}^{-is\alpha_4}\Big)\hat{v}_{k_1}\hat{v}_{k_2}\hat{v}_{k_3}\,ds,$$

then by using similar methods as in Lemma 4.1 in [24], we can obtain a fully discrete scheme which converges at the second order.

4. Proof of Theorem 2.1

In this section, we present the rigorous proof of Theorem 2.1, including the error estimate and the stability estimate. First, taking the difference between (3.27) and (3.29), we have

$$v^{n+1}_{\tau,N} - v(t_{n+1}) = \mathcal{L}^n + \Phi^n_{\tau,N}\big(v^n_{\tau,N}\big) - \Phi^n_{\tau,N}\big(v(t_n)\big), \tag{4.1}$$

where $\mathcal{L}^n$ is the local error

$$\mathcal{L}^n = -\frac{1}{2}\sum_{j=1}^{8}\mathcal{R}_j(v).$$

Next we estimate the local error as follows.

LEMMA 4.1. *Let $\gamma > \frac{1}{2}$ and $0\le\tau\lesssim 1$, under the assumption of $N\ge 1/\tau$, we have the following estimate*

$$\big\|\mathcal{L}^n\big\|_{H^\gamma} \le C(\tau^2 + \tau N^{-1}),$$

where the constant C depends only on T and $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

Proof. By (3.3), we rewrite $\mathcal{R}_1(v)$ as

$$\begin{aligned}\mathcal{R}_1(v)=&\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\right)^2 ds\\&-\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\left(v(t_n)+\frac{1}{2}F_n\big(v(t_n),s\big)\right)\right)^2 ds\\&+\frac{1}{4}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}F_n\big(v(t_n),s\big)\right)^2 ds.\end{aligned}$$

Then, as was shown in [24], we have the estimate

$$\|\mathcal{R}_1(v)\|_{H^\gamma}\lesssim \tau^2\|v(t)\|^4_{L^\infty((0,T);H^{\gamma+1})}+\tau^3\|v(t)\|^5_{L^\infty((0,T);H^{\gamma+1})}.$$

From the definition of $\mathcal{R}_2(v)$ in (3.5) and by Lemma 2.2 (1), we derive

$$\mathcal{R}_2(v)=\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_{>N}\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\right)^2-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}P_{>N}\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\right)^2.$$

Therefore, under the assumption of CFL condition, the following inequality holds

$$\|\mathcal{R}_2(v)\|_{H^\gamma}\lesssim N^{-2}\|\partial_x^{-1}v(t_n)\|^2_{H^{\gamma+2}}\lesssim \tau N^{-1}\|v(t_n)\|^2_{H^{\gamma+1}}.$$

The above inequality is valid because the frequency is limited to $|k|>N$.

For $\mathcal{R}_3(v)$ in (3.9), using Lemma 2.2 (1) again we have

$$\begin{aligned}\mathcal{R}_3(v)=&\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}P_{>N}\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot \mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right)\\&-\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}P_{>N}\left(\mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}v(t_n)\cdot \mathrm{e}^{-t_n\partial_x^3}\partial_x^{-1}F_n\big(v(t_n),0\big)\right)\\&-\frac{1}{3}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}P_{>N}\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}\partial_sF_n\big(v(t_n),s\big)\right)ds.\end{aligned}$$

By the Hölder inequality we get

$$\begin{aligned}\|\mathcal{R}_3(v)\|_{H^\gamma}\lesssim& N^{-1}\left\|\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}v(t_n)\cdot \mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right\|_{H^{\gamma+1}}\\&+N^{-1}\int_0^\tau\left\|\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}v(t_n)\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x^{-1}\partial_sF_n\big(v(t_n),s\big)\right\|_{H^{\gamma+1}}ds.\end{aligned}$$

Moreover, from the definition (3.2) of F_n, for any $0\le s\le\tau$, we have

$$\left\|F_n\big(v(t_n),s\big)\right\|_{H^\gamma}\lesssim \tau\|v\|^2_{L^\infty((0,T);H^{\gamma+1})},\tag{4.2}$$

$$\left\|\partial_sF_n\big(v(t_n),s\big)\right\|_{H^\gamma}\lesssim \|v\|^2_{L^\infty((0,T);H^{\gamma+1})}.\tag{4.3}$$

