

GLOBAL-IN-TIME STABILITY OF GROUND STATES OF A PRESSURELESS HYDRODYNAMIC MODEL OF COLLECTIVE BEHAVIOUR*

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Abstract. We consider a pressureless hydrodynamic model of collective behaviour, which is concerned with a density function ρ and a velocity field v on the torus, and is described by the continuity equation for ρ , $\rho_t + \operatorname{div}(v\rho) = 0$, and a compressible hydrodynamic equation for v , $\rho v_t + \rho v \cdot \nabla v - \Delta v = -\rho \nabla K \rho$ with a forcing modelling collective behaviour related to the density ρ , where K stands for the repulsive interaction potential, defined as the solution to the Poisson equation on \mathbb{T}^d . We show global-in-time stability of the ground state $(\rho, v) = (1, 0)$ if the perturbation $(\rho_0 - 1, v_0)$ satisfies $\|v_0\|_{B_{p,1}^{d/p-1}(\mathbb{T}^d)} + \|\rho_0 - 1\|_{B_{p,1}^{d/p}(\mathbb{T}^d)} \leq \epsilon$, where $p \in (\min(d/2, 2), d)$ and $\epsilon > 0$ is sufficiently small.

Keywords. Pressureless hydrodynamic model; stability; collective behaviour; Besov spaces; repulsive system.

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1. Introduction

We are concerned with a model of a pressureless gas driven by the following set of laws:

$$\begin{aligned} \rho_t + \operatorname{div}(v\rho) &= 0, \\ \rho v_t + \rho v \cdot \nabla v - \Delta v &= -\rho \nabla K \rho \quad \text{in } [0, \infty) \times \mathbb{T}^d, \end{aligned} \tag{1.1}$$

considered, for simplicity, on the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$. The first equation is the mass conservation, prescribing the dynamics of the density ρ under the flow v , and the second one is the momentum equation. Here K is the operator $(-\Delta)^{-1}$ restricted to the periodic functions with zero mean; namely $\Psi := K(\rho - \{\rho\})$ satisfies

$$-\Delta \Psi = \rho - \{\rho\}, \quad \text{where } \{\rho\} := \int \rho dx \quad \text{and} \quad \int \Psi = 0. \tag{1.2}$$

Here and below we use the short-hand notation $\int \equiv \int_{\mathbb{T}^d}$, and we will often omit “ dx ”, for brevity. We note that, since the average $\{\rho\}$ is preserved by the flow, we assume, without loss of generality, that $\{\rho\} = 1$ for all times.

Model (1.1) arises as a nonlinear repulsion model of collective behaviour [5, 9, 28]. For example, (1.1) can be obtained as a simplification of the hydrodynamic system [9, (1.1)] by neglecting some forcing terms and taking $m = 0$, (which makes it a pressureless system). It can also be viewed as a dissipative version of [5, (14)].

A common feature of these models is the presence of nonlocal interactions, which is captured by the term $-\rho \nabla K(\rho - 1)$. In the case of a repulsive system, this term is expected to cause the system to avoid the states with high local concentrations [5, 9, 24].

On the other hand, in hydrodynamic models such behaviour is often expected to be caused by the pressure function. For example, the incompressible Euler equations,

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$D_t u = -\nabla p$ (where “ D_t ” denotes the material derivative $\partial_t + u \cdot \nabla$), show that the motion of an inviscid and incompressible fluid can be described, roughly speaking, by the fluid particles “avoiding” the regions of large pressure. As for compressible models, the precise role of the pressure function is less clear; a natural assumption is the presence of a pressure function of type $\nabla p(\rho)$, such as the polytropic equation of state of the form $p(\rho) = \rho^m$ for some $m > 0$. Within such framework, the Euler-Poisson system is one of the models related to (1.1) and it models a compressible gas with an interaction force between particles. It can be used to describe the evolution of gaseous stars, which include the classical nonexistence result provided by Makino and Perthame [35] in the case of attractive forces, and Perthame [41] in the case of repulsive forces. We refer the reader to [47] for a study on the singularity formation in the model with diffusion and relaxation, and to [22] (and the references therein) for an excellent review of the subject. We also note the recent developments in the study of stationary and self-similar solutions by [27, 30–32], see also [19], as well as global-in-time existence results [25, 26, 29]. On the other hand, the authors of [9] consider a viscous model of collective behaviour, and prove global-in-time existence of weak solutions.

These results show the need to further understand the role of the pressure function in models involving nonlocal interactions via repulsive or attractive kernels.

In fact, the pressureless models, which correspond to the case $m = 0$, are closely related to models of collective behaviour. For example, they can be used as a macroscopic description of flocking and also model finite-time blow-up phenomena, see [5, 28] and the references therein for details. Moreover, Carrillo, Choi and Zatorska [6] consider a one-dimensional pressureless Euler-Poisson system in which, instead of dissipation, a friction term is taken into account. What is interesting from the mathematical viewpoint is that the friction somehow gives better stability properties than dissipation, even in the case of a bounded domain. We also refer the reader to the extensive article [22] (see also [2]) on the one-dimensional pressureless Euler-Poisson system, which discusses the model under various assumptions regarding background terms, relaxation, and viscosity. The multidimensional case is more complicated and requires some modifications, see [34, 45] for details, as well as [22] for the particular case of geometrical symmetry. Here also we shall mention about results concerning the Euler alignment system [17, 21, 44], where equations are a hydrodynamical limit of systems describing flocking phenomena like for the Cucker-Smale model [10]. The key difference between the Euler alignment system and (1.1) is that the Euler alignment system involves a weighted fractional Laplacian instead of the Laplace operator.

We also note that one of the interesting themes occurring in the pressureless models is the existence of a *critical threshold* [2, 3, 34, 45], which distinguishes the supercritical and subcritical regimes of the system.

These phenomena of pressureless systems are related to the issue of stability of models of collective behaviour, and are one of our motivations to study (1.1). In fact (1.1) appears to be the simplest pressureless model that involves dissipation and nonlocal repulsive interactions, given by $-\rho \nabla K(\rho - 1)$. This raises the question of stability of (1.1), which we address in this article.

In fact, from the mathematical perspective, the pressureless models, such as (1.1), lie at the borderline of the modern PDE techniques, which makes them exciting. To be more precise, we now describe some of the remarkable features of (1.1).

First of all, although we do obtain global-in-time existence of unique solutions of (1.1) for small data (see Theorem 1.1 below), any exponential decay as $t \rightarrow \infty$ cannot be expected. This is a consequence of the fact that the spectrum of an operator coming from

the linearization of (1.1) is not separated from zero, see (3.10) below for details. This shows that, at least in the case of the torus \mathbb{T}^d , the repulsive interactions modelled by (1.1) can, roughly speaking, cause the particles to “spread uniformly over \mathbb{T}^d ” as $t \rightarrow \infty$, but not exponentially fast. It is a similar effect as the convergence to the equilibrium of the three-dimensional plasma fluid modelled by the Euler-Poisson system, which was proved in [24]. It also shows that one should not expect stability under attractive interactions (for example in the case without the minus sign on the right-hand side of (1.1)), as in that case the spectrum has a positive real part.

Secondly, a key difficulty of (1.1) is the lack of the effective viscous flux which relates the divergence of the velocity v with the pressure $p(\rho)$ in the form

$$\operatorname{div} v - p(\rho).$$

This quantity is often used in the theory of compressible Navier-Stokes system to smooth out the density, by proving its decay or integrability in time, which in turns gives enough compactness to yield existence of weak solutions [23, 33]. In the case of (1.1), there is no such simple quantity with fine properties.

In order to analyze (1.1) one can consider the viewpoint of the energy, in analogy to the compressible Navier-Stokes equation. Indeed, we can test the momentum equation with v to get

$$\frac{1}{2} \frac{d}{dt} \int (\rho |v|^2 + \rho K \rho) dx + \mu \int |\nabla v|^2 dx = 0. \tag{1.3}$$

To understand the meaning of the term involving K , we note that $\Psi := K(\rho - 1)$ has zero average, which implies that

$$\int \rho K \rho dx = \int (\rho - 1) K (\rho - 1) dx = \int (-\Delta \Psi) \Psi dx = \int |\nabla \Psi|^2 dx = \|\rho - 1\|_{H^{-1}}^2.$$

This shows that the energy $\int \rho |v|^2 dx + \|\rho - 1\|_{H^{-1}}^2$ decreases in time, which suggests that system (1.1) is stable, at least around static solutions. This fact can be interpreted as the repulsive nature of the interaction force $-\rho \nabla K \rho$ mentioned above. We also note that the case $K \equiv 0$ can be analyzed in a nonstandard framework of the Lorentz spaces and time-weighted norms, see [18].

On the other hand, one can study (1.1) by considering its quasi-stationary approximation, which leads to the following aggregation type equation

$$\rho_t - \operatorname{div}(\rho(-\Delta)^{-1}(\rho \nabla K \rho)) = 0. \tag{1.4}$$

An analysis of (1.4) could deliver possible stationary solutions to (1.1). One of them is the case of constant ρ , which leads to the stationary solution

$$(\rho, v) = (1, 0) \tag{1.5}$$

of (1.1). We note that $(\rho, v) = (1, V)$ is also a steady state of (1.1) for every constant $V \in \mathbb{R}^d$, but using the Galilean transformation $(t, x) \mapsto (t, x - tV)$ it can be reduced to (1.5).

The main purpose of this paper is to establish the first stability result of the steady state (1.5) of (1.1). We consider a particular functional setting, which we now motivate. To retain positivity of the density ρ as well as the finiteness of the total mass for all times, we consider the case of a bounded domain. For simplicity we consider only the

case of the torus \mathbb{T}^d . To obtain the stability result, we will make use of some tools from the theory of regular solutions of the compressible Navier-Stokes system. The first approach to such a system is based on the L^2 setting [37]. However we will extend the techniques from [38–40], developed for the L^p spaces, as well as from [11], which focuses on the Besov space setting on the whole space \mathbb{R}^3 . These methods can be further developed to yield the following.

