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DISSIPATIVE SOLUTIONS TO THE COMPRESSIBLE ISENTROPIC NAVIER-STOKES EQUATIONS*

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Abstract. The dissipative solutions to the compressible isentropic Navier-Stokes equations are introduced in this paper. This notion was inspired by the concept of dissipative solutions to the incompressible Euler equations of Lions ([P.-L. Lions, Oxford Science Publication, Oxford, 1996], Section 4.4). We establish the existence of the dissipative solutions for the compressible Navier-Stokes equations, which is carried out by an approximate scheme for a modified Brenner model with artificial diffusion and artificial pressure at the same level. Moreover, we prove that the weak solution of the compressible isentropic Navier-Stokes equations is a dissipative solution.

Keywords. Compressible isentropic Navier-Stokes equations; dissipative solutions; weak-strong uniqueness.

AMS subject classifications. 35Q30; 76N10.

1. Introduction

This paper aims to study the existence of dissipation solutions to the compressible Navier-Stokes equations, which was inspired by the work of Lions [15]. In particular, Lions introduced the concept of dissipation solutions to the incompressible Euler equations and proved its existence. This can imply the property of weak-strong uniqueness of the Euler equations. In this paper, we are particularly interested in extending these results to the compressible Navier-Stokes equations.

Thus, we consider the following compressible isentropic Navier-Stokes equations over $\mathbb{R}^+\times\Omega$ $(\Omega\subset\mathbb{R}^3)$:

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0, \\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla \mathrm{p}(\rho)=\operatorname{div}\mathbb{S}(\nabla\mathbf{u}), \end{cases} \tag{1.1}$$

where ρ denotes the density, $\mathbf{u}\in\mathbb{R}^3$ the velocity and $\mathrm{p}(\rho)=A\rho^\gamma$ the pressure with the constant $A>0$ and the adiabatic exponent $\gamma>1$, respectively. The viscous stress tensor $\mathbb{S}$ satisfies the Newton's rheological law:

$$\mathbb{S}(\nabla\mathbf{u})=\mu(\nabla\mathbf{u}+\nabla^\top\mathbf{u})+\lambda(\operatorname{div}\mathbf{u})\mathbf{I}_3,$$

where the constants μ and λ are the Lamé viscosity coefficients of the flow satisfying $\mu>0$ and $2\mu+3\lambda\geq 0$, and $\mathbf{I}_3$ is the 3×3 identity matrix. The equations (1.1) are supplemented with the initial data

$$(\rho,\rho\mathbf{u})|_{t=0}=(\rho_0,\mathbf{m}_0), \tag{1.2}$$

and one of the following boundary conditions:
(1) the periodic case

$$\Omega=\mathbb{T}^3=\mathbb{R}^3/2\pi\mathbb{Z}^3; \tag{1.3}$$

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(2) the Dirichlet boundary condition

$$\mathbf{u}|_{\partial\Omega}=0, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^3$.

For the weak solutions of the compressible Navier-Stokes equations, Lions [16] introduced the concept of renormalized solutions. This allows him to establish the global existence of weak solutions with large data for any $\gamma\geq\frac{9}{5}$. Later, his result was improved in [10] by extending the value of the adiabatic exponent to $\gamma>\frac{3}{2}$. Improving the range of γ is an interesting and fundamental problem, which certainly is from a physical viewpoint. It is also a challenging problem in mathematics, since the restriction on $\gamma>\frac{3}{2}$ is absolutely essential to the analysis in [10]. This current paper aims to build up solutions in a weaker sense than in the renormalized sense, which was inspired by the concept of dissipation solutions introduced in Lions [15].

DiPerna and Majda [5] proposed a measure-valued solution, by a generalized Young measure and proved the global existence of such solution to the incompressible Euler equations with any initial data. However, they have not investigated the weak-strong uniqueness structure. The concept of dissipation solutions to the incompressible Euler equations was introduced and its existence was given in [15]. This solution can imply the weak-strong uniqueness property. Meanwhile, Bellout et al. [1] also proposed a *very weak* L^2 solution. Later, Brenier et al. [2] established the weak-strong uniqueness of the admissible measure-valued solutions to the incompressible Euler equations, and showed that the admissible measure-valued solution is also a dissipative solution in the sense of Lions.

The admissible assumption is to say that the kinetic energy is always less than or equal to the initial energy, which plays a key role in [2]. Under the admissible assumption, De Lellis and Székelyhidi ([4], Proposition 1) proved that the weak solution of incompressible Euler equations is a dissipative solution in the sense of Lions. Gwiazda et al. [12] extended the measure-valued solutions to some compressible fluid models and proved the weak-strong uniqueness of the admissible measure-valued solutions to the isentropic Euler equations in any space dimension. And afterwards, Feireisl et al. [7] introduced a dissipative measure-valued solution to the compressible barotropic Navier-Stokes system, and proved the existence for the adiabatic exponent $\gamma>1$ and the weak-strong uniqueness property of the dissipative measure-valued solutions. Recently, Kwon and Novotný [14] studied the dissipative weak solutions to compressible Navier-Stokes equations with general inflow-outflow data, and verified the existence, stability and weak-strong uniqueness for $\gamma>\frac{3}{2}$ in the framework of weak solutions.

The main contribution of this paper is to extend the notion of dissipative solutions to the incompressible Euler equations introduced in [15] to the equations of the compressible fluids. Compared to the incompressible case, we have to consider the density effect and the mass equation. To this end, we define the following two smooth functions

$$E_1(r,\mathbf{v})=\partial_t r+\operatorname{div}(r\mathbf{v}),\quad E_2(r,\mathbf{v})=r\partial_t\mathbf{v}+r\mathbf{v}\cdot\nabla\mathbf{v}+\nabla \mathrm{p}(r)-\operatorname{div}\mathbb{S}(\nabla\mathbf{v}), \tag{1.5}$$

where r (has a positive lower bound r_1) and $\mathbf{v}$ be two smooth functions on $[0,\infty)\times\Omega$. This allows us to derive a priori relative entropy (2.17), which is crucial to give the definition of dissipative solutions to the compressible Navier-Stokes equations, and to obtain the weak-strong uniqueness property.