Then from (4.2) and (4.3), we finally obtain

$$\|\mathcal{R}_3(v)\|_{H^\gamma}\lesssim \tau N^{-1}\|v(t_n)\|^3_{H^{\gamma+1}}.$$

For $\mathcal{R}_4(v)$ in (3.13), by the Hölder inequality, we observe that

$$\|\mathcal{R}_4(v)\|_{H^\gamma}\lesssim\|v(t_n)\|_{H^\gamma}\cdot\left\|\partial_x^{-1}P_{>N}F_n\big(v(t_n),\tau\big)\right\|_{H^\gamma}\lesssim N^{-1}\|v(t_n)\|_{H^\gamma}\cdot\left\|\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right\|_{H^{\gamma+1}}.\tag{4.4}$$

Plugging (4.2) into the above inequality to obtain

$$\|\mathcal{R}_4(v)\|_{H^\gamma} \lesssim \tau N^{-1}\|v(t_n)\|^3_{H^{\gamma+1}}.$$

A similar calculation shows that

$$\|\mathcal{R}_5(v)\|_{H^\gamma} \lesssim \tau N^{-1}\|v(t_n)\|^3_{H^{\gamma+1}}.$$

For $\mathcal{R}_6(v)$ in (3.22), noting $k=k_1+k_2+k_3+k_4$, we have

$$\alpha_4 = 3kk_1k_2 + 3kk_1k_3 + 3kk_2k_3 - 3k_1k_2k_3,$$

and thus

$$\left|\frac{\alpha_4}{kk_1k_2k_3}\right| \lesssim 1.$$

Hence, we have the following inequality

$$\begin{aligned}\left|\hat{\mathcal{R}}_6(v,k)\right| &\lesssim \tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|>N}} |\hat{v}_{k_1}||\hat{v}_{k_2}||\hat{v}_{k_3}|\\ &\lesssim \tau \mathcal{F} P_N\left(\mathcal{F}^{-1}|\hat{v}_{k_1}|\cdot P_{>N}\left(\mathcal{F}^{-1}|\hat{v}_{k_2}|\cdot\mathcal{F}^{-1}|\hat{v}_{k_3}|\right)\right),\end{aligned}$$

where $\mathcal{F}^{-1}$ is defined in (2.1).

Using Parseval's identity we obtain

$$\begin{aligned}\left\|\mathcal{R}_6(v)\right\|_{H^\gamma} &\lesssim \tau\left\|\mathcal{F}^{-1}|\hat{v}_{k_1}|\right\|_{H^\gamma}\cdot\left\|P_{>N}\left(\mathcal{F}^{-1}|\hat{v}_{k_2}|\cdot\mathcal{F}^{-1}|\hat{v}_{k_3}|\right)\right\|_{H^\gamma}\\ &\lesssim \tau N^{-1}\|v(t_n)\|^3_{H^{\gamma+1}}.\end{aligned}$$

For $\mathcal{R}_7(v)$ in (3.25), since $\left|\frac{1}{k}\right| \le 1$, we use the same method as $\mathcal{R}_6(v)$ to get

$$\left\|\mathcal{R}_7(v)\right\|_{H^\gamma} \lesssim \tau N^{-1}\|v(t_n)\|^3_{H^{\gamma+1}}.$$

For $\mathcal{R}_8(v)$, using similar method as in Lemma 3.5 in [24], we obtain

$$\left\|\mathcal{R}_8(v)\right\|_{H^\gamma} \lesssim \tau^2\|v(t_n)\|^3_{H^{\gamma+1}}.$$

Collecting together with the above estimates, we finish the proof of the lemma. ∎

Now we show the stability of the scheme.

LEMMA 4.2. *Let $\gamma > \frac{1}{2}$ and $N \ge 1/\tau$, then*

$$\begin{aligned}\left\|\Phi^n_{\tau,N}(v^n_{\tau,N}) - \Phi^n_{\tau,N}(v(t_n))\right\|_{H^\gamma} \le (1+C\tau)\left\|v^n_{\tau,N}-v(t_n)\right\|_{H^\gamma} + C\tau\left\|v^n_{\tau,N}-v(t_n)\right\|^5_{H^\gamma}\\ + C\tau N^{-1} + C\sqrt{\tau}\left\|v^n_{\tau,N}-v(t_n)\right\|^2_{H^\gamma},\end{aligned}$$

where the constant C depends only on T and $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