THEOREM 1.1 (Main result). *Given $d \geq 3$, $p \in (\min(d/2, 2), d)$ there exists $\epsilon > 0$ with the following property. For every $\rho_0 - 1 \in B_{p,1}^{d/p}(\mathbb{T}^d)$ and $v_0 \in B_{p,1}^{d/p-1}(\mathbb{T}^d)$ such that $\int \rho_0 v_0 = 0$ and*

$$\|v_0\|_{B_{p,1}^{d/p-1}(\mathbb{T}^d)} + \|\rho_0 - 1\|_{B_{p,1}^{d/p}(\mathbb{T}^d)} \leq \epsilon$$

there exists a unique global-in-time solution (ρ, v) of (1.1) such that

$$\rho - 1 \in C_b([0, \infty); B_{p,1}^{d/p}(\mathbb{T}^d)), \quad v_t, \nabla^2 v \in L^1((0, \infty); B_{p,1}^{d/p-1}(\mathbb{T}^d)),$$

with

$$\|\rho - 1\|_{L^\infty((0, \infty); B_{p,1}^{d/p}(\mathbb{T}^d))} + \|v_t\|_{L^1((0, \infty); B_{p,1}^{d/p-1}(\mathbb{T}^d))} + \|v\|_{L^1((0, \infty); B_{p,1}^{d/p+1}(\mathbb{T}^d))} \leq C\epsilon,$$

where $C = C(d, p) > 1$ is a constant.

We note that the case $d = 2$ remains an interesting open problem; it is not covered by the above theorem, as then the range of p 's is empty. In order to consider that case, the energy methods appear more well-suited, see [11, 16] for example.

Here $C_b(I; X)$ denotes the space of continuous and bounded functions from an interval I to a Banach space X . Moreover $B_{p,1}^s(\mathbb{T}^d)$ stands for the Besov space on the torus (see Section 2 for details).

In order to motivate the functional framework considered in Theorem 1.1, we first observe that the transport equation for ρ gives the a priori bound

$$\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \exp\left(\int_0^t \|\operatorname{div} v(s)\|_{L^\infty} ds\right). \tag{1.6}$$

This suggests that the condition

$$\operatorname{div} v \in L^1((0, \infty); L^\infty) \tag{1.7}$$

is necessary for any global well-posedness result. On the other hand, in order to construct the solution claimed by the above theorem, one would consider a linearization of (1.1) around the ground state. In order to effectively analyze such linearization one would need to use maximal regularity of the heat equation $u_t - \Delta u = f$ in a space of the form $L^1((0, \infty); X)$ where X is some Banach space. However, it is well-known that if X is reflexive (more precisely *UMD*, see [20]) then such maximal regularity holds in $L^q((0, T); X)$ only for $1 < q < \infty$. Thus, in order to reach the borderline case $q = 1$, we need to find a non-reflexive Banach space X . This suggests that we should leave the classical L^p framework and enter the universe of the Besov spaces. It naturally leads us to consider the spaces of the form $L^1((0, \infty); B_{p,1}^s)$, considered by [14, 15]. In fact, maximal regularity of the heat equation holds in $L^1((0, \infty); B_{p,1}^s)$ (see (2.6) below), which is

one of the most significant properties of Besov spaces with the last index one. Moreover, $B_{p,1}^s \subset L^\infty$ for $s \geq d/p$, which, in light of (1.7), suggests that we should consider

$$\nabla v \in L^1\left((0, \infty); B_{p,1}^{d/p}\right),$$

which naturally leads us to the functional setting considered in Theorem 1.1. Another advantage of this marginal functional framework is that no explicit decay of $\|\nabla v\|_{L^\infty}$ is required.

Thanks to this choice of functional setting, the hyperbolic character of the continuity equation for ρ can be removed. Moreover, due to the smallness assumption of Theorem 1.1, we expect that

$$\left\| \int_0^t |\nabla v(s, x)| ds \right\|_{L^\infty} < \frac{1}{2}$$

for all $t > 0$, which suggests that the Lagrangian coordinates should be well-defined. In fact, the problem (1.1) becomes simpler in such coordinates, which we now introduce in order to describe the main difficulties.

Let $X(t, y)$ be the solution of the system

$$\frac{dX(t, y)}{dt} = v(t, X(t, y)), \quad X|_{t=0} = y.$$

That is we define the Lagrangian coordinates y by the relation

$$X(t, y) := y + \int_0^t v(\tau, X(\tau, y)) d\tau. \tag{1.8}$$

We set

$$\eta(t, y) := \rho(t, X(t, y)), \quad u(t, y) := v(t, X(t, y)). \tag{1.9}$$

The transformation matrix reads

$$A := \left(\frac{\partial X}{\partial y}\right)^{-1} = \left(I + \int_0^t \nabla u\right)^{-1}. \tag{1.10}$$

Since we assumed that $\int \rho dx = 1$, and we aim at analysis of the flow around $\rho = 1$, we introduce a by

$$\eta = 1 + a. \tag{1.11}$$

The Equations (1.1) in Lagrangian coordinates become

$$\begin{aligned} \eta_t + \eta \operatorname{div}_u u &= 0, \\ \eta u_t - \Delta_u u &= -\eta \nabla_u (-\Delta_u)^{-1} a \quad \text{in } [0, \infty) \times \mathbb{T}^d, \end{aligned} \tag{1.12}$$

with initial conditions $\eta|_{t=0} = \rho_0$, $u|_{t=0} = v_0$, where we have denoted by ∇_u , div_u and Δ_u the gradient, the divergence and the Laplace operator, respectively, with respect to the x variable, that is in the Eulerian setting. For example

$$\operatorname{div}_u u(t, y) = [\operatorname{div} v(t, x)]_{x=X(t, y)} \quad \text{and} \quad \Delta_u u(t, y) = [\Delta v(t, x)]_{x=X(t, y)}.$$

We will use the notation div , ∇ and Δ to denote the respective differential operators of u , a or η with respect to their spatial variable, namely y . Similarly, we denote by $(-\Delta)^{-1}$ the solution operator of (1.2) in variable y , and we have also denoted by $(-\Delta_u)^{-1}g(y)$ the solution operator in the Eulerian variables. Namely, since $a(t, y) = [\rho(t, x) - 1]_{x=X(t, y)}$ we have

$$\nabla_u(-\Delta_u)^{-1}a = \nabla_u [(-\Delta)^{-1}(\rho - 1)]_{x=X(t, y)}.$$

We will show (in Section 4 below) that Theorem 1.1 is equivalent to the following restatement in the Lagrangian coordinates.

PROPOSITION 1.1 (Main result in the Lagrangian coordinates). *Given $d \geq 3$, $p \in (\min(d/2, 2), d)$ there exists $\epsilon > 0$ with the following property. For every $a_0 \in B_{p,1}^{d/p}(\mathbb{T}^d)$ and $u_0 \in B_{p,1}^{d/p-1}(\mathbb{T}^d)$ with*

$$\|a_0\|_{B_{p,1}^{d/p}(\mathbb{T}^d)} + \|u_0\|_{B_{p,1}^{d/p-1}(\mathbb{T}^d)} \leq \epsilon,$$

there exists a unique Lagrangian map X and a pair (a, u) such that $X(t) - \operatorname{id} = \int_0^t u(\tau) d\tau$ holds for $t > 0$, the Equations (1.12) hold on $(0, \infty) \times \mathbb{T}^d$ with initial conditions (a_0, u_0) , and

$$\begin{aligned} & \|a\|_{L^\infty((0, \infty); B_{p,1}^{d/p}(\mathbb{T}^d))} + \|a_t\|_{L^1((0, \infty); B_{p,1}^{d/p}(\mathbb{T}^d))} + \|a - \{a\}\|_{L^1((0, \infty); B_{p,1}^{d/p-2}(\mathbb{T}^d))} \\ & + \|u\|_{L^\infty((0, \infty); B_{p,1}^{d/p-1}(\mathbb{T}^d))} + \|u_t\|_{L^1((0, \infty); B_{p,1}^{d/p-1}(\mathbb{T}^d))} \\ & + \|u - \{u\}\|_{L^1((0, \infty); B_{p,1}^{d/p+1}(\mathbb{T}^d))} + \|\nabla X - I\|_{L^\infty((0, \infty); B_{p,1}^{d/p}(\mathbb{T}^d))} \leq C\epsilon, \end{aligned}$$

where $C = C(d, p) > 1$ is a constant.

We note that in Proposition 1.1 we only obtain the smallness of the averages $\{a\}, \{u\}$ of a, u that is uniform in time, while in the Eulerian coordinates (i.e. in Theorem 1.1) we obtain L^1 control, which is a consequence of the conservation of mass $\int \rho$ and the conservation of momentum $\int \rho v$, see (4.9) for details.

We note that the claim of Proposition 1.1 implies in particular that

$$\nabla u \in L^1((0, \infty); L^\infty) \quad \text{and} \quad a \in L^\infty((0, \infty) \times \mathbb{T}^d).$$

To prove Proposition 1.1 we consider the linearization of the system (1.12) around the ground state $(a, u) = (0, 0)$, see (3.1) below. We establish well-posedness of the linearization in Lemma 3.1. To this end, we apply the explicit formula for the linear system, and we use a multiplier theorem, which is a version of the 1939 Marcinkiewicz theorem [36]. We find the maximal regularity estimate, which then determines the regularity framework used in Proposition 1.1.

In order to consider the nonlinear problem (1.12), we note that the main difficulty of Proposition 1.1 is the appearance of the inverse Laplacian $(-\Delta_u)^{-1}$ in the Eulerian coordinates in (1.12). This term, i.e. $\nabla_u(-\Delta_u)^{-1}a$, needs to be estimated in Besov spaces in Lagrangian coordinates, and this can be achieved by showing that elliptic estimates are stable with respect to the Lagrangian mapping $y \mapsto X(t, y)$, given smallness of the initial data, see (3.31) and (3.35) for example. Using this trick we prove Proposition 1.1 by linearizing (1.12) and then applying Banach Contraction Theorem to obtain a unique global-in-time solution for small data obtaining the existence and uniqueness at once.

We emphasize that the use of the Besov spaces seems irreplaceable, since, although we are working in a bounded domain, we are not able to obtain any exponential time decay. In fact, as mentioned above, the real spectrum of the linearized system (3.1) is not separated from zero, see (3.10).

Finally we discuss possible directions coming from our result. Firstly, it seems possible to extend this analysis to other operators K . Here we consider a very particular one, but there is a zoo of other interesting and physically relevant examples, see [1, 4, 7, 8] for example. The next problem is to consider the case of the whole space \mathbb{R}^3 , which seems to be more natural for problems arising from collective behaviors. Such setting, in comparison with the standard definition of Besov spaces, requires a more subtle functional setting, which is related to a number of mathematical challenges which remain to be addressed. One of them is a natural assumption of finite mass $\int_{\mathbb{R}^d} \rho \, dx < \infty$, which implies decay of ρ at spatial infinity. It is not clear what decay of ρ should be assumed, but assuming compact support of ρ , the system suffers an elliptic degeneration, namely the term ρv_t just disappears.

We also note that even the change of the sign of K in (1.1) results in an unstable system (recall (1.3), see also (3.8) below). In that case it seems natural to expect the density to converge to a single point, namely to a Dirac delta.