The main idea is to build up the solutions of the modified Brenner model (4.1), and to show that this approximated solution is also a dissipative solution to (4.1). Then we can recover our solution by passing to the limits from this dissipative solution, thanks to the compactness argument.

This paper is organized as follows. In Section 2, we derive some a priori estimates, introduce the definition of dissipative solutions to the compressible isentropic Navier-Stokes equations and state our main results: the existence of dissipative solutions (Theorem 2.1) and the weak solution is also a dissipative solution (Theorem 2.3). In Section 3, we show that the smooth functions r and $\mathbf{v}$ can be replaced by a class of functions with lower regularities. This can be done through the regularization procedure. In Section 4, we give the proof of Theorem 2.1 by using compactness analysis on the approximated solutions. Section 5 is devoted to the proof of Theorem 2.2. Note that Corollary 2.1 and Corollary 2.2 are direct conclusions of Theorem 2.1 and Theorem 2.2, respectively, thus we omit their proofs. Finally, we collect some auxiliary lemmas in an appendix.

2. Definition of dissipative solutions and the main results

The main goal of this section is to define our dissipative solutions to the compressible Navier-Stokes equations and address our main results. To this end, we start with deriving some a priori estimates which are crucial to our definition. Thus, we assume that the solutions are smooth here.

Multiplying the continuity Equation $(1.1)_1$ by $\frac{1}{2}|\mathbf{v}|^2-\mathrm{P}'(r)$ $(\mathrm{P}'(r)=\frac{A\gamma}{\gamma-1}r^{\gamma-1})$, integrating the result over Ω, taking the inner product of the momentum Equation $(1.1)_2$ with $\mathbf{u}-\mathbf{v}$, and adding them up gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int \frac{1}{2}\rho|\mathbf{u}-\mathbf{v}|^2+\mathrm{P}(\rho)-\mathrm{P}'(r)\rho\,\mathrm{d}x+\int \mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}x\\
&=-\int \rho(\mathbf{u}-\mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x+\int\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\\
&\quad-\int\rho\partial_t\mathrm{P}'(r)\,\mathrm{d}x-\int\rho\mathbf{u}\cdot\nabla\mathrm{P}'(r)\,\mathrm{d}x-\int\mathrm{p}(\rho)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x+\int\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{v}\,\mathrm{d}x, \qquad (2.1)
\end{aligned}$$

where $\mathrm{P}(s)=\frac{A}{\gamma-1}s^\gamma$. Note that $\mathrm{P}'(r)r-\mathrm{P}(r)=\mathrm{p}(r)=Ar^\gamma$ and $\mathrm{P}''(r)r=\mathrm{p}'(r)$, this gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\int \mathrm{P}'(r)r-\mathrm{P}(r)\,\mathrm{d}x&=\int\partial_t\mathrm{p}(r)+\mathrm{div}\,(\mathrm{p}(r)\mathbf{v})\,\mathrm{d}x\\
&=\int \mathrm{p}'(r)\partial_t r+\mathbf{v}\cdot\nabla\mathrm{p}(r)+\mathrm{p}(r)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x\\
&=\int r\partial_t\mathrm{P}'(r)\,\mathrm{d}x+\int r\mathbf{v}\cdot\nabla\mathrm{P}'(r)+\mathrm{p}(r)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x. \qquad (2.2)
\end{aligned}$$

Using (1.5), (2.1) and (2.2), one obtains that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int\frac{1}{2}\rho|\mathbf{u}-\mathbf{v}|^2+\mathrm{P}(\rho)-\mathrm{P}'(r)(\rho-r)-\mathrm{P}(r)\,\mathrm{d}x+\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\\
&=-\int\rho(\mathbf{u}-\mathbf{v})\cdot\mathbb{D}(\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x-\int(\mathrm{p}(\rho)-\mathrm{p}'(r)(\rho-r)-\mathrm{p}(r))\mathrm{div}\,\mathbf{v}\,\mathrm{d}x\\
&\quad+\int(r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v})\,\mathrm{d}x+\int\tfrac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x+\int\tfrac{\rho-r}{r}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x,
\end{aligned} \qquad (2.3)$$

where $\mathbb{D}(\mathbf{v})=\frac{1}{2}(\nabla\mathbf{v}+\nabla^\top\mathbf{v})$, and we have used the identity

$$(\mathbf{b}\cdot\nabla)\mathbf{a}\cdot\mathbf{b}=\frac{1}{2}\mathbf{b}\cdot(\nabla\mathbf{a}+\nabla^\top\mathbf{a})\cdot\mathbf{b},$$

for the vectors **a** and **b**.

Next we look at the last term on the left-hand side of (2.3) for the two boundary cases (1.3) and (1.4), respectively.

For the periodic case $\Omega=\mathbb{T}^3$, using Lemma A.2 in the Appendix gives

$$\|\mathbf{u}-\mathbf{v}\|_{H^1}\le c_1\big(\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2}+\|\sqrt{\rho}(\mathbf{u}-\mathbf{v})\|_{L^2}\big), \tag{2.4}$$

where $c_1>0$ is a constant dependent on γ, and $\gamma\ge\frac{6}{5}$ for $n=3$ from Lemma A.2.

For the bounded domain Ω with Dirichlet boundary condition, it needs the extra restriction $\mathbf{v}|_{\partial\Omega}=0$. The Poincaré's inequality yields

$$\|\mathbf{u}-\mathbf{v}\|_{H^1}\le c_2\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2}, \tag{2.5}$$

where $c_2>0$ is a constant. Thus, we only need $\gamma>1$ in this situation.

By integration by parts, one has

$$\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x=\mu\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\,(\mathbf{u}-\mathbf{v})\|_{L^2}^2. \tag{2.6}$$

Using Sobolev embedding $H^1\hookrightarrow L^6$ with the generic constant $\hat{c}$, and the estimates (2.4), (2.5) and (2.6), we have

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|_{L^6}^2\le&\hat{c}\|\mathbf{v}-\mathbf{u}\|_{H^1}^2\le 2\hat{c}c_\gamma^2\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2}^2+2\hat{c}c_\gamma^2\|\sqrt{\rho}(\mathbf{u}-\mathbf{v})\|_{L^2}^2\\ \le&\frac{2\hat{c}c_\gamma^2}{\mu}\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x+2\hat{c}c_\gamma^2\|\sqrt{\rho}(\mathbf{u}-\mathbf{v})\|_{L^2}^2\end{aligned} \tag{2.7}$$

for the periodic case (1.3), and

$$\|\mathbf{v}-\mathbf{u}\|_{L^6}^2\le\hat{c}\|\mathbf{v}-\mathbf{u}\|_{H^1}^2\le\frac{2\hat{c}c_\gamma^2}{\mu}\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x \tag{2.8}$$

for the Dirichlet boundary case (1.4). Here, $c_\gamma=\max\{c_1,c_2\}$.