Proof. As shown in the analysis in Section 3, by the definition in (3.28), $\Phi^n_{\tau,N}(f)$ can be written as the following form:

$$\Phi^n_{\tau,N}(f) = f + \Phi^{n,1}_{\tau,N}(f) + \Phi^{n,2}_{\tau,N}(f) + \Phi^{n,3}_{\tau,N}(f) + \Phi^{n,4}_{\tau,N}(f) + \Phi^{n,5}_{\tau,N}(f),$$

where

$$\Phi^{n,1}_{\tau,N}(f)=\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}f\right)^2 ds;$$
$$\Phi^{n,2}_{\tau,N}(f)=\frac{1}{6}\mathbb{P}\left[\mathrm{e}^{t_{n+1}\partial_x^3}P_N\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}f\cdot \mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}P_N F_n(f,\tau)\right)\right];$$
$$\Phi^{n,3}_{\tau,N}(f)=\frac{\tau}{12\pi}\partial_x^{-1}P_N f\int_{\mathbb{T}}(f)^2\,dx;$$
$$\Phi^{n,4}_{\tau,N}(f)=\frac{i}{54}\mathcal{F}^{-1}\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \mathrm{e}^{-i(t_n+s)\alpha_4}\frac{\alpha_4}{kk_1k_2k_3}\hat{f}_{k_1}\hat{f}_{k_2}\hat{f}_{k_3}\,ds;$$
$$\Phi^{n,5}_{\tau,N}(f)=\frac{i}{18}\mathcal{F}^{-1}\int_0^\tau \sum_{\substack{k=k_1+k_2+k_3\\ |k|\le N,\ |k_2+k_3|\le N}} \mathrm{e}^{-it_n\alpha_4}\frac{1}{k}\hat{f}_{k_1}\hat{f}_{k_2}\hat{f}_{k_3}\,ds.$$

Then, we have

$$\Phi^n_{\tau,N}(v^n_{\tau,N})-\Phi^n_{\tau,N}(v(t_n))=e_n+\Phi^n_1+\Phi^n_2+\Phi^n_3+\Phi^n_4+\Phi^n_5,$$

where e_n and Φ^n_j are defined by

$$e_n=v^n_{\tau,N}-v(t_n);\qquad \Phi^n_j=\Phi^{n,j}_{\tau,N}(v^n_{\tau,N})-\Phi^{n,j}_{\tau,N}(v(t_n)),\quad j=1,2,\cdots,5.$$

Then we obtain

$$\begin{aligned}&\left\|\Phi^n_{\tau,N}\left(v^n_{\tau,N}\right)-\Phi^n_{\tau,N}\left(v(t_n)\right)\right\|^2_{H^\gamma}\\ \le{}&\|e_n\|^2_{H^\gamma}+2\left\langle J^\gamma\Phi^n_1,J^\gamma e_n\right\rangle+2\|e_n\|_{H^\gamma}\left\|\Phi^n_2\right\|_{H^\gamma}+\cdots+2\|e_n\|_{H^\gamma}\left\|\Phi^n_5\right\|_{H^\gamma}\\ &+5\left\|\Phi^n_1\right\|^2_{H^\gamma}+\cdots+5\left\|\Phi^n_5\right\|^2_{H^\gamma}.\end{aligned}$$

First, by the definition of Φ^n_1, we get

$$\begin{aligned}\Phi^n_1&=\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left[\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v^n_{\tau,N}\right)^2-\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^2\right]ds\\ &=\frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}e_n\right)^2 ds\\ &\quad+\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}e_n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)ds.\end{aligned}\tag{4.5}$$

Then, we get

$$\begin{aligned}\left\langle J^\gamma\Phi^n_1,J^\gamma e_n\right\rangle&=\frac{1}{2}\left\langle\int_0^\tau J^\gamma \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}e_n\right)^2 ds,J^\gamma e_n\right\rangle\\ &\quad+\left\langle\int_0^\tau J^\gamma \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}e_n\cdot \mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)ds,J^\gamma e_n\right\rangle\\ &\triangleq Q_1+Q_2.\end{aligned}$$