The structure of the note is as follows. In the next section we introduce the notion of Besov spaces on the torus, $B_{p,q}^s(\mathbb{T}^d)$. We discuss some basic properties of such spaces, including the Nikol'skij inequality (2.3), embeddings (2.4), as well as maximal regularity of the heat equation (2.6), multiplier properties (2.11), product laws (2.12) and diffeomorphism invariance (2.13). Some of the results we could not find in the literature, and so we provide proofs for the sake of completeness. In Section 3 we first discuss a well-posedness result (Lemma 3.1) of the linearization of the equations in the Lagrangian form (1.12) and then prove Proposition 1.1. Section 4 is devoted to the proof of the claimed equivalence of the Eulerian setting (Theorem 1.1) and the Lagrangian setting (Proposition 1.1).

2. Preliminaries

We denote by $A \lesssim B$ the inequality $A \lesssim CB$, where $C > 0$ is a universal constant. If C depends on some parameters, we denote those using subscripts. By “ \sim ” we mean “ \lesssim and \gtrsim ”. We will use the standard notation of the Sobolev space by $H^\beta := H^\beta(\mathbb{R}^d) \equiv W^{\beta,2}(\mathbb{R}^d)$. We will also write $L^p \equiv L^p(\mathbb{T}^d)$, and $\|\cdot\|_p \equiv \|\cdot\|_{L^p}$.

Given $u: \mathbb{T}^d \rightarrow \mathbb{R}$ and $k \in \mathbb{Z}^d$ we denote its k -th Fourier mode by $u_k := \int_{\mathbb{T}^d} u(x) e^{ik \cdot x} dx$. In the case when u also depends on time, for the sake of brevity of the notation, we will abuse the notation to denote its time derivative by u_t ; in such case we will denote the time derivatives of the k -th Fourier mode by u'_k .

Let $M: \mathbb{R}^d \rightarrow \mathbb{R}$ be such that $M \in H^\beta$ for some $\beta > d/2$. Letting $\Lambda \subset \mathbb{Z}^d$ be a finite set, and letting $d_\Lambda := \max_{k,l \in \Lambda} |k-l|$, we recall a Fourier multiplier inequality

$$\left\| \sum_{k \in \Lambda} M(k) u_k e^{ik \cdot x} \right\|_p \leq C \|M_{d_\Lambda}\|_{H^\beta} \|u\|_p, \quad p \in [1, \infty], \beta > d/2, \tag{2.1}$$

where $C = C(d, p, \beta)$ and $M_\lambda := M(\lambda \cdot)$ denotes the λ -dilation of M . We refer the reader to Section 3.3.4 in [43] for a proof of (2.1). We recall the Nikol'skij inequality,

$$\|f\|_q \lesssim_{p,q} d_\Lambda^{\frac{d}{p} - \frac{d}{q}} \|f\|_p \tag{2.2}$$

for $p, q \in [1, \infty]$ such that $q \geq p$ and for $f \in L^p$ such that $f_k = 0$ for $k \notin \Lambda$, see Section 3.3.2 in [43] for a proof.

To define Besov spaces $B_{p,q}^s(\mathbb{T}^d)$ we first let $\chi \in C^\infty(\mathbb{R}; [0, 1])$ be such that $\chi(z) = 1$ for $z \leq 1$ and $\chi(z) = 0$ for $z \geq 2$, and we set

$$\phi_1(x) := \chi(|x|/2) - \chi(|x|), \quad \phi_j := \phi_1(2^{-j} \cdot) \quad \text{for } j \geq 2, \quad \phi_0(x) := \chi(|x|)$$

for $x \in \mathbb{R}^d$. We also set

$$\psi_j := \phi_1(2^{-j} \cdot) \quad \text{and} \quad \psi_{j \pm k} := \sum_{l=j-k}^{j+k} \psi_l \quad \text{for } j \in \mathbb{Z}, k \geq 0.$$

For $m \geq 0$ we set

$$P_m u := \sum_{k \in \mathbb{Z}^d} \phi_m(k) u_k e^{ik \cdot x}.$$

We note that

$$\|P_m u\|_q \lesssim_{p,q} 2^{\frac{md}{p} - \frac{mq}{q}} \|P_m u\|_p, \tag{2.3}$$

by the Nikol'skij inequality (2.2). Given $p, q \in [1, \infty]$, $s \in \mathbb{R}$ we let $B_{p,q}^s$ denote the space of distributions u such that

$$\|u\|_{B_{p,q}^s}^q := \sum_{m \geq 0} 2^{smq} \|P_m u\|_p^q < \infty,$$

recall [43, Definition 3.5.1(i)]. In this work we will only be concerned with functions with vanishing mean, i.e. $\int_{\mathbb{T}^d} f = 0$, for which the above sum can be taken over $m \geq 1$.

Note that $B_{p,q}^s$ is a Banach space by Theorem 1 in Section 3.5.1 in [43]. We recall the embedding

$$B_{p,1}^{d/p+\delta}(\mathbb{T}^d) \subset B_{p,1}^{d/p}(\mathbb{T}^d) \subset C^0(\mathbb{T}^d) \tag{2.4}$$

for every $\delta > 0$, $p \in [1, \infty]$, see [43, p. 170].

Suppose that $f, g \in L_{loc}^1((0, \infty); L^1)$ are such that $\int_{\mathbb{T}^d} f(t) = \int_{\mathbb{T}^d} g(t) = 0$ for each $t > 0$ and that

$$f_t - \Delta f = g \tag{2.5}$$

hold in the sense of distributions in $\mathbb{T}^d \times \mathbb{R}_+$. Then

$$\|f_t\|_{L^1 B_{p,1}^s} + \|f\|_{L^1 B_{p,1}^{s+2}} \lesssim \|g\|_{L^1 B_{p,1}^s} \tag{2.6}$$

for every $p \in [1, \infty]$, $s \in \mathbb{R}$. In order to verify (2.6) we first note that the solution f of (2.5) can be characterized in terms of its Fourier coefficients,

$$f_k(t) = \int_0^t e^{-k^2(t-s)} g_k(s) ds \tag{2.7}$$

for every $k \in \mathbb{Z}^d \setminus \{0\}$. Secondly, for every $\alpha > 0$

$$\left\| P_m \left(\sum_{k \in \mathbb{Z}^d} e^{-\alpha k^2} g_k e^{ik \cdot x} \right) \right\|_p \lesssim_p e^{-c\alpha 2^{2m}} \|P_m g\|_p, \tag{2.8}$$

where $c > 0$ is a constant.

Let $m \geq 1$. We take $u := P_m g$, $M(\xi) := e^{-\alpha|\xi|^2} \phi_{m\pm 1}(\xi)$. We take $N := [d/2] + 1$, $\Lambda := \{k \in \mathbb{Z}^d : 2^{m-1} \leq |k| \leq 2^{m+1}\}$. We have $d_\Lambda \sim 2^m$, which gives that

$$\begin{aligned} \|M_{d_\Lambda}\|_{HN}^2 &= \sum_{|\gamma| \leq N} d_\Lambda^{2|\gamma|} \int |D^\gamma M(d_\Lambda \xi)|^2 d\xi \lesssim \sum_{|\gamma| \leq N} 2^{(2|\gamma|-d)m} \int |D^\alpha M|^2 \\ &\lesssim \sum_{|\gamma| \leq N} 2^{(2|\gamma|-d)m} \int_{\{2^{m-2} \leq |\xi| \leq 2^{m+2}\}} \left((2^{-m} + |\alpha\xi| + \dots + |\alpha\xi|^{|\gamma|}) e^{-\alpha|\xi|^2} \right)^2 d\xi \\ &\lesssim e^{-\alpha 2^{2m-4}} \int \left(Q_N(\alpha|\xi|^2) e^{-\frac{\alpha|\xi|^2}{2}} \right)^2 d\xi \lesssim e^{-\alpha 2^{2m-4}}, \end{aligned} \tag{2.9}$$

where Q_N denotes a polynomial of order N and, in the second inequality, we obtained the term “ 2^{-m} ” in the case when all derivatives fall onto $\phi_{m\pm 1}$. If k derivatives fall on “ $e^{-\alpha|\xi|^2}$ ” we obtain “ $|\alpha\xi|^k$ ”, and then each of the other $|\gamma| - k$ derivatives give factors of 2^{-m} , as $|\xi| \sim 2^m$ and $m \geq 1$. In the third inequality above we also used our choice of N , which implies that $2|\gamma| - d \leq 2$. Applying the multiplier inequality (2.1) gives (2.8), as required.

Multiplying (2.7) by $\phi_m e^{ik \cdot x}$, summing in $k \in \mathbb{Z}^d$ and taking the L^p norm we obtain

$$\|P_m f(t)\|_p \leq \int_0^t \left\| P_m \left(\sum_{k \in \mathbb{Z}^d} e^{-(t-s)k^2} g_k(s) e^{ik \cdot x} \right) \right\|_p ds \lesssim \int_0^t e^{-c(t-s)2^{2m}} \|P_m g(s)\|_p ds,$$

where we used (2.8) with $\alpha := t - s$ in the last step. Integration over $t \in (0, \infty)$ and using Young’s inequality $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ gives

$$\int_0^\infty \|P_m f(t)\|_p dt \lesssim 2^{-2m} \int_0^\infty \|P_m g(t)\|_p dt$$

for every m . Multiplying both sides by 2^{2m} , summing in m and applying the Tonelli theorem gives that $\|f\|_{L^1 B_{p,1}^{s+2}} \lesssim \|g\|_{L^1 B_{p,1}^s}$. This and (2.5) prove (2.6), as required.

As a simple corollary we note that an argument analogous to (2.9) shows that

$$\|P_m f\|_p \lesssim_p \|f\|_p, \quad \|P_m D^\gamma f\|_p \lesssim_p 2^{|\gamma|m} \|P_m f\|_p, \quad \|P_m \Delta f\|_p \sim_p 2^{2m} \|P_m f\|_p \tag{2.10}$$

for every $m \geq 1$, $p \in [1, \infty]$ (by taking taking, respectively, $M(\xi) = \phi_m(\xi)$, $M(\xi) = \xi^\gamma \phi_{m\pm 1}(\xi)$, $M(\xi) = |\xi|^2 \phi_{m\pm 1}(\xi)$ and $M(\xi) = |\xi|^{-2} \phi_{m\pm 1}(\xi)$). In particular

$$\|D^\gamma f\|_{B_{p,q}^s} \lesssim \|f\|_{B_{p,q}^{s+|\gamma|}}, \quad \|\Delta f\|_{B_{p,q}^s} \sim \|f\|_{B_{p,q}^{s+2}} \tag{2.11}$$

for every $p, q \in [1, \infty]$, $s \in \mathbb{R}$, given $\int f = 0$.