In view of the convexity of $\mathrm{P}(\rho)$ with $\gamma>1$ and $r\ge r_1>0$, it holds that

$$\mathrm{P}(\rho)-\mathrm{P}'(r)(\rho-r)-\mathrm{P}(r)\ge\begin{cases}C|\rho-r|^2, & \frac{1}{2}r\le\rho\le\frac{3}{2}r,\\ C(1+|\rho-r|^\gamma), & 0\le\rho<\frac{1}{2}r,\ \rho>\frac{3}{2}r.\end{cases} \tag{2.9}$$

For the convenience of notations, letting f be a given function and the bold $\mathbf{1}$ be characteristic function, we denote

$$\begin{aligned}&\Omega_{\mathrm{ess}}=\Big\{x\in\Omega:|\rho-r|\le\frac{1}{2}r\Big\},\quad \Omega_{\mathrm{res}}=\Big\{x\in\Omega:0\le\rho<\frac{1}{2}r,\rho>\frac{3}{2}r\Big\},\\ &f=[f]_{\mathrm{ess}}+[f]_{\mathrm{res}},\quad [f]_{\mathrm{ess}}=f\mathbf{1}_{\Omega_{\mathrm{ess}}},\quad [f]_{\mathrm{res}}=f\mathbf{1}_{\Omega_{\mathrm{res}}}.\end{aligned}$$

Utilizing the above notations, and by Hölder's inequality, one deduces that

$$\begin{aligned}&\int\frac{\rho-r}{r}\mathrm{div}\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\\ =&\int\frac{1}{r}(\sqrt{\rho}-\sqrt{r})(\sqrt{\rho}+\sqrt{r})\mathrm{div}\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\\ =&\int_{\Omega_{\mathrm{ess}}\cup\Omega_{\mathrm{res}}}\frac{\sqrt{\rho}-\sqrt{r}}{\sqrt{r}}\mathrm{div}\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega_{\rm ess}\cup\Omega_{\rm res}}\frac{\sqrt{\rho}-\sqrt{r}}{r}{\rm div}\mathbb{S}(\nabla\mathbf{v})\cdot\sqrt{\rho}(\mathbf{v}-\mathbf{u})\ {\rm d}x\\
\leq&\int_{\Omega_{\rm ess}\cup\Omega_{\rm res}}\frac{1}{\sqrt{r}}|\rho-r|^{\frac12}|{\rm div}\mathbb{S}(\nabla\mathbf{v})||\mathbf{v}-\mathbf{u}|\ {\rm d}x\\
&+\int_{\Omega_{\rm ess}\cup\Omega_{\rm res}}\frac{1}{r}|\rho-r|^{\frac12}|{\rm div}\mathbb{S}(\nabla\mathbf{v})||\sqrt{\rho}(\mathbf{v}-\mathbf{u})|\ {\rm d}x\\
\leq&\frac{\sqrt{2}}{2}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac65}}\|\mathbf{v}-\mathbf{u}\|_{L^6}+\left\|\frac{1}{\sqrt{r}}\right\|_{L^\infty}\|[\rho-r]_{\rm res}\|^{\frac12}_{L^\gamma}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{6\gamma}{5\gamma-3}}}\|\mathbf{v}-\mathbf{u}\|_{L^6}\\
&+\left\|\frac{1}{\sqrt{2r}}\right\|_{L^\infty}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^2}\|\sqrt{\rho}(\mathbf{v}-\mathbf{u})\|_{L^2}\\
&+\left\|\frac{1}{r}\right\|_{L^\infty}\|[\rho-r]_{\rm res}\|^{\frac12}_{L^\gamma}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{2\gamma}{\gamma-1}}}\|\sqrt{\rho}(\mathbf{v}-\mathbf{u})\|_{L^2},
\end{aligned}
\tag{2.10}
$$

where we have also employed the elementary inequality

$$|\rho^\theta-r^\theta|\leq|\rho-r|^\theta,\ 0\leq\theta\leq1,$$

with the special one $\theta=\frac12$. Noticing that $r\geq r_1>0$, with the help of (2.7), (2.8), (2.9) and Young's inequality, we have

$$
\begin{aligned}
&\int\frac{\rho-r}{r}{\rm div}\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ {\rm d}x\\
\leq&\frac{\mu}{8\hat{c}c_\gamma^2}\|\mathbf{v}-\mathbf{u}\|^2_{L^6}+\frac{\hat{c}c_\gamma^2}{\mu}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|^2_{L^{\frac65}}\\
&+\frac{\mu}{8\hat{c}c_\gamma^2}\|\mathbf{v}-\mathbf{u}\|^2_{L^6}+\frac{2\hat{c}c_\gamma^2}{\mu r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|^2_{L^{\frac{6\gamma}{5\gamma-3}}}\|[\rho-r]_{\rm res}\|_{L^\gamma}\\
&+\frac{1}{2\sqrt{2r_1}}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^2}(1+\|\sqrt{\rho}(\mathbf{v}-\mathbf{u})\|^2_{L^2})\\
&+\frac{1}{r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{2\gamma}{\gamma-1}}}(\|\sqrt{\rho}(\mathbf{v}-\mathbf{u})\|^2_{L^2}+\|[\rho-r]_{\rm res}\|_{L^\gamma})\\
\leq&\frac12\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ {\rm d}x+\frac{\mu}{2}\int\rho|\mathbf{u}-\mathbf{v}|^2\ {\rm d}x\\
&+C\frac{\hat{c}c_\gamma^2}{\mu r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|^2_{L^{\frac{6\gamma}{5\gamma-3}}}\int{\rm P}(\rho)-{\rm P}'(r)(\rho-r)-{\rm P}(r)\ {\rm d}x\\
&+C\frac{1+\sqrt{r_1}}{r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{2\gamma}{\gamma-1}}}\int\frac12\rho|\mathbf{u}-\mathbf{v}|^2+{\rm P}(\rho)-{\rm P}'(r)(\rho-r)-{\rm P}(r)\ {\rm d}x
\end{aligned}
\tag{2.11}
$$