By Lemma 2.2 (2) and the Hölder inequality, Q_1 can be controlled as

$$|Q_1|\lesssim\left\|\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x P_N\left(\mathrm{e}^{-(t_n+s)\partial_x^3}e_n\right)^2 ds\right\|_{H^\gamma}\|e_n\|_{H^\gamma}\lesssim\sqrt{\tau}\|e_n\|^3_{H^\gamma}.\tag{4.6}$$

We divide e_n into two parts:

$$e_n = P_N e_n + P_{>N} v(t_n).$$

Then, we have

$$\begin{aligned}
Q_2 &= \Big\langle \int_0^\tau J^\gamma \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} e_n \cdot \mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n)\Big) ds, J^\gamma P_N e_n \Big\rangle \\
&= \Big\langle \int_0^\tau J^\gamma \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} P_N e_n \cdot \mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n)\Big) ds, J^\gamma P_N e_n \Big\rangle \\
&\quad + \Big\langle \int_0^\tau J^\gamma \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} P_{>N} v(t_n) \cdot \mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n)\Big) ds, J^\gamma P_N e_n \Big\rangle \\
&\triangleq Q_3 + Q_4.
\end{aligned}$$

By (2.7), we obtain directly

$$|Q_3| \lesssim \tau \|e_n\|_{H^\gamma}^2 \|v(t_n)\|_{H^{\gamma+1}}. \tag{4.7}$$

For Q_4, by Lemma 2.2 (1) we write it in the point-wise form:

$$\begin{aligned}
Q_4 = &\frac{1}{3} \Big\langle J^\gamma \mathrm{e}^{t_{n+1}\partial_x^3} \Big(\mathrm{e}^{-t_{n+1}\partial_x^3} P_{>N} \partial_x^{-1} v(t_n) \cdot \mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} v(t_n)\Big), J^\gamma P_N e_n \Big\rangle \\
&- \frac{1}{3} \Big\langle J^\gamma \mathrm{e}^{t_n\partial_x^3} \Big(\mathrm{e}^{-t_n\partial_x^3} P_{>N} \partial_x^{-1} v(t_n) \cdot \mathrm{e}^{-t_n\partial_x^3} \partial_x^{-1} v(t_n)\Big), J^\gamma P_N e_n \Big\rangle.
\end{aligned}$$

Applying the Hölder inequality yields

$$\begin{aligned}
|Q_4| &\lesssim \|P_{>N} \partial_x^{-1} v(t_n)\|_{H^\gamma} \|v(t_n)\|_{H^\gamma} \|e_n\|_{H^\gamma} \\
&\lesssim N^{-2} \|v(t_n)\|_{H^{\gamma+1}}^2 \|e_n\|_{H^\gamma}.
\end{aligned}$$

Under the assumption of CFL condition, we get

$$|Q_4| \lesssim \tau N^{-1} \|v(t_n)\|_{H^{\gamma+1}}^2 \|e_n\|_{H^\gamma}. \tag{4.8}$$

Combining with (4.6), (4.7) and (4.8) yields

$$\Big\langle J^\gamma \Phi_1^n, J^\gamma e_n \Big\rangle \le C\big(\tau \|e_n\|_{H^\gamma}^2 + \tau N^{-1} \|e_n\|_{H^\gamma} + \sqrt{\tau} \|e_n\|_{H^\gamma}^3\big), \tag{4.9}$$

where the constant C depends on $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

From (4.5) and Lemma 2.2 (2), we get the following estimate

$$\|\Phi_1^n\| \le C\sqrt{\tau}\big(\|e_n\|_{H^\gamma}^2 + \|e_n\|_{H^\gamma} \|v(t_n)\|_{H^\gamma}\big),$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

From the definition of Φ_2^n, we obtain

$$\begin{aligned}
\Phi_2^n = &\frac{1}{6} \mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3} P_N \Big(\mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} e_n \cdot \mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} P_N F_n\big(v(t_n), \tau\big)\Big)\Big] \\
&+ \frac{1}{6} \mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3} P_N \Big(\mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} e_n \cdot \mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} P_N \big(F_n\big(v_{\tau,N}^n, \tau\big) - F_n\big(v(t_n), \tau\big)\big)\Big)\Big] \\
&+ \frac{1}{6} \mathbb{P}\Big[\mathrm{e}^{t_{n+1}\partial_x^3} P_N \Big(\mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} v(t_n) ` \cdot \mathrm{e}^{-t_{n+1}\partial_x^3} \partial_x^{-1} P_N \big(F_n\big(v_{\tau,N}^n, \tau\big) - F_n\big(v(t_n), \tau\big)\big)\Big)\Big].
\end{aligned}$$