We note that

$$\|fg\|_{B_{p,1}^s} \lesssim \|f\|_{B_{p,1}^{d/p}} \|g\|_{B_{p,1}^s} \tag{2.12}$$

for $p \in [2, d]$, $s \in (-d/p, d/p]$, see Proposition 8.1 in [13]; it can also be proved in the same way as the analogous claim for nonhomogeneous Besov spaces on \mathbb{R}^3 , see Theorem 2(i) in Section 4.6.1 in [42].

Finally we note that the $B_{p,1}^s$, for $s \in (0,1)$ norm is equivalent to the Lipschitz norm,

$$\|f\|_{B_{p,1}^s} \sim_{d,s,p} \|f\|_p + \int \left(\int \frac{|f(y) - f(x)|^p}{|y - x|^{p(d+s)}} dy \right)^{\frac{1}{p}} dx$$

for $s \in (0,1)$, $p \in [1, \infty)$, see (18) on p. 169 in [43]. See also (4) on p. 110 in [46].

We can deduce from it that $B_{p,1}^s$ is invariant under diffeomorphisms for $s \in (0,1)$. Namely, given a diffeomorphism $Z: \mathbb{T}^d \rightarrow \mathbb{T}^d$ we have

$$\|f \circ Z\|_{B_{p,1}^s} \sim_{s,d,p} C(\|\nabla Z\|_\infty, \|(\nabla Z)^{-1}\|_\infty) \|f\|_{B_{p,1}^s}, \tag{2.13}$$

for $s \in (0,1)$, $p \in [1, \infty]$, by applying the change of variable $y \mapsto Z(y)$, using the Mean Value Theorem and estimating the Jacobian by the L^∞ norms of ∇Z and $(\nabla Z)^{-1}$, see Lemma 2.1.1 in [15], or Proposition A.1 in [12], for details. In what follows we will apply (2.13) for $s := d/p - 1$, which belongs to $(0,1)$, due to our restriction on p , namely $p \in (\min(d/2, 2), d)$.

In what follows we will use a shorthand notation

$$L^p B^s \equiv L^p((0, \infty); B_{p,1}^s(\mathbb{T}^d)).$$

3. Proof of Theorem 1.1

In this section we prove Proposition 1.1, which is equivalent to Theorem 1.1 (see Section 4 below). We first consider the following compressible Stokes system,

$$\begin{aligned} a_t + \operatorname{div} u &= h && \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ u_t - \nu \Delta u + \nabla(Ka) &= g && \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ a|_{t=0} &= a_0, \quad u|_{t=0} = u_0 && \text{in } \mathbb{T}^d, \end{aligned} \tag{3.1}$$

where g, h are given. This system is a linearization of (1.12), and the following lemma determines the types of spaces which we will use to estimate u and a .

LEMMA 3.1 (Solution of the linear system). *Given $s \in \mathbb{R}$, $p \in [1, \infty]$, $a_0 \in B_{p,1}^{s+1}$, $u_0 \in B_{p,1}^s$, $g \in L^1 B_{p,1}^s$, $h \in L^1 B_{p,1}^{s+1}$ the system (3.1) admits a unique solution (a, u) such that*

$$\begin{aligned} & \|a\|_{L^\infty B_{p,1}^{s+1}} + \|a_t\|_{L^1 B_{p,1}^{s+1}} + \|a - \{a\}\|_{L^1 B_{p,1}^{s-1}} + \|u\|_{L^\infty B_{p,1}^s} \\ & + \|u_t\|_{L^1 B_{p,1}^s} + \|u - \{u\}\|_{L^1 B_{p,1}^{s+2}} \\ & \leq c_\nu \|a_0\|_{B_{p,1}^{s+1}} + \|u_0\|_{B_{p,1}^s} + \|h\|_{L^1 B_{p,1}^{s+1}} + \|g\|_{L^1 B_{p,1}^s}. \end{aligned} \tag{3.2}$$

The lemma can be proved by first taking div of the second equation to obtain an evolution equation for $d := \operatorname{div} u$. Taking the time derivative of the resulting PDE and substituting a_t from the first equation we obtain an autonomous PDE on d , which we can solve by translating it into a family of second order ODEs for the Fourier coefficients of d . This allows us to find a from the first equation. We can then use it to find u from the second equation.

Proof. We first assume that $g = 0$ and $u_0 = 0$. Indeed, otherwise, we denote by \tilde{u} the solution of the heat equation with initial data u_0 and forcing g , i.e. we set

$$\tilde{u}_k(t) := \int_0^t g_k e^{-\nu k^2(t-s)} ds + e^{-\nu k^2 t} u_{0k}. \tag{3.3}$$

Maximal regularity (2.6) gives that

$$\|\tilde{u}_t\|_{L^1 B_{p,1}^s} + \|\Delta\tilde{u}\|_{L^1 B_{p,1}^s} \lesssim \|g\|_{L^1 B_{p,1}^s} + \|u_0\|_{B_{p,1}^s}. \tag{3.4}$$

Then $(a, u - \tilde{u})$ satisfies (3.1) with $g=0$, $(u - \tilde{u})_{t=0} = 0$ and the right-hand side of the equation for a equals

$$\tilde{h} := h - \operatorname{div} \tilde{u}. \tag{3.5}$$

We note that

$$\|\tilde{h}\|_{L^1 B_{p,1}^s} \lesssim \|h\|_{L^1 B_{p,1}^s} + \|\Delta\tilde{u}\|_{L^1 B_{p,1}^{s-1}} \lesssim \|h\|_{L^1 B_{p,1}^s} + \|g\|_{L^1 B_{p,1}^{s-1}} + \|u_0\|_{B_{p,1}^{s-1}}$$

for all s . Thus, if the lemma is valid in the homogeneous case $g=0$, $u_0=0$, then it is also valid in the inhomogeneous case. We can thus assume that $g=0$ and $u_0=0$.

Taking div of the second equation of (3.1) and setting $d := \operatorname{div} u$ we obtain

$$d_t - \nu \Delta d - (a - \{a\}) = 0, \tag{3.6}$$

where we also used the fact that $\Delta(Ka) = -(a - \{a\})$. Taking the time derivative and recalling that $a_t = h - d$ we obtain

$$d_{tt} - \nu \Delta d_t + d = -(h - \{h\})$$

with initial data $d|_{t=0} = 0$ and from (3.6) $d_t|_{t=0} = a_0$.

In terms of Fourier coefficients we obtain a second order ODE

$$d_k'' + \nu k^2 d_k' + d_k = -h_k \tag{3.7}$$

for $k \neq 0$. (Note that $d_0 = \int \operatorname{div} u = 0$.) The roots of the characteristic polynomial $\lambda^2 + \nu k^2 \lambda + 1$ are

$$\lambda_k^\pm = (-\nu k^2 \pm \sqrt{\nu^2 k^4 - 4})/2, \tag{3.8}$$

where $k^2 := k_1^2 + k_2^2 + \dots + k_d^2$. Let us first assume that $\nu^2 k^4 \neq 4$ for all $k \in \mathbb{Z}^d \setminus \{0\}$. Then there exists $C_\nu > 0$ such that

$$|\lambda_k^+ - \lambda_k^-| \geq C_\nu k^2 \quad \text{for } k \in \mathbb{Z}^d \setminus \{0\}. \tag{3.9}$$

Note also that $\operatorname{Re} \lambda_k^+, \operatorname{Re} \lambda_k^- < 0$ with

$$|\lambda_k^+| \sim |\operatorname{Re} \lambda_k^+| \sim C_\nu k^{-2}, \quad |\lambda_k^-| \sim |\operatorname{Re} \lambda_k^-| \sim C_\nu k^2 \tag{3.10}$$

for $k \neq 0$. We note that the above behavior of the roots λ_k^+, λ_k^- determines the properties of the spectrum of the operator coming from the linear system (3.1). In particular, as mentioned in the introduction, we emphasize that, although the d -dimensional torus \mathbb{T}^d is bounded, the spectrum is not separated from zero, which excludes the possibility of the exponential decay of solutions.

We now write the explicit form of the solution,

$$d_k(t) = A_k e^{\lambda_k^+ t} + B_k e^{\lambda_k^- t} + \frac{1}{\lambda_k^+ - \lambda_k^-} \int_0^t h_k(s) \left(e^{\lambda_k^+(t-s)} - e^{\lambda_k^-(t-s)} \right) ds, \tag{3.11}$$

where $A_k, B_k \in \mathbb{R}$ are such that

$$\begin{pmatrix} 1 & 1 \\ \lambda_k^+ & \lambda_k^- \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} 0 \\ a_{0k} \end{pmatrix}.$$

In particular, (3.9) gives that

$$|A_k|, |B_k| \leq C_\nu k^{-2} |a_{0k}|. \tag{3.12}$$

If $\nu k^2 = 2$ for some $k \in \mathbb{Z}^d \setminus \{0\}$ then

$$d_k(t) = A_k e^{-t} + B_k t e^{-t} + \int_0^t \int_0^s h_k(\tau) e^{-(t-\tau)} d\tau ds$$

for such k , where $A_k := 0, B_k := a_{0k}$. In particular (3.12) follows in this case as well.

Thus considering the modes $k \sim 2^m$ we can use (2.8) to obtain

$$\begin{aligned} \int_0^\infty \|P_m d(t)\|_p dt &\leq c_\nu 2^{-2m} \|P_m a_0\|_p \int_0^\infty \left(e^{-c2^{-2m}t} + e^{-c2^{2m}t} \right) dt \\ &\quad + 2^{-2m} \left\| \|P_m h\|_p * e^{-c2^{-2m}t} + \|P_m h\|_p * e^{-c2^{2m}t} \right\|_{L^1_t} \\ &\leq c \|P_m a_0\|_p + \int_0^\infty \|P_m h(t)\|_p dt \end{aligned} \tag{3.13}$$

for $m \geq 1$, with $\|P_0 d(t)\|_p \sim d_0(t) = 0$ for all $t > 0$.