under the case (1.3), and

$$
\begin{aligned}
&\int\frac{\rho-r}{r}{\rm div}\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ {\rm d}x\\
\leq&\frac12\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ {\rm d}x\\
&+C\frac{\hat{c}c_\gamma^2}{\mu r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|^2_{L^{\frac{6\gamma}{5\gamma-3}}}\int{\rm P}(\rho)-{\rm P}'(r)(\rho-r)-{\rm P}(r)\ {\rm d}x\\
&+C\frac{1+\sqrt{r_1}}{r_1}\|{\rm div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{2\gamma}{\gamma-1}}}\int\frac12\rho|\mathbf{u}-\mathbf{v}|^2+{\rm P}(\rho)-{\rm P}'(r)(\rho-r)-{\rm P}(r)\ {\rm d}x
\end{aligned}
\tag{2.12}
$$

under the case (1.4).

Putting (2.11) and (2.12) into (2.3), respectively, and integrating over $(0,t)$, it follows that

$$\begin{aligned}&\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})+\frac{1}{2}\int_0^t\int \mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s\\ \le&\mathcal{E}_0(\rho_0,\mathbf{m}_0;r_0,\mathbf{v}_0)+\int_0^t C_0\Lambda(\mathbf{v})\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})\ \mathrm{d}s\\ &+\int_0^t\int \left|(r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v})\right|+\left|\frac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\right|\ \mathrm{d}x\ \mathrm{d}s,\end{aligned}\tag{2.13}$$

where $C_0>0$ is a generic constant, $(r_0,\mathbf{v}_0)=(r,\mathbf{v})\big|_{t=0}$,

$$\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})=\int\frac{1}{2}\rho|\mathbf{u}-\mathbf{v}|^2+\mathrm{P}(\rho)-\mathrm{P}'(r)(\rho-r)-\mathrm{P}(r)\ \mathrm{d}x,\tag{2.14}$$

$$\mathcal{E}_0(\rho_0,\mathbf{m}_0;r_0,\mathbf{v}_0)=\int\frac{1}{2}\rho_0\left|\frac{\mathbf{m}_0}{\rho_0}-\mathbf{v}_0\right|^2+\mathrm{P}(\rho_0)-\mathrm{P}'(r_0)(\rho_0-r_0)-\mathrm{P}(r_0)\ \mathrm{d}x,\tag{2.15}$$

and

$$\Lambda(\mathbf{v})=\begin{cases}\big(\mu+\Lambda_0(\mathbf{v})\big), & \text{for the case (1.3)},\\ \big(\Lambda_0(\mathbf{v})\big), & \text{for the case (1.4)},\end{cases}\tag{2.16}$$

with

$$\Lambda_0(\mathbf{v})=\|\mathbb{D}(\mathbf{v})\|_{L^\infty}+\frac{\hat{c}c_\gamma^2}{\mu r_1}\|\mathrm{div}\mathbb{S}(\nabla\mathbf{v})\|^2_{L^{\frac{6\gamma}{5\gamma-3}}}+\frac{1+\sqrt{r_1}}{r_1}\|\mathrm{div}\mathbb{S}(\nabla\mathbf{v})\|_{L^{\frac{2\gamma}{\gamma-1}}}.$$

Applying Grönwall's inequality (integral form) to (2.13), we obtain that, for all $t\ge0$,

$$\begin{aligned}\mathrm{LS}(\rho,\mathbf{u};r,\mathbf{v}):=&\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})+\frac{1}{2}\int_0^t\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s\\ &\le\Big(\int_0^t C_0\Lambda(\mathbf{v})\exp\Big(\int_s^t C_0\Lambda(\mathbf{v})\ \mathrm{d}s\Big)\ \mathrm{d}s+1\Big)\Big\{\mathcal{E}_0(\rho_0,\mathbf{m}_0;r_0,\mathbf{v}_0)\\ &\quad+\int_0^t\int\left|(r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v})\right|+\left|\frac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\right|\ \mathrm{d}x\ \mathrm{d}s\Big\}\\ &=:\mathrm{RS}(\rho,\mathbf{u};r,\mathbf{v}).\end{aligned}\tag{2.17}$$

With the above a priori estimates at hand, we are ready to define the dissipative solutions to the compressible isentropic Navier-Stokes (1.1) in the following sense.

Definition 2.1. *Let $\rho\in L^\infty(0,T;L^\gamma)\cap C([0,T];L^\gamma-w)$, $\sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2)$ and $\mathbf{u}\in L^2(0,T;H^1)$ for any fixed $T>0$. We call $(\rho,\mathbf{u})$ a dissipative solution of the problem (1.1)-(1.3) or (1.1), (1.2) and (1.4), if (2.17) holds for $(r,\mathbf{v})$ satisfying*

$$\begin{cases} r\in C([0,T];L^\gamma),\ r\ge r_1>0,\ \mathbf{v}\in C([0,T];L^{\frac{2\gamma}{\gamma-1}}),\ \mathbb{D}(\mathbf{v})\in L^1(0,T;L^\infty),\\ \nabla\mathbf{v}\in L^2(0,T;L^2),\ \mathrm{div}\mathbb{S}(\nabla\mathbf{v})\in L^2(0,T;L^{\frac{6\gamma}{5\gamma-3}})\cap L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}),\\ \mathrm{P}''(r)E_1(r,\mathbf{v})\in L^1(0,T;L^{\frac{\gamma}{\gamma-1}}),\ \frac{1}{r}E_2(r,\mathbf{v})\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}),\end{cases}\tag{2.18}$$

where r_1 is a positive constant. Note that it needs the extra condition $\mathbf{v}|_{\partial\Omega}=0$ for the bounded domain case (1.4).