Therefore, we get the following inequality

$$\begin{aligned}\|\Phi_2^n\|_{H^\gamma} \lesssim \|e_n\|_{H^\gamma}\left\|\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right\|_{H^\gamma} \\ +\|e_n\|_{H^\gamma}\left\|\partial_x^{-1}\big(F_n\big(v_{\tau,N}^n,\tau\big)-F_n\big(v(t_n),\tau\big)\big)\right\|_{H^\gamma} \\ +\|v(t_n)\|_{H^\gamma}\left\|\partial_x^{-1}\big(F_n\big(v_{\tau,N}^n,\tau\big)-F_n\big(v(t_n),\tau\big)\big)\right\|_{H^\gamma}.\end{aligned}$$

From (3.2) and (2.6), we get

$$\left\|\partial_x^{-1}F_n\big(v(t_n),\tau\big)\right\|_{H^\gamma} \lesssim \tau\|v(t_n)\|_{H^\gamma}^2;$$

$$\left\|\partial_x^{-1}\big(F_n\big(v_{\tau,N}^n,\tau\big)-F_n\big(v(t_n),\tau\big)\big)\right\|_{H^\gamma} \lesssim \tau\Big(\|e_n\|_{H^\gamma}^2+\|e_n\|_{H^\gamma}\|v(t_n)\|_{H^\gamma}\Big).$$

Use the above inequalities to derive

$$\|\Phi_2^n\| \lesssim \tau\big(\|e_n\|_{H^\gamma}\|v(t_n)\|_{H^\gamma}^2+\|e_n\|_{H^\gamma}^2\|v(t_n)\|_{H^\gamma}+\|e_n\|_{H^\gamma}^3\big).$$

A similar calculation yields that

$$\|\Phi_j^n\| \le C\tau\big(\|e_n\|_{H^\gamma}+\|e_n\|_{H^\gamma}^3\big), \qquad j=3,4,5.$$

Putting these estimates together yields

$$\begin{aligned}&\left\|\Phi_{\tau,N}^n(v_{\tau,N}^n)-\Phi_{\tau,N}^n(v(t_n))\right\|_{H^\gamma}^2 \\ \le&(1+C\tau)\left\|e_n\right\|_{H^\gamma}^2+C\tau\left\|e_n\right\|_{H^\gamma}^6+C\tau N^{-1}\left\|e_n\right\|_{H^\gamma}+C\sqrt{\tau}\left\|e_n\right\|_{H^\gamma}^3.\end{aligned}$$

That implies

$$\begin{aligned}&\left\|\Phi_{\tau,N}^n(v_{\tau,N}^n)-\Phi_{\tau,N}^n(v(t_n))\right\|_{H^\gamma} \\ \le&\left\|e_n\right\|_{H^\gamma}\sqrt{1+C\tau+C\tau\left\|e_n\right\|_{H^\gamma}^4+\frac{C\tau N^{-1}}{\left\|e_n\right\|_{H^\gamma}}+C\sqrt{\tau}\left\|e_n\right\|_{H^\gamma}}.\end{aligned}$$

Since $\sqrt{1+x} \le 1+\frac{1}{2}x$ for $x \ge 0$, we obtain

$$\begin{aligned}&\left\|\Phi_{\tau,N}^n(v_{\tau,N}^n)-\Phi_{\tau,N}^n(v(t_n))\right\|_{H^\gamma} \\ \le&(1+C\tau)\left\|e_n\right\|_{H^\gamma}+C\tau\left\|e_n\right\|_{H^\gamma}^5+C\tau N^{-1}+C\sqrt{\tau}\left\|e_n\right\|_{H^\gamma}^2.\end{aligned}$$

Finally we get the desired result. □

Now we prove Theorem 2.1. Combining (4.1) with the error and the stability estimates, we obtain

$$\begin{aligned}\|v(t_{n+1},\cdot)-v_{\tau,N}^{n+1}\|_{H^\gamma} \le\ & (1+C\tau)\left\|v_{\tau,N}^n-v(t_n)\right\|_{H^\gamma}+C\tau\left\|v_{\tau,N}^n-v(t_n)\right\|_{H^\gamma}^5 \\ &+C\sqrt{\tau}\left\|v_{\tau,N}^n-v(t_n)\right\|_{H^\gamma}^2+C(\tau^2+\tau N^{-1}), \quad 0 \le (n+1)\tau \le T.\end{aligned}$$