Similarly,

$$d'_k(t) = A_k \lambda_k^+ e^{\lambda_k^+ t} + B_k \lambda_k^- e^{\lambda_k^- t} + \frac{1}{\lambda_k^+ - \lambda_k^-} \int_0^t h(s) \left(\lambda_k^+ e^{\lambda_k^+(t-s)} - \lambda_k^- e^{\lambda_k^-(t-s)} \right) ds$$

for every $k \in \mathbb{Z}^3 \setminus \{0\}$, and, analogously to (3.13), (2.8) gives that ($k \sim 2^m$)

$$\int_0^\infty \|P_m d_t(t)\|_p dt \lesssim 2^{-2m} \|P_m a_0\|_p + 2^{-2m} \int_0^\infty \|P_m h(t)\|_p dt$$

for $m \geq 1$. This and (3.13) implies that

$$\begin{aligned} \|d\|_{L^1 B_{p,1}^s} &\lesssim_\nu \|a_0\|_{B_{p,1}^s} + \|h\|_{L^1 B_{p,1}^s}, \\ \|d\|_{L^\infty B_{p,1}^s} + \|d_t\|_{L^1 B_{p,1}^s} &\lesssim_\nu \|a_0\|_{B_{p,1}^{s-2}} + \|h\|_{L^1 B_{p,1}^{s-2}}. \end{aligned} \tag{3.14}$$

Moreover, using (3.6) we see that $a - \{a\} = d_t - \Delta d$, which implies that

$$\|a - \{a\}\|_{L^1 B_{p,1}^s} \lesssim_\nu \|d_t\|_{L^1 B_{p,1}^s} + \|d\|_{L^1 B_{p,1}^{s+2}} \lesssim_\nu \|a_0\|_{B_{p,1}^{s+2}} + \|h\|_{L^1 B_{p,1}^{s+2}}, \tag{3.15}$$

where we applied (2.4) to write $\|d_t\|_{L^1 B_{p,1}^s} \lesssim \|d_t\|_{L^1 B_{p,1}^{s+4}}$, and used (3.14). On the other hand, $a_t = -d + h$, which gives that

$$\|a\|_{L^\infty B_{p,1}^s} + \|a_t\|_{L^1 B_{p,1}^s} \lesssim \|a_0\|_{B_{p,1}^s} + \|d\|_{L^1 B_{p,1}^s} + \|h\|_{L^1 B_{p,1}^s} \lesssim \|a_0\|_{B_{p,1}^s} + \|h\|_{L^1 B_{p,1}^s}. \tag{3.16}$$

Moreover, recalling that $f \mapsto \nabla K f$ is an operator of order -1 , we can use maximal regularity (2.6) of the second equation of (3.1),

$$u_t - \Delta u = -\nabla K a \tag{3.17}$$

to obtain

$$\|u\|_{L^\infty B_{p,1}^s} + \|u_t\|_{L^1 B_{p,1}^s} + \|\Delta u\|_{L^1 B_{p,1}^s} \lesssim_\nu \|a\|_{L^1 B_{p,1}^{s-1}} \lesssim_\nu \|a_0\|_{B_{p,1}^{s+1}} + \|h\|_{L^1 B_{p,1}^{s+1}},$$

where we used (3.15) in the last inequality. This, (3.15) and (3.16) give (3.2), as required.

The estimates (3.2) prove uniqueness of solutions. As for existence, we first define d by (3.11) and then we set $a := \{a_0\} + \int_0^t \{h(s)\} ds + d_t - \Delta d$. \square

We can now prove Proposition 1.1

Proof. (Proof of Proposition 1.1.) We rewrite (1.12) in the form

$$\begin{aligned} a_t + \operatorname{div} u &= -a \operatorname{div} u + (1+a)(\operatorname{div} - \operatorname{div}_u)u \\ &=: h, \\ u_t - \Delta u + \nabla(-\Delta)^{-1}a &= a \nabla(-\Delta)^{-1}a - a u_t + (\Delta_u - \Delta)u \\ &\quad + (1+a)(\nabla(-\Delta)^{-1} - \nabla_u(-\Delta_u)^{-1})a \\ &=: g, \end{aligned} \tag{3.18}$$

and note that Lemma 3.1 gives that

$$\begin{aligned} &\|a\|_{L^\infty B^{d/p}} + \|a_t\|_{L^1 B^{d/p}} + \|a - \{a\}\|_{L^1 B^{d/p-2}} \\ &\quad + \|u\|_{L^\infty B^{d/p-1}} + \|u_t\|_{L^1 B^{d/p-1}} + \|u - \{u\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \|a_0\|_{B^{d/p}} + \|u_0\|_{B^{d/p-1}} + \|h\|_{L^1 B^{d/p}} + \|g\|_{L^1 B^{d/p-1}}. \end{aligned} \tag{3.19}$$

Assuming that

$$\|u - \{u\}\|_{L^1 B^{d/p+1}} \leq \gamma, \tag{3.20}$$

where $\gamma \in (0, 1)$ is a sufficiently small constant, we show in Step 1 below that

$$\|h\|_{L^1 B^{d/p}} \lesssim \|a\|_{L^\infty B^{d/p}} \|u - \{u\}\|_{L^1 B^{d/p}} + (1 + \|a\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}}^2 \tag{3.21}$$

and in Step 2 that

$$\begin{aligned} \|g\|_{L^1 B^{d/p-1}} &\lesssim \|a\|_{L^\infty B^{d/p}} (\|a - \{a\}\|_{L^1 B^{d/p-2}} + \|u_t\|_{L^1 B^{d/p-1}}) + \|u - \{u\}\|_{L^1 B^{d/p+1}}^2 \\ &\quad + (1 + \|a\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}} \|a - \{a\}\|_{L^1 B^{d/p-2}}. \end{aligned} \tag{3.22}$$

Thanks to these estimates we can use (3.19) to obtain the a priori bound

$$\begin{aligned} &\|a\|_{L^\infty B^{d/p}} + \|a_t\|_{L^1 B^{d/p}} + \|a - \{a\}\|_{L^1 B^{d/p-2}} \\ &\quad + \|u\|_{L^\infty B^{d/p-1}} + \|u_t\|_{L^1 B^{d/p-1}} + \|u - \{u\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \|a_0\|_{B^{d/p}} + \|u_0\|_{B^{d/p-1}} + \|h\|_{L^1 B^{d/p}} + \|g\|_{L^1 B^{d/p-1}} \\ &\lesssim \|a_0\|_{B^{d/p}} + \|u_0\|_{B^{d/p-1}} \\ &\quad + \|a\|_{L^\infty B^{d/p}} (\|a - \{a\}\|_{L^1 B^{d/p-2}} + \|u_t\|_{L^1 B^{d/p-1}} + \|u - \{u\}\|_{L^1 B^{d/p+1}}) \\ &\quad + (1 + \|a\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}} (\|a - \{a\}\|_{L^1 B^{d/p-2}} + \|u - \{u\}\|_{L^1 B^{d/p+1}}). \end{aligned} \tag{3.23}$$

Note that the right-hand sides of (3.21), (3.22) are at least quadratic in (a, u) . This allows us to use the a priori bound to prove claim using Banach Contraction Theorem, which we discuss in Step 3 for the sake of completeness.

Step 1. We prove (3.21).

We note that

$$\|a \operatorname{div} u\|_{L^1 B^{d/p}} \lesssim \|a\|_{L^\infty B^{d/p}} \|u - \{u\}\|_{L^1 B^{d/p+1}}. \tag{3.24}$$

As for the other ingredient of h , we first use (1.10) to expand A as the Neumann series

$$A - I = \sum_{k \geq 1} \left(- \int_0^t \nabla u \right)^k, \tag{3.25}$$

where I denotes the $d \times d$ identity matrix. Taking the $L^\infty B^{d/p}$ norm we see that

$$\begin{aligned} \|A - I\|_{L^\infty B^{d/p}} &\leq \sum_{k \geq 1} \left\| \int_0^t \nabla u \right\|_{L^\infty B^{d/p}}^k \\ &\leq \sum_{k \geq 1} \|u - \{u\}\|_{L^1 B^{d/p+1}}^k \leq 2 \|u - \{u\}\|_{L^1 B^{d/p+1}} \leq 2\gamma, \end{aligned} \tag{3.26}$$

provided $\gamma < 1/2$, where we used (3.20). We note that, since $\nabla = \left(\frac{\partial X}{\partial y}\right)^T \nabla_u$ we have $\nabla_u = A^T \nabla$. Thus

$$(\operatorname{div} - \operatorname{div}_u)u = (\delta_{ij} - A_{ji})\partial_j u_i, \tag{3.27}$$

and consequently

$$\begin{aligned} \|(1+a)(\operatorname{div} - \operatorname{div}_u)u\|_{L^1 B^{d/p}} &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) \|(\delta_{ij} - A_{ji})\partial_j u_i\|_{L^1 B^{d/p}} \\ &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) \|I - A\|_{L^\infty B^{d/p}} \|u - \{u\}\|_{L^1 B^{d/p+1}} \\ &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}}^2, \end{aligned} \tag{3.28}$$

as required, where we used (3.26) in the last step.

Step 2. We prove (3.22).

As for the first two ingredients of g we obtain

$$\|a \nabla (-\Delta)^{-1} a - a u_t\|_{L^1 B^{d/p-1}} \lesssim \|a\|_{L^\infty B^{d/p}} (\|a - \{a\}\|_{L^1 B^{d/p-2}} + \|u_t\|_{L^1 B^{d/p-1}}), \tag{3.29}$$

as required.

As for the remaining two ingredients we first note that

$$\begin{aligned} \Delta_u - \Delta &= \operatorname{div}_u \nabla_u - \operatorname{div} \nabla = \operatorname{div}(\nabla_u - \nabla) + (\operatorname{div}_u - \operatorname{div})\nabla_u \\ &= \partial_j ((A_{ij} - \delta_{ij})\partial_i) + (A_{ji} - \delta_{ij})\partial_j (\partial_i + (\delta_{ik} - A_{ki})\partial_k) \end{aligned} \tag{3.30}$$

which gives that

$$\begin{aligned} \|(\Delta_u - \Delta)u\|_{B^s} &\leq \|(A^T - I)\nabla u\|_{B^{s+1}} + \|A - I\|_{B^{d/p}} \|\nabla u + (I - A^T)\nabla u\|_{B^{s+1}} \\ &\lesssim \|A - I\|_{B^{d/p}} (1 + \|A - I\|_{B^{d/p}}) \|u - \{u\}\|_{B^{s+2}} \end{aligned} \tag{3.31}$$

for each time and $s = d/p - 2, d/p - 1$, where we used (2.12) in the first line. We note in passing that (3.31) is the main reason for our restriction on the range of p , due to

the order restriction in the product law (2.12). Taking $s = d/p - 1$ we can estimate the third ingredient of g ,

$$\begin{aligned} \|(\Delta_u - \Delta)u\|_{L^1 B^{d/p-1}} &\leq \|A - I\|_{L^\infty B^{d/p}} (1 + \|A - I\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \|u - \{u\}\|_{L^1 B^{d/p+1}}^2, \end{aligned} \tag{3.32}$$

where we used (3.26) and (3.20).