Remark 2.1. If $(r,\mathbf{v})$ satisfies (2.18), then (2.17) is well-defined. We observe that $\mathbf{v}\in L^1(0,T;W^{1,p})$ for any $1<p<\infty$ from (2.18). Indeed, in view of Korn's inequality ([9], Theorem 11.21)

$$\|\mathbf{v}\|_{W^{1,p}}\le C\Big(\|\mathbb{D}(\mathbf{v})\|_{L^p}+\int|\mathbf{v}|\,\mathrm{d}x\Big),\ 1<p<\infty,$$

it follows that

$$\|\mathbf{v}\|_{L^1(0,T;W^{1,p})}\le C\Big(\|\mathbb{D}(\mathbf{v})\|_{L^1(0,T;L^\infty)}+\|\mathbf{v}\|_{C([0,T];L^{\frac{2\gamma}{\gamma-1}})}\Big),\ 1<p<\infty. \tag{2.19}$$

Here we address our first main result on the existence of dissipative solution in the sense of Definition 2.1.

Theorem 2.1. *Suppose that the assumptions of Definition 2.1 hold. In addition, we assume that*

$$\begin{cases} r\le r_2 \text{ if } \gamma>2,\ \nabla r\in L^\infty(0,T;L^{\frac{2\gamma}{\gamma-1}}),\ \nabla\mathbf{v}\in L^\infty(0,T;L^q) \text{ for } q>1,\\ \partial_t r\in L^1(0,T;L^{\frac{\gamma}{\gamma-1}}),\ \partial_t\mathbf{v}\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}),\end{cases} \tag{2.20}$$

where r_2 is a positive constant.

(1) *For the periodic case (1.3): Let $\gamma\ge\frac{6}{5}$. Assume that the initial data (1.2) satisfy*

$$\rho_0\in L^\gamma,\ \rho_0\ge0,\ \int\rho_0\,\mathrm{d}x\ge C_{\rho_0},\ \mathbf{m}_0=0 \ a.e.\ in\ \{x\in\Omega:\rho_0=0\},\ \frac{\mathbf{m}_0^2}{\rho_0}\in L^1, \tag{2.21}$$

for some positive constant C_{ρ_0}. Then, there exists a dissipative solution of the compressible isentropic Navier-Stokes system (1.1) with (1.2) and (1.3).

(2) *For the Dirichlet boundary case (1.4): Assume the boundary $\partial\Omega$ is C^2. Let $\gamma>1$. Assume the initial data (1.2) satisfy*

$$\rho_0\in L^\gamma,\ \rho_0\ge0,\ \mathbf{m}_0=0 \ a.e.\ in\ \{x\in\Omega:\rho_0=0\},\ \frac{\mathbf{m}_0^2}{\rho_0}\in L^1. \tag{2.22}$$

Then, there exists a dissipative solution of the compressible isentropic Navier-Stokes system (1.1) with (1.2) and (1.4).

Remark 2.2. Since $\nabla\mathbf{v}\in L^\infty(0,T;L^q)$ for $q>1$, by (2.19), it follows from Lemma A.5 that $\nabla\mathbf{v}\in L^1(0,T;L^\infty)$. Furthermore, it infers that $\mathbf{v}\cdot\nabla\mathbf{v}\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}})$.

Remark 2.3. Feireisl, Novotný and Sun [11] introduced a class of suitable weak solutions to the compressible barotropic Navier-Stokes equations and proved that the solution satisfies the relative entropy inequality for $\gamma>\frac{3}{2}$. Compared with [11], we define the dissipative solution satisfying the inequality (2.17). We give the process of the density arguments in Section 3. In addition, the adiabatic exponent can reduce to $\gamma\ge\frac{6}{5}$ for the periodic case and $\gamma>1$ for the Dirichlet boundary case.

Applying Theorem 2.1 directly, we have the following result on the relationship of the strong solution and dissipative solution.

COROLLARY 2.1. *Assume that* $(r,\mathbf{v})$ *is a strong solution of* (1.1) *and the initial data* $(r,\mathbf{v})|_{t=0}=(r_0,\mathbf{v}_0)$ *together with* (1.3) *or* (1.4)*, satisfying the regularities*

$$\begin{cases} r\in C([0,T];L^\gamma),\ r_1\le r\le r_2,\ \mathbf{v}\in C([0,T];L^{\frac{2\gamma}{\gamma-1}}),\ \mathbb{D}(\mathbf{v})\in L^1(0,T;L^\infty) \\ \partial_t r\in L^1(0,T;L^{\frac{\gamma}{\gamma-1}}),\ \partial_t\mathbf{v}\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}),\ \nabla r\in L^\infty(0,T;L^{\frac{2\gamma}{\gamma-1}}), \\ \operatorname{div}\mathbb{S}(\nabla\mathbf{v})\in L^2(0,T;L^{\frac{6\gamma}{5\gamma-3}})\cap L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}),\ \nabla\mathbf{v}\in L^\infty(0,T;L^q)\ \text{for } q>1, \end{cases} \tag{2.23}$$

for some positive constants r_1 *and* r_2*, and any* $T\in(0,T_{\max})$*, where* $T_{\max}$ *is the maximal existence time. Let* $(\rho,\mathbf{u})$ *be a dissipative solution of the system* (1.1)-(1.2) *together with* (1.3) *or* (1.4)*, and the initial data satisfy*

$$\int \frac{1}{2}\rho_0\Big|\frac{\mathbf{m}_0}{\rho_0}-\mathbf{v}_0\Big|^2+\mathrm{P}(\rho_0)-\mathrm{P}'(r_0)(\rho_0-r_0)-\mathrm{P}(r_0)\ \mathrm{d}x=0.$$

Then, the dissipative solution $(\rho,\mathbf{u})$ *is equal to* $(r,\mathbf{v})$ *on a.e.* $(t,x)\in[0,T]\times\Omega$*.*