Now we claim that there exists some constant τ_0 such that for any $0<\tau\le\tau_0$ we have

$$\left\|v(t_n)-v_{\tau,N}^n\right\|_{H^\gamma} \le C(\tau^2+\tau N^{-1})\sum_{j=0}^{n}(1+2C\tau)^j, \quad 0 \le n\tau \le T.$$

It is trivial that it holds for $n=0$. We assume it holds for any $0 \le n \le n_0$, i.e.,

$$\left\|v(t_n)-v_{\tau,N}^n\right\|_{H^\gamma} \le C(\tau^2+\tau N^{-1})\sum_{j=0}^{n}(1+2C\tau)^j, \quad 0 \le n \le n_0.$$

Then, for any $0 \le n \le n_0$, we have

$$\left\|v(t_n)-v_{\tau,N}^n\right\|_{H^\gamma} \le \tilde{C}(\tau+N^{-1}), \quad 0\le n\le n_0,$$

where $\tilde{C}=CT\mathrm{e}^{2CT}$. Therefore, for $n=n_0+1$, the following inequality holds

$$\begin{aligned}\|v(t_{n_0+1},\cdot)-v_{\tau,N}^{n_0+1}\|_{H^\gamma} \le \Big(1+C\tau+C\tau\tilde{C}^4(\tau+N^{-1})^4+C\sqrt{\tau}\tilde{C}(\tau+N^{-1})\Big)\\ \cdot C(\tau^2+\tau N^{-1})\sum_{j=0}^{n_0}(1+2C\tau)^j+C(\tau^2+\tau N^{-1}).\end{aligned}$$

Since $N\ge 1/\tau$, by choosing τ_0 small enough such that

$$\tilde{C}^4(\tau_0+N^{-1})^4+\sqrt{\tau_0}\tilde{C}\le 1,$$

we obtain for any $\tau\in(0,\tau_0]$,

$$\|v(t_{n_0+1},\cdot)-v_{\tau,N}^{n_0+1}\|_{H^\gamma}\le C(\tau^2+\tau N^{-1})\sum_{j=0}^{n_0+1}(1+2C\tau)^j.$$

This finishes the induction and thus for any $0\le n\tau\le T$,

$$\left\|v(t_n)-v_{\tau,N}^n\right\|_{H^\gamma}\le C(\tau+N^{-1}).$$

The proof of Theorem 2.1 is completed.

5. Numerical experiments

In this section we present numerical experiments to illustrate the convergence result given in Theorem 2.1. We choose two different initial data of the nonlinear KdV equation (1.1) in the following way:

$$u^0(x)=\sum_{0\ne k\in\mathbb{Z}}|k|^{-\frac12-\gamma-\varepsilon}\mathrm{e}^{ikx}, \qquad \tilde{u}^0=\sum_{0\ne k\in\mathbb{Z}}|k|^{-\frac12-\gamma-\varepsilon}a_k\mathrm{e}^{ikx}, \tag{5.1}$$

where ε is arbitrarily small, a_k are uniformly distributed random variables in $[-2,2]+i[-2,2]$. These choices allow us to get our desired regularity $u^0\in H^\gamma$, $\tilde{u}^0\in H^\gamma$.

We present the numerical experiments with the spatial discretization errors in the Figures 5.1 and 5.2. We choose two sufficiently small time steps τ and two different initial data u^0 and $\tilde{u}^0$. Furthermore, we measure the error $u(t_n,\cdot)-u_{\tau,N}^n$ in the discrete H^γ-norm

$$\|w\|_{H_N^\gamma}^2=\frac{2\pi}{N}\sum_{k=-N}^{N}|k|^{2\gamma}|\hat{w}_k|^2.$$

As we can see in Figure 5.1, it illustrates the error in H^1-norm of the first-order scheme by using different N. It can be clearly observed that the scheme is of the first-order convergence in H^1 for the initial data in H^2. This agrees well with the corresponding result in Theorem 2.1. In the same way, Figure 5.2 validates the convergence of the scheme in H^2-norm with H^3 initial data.