On the other hand, taking $s = d/p - 2$ in (3.31) gives an elliptic estimate

$$\|(-\Delta_u)^{-1}w\|_{B^{d/p}} \lesssim \|w\|_{B^{d/p-2}} \tag{3.33}$$

for $w \in B^{d/p-2}$ with $\int w = 0$. Indeed letting $f := (-\Delta_u)^{-1}w$ we see that f satisfies the Poisson equation

$$-\Delta f = -\Delta_u f + (\Delta_u - \Delta)f = w + (\Delta_u - \Delta)f$$

on the torus, which, after noting that $w = (-\Delta)(-\Delta)^{-1}w$, gives

$$-\Delta((-\Delta_u)^{-1} - (-\Delta)^{-1})w = (\Delta_u - \Delta)f.$$

Taking the $B^{d/p-2}$ norm gives

$$\begin{aligned} &\|((-\Delta_u)^{-1} - (-\Delta)^{-1})w\|_{B^{d/p}} \\ &\lesssim \|(\Delta_u - \Delta)f\|_{B^{d/p-2}} \lesssim \|A - I\|_{B^{d/p}} (1 + \|A - I\|_{B^{d/p}}) \|f\|_{B^{d/p}}, \end{aligned} \tag{3.34}$$

where we used (3.31) with $s = d/p - 2$ in the second inequality. In particular

$$\|f\|_{B^{d/p}} \lesssim \|w\|_{B^{d/p-2}} + \|(-\Delta_u)^{-1}w - (-\Delta)^{-1}w\|_{B^{d/p}} \lesssim \|w\|_{B^{d/p-2}} + \gamma \|f\|_{B^{d/p}},$$

where we used (3.26) in the last inequality. The elliptic estimate (3.33) follows if γ is chosen sufficiently large so that the last term can be absorbed by the left-hand side.

The last ingredient of g can now be estimated by noting the identity

$$\nabla(-\Delta)^{-1} - \nabla_u(-\Delta_u)^{-1} = \nabla((-\Delta)^{-1} - (-\Delta_u)^{-1}) - (\nabla_u - \nabla)(-\Delta_u)^{-1},$$

which gives that

$$\begin{aligned} &\|(1 + a)(\nabla(-\Delta)^{-1} - \nabla_u(-\Delta_u)^{-1})a\|_{L^1 B^{d/p-1}} \\ &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) (\|((-\Delta)^{-1} - (-\Delta_u)^{-1})(a - \{a\})\|_{L^1 B^{d/p}} \\ &\quad + \|(\nabla_u - \nabla)(-\Delta_u)^{-1}(a - \{a\})\|_{L^1 B^{d/p-1}}) \\ &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) \|I - A\|_{L^\infty B^{d/p}} (1 + \|A - I\|_{L^\infty B^{d/p}}) \|(-\Delta_u)^{-1}(a - \{a\})\|_{L^1 B^{d/p}} \\ &\lesssim (1 + \|a\|_{L^\infty B^{d/p}}) \|u - \{u\}\|_{L^1 B^{d/p+1}} \|a - \{a\}\|_{L^1 B^{d/p-2}}, \end{aligned} \tag{3.35}$$

as required, where we used (3.34), the product rule (2.12) and the fact that $\nabla_u - \nabla = (A^T - I)\nabla$ in the second inequality, as well as (3.26) in the last line.

Step 3. We prove the claim. We set

$$\begin{aligned} \|(u, a)\| &:= \|a\|_{L^\infty B^{d/p}} + \|a_t\|_{L^1 B^{d/p}} + \|a - \{a\}\|_{L^1 B^{d/p-2}} \\ &\quad + \|u\|_{L^\infty B^{d/p-1}} + \|u_t\|_{L^1 B^{d/p-1}} + \|u - \{u\}\|_{L^1 B^{d/p+1}} \end{aligned}$$

and

$$V := \{(u, a) : \|(u, a)\| < \infty\}.$$

Then V equipped with the norm $\|\cdot\|$ is a Banach space. Given $(v, \alpha) \in V$, we let $S(v, \alpha)$ denote the solution of the linear system (3.1) with

$$\begin{aligned} h &:= -\alpha \operatorname{div} v + (1 + \alpha)(\operatorname{div} - \operatorname{div}_v)v, \\ g &:= \alpha \nabla(-\Delta)^{-1}\alpha - \alpha v_t + (\Delta_v - \Delta)v + (1 + \alpha)(\nabla(-\Delta)^{-1} - \nabla_v(-\Delta_v)^{-1})\alpha. \end{aligned}$$

By (3.23) we see that $S: V \rightarrow V$. By Lemma 3.1 there exists $C > 0$ such that $\|S(0, 0)\| \leq C(\|a_0\|_{B^{d/p}} + \|u_0\|_{B^{d/p-1}})/2 \leq C\epsilon/2$. By the a priori estimate (3.23) we obtain that $\|S(v, \alpha) - S(0, 0)\| \lesssim \|(v, \alpha)\|^2 \leq C^2\epsilon^2$ for every $(v, \alpha) \in B := B(0, C\epsilon)$. Thus S maps B into itself for sufficiently small $\epsilon > 0$, as $\|S(v, \alpha)\| \leq \frac{C\epsilon}{2} + \|S(v, \alpha) - S(0, 0)\| \leq C\epsilon$ for $(v, \alpha) \in B$. We show below that S is a contraction on B , namely that

$$\|S(v, \alpha) - S(w, \beta)\| \leq \frac{1}{2} \|(v - w, \alpha - \beta)\| =: \frac{1}{2}d \tag{3.36}$$

for all $(v, \alpha), (w, \beta) \in B$, given $\epsilon > 0$ is chosen sufficiently small. Banach Contraction Theorem then gives the claimed existence and uniqueness result.

Letting $(u, a) := S(v, \alpha) - S(w, \beta)$ we see that (u, a) is a solution to the problem

$$\begin{aligned} a_t + \operatorname{div} u &= \delta h && \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ u_t - \nu \Delta u + \nabla(Ka) &= \delta g && \text{in } \mathbb{R}_+ \times \mathbb{T}^d \end{aligned} \tag{3.37}$$

with homogeneous initial conditions $a(0) = 0, u(0) = 0$, where

$$\begin{aligned} \delta h &:= -\alpha \operatorname{div} v + (1 + \alpha)(\operatorname{div} - \operatorname{div}_v)v - \beta \operatorname{div} w + (1 + \beta)(\operatorname{div} - \operatorname{div}_w)w \\ &= -(\alpha - \beta) \operatorname{div} v - w \operatorname{div}(v - w) + (\alpha - \beta)(\operatorname{div} - \operatorname{div}_v)v \\ &\quad + (1 + \beta)((\operatorname{div} - \operatorname{div}_v)v - (\operatorname{div} - \operatorname{div}_w)w) \\ &= -(\alpha - \beta) \operatorname{div} v - \beta \operatorname{div}(v - w) + (\alpha - \beta)(\operatorname{div} - \operatorname{div}_v)v \\ &\quad + (1 + \beta)((\operatorname{div} - \operatorname{div}_w)(v - w)) + (1 + \beta)((\operatorname{div}_w - \operatorname{div}_v)v) \end{aligned}$$

and

$$\begin{aligned} \delta g &:= \alpha \nabla(-\Delta)^{-1}\alpha - \alpha v_t + (\Delta_v - \Delta)v + (1 + \alpha)(\nabla(-\Delta)^{-1} - \nabla_v(-\Delta_v)^{-1})\alpha \\ &\quad - (\beta \nabla(-\Delta)^{-1}\beta - \beta w_t + (\Delta_w - \Delta)w + (1 + \beta)(\nabla(-\Delta)^{-1} - \nabla_w(-\Delta_w)^{-1})\beta) \\ &= \underbrace{(\alpha - \beta) \nabla(-\Delta)^{-1}\alpha + \beta \nabla(-\Delta)^{-1}(\alpha - \beta)}_{=: \delta g_1} - \underbrace{(\alpha - \beta)v_t - \beta(v - w)_t}_{=: \delta g_2} \\ &\quad + \underbrace{(\Delta_v - \Delta)(v - w)}_{=: \delta g_3} + \underbrace{(\Delta_v - \Delta_w)w}_{=: \delta g_4} \\ &\quad + \underbrace{(\alpha - \beta)(\nabla(-\Delta)^{-1} - \nabla_v(-\Delta_v)^{-1})\alpha}_{=: \delta g_5} + \underbrace{(1 + \beta)(\nabla(-\Delta)^{-1} - \nabla_v(-\Delta_v)^{-1})(\alpha - \beta)}_{=: \delta g_6} \\ &\quad + \underbrace{(1 + \beta)(\nabla_w(-\Delta_w)^{-1} - \nabla_v(-\Delta_v)^{-1})\beta}_{=: \delta g_7}. \end{aligned}$$

In the remainder of the proof we verify that

$$\|\delta h\|_{L^1 B^{d/p}} + \|\delta g\|_{L^1 B^{d/p-1}} \lesssim \epsilon d$$

whenever $(v, \alpha), (w, \beta) \in B$. This and Lemma 3.1 proves the required contraction property (3.36) if $\epsilon > 0$ is chosen sufficiently small.

Looking at the structure of δh we see that the first four terms can be bounded in $\|\cdot\|_{L^1 B^{d/p}}$ in the same way as in Step 1 above (recall (3.24) and (3.28)), to give the upper bound

$$\begin{aligned} & \|\alpha - \beta\|_{L^\infty B^{d/p}} \|v - \{v\}\|_{L^1 B^{d/p+1}} + \|\beta\|_{L^\infty B^{d/p}} \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \\ & + \|\alpha - \beta\|_{L^\infty B^{d/p}} \|I - \bar{A}\|_{L^\infty B^{d/p}} \|v - \{v\}\|_{L^1 B^{d/p+1}} \\ & + (1 + \|\beta\|_{L^\infty B^{d/p}}) \|I - \bar{\bar{A}}\|_{L^\infty B^{d/p}} \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \\ & \lesssim \epsilon (\|\alpha - \beta\|_{L^\infty B^{d/p}} + \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}}) \lesssim \epsilon d, \end{aligned}$$

where

$$\bar{A} := I + \sum_{k \geq 1} \left(- \int_0^t \nabla v \right)^k \quad \text{and} \quad \bar{\bar{A}} := I + \sum_{k \geq 1} \left(- \int_0^t \nabla w \right)^k,$$

recall the Neumann expansion (3.25).