REMARK 2.4. We point out that the condition $\mathbb{D}(\mathbf{v})\in L^1(0,T;L^\infty)$ corresponds to the blow up criteria of the Navier-Stokes equations given by Huang et al. [13]. Under the condition $\mathbb{D}(\mathbf{v})\in L^1(0,T;L^\infty)$, the regularities (2.23) can be replaced by placing a restriction on the initial data and γ in some situations. For example, endowing the initial data

$$0<\underline{r}\le r_0\le\overline{r},\quad r_0\in W^{1,p}\ \text{ for }\ p>6,\quad \mathbf{v}_0\in H^2, \tag{2.24}$$

for some positive constants $\underline{r}$ and $\overline{r}$, and making use of the condition $\mathbb{D}(\mathbf{v})\in L^1(0,T;L^\infty)$, a direct conclusion from [13] shows that there exists a global strong solution $(r,\mathbf{v})$ with

$$\begin{cases} r_1\le r\le r_2,\quad r\in C([0,T];W^{1,6}),\ \partial_t r\in C([0,T];L^6), \\ \mathbf{v}\in C([0,T];H^2)\cap L^2(0,T;W^{2,6}),\ \partial_t\mathbf{v}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1), \end{cases} \tag{2.25}$$

for any $T>0$. We see that $(r,\mathbf{v})$ with (2.25) meets the requirement of (2.23) for $\gamma\ge\frac{3}{2}$. It says that (2.23) can be substituted by (2.24) for $\gamma\ge\frac{3}{2}$.

Our next goal is to show the weak solution of the compressible isentropic Navier-Stokes equations is also a dissipative solution. We first recall the definition of weak solution as follows.

DEFINITION 2.2. *A pair* $(\rho,\mathbf{u})$ *is a weak solution of the problem* (1.1)-(1.3) *or* (1.1), (1.2) *and* (1.4) *provided that, for any fixed* $T>0$,

$$\rho\in L^\infty(0,T;L^\gamma),\quad \sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2),\quad \mathbf{u}\in L^2(0,T;H^1), \tag{2.26}$$

and the continuity Equation $(1.1)_1$ *and the momentum Equation* $(1.1)_2$ *are satisfied in* $\mathcal{D}'([0,T]\times\Omega)$*, that is, for* $\Psi\in C_c^\infty([0,T]\times\Omega)$*,* $\Phi\in C_c^\infty([0,T]\times\Omega)$*,*

$$\int\rho(T,x)\Psi(T,x)\ \mathrm{d}x-\int\rho_0\Psi(0,x)\ \mathrm{d}x=\int_0^T\int\rho\partial_t\Psi+\rho\mathbf{u}\cdot\nabla\Psi\ \mathrm{d}x\ \mathrm{d}t, \tag{2.27}$$

$$\int\rho(T,x)\mathbf{u}(T,x)\cdot\Phi(T,x)\ \mathrm{d}x-\int\mathbf{m}_0\cdot\Phi(0,x)\ \mathrm{d}x$$
$$=\int_0^T\int\rho\mathbf{u}\cdot\partial_t\Phi+\rho\mathbf{u}\otimes\mathbf{u}:\nabla\Phi+A\rho^\gamma\operatorname{div}\Phi-\mathbb{S}(\nabla\mathbf{u}):\nabla\Phi\ \mathrm{d}x\ \mathrm{d}t, \tag{2.28}$$

and the energy inequality holds

$$\int \frac{1}{2}\rho|\mathbf{u}|^2+\frac{A}{\gamma-1}\rho^\gamma\ \mathrm{d}x+\int_0^t\int \mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t\le\int \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0}+\frac{A}{\gamma-1}\rho_0^\gamma\ \mathrm{d}x, \tag{2.29}$$

for almost every (a.e.) $t\in[0,T]$.

Then, we write our next result in the following.

THEOREM 2.2. *Suppose that the assumptions of Theorem 2.1 hold. If* $(\rho,\mathbf{u})$ *is a weak solution of the problem* (1.1)-(1.3) *or* (1.1), (1.2) *and* (1.4)*, then, the weak solution is a dissipative solution in the sense of Definition 2.1.*

The weak solution of the compressible isentropic Navier-Stokes equations also has the weak-strong uniqueness property, see [8]. By means of Theorem 2.2 and Corollary 2.1, it can directly give another version of the proof of the weak-strong uniqueness property for the weak solution of the compressible isentropic Navier-Stokes equations. We state the conclusion by a corollary as follows.

COROLLARY 2.2. *Under the same assumptions of Corollary 2.1, if we assume that* $(\rho,\mathbf{u})$ *is a weak solution of the problem* (1.1)-(1.3) *or* (1.1), (1.2) *and* (1.4)*, then* $(\rho,\mathbf{u})=(r,\mathbf{v})$ *on a.e.* $(t,x)\in[0,T]\times\Omega$.

3. Regularization of r and $\mathbf{v}$

This section is devoted to showing that the smooth function r and $\mathbf{v}$ can be replaced by the functions with regularities given in (2.18) and (2.20). Thus, the following is the main result of this section.

PROPOSITION 3.1. *If* (2.17) *holds for smooth functions* r *and* $\mathbf{v}$*, then it also holds for the functions satisfying* (2.18) *and* (2.20).

Proof. By the extension theorem (Theorem 1 in Section 5.4 of [6]), we can extend r and $\mathbf{v}$ on $[-\delta_0,T+\delta_0]\times\widetilde{\Omega}$ with small $\delta_0>0$ and $\Omega\subset\subset\widetilde{\Omega}$, which still satisfy the regularities (2.18) and (2.20). Here, it requires that the boundary $\partial\Omega$ is C^2 for the bounded domain case. Let

$$\eta(x)\in C_c^\infty(\mathbb{R}^3),\ \mathrm{supp}\ \eta=\{x\in\mathbb{R}^3:|x|\le 1\},\ \int_{\mathbb{R}^3}\eta(x)\ \mathrm{d}x=1,\ \eta^\delta(x)=\frac{1}{\delta^3}\eta\big(\frac{x}{\delta}\big), \tag{3.1}$$

$$\tilde\eta(t)\in C_c^\infty(\mathbb{R}),\ \mathrm{supp}\ \tilde\eta=\{t\in\mathbb{R}:|t|\le 1\},\ \int_{\mathbb{R}}\tilde\eta(t)\ \mathrm{d}t=1,\ \tilde\eta^\delta(t)=\frac{1}{\delta}\tilde\eta\big(\frac{t}{\delta}\big), \tag{3.2}$$

where $0<\delta\le 1$. We mollify r and $\mathbf{v}$ with respect to space and time in the following way:

$$r_{t,x}^\delta(t,x)=(r*\eta^\delta)*\tilde\eta^\delta(t,x)=\int_{\mathbb{R}}\int_{\mathbb{R}^n}r(t-s,x-y)\eta^\delta(y)\ \mathrm{d}y\tilde\eta^\delta(s)\ \mathrm{d}s,$$
$$\mathbf{v}_{t,x}^\delta(t,x)=(\mathbf{v}*\eta^\delta)*\tilde\eta^\delta(t,x)=\int_{\mathbb{R}}\int_{\mathbb{R}^n}\mathbf{v}(t-s,x-y)\eta^\delta(y)\ \mathrm{d}y\tilde\eta^\delta(s)\ \mathrm{d}s.$$