The numerical experiments with the temporal discretization errors are shown in Figure 5.3. In a similar way, we choose $N=2^{12}$ large enough and two different initial data u^0 and $\tilde{u}^0$. The results are presented by using different τ and illustrate that the scheme has the first-order convergence rate in time in H^1-norm with H^2-initial data. In the same way, we choose $N = 2^{10}$ in Figure 5.4 and performed the numerical experiments in H^2-norm with H^3-initial data.

In particular, numerical experiments show that the scheme is convergent in H^γ for initial data in $H^{\gamma+1}$ without CFL condition. However, the proof of Theorem 2.1, such as the estimate of the term $\mathcal{R}_2$ in Lemma 4.1 and the estimate of the term Q_4 in Lemma 4.2, rely on the CFL condition $N \geq 1/\tau$. It is reasonable to believe that there is no CFL condition in the analysis, but we require more technical analysis and will address it in future work.

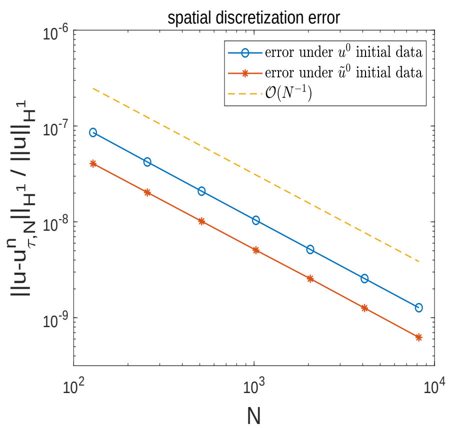

Fig. 5.1: *Spatial convergence for* $\tau = 2^{-11}$ *at* $T = 1$ *for* H^2 *initial data*

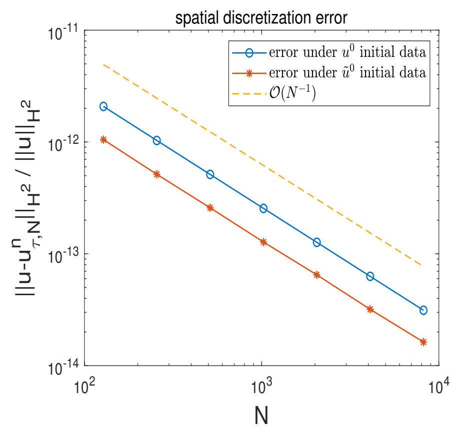

Fig. 5.2: *Spatial convergence for* $\tau = 2^{-10}$ *at* $T = 1$ *for* H^3 *initial data*

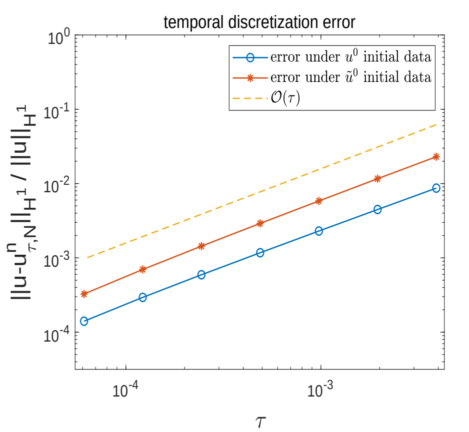

Fig. 5.3: *Temporal error for* $N = 2^{12}$ *at* $T = 1$ *for* H^2 *initial data*

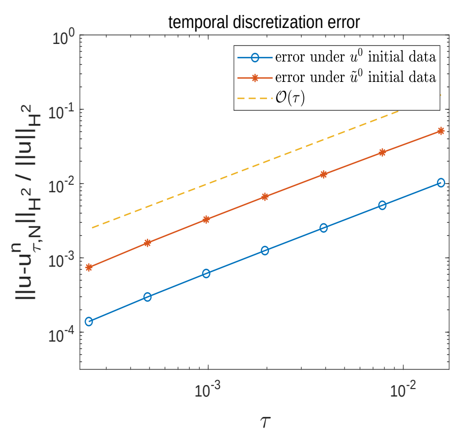

Fig. 5.4: *Temporal error for* $N = 2^{10}$ *at* $T = 1$ *for* H^3 *initial data*

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