As for the last ingredient of δh we recall the Neumann series (3.25) and the algebraic identity $a^k - b^k = (a - b) \sum_{m=0}^{k-1} a^{k-m} b^m$ to write

$$\begin{aligned} \bar{\bar{A}} - \bar{A} &= \sum_{k \geq 1} \left(\left(- \int_0^t \nabla w \right)^k - \left(- \int_0^t \nabla v \right)^k \right) \\ &= \left(- \int_0^t \nabla(w - v) \right) \sum_{k \geq 1} \sum_{m=0}^{k-1} \left(- \int_0^t \nabla w \right)^{k-m} \left(- \int_0^t \nabla v \right)^m. \end{aligned}$$

Thus, taking the $L^\infty B^{d/p}$ norm gives

$$\begin{aligned} \|\bar{\bar{A}} - \bar{A}\|_{L^\infty B^{d/p}} &\leq \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \\ &\quad \times \sum_{k \geq 1} \sum_{m=0}^{k-1} \|w - \{w\}\|_{L^1 B^{d/p+1}}^{k-1-m} \|v - \{v\}\|_{L^1 B^{d/p+1}}^m \\ &\lesssim \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \sum_{k \geq 1} \sum_{m=0}^{k-1} \epsilon^{k-1-m} \epsilon^m \\ &\lesssim \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \sum_{k \geq 1} k \epsilon^{k-1} \lesssim \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}}. \end{aligned} \tag{3.38}$$

Hence, recalling (3.27), we obtain

$$\begin{aligned} \|(1 + \beta)(\operatorname{div}_w - \operatorname{div}_v)v\|_{L^1 B^{d/p}} &\leq (1 + \|\beta\|_{L^\infty B^{d/p}}) \|A_w - A_v\|_{L^\infty B^{d/p}} \|v - \{v\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \|v - \{v\}\|_{L^1 B^{d/p+1}} \lesssim \epsilon d, \end{aligned} \tag{3.39}$$

as required.

As for δg we have

$$\begin{aligned} \|\delta g_1 + \delta g_2\|_{L^1 B^{d/p-1}} &\lesssim \|(\alpha - \beta)\nabla(-\Delta)^{-1}\alpha\|_{L^1 B^{d/p-1}} + \|\beta\nabla(-\Delta)^{-1}(\alpha - \beta)\|_{L^1 B^{d/p-1}} \\ &\quad + \|(\alpha - \beta)v_t\|_{L^1 B^{d/p-1}} + \|\beta(v - w)_t\|_{L^1 B^{d/p-1}} \\ &\lesssim \|\alpha - \beta\|_{L^\infty B^{d/p}} (\|\alpha - \{\alpha\}\|_{L^1 B^{d/p-2}} + \|\beta - \{\beta\}\|_{L^1 B^{d/p-2}} + \|v_t\|_{L^1 B^{d/p-1}}) \\ &\quad + \|\beta\|_{L^\infty B^{d/p}} \|v_t - w_t\|_{L^1 B^{d/p-1}} \\ &\lesssim \epsilon \|(v, \alpha) - (w, \beta)\| \end{aligned}$$

as in (3.29). On the other hand, the first inequality in (3.32) gives

$$\begin{aligned} \|\delta g_3\|_{L^1 B^{d/p-1}} &\lesssim \|\bar{A} - I\|_{L^\infty B^{d/p}} (1 + \|\bar{A} - I\|_{L^\infty B^{d/p}}) \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \epsilon \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}}. \end{aligned}$$

As for δg_4 we have, as in (3.30)

$$\begin{aligned} \Delta_v - \Delta_w &= \operatorname{div}_v(\nabla_v - \nabla_w) + (\operatorname{div}_v - \operatorname{div}_w)\nabla w \\ &= \bar{A}_{ji}\partial_j \left((\bar{A}_{ki} - \bar{\bar{A}}_{ki})\partial_k \right) + (\bar{A}_{ji} - \bar{\bar{A}}_{ji})\partial_j\partial_i, \end{aligned} \tag{3.40}$$

which implies (as in (3.31)) that

$$\begin{aligned} \|(\Delta_v - \Delta_w)w\|_{B^s} &\leq \|\bar{A}\|_{B^{d/p}} \left\| (\bar{A} - \bar{\bar{A}})\nabla w \right\|_{B^{s+1}} + \left\| (\bar{A} - \bar{\bar{A}}) : D^2 w \right\|_{B^s} \\ &\lesssim (1 + \|\bar{A} - I\|_{B^{d/p}}) \|\bar{A} - \bar{\bar{A}}\|_{B^{d/p}} \|w - \{w\}\|_{B^{s+2}} \end{aligned} \tag{3.41}$$

for $s = d/p - 2, d/p - 1$ and each fixed time, which in turn (similarly to (3.32)) gives that

$$\begin{aligned} \|\delta g_4\|_{L^1 B^{d/p-1}} &= \|(\Delta_v - \Delta_w)w\|_{L^1 B^{d/p-1}} \lesssim \left\| \bar{A} - \bar{\bar{A}} \right\|_{L^\infty B^{d/p}} \|w - \{w\}\|_{L^1 B^{d/p+1}} \\ &\lesssim \epsilon \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}}, \end{aligned}$$

where we used (3.38) in the last inequality.

As for δg_5 and δg_6 (3.35) gives that

$$\begin{aligned} \|\delta g_5 + \delta g_6\|_{L^1 B^{d/p-1}} &\lesssim \|\alpha - \beta\|_{L^\infty B^{d/p}} \|v - \{v\}\|_{L^1 B^{d/p+1}} \|\alpha - \{\alpha\}\|_{L^1 B^{d/p-2}} \\ &\quad + (1 + \|\beta\|_{L^\infty B^{d/p}}) \|v - \{v\}\|_{L^1 B^{d/p+1}} \|\alpha - \beta - \{\alpha - \beta\}\|_{L^1 B^{d/p-2}} \\ &\lesssim \|v - \{v\}\|_{L^1 B^{d/p+1}} (\|\alpha - \beta\|_{L^\infty B^{d/p}} + \|\alpha - \beta - \{\alpha - \beta\}\|_{L^1 B^{d/p-2}}). \end{aligned}$$

Finally, $\delta g_7 = (1 + \beta)(\nabla_w(-\Delta_w)^{-1} - \nabla_v(-\Delta_v)^{-1})\beta$ is the most challenging term, where, as in (3.39) above, we need to extract $v - w$ from the difference of differential operators $\nabla_w(-\Delta_w)^{-1} - \nabla_v(-\Delta_v)^{-1}$. To this end we need to explore the steps leading to (3.35) a bit further. Namely, setting

$$f := (-\Delta_w)^{-1}(\beta - \{\beta\}), \quad g := (-\Delta_v)^{-1}(\beta - \{\beta\}),$$

we see that

$$\delta g_7 = (1 + \beta)(\nabla_w(f - g) + (\nabla_w - \nabla_v)g),$$

recalling (1.10) that $\nabla_w = \bar{\bar{A}}^T \nabla$ we obtain

$$\begin{aligned} \|\delta g_7\|_{B^{d/p-1}} &\lesssim \underbrace{(1 + \|\beta\|_{B^{d/p}})}_{\lesssim 1} \|\nabla_w(f - g) + (\nabla_w - \nabla_v)g\|_{B^{d/p-1}} \\ &\lesssim \underbrace{\|\bar{\bar{A}}\|_{B^{d/p}}}_{\lesssim 1} \|f - g\|_{B^{d/p}} + \|\bar{\bar{A}} - \bar{A}\|_{B^{d/p}} \|g\|_{B^{d/p}}. \end{aligned} \tag{3.42}$$

In order to estimate the first term on the right-hand side we note that

$$-\Delta(f - g) = (\Delta_w - \Delta)f - (\Delta_v - \Delta)g = (\Delta_w - \Delta_v)f + (\Delta_v - \Delta)(f - g).$$

This lets us use (3.40) with $s = -1$ (and $(\Delta_v - \Delta_w)w$ replaced by $(\Delta_w - \Delta_v)f$) and (3.31) with $s = -1$ (and $(\Delta_u - \Delta)v$ replaced by $(\Delta_v - \Delta)(f - g)$) to obtain

$$\begin{aligned} \|f - g\|_{B^{d/p}} &\lesssim \|(\Delta_w - \Delta_v)f\|_{B^{d/p-2}} + \|(\Delta_v - \Delta)(f - g)\|_{B^{d/p-2}} \\ &\lesssim \underbrace{\left(1 + \|\bar{A} - I\|_{B^{d/p}}\right)}_{\lesssim 1} \|\bar{A} - \bar{A}\|_{B^{d/p}} \|f\|_{B^{d/p}} \\ &\quad + \underbrace{\|\bar{A} - I\|_{B^{d/p}}}_{\lesssim \epsilon} \underbrace{\left(1 + \|\bar{A} - I\|_{B^{d/p}}\right)}_{\lesssim 1} \|f - g\|_{B^{d/p}}. \end{aligned}$$

Thus, for sufficiently small $\epsilon > 0$ we can absorb the last term on the left-hand side to obtain that

$$\|f - g\|_{B^{d/p}} \lesssim \|\bar{A} - \bar{A}\|_{B^{d/p}} \|f\|_{B^{d/p}}$$

at each time. Applying this in (3.42) and integrating in time we obtain

$$\begin{aligned} \|\delta g_T\|_{L^1 B^{d/p-1}} &\lesssim \|\bar{A} - \bar{A}\|_{L^\infty B^{d/p}} (\|f\|_{L^1 B^{d/p}} + \|g\|_{L^1 B^{d/p}}) \\ &\lesssim \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}} \|\beta - \{\beta\}\|_{L^1 B^{d/p-2}} \lesssim \epsilon \|v - w - \{v - w\}\|_{L^1 B^{d/p+1}}, \end{aligned}$$

as required, where we used (3.38) and (3.33) in the second inequality. □

4. Equivalence of the Eulerian and Lagrangian formulations

In this section we show the equivalence of Theorem 1.1 and Proposition 1.1.

We first show that Theorem 1.1 \Rightarrow Proposition 1.1.

To this end, given a solution (ρ, v) in the Eulerian coordinates, we need to construct a Lagrangian map $X = X(t, y)$, so that $a(t, y) := \rho(t, X(t, y)) - 1$, $u(t, y) := v(t, X(t, y))$ is a solution in the Lagrangian coordinates (that is a solution of (1.12)).

In order to construct the Lagrangian map, we first prove the following a priori estimate: given $X(t)$ exists for all $t \geq 0$, and $\|\nabla X - I\|_{L^\infty B^{d/p}}$ is sufficiently small (see (4.5) below) then

$$\|\nabla X - I\|_{L^\infty B^{d/p+1}} \leq C(\|X\|_{L^\infty B^{d/p+1}}) \|v\|_{L^1 B^{d/p+1}} \lesssim \epsilon. \tag{4.1}$$

Given (4.1), one can use a Picard iteration to construct X . In particular an appropriate choice of small $\epsilon > 0$ guarantees the assumed smallness of $\|\nabla X - I\|_{L^\infty B^{d/p}}$.