Since

$$\nabla r\in L^1(-\delta_0,T+\delta_0;L^{\frac{2\gamma}{\gamma-1}}(\widetilde{\Omega})),\quad \partial_t r\in L^1(-\delta_0,T+\delta_0;L^{\frac{\gamma}{\gamma-1}}(\widetilde{\Omega})),$$
$$\nabla\mathbf{v}\in L^2(-\delta_0,T+\delta_0;L^2(\widetilde{\Omega})),\quad \mathrm{div}\,\mathbf{v}\in L^2(-\delta_0,T+\delta_0;L^2(\widetilde{\Omega})),$$

by Lemmas A.6 and A.7, then, as $\delta\to 0$,

$$\|\nabla r^\delta_{t,x}-\nabla r\|_{L^1(0,T;L^{\frac{2\gamma}{\gamma-1}})}=\|(\nabla r)^\delta_{t,x}-\nabla r\|_{L^1(0,T;L^{\frac{2\gamma}{\gamma-1}})}\to 0, \tag{3.3}$$

$$\|\partial_t r^\delta_{t,x}-\partial_t r\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}})}=\|(\partial_t r)^\delta_{t,x}-\partial_t r\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}})}\to 0, \tag{3.4}$$

$$\|\nabla \mathbf{v}^\delta_{t,x}-\nabla \mathbf{v}\|_{L^2(0,T;L^2)}=\|(\nabla \mathbf{v})^\delta_{t,x}-\nabla \mathbf{v}\|_{L^2(0,T;L^2)}\to 0, \tag{3.5}$$

$$\|\operatorname{div}\mathbf{v}^\delta_{t,x}-\operatorname{div}\mathbf{v}\|_{L^2(0,T;L^2)}=\|(\operatorname{div}\mathbf{v})^\delta_{t,x}-\operatorname{div}\mathbf{v}\|_{L^2(0,T;L^2)}\to 0. \tag{3.6}$$

For $\mathbf{v}\in C([-\delta_0,T+\delta_0];L^{\frac{2\gamma}{\gamma-1}}(\widetilde{\Omega}))$, $\partial_t\mathbf{v}\in L^1(-\delta_0,T+\delta_0;L^{\frac{2\gamma}{\gamma-1}}(\widetilde{\Omega}))$, by Lemma A.8, then, as $\delta\to 0$,

$$\|\mathbf{v}^\delta_{t,x}-\mathbf{v}\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma-1}})}\to 0. \tag{3.7}$$

For $\partial_t r\in L^1(-\delta_0,T+\delta_0;L^{\frac{\gamma}{\gamma-1}}(\widetilde{\Omega}))$ and $\nabla r\in L^\infty(-\delta_0,T+\delta_0;L^{\frac{2\gamma}{\gamma-1}}(\widetilde{\Omega}))$, by Lemma A.8, we know that, as $\delta\to 0$,

$$\|r^\delta_{t,x}-r\|_{L^\infty(0,T;L^\infty)}\to 0. \tag{3.8}$$

In addition, since $r\ge r_1$ if $1<\gamma\le 2$ and $r_1\le r\le r_2$ if $\gamma>2$, and noticing that $\int_{\mathbb{R}^3}\eta^\delta(y)\,\mathrm{d}y=1$ and $\int_{\mathbb{R}}\tilde\eta^\delta(s)\,\mathrm{d}s=1$, then we have

$$r^\delta_{t,x}\ge r_1 \text{ if } 1<\gamma\le 2,\quad r_1\le r^\delta_{t,x}(t,x)\le r_2 \text{ if } \gamma>2. \tag{3.9}$$

By Lagrange mean value theorem, there exists a $\theta_1\in(0,1)$ such that

$$\begin{aligned}\mathrm{P}''(r^\delta_{t,x})-\mathrm{P}''(r)=&\mathrm{P}'''(\theta_1 r^\delta_{t,x}+(1-\theta_1)r)(r^\delta_{t,x}-r)\\ =&A\gamma(\gamma-2)(\gamma-3)[\theta_1 r^\delta_{t,x}+(1-\theta_1)r]^{\gamma-3}(r^\delta_{t,x}-r).\end{aligned}$$

Combining (3.8) and (3.9), then the above equality gives, as $\delta\to 0$,

$$\|\mathrm{P}''(r^\delta_{t,x})-\mathrm{P}''(r)\|_{L^\infty(0,T;L^\infty)}\to 0. \tag{3.10}$$

By the same argument, we have, as $\delta\to 0$,

$$\|\mathrm{p}'(r^\delta_{t,x})-\mathrm{p}'(r)\|_{L^\infty(0,T;L^{\frac{\gamma}{\gamma-1}})}\to 0. \tag{3.11}$$

Since

$$\nabla\mathrm{P}'(r^\delta_{t,x})-\nabla\mathrm{P}'(r)=\mathrm{P}''(r^\delta_{t,x})(\nabla r^\delta_{t,x}-\nabla r)+[\mathrm{P}''(r^\delta_{t,x})-\mathrm{P}''(r)]\nabla r,$$

along with (3.3), (3.9) and (3.10), then it holds, as $\delta\to 0$,

$$\|\nabla\mathrm{P}'(r^\delta_{t,x})-\nabla\mathrm{P}'(r)\|_{L^1(0,T;L^{\frac{2\gamma}{\gamma-1}})}\to 0. \tag{3.12}$$