In order to prove (4.1), we first note that, since $X(t) : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is a diffeomorphism, we have

$$\int (\nabla X(t) - I) dy = 0$$

for each $t \geq 0$. Thus, in light of (2.11), in order to show (4.1) it suffices to verify that

$$\Delta X(t, y) = \int_0^t \Delta_y v(s, X(s, y)) ds \tag{4.2}$$

remains small in $B^{d/p-1}$ for all times, where $\Delta_y = \nabla_y \cdot \nabla_y$ and we denoted by ∇_y the derivative with respect to the y variable. Since

$$\Delta_y v(t, X(t, y)) = \partial_{y_i} (\partial_j v \circ X \partial_i X_j) = \partial_k \partial_j v \circ X \partial_i X_k \partial_i X_j + \partial_j v \circ X \Delta X_j,$$

we have

$$\|\Delta_y(v \circ X)\|_{B^{d/p-1}} \leq \|D^2 v \circ X\|_{B^{d/p-1}} \|\nabla X\|_{B^{d/p}}^2 + \|\partial_j v \circ X\|_{B^{d/p}} \|\Delta X_j\|_{B^{d/p-1}}. \tag{4.3}$$

At this point we would like to estimate $\|\partial_j v \circ X\|_{B^{d/p}}$ by $\|\nabla_y(\partial_j v \circ X)\|_{B^{d/p-1}}$, so that we could estimate it by $\|D^2 v \circ X\|_{B^{d/p-1}} \|\nabla X\|_{B^{d/p}}$, that is the same as the first term on the right-hand side above.

However, this is not immediate as the average of $\partial_j v \circ X$ does not necessarily vanish (recall (2.11)). Instead by adding and subtracting the average we obtain

$$\|\partial_j v \circ X\|_{B^{d/p}} \lesssim \|D^2 v \circ X\|_{B^{d/p-1}} \|\nabla X\|_{B^{d/p}} + \{\partial_j v \circ X\},$$

and by recalling the fact that $\nabla = A^T \nabla_y$ (see (3.27), for example) we can estimate the average,

$$\begin{aligned} \int \partial_j v(t, X(t, y)) dy &= \int A_{kj} \partial_{y_k} v(t, X(t, y)) dy \\ &= \int (A_{kj} - \delta_{kj}) \partial_{y_k} v(t, X(t, y)) dy \\ &\leq \|A - I\|_\infty \|\nabla_y(v \circ X)\|_\infty, \end{aligned}$$

where $A := (\nabla X)^{-1}$. This gives

$$\|\nabla v \circ X\|_{B^{d/p}} \lesssim \|D^2 v \circ X\|_{B^{d/p-1}} \|\nabla X\|_{B^{d/p}} + \|A - I\|_{B^{d/p}} \|\nabla v \circ X\|_{B^{d/p}} \|\nabla X\|_{B^{d/p}}. \tag{4.4}$$

Thus if we suppose that

$$\|\nabla X - I\|_{B^{d/p}} \leq 1/8C \|I\|_{B^{d/p}}, \tag{4.5}$$

where $C > 1$ is the implicit constant in (4.4) then $\|\nabla X\|_{B^{d/p}} \leq 2\|I\|_{B^{d/p}}$ and the Neumann expansion gives (as in (3.25))

$$\|A - I\|_{B^{d/p}} \leq \sum_{k \geq 1} \|\nabla X - I\|_{B^{d/p}}^k \leq \frac{1}{\|I\|_{B^{d/p}} C} \sum_{k \geq 1} 8^{-k} \leq \frac{1}{4\|I\|_{B^{d/p}} C}. \tag{4.6}$$

Thus the last term in (4.4) could be absorbed by the left-hand side to give

$$\|\nabla v \circ X\|_{B^{d/p}} \lesssim \|D^2 v \circ X\|_{B^{d/p-1}} \|\nabla X\|_{B^{d/p}}.$$

Applying this in (4.3) and taking the $\|\cdot\|_{L^\infty((0,\infty);B^{d/p-1})}$ norm of (4.2) gives

$$\begin{aligned} \|\Delta X\|_{L^\infty((0,\infty);B^{d/p-1})} &\leq \|\Delta_y(v \circ X)\|_{L^1 B^{d/p-1}} \\ &\leq C(\|X\|_{L^\infty B^{d/p+1}}) \int_0^\infty \|D^2 v \circ X\|_{B^{d/p-1}} \\ &\leq C(\|X\|_{L^\infty B^{d/p+1}}) \int_0^\infty \|D^2 v\|_{B^{d/p-1}}, \end{aligned} \tag{4.7}$$

proving the a priori estimate (4.1).

We note that, given X , we also obtain (as in (4.6)) that

$$\|A - I\|_{L^\infty B^{d/p}} \lesssim \epsilon.$$

Setting $u(t, y) := v(t, X(t, y))$, $a := \rho(t, X(t, y)) - 1$ we can use (2.13) to obtain that

$$\|u\|_{L^1 B^{d/p+1}} \lesssim \|v\|_{L^1 B^{d/p+1}} \lesssim \epsilon,$$

by the last two lines of (4.7), as well as

$$\|\nabla a\|_{L^\infty B^{d/p-1}} = \|\nabla(\rho \circ X)\|_{L^\infty B^{d/p-1}} \lesssim \|\nabla \rho \circ X\|_{L^\infty B^{d/p-1}} \|\nabla X\|_{L^\infty B^{d/p}} \lesssim \epsilon,$$

by (4.1) and the assumption $\|\rho - 1\|_{L^\infty B^{d/p}} \lesssim \epsilon$. (Recall that the diffeomorphism property (2.13) is only valid for $s \in (0, 1)$.) This and the fact that $\|\{a\}\|_{L^\infty(0, \infty)} \lesssim \|\rho - 1\|_{L^\infty L^\infty} \lesssim \epsilon$ imply that

$$\|a\|_{L^\infty B^{d/p}} \lesssim \epsilon.$$

Another application of (2.13) and the chain rule gives that

$$\|u_t\|_{L^1 B^{d/p-1}} \leq \|v_t\|_{L^1 B^{d/p-1}} + \|\nabla v\|_{L^1 B^{d/p}} \|\nabla X\|_{L^\infty B^{d/p}} \lesssim \epsilon.$$

As for a_t we use the a priori estimates (3.19), (3.21), (3.22) to obtain that

$$\|a_t\|_{L^1 B^{d/p}} \lesssim \epsilon,$$

as required. This completes the proof of Proposition 1.1.

We now prove that Proposition 1.1 \Rightarrow Theorem 1.1.

To this end, one defines $v(t, x) := u(t, X^{-1}(x))$, $\rho(t, x) := 1 + a(t, X^{-1}(t, x))$, and notes that the Lagrangian trajectory X satisfying $\|\nabla X - I\|_{L^\infty B^{d/p}} \lesssim \epsilon$ is already given by Proposition 1.1. Moreover, in the Eulerian coordinates the structure of the Equations (1.1) allows us to control the L^1 norm in time of $\{v\}$.

To be more precise we first note that $\|A - I\|_{L^\infty B^{d/p}} \lesssim \epsilon$, by (3.26), and so in particular $\|\nabla X^{-1} - I\|_{L^\infty L^\infty} \lesssim \epsilon$. Thus (2.13) implies that

$$\|\rho - 1\|_{L^\infty B^{d/p-1}} = \|a \circ X^{-1}\|_{L^\infty B^{d/p-1}} \lesssim \|a\|_{L^\infty B^{d/p-1}} \lesssim \epsilon,$$

while continuity of $\rho - 1$ in time with values in $B^{d/p-1}$ follows from the continuity of a , a consequence of $\|a_t\|_{L^1 B^{d/p-1}} \lesssim \epsilon$.

Similarly (2.13) implies that

$$\begin{aligned} \|\Delta v\|_{L^1 B^{d/p-1}} &= \|(\Delta_u u) \circ X^{-1}\|_{L^1 B^{d/p-1}} \lesssim \|\Delta_u u\|_{L^1 B^{d/p-1}} \\ &\lesssim \|\Delta u\|_{L^1 B^{d/p-1}} + \|A - I\|_{B^{d/p}} (1 + \|A - I\|_{B^{d/p}}) \|\Delta u\|_{B^{d/p-1}} \lesssim \epsilon, \end{aligned} \tag{4.8}$$

where we also used (3.31) in the second inequality.

We now note that

$$\{v\} = \int \{v\} \rho = \int (\{v\} - v) \rho,$$

where we used the mass conservation $\int \rho = \int \rho_0 = 1$ in the first equality and, in the second equality, we used the assumption $\int \rho_0 v_0 = 0$ and the momentum conservation,

$$\frac{d}{dt} \int \rho v = - \int \rho \nabla \Psi = - \frac{1}{2} \int \nabla |\nabla \Psi|^2 = 0,$$

which follows from (1.1), where $\Psi := K\rho$. Thus

$$\|\{v\}\| \lesssim \|\rho\|_{L^\infty} \|v - \{v\}\|_{L^p} \lesssim \|\rho\|_{L^\infty} \|\Delta v\|_{B^{d/p-1}}, \tag{4.9}$$

due to (2.11). This and (4.8) implies that $\|v\|_{L^1 B^{d/p+1}} \lesssim \epsilon$, as required.

It remains to show the estimate for v_t in $L^1 B^{d/p-1}$. To this end we note that $u_t = v_t \circ X + (v \circ X) \cdot (\nabla v \circ X)$, which implies that

$$v_t = u_t \circ X^{-1} - (u \cdot \nabla_u u) \circ X^{-1}.$$

Thus (2.13) gives

$$\begin{aligned} \|v_t\|_{L^1 B^{d/p-1}} &\leq \|u_t \circ X^{-1}\|_{L^1 B^{d/p-1}} + \|(u \cdot \nabla_u u) \circ X^{-1}\|_{L^1 B^{d/p-1}} \\ &\lesssim \|u_t\|_{L^1 B^{d/p-1}} + \|u \cdot \nabla_u u\|_{L^1 B^{d/p-1}} \\ &\lesssim \epsilon + \|u\|_{L^\infty B^{d/p-1}} \|\nabla_u u\|_{L^1 B^{d/p}} \lesssim \epsilon(1 + \|A - I\|_{L^\infty B^{d/p}} \|\Delta u\|_{L^1 B^{d/p-1}}) \lesssim \epsilon, \end{aligned}$$

where we used (2.12) twice in the third line.

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