Next, we will show that $\mathrm{P}''(r)E_1(r,\mathbf{v})$ can be approximated by $\mathrm{P}''(r^\delta_{t,x})E_1(r^\delta_{t,x},\mathbf{v}^\delta_{t,x})$. Recall the definition $E_1(r,\mathbf{v})$ in (1.5) and the relation $r\mathrm{P}''(r)=\mathrm{p}'(r)$, then,

$$\begin{aligned}\mathrm{P}''(r^\delta_{t,x})E_1(r^\delta_{t,x},\mathbf{v}^\delta_{t,x})=&\partial_t\mathrm{P}'(r^\delta_{t,x})+\mathbf{v}^\delta_{t,x}\cdot\nabla\mathrm{P}'(r^\delta_{t,x})+\mathrm{p}'(r^\delta_{t,x})\operatorname{div}\mathbf{v}^\delta_{t,x}\\ =&(\mathrm{P}''(r)E_1(r,\mathbf{v}))^\delta_{t,x}+\partial_t\mathrm{P}'(r^\delta_{t,x})-(\partial_t\mathrm{P}'(r))^\delta_{t,x}+\mathbf{v}^\delta_{t,x}\cdot\nabla\mathrm{P}'(r^\delta_{t,x})\\ &-(\mathbf{v}\cdot\nabla\mathrm{P}'(r))^\delta_{t,x}+\mathrm{p}'(r^\delta_{t,x})\operatorname{div}\mathbf{v}^\delta_{t,x}-(\mathrm{p}'(r)\operatorname{div}\mathbf{v})^\delta_{t,x}.\end{aligned} \tag{3.13}$$

By Lemma A.7, we have, as $\delta\to 0$,

$$(\mathrm{P}''(r)E_1(r,\mathbf{v}))^{\delta}_{t,x}\to \mathrm{P}''(r)E_1(r,\mathbf{v}) \quad \text{strongly in } L^1(0,T;L^{\frac{\gamma}{\gamma-1}}). \tag{3.14}$$

For the term $\partial_t\mathrm{P}'(r^{\delta}_{t,x})-(\partial_t\mathrm{P}'(r))^{\delta}_{t,x}$, it can be rewritten as

$$\begin{aligned}\partial_t\mathrm{P}'(r^{\delta}_{t,x})-(\partial_t\mathrm{P}'(r))^{\delta}_{t,x} =&\mathrm{P}''(r)(\partial_t r^{\delta}_{t,x}-\partial_t r)+[\mathrm{P}''(r^{\delta}_{t,x})-\mathrm{P}''(r)]\partial_t r^{\delta}_{t,x}\\ &+\partial_t\mathrm{P}'(r)-(\partial_t\mathrm{P}'(r))^{\delta}_{t,x}.\end{aligned} \tag{3.15}$$

By (3.4), (3.10) and Lemma A.7, we have, as $\delta\to 0$,

$$\partial_t\mathrm{P}'(r^{\delta}_{t,x})-(\partial_t\mathrm{P}'(r))^{\delta}_{t,x}\to 0 \quad \text{strongly in } L^1(0,T;L^{\frac{\gamma}{\gamma-1}}). \tag{3.16}$$

Similarly, we have

$$\mathbf{v}^{\delta}_{t,x}\cdot\nabla\mathrm{P}'(r^{\delta}_{t,x})-(\mathbf{v}\cdot\nabla\mathrm{P}'(r))^{\delta}_{t,x}\to 0 \quad \text{strongly in } L^1(0,T;L^{\frac{\gamma}{\gamma-1}}). \tag{3.17}$$

Now, we turn to deal with the term $\mathrm{p}'(r^{\delta}_{t,x})\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}-(\mathrm{p}'(r)\mathrm{div}\,\mathbf{v})^{\delta}_{t,x}$, which can be written as

$$\begin{aligned}&\mathrm{p}'(r^{\delta}_{t,x})\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}-(\mathrm{p}'(r)\mathrm{div}\,\mathbf{v})^{\delta}_{t,x}\\
=&\Big(\mathrm{p}'(r^{\delta}_{t,x})-[\mathrm{p}'(r)]^{\delta}_{t,x}\Big)\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}+[\mathrm{p}'(r)]^{\delta}_{t,x}\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}-(\mathrm{p}'(r)\mathrm{div}\,\mathbf{v})^{\delta}_{t,x}\\
=&\Big(\mathrm{p}'(r^{\delta}_{t,x})-[\mathrm{p}'(r)]^{\delta}_{t,x}\Big)\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}+\Big([\mathrm{p}'(r)]^{\delta}_{t,x}-\mathrm{p}'(r)\Big)(\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}-\mathrm{div}\,\mathbf{v})\\
&-\int_{\mathbb{R}}\int_{\mathbb{R}^n}\Big([\mathrm{p}'(r)](t,x)-[\mathrm{p}'(r)](t-s,x-y)\Big)\mathrm{div}\,(\mathbf{v}(t,x)\\
&\qquad-\mathbf{v}(t-s,x-y))\eta^{\delta}(y)\ \mathrm{d}y\tilde{\eta}^{\delta}(s)\ \mathrm{d}s\\
=&\Big(\mathrm{p}'(r^{\delta}_{t,x})-\mathrm{p}'(r)\Big)\mathrm{div}\,\mathbf{v}^{\delta}_{t,x}-\Big([\mathrm{p}'(r)]^{\delta}_{t,x}-\mathrm{p}'(r)\Big)\mathrm{div}\,\mathbf{v}\\
&-\int_{\mathbb{R}}\int_{\mathbb{R}^n}\Big([\mathrm{p}'(r)](t,x)-[\mathrm{p}'(r)](t-\delta s,x-\delta y)\Big)\mathrm{div}\,(\mathbf{v}(t,x)\\
&\qquad-\mathbf{v}(t-\delta s,x-\delta y))\eta(y)\ \mathrm{d}y\tilde{\eta}(s)\ \mathrm{d}s\\
=:&I_1+I_2+I_3.\end{aligned}$$

For the term I_1, by (3.11), we get, as $\delta\to 0$,

$$\|I_1\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}})}\le C\|\mathrm{div}\,\mathbf{v}\|_{L^1(0,T;L^\infty)}\|\mathrm{p}'(r^{\delta}_{t,x})-\mathrm{p}'(r)\|_{L^\infty(0,T;L^{\frac{\gamma}{\gamma-1}})}\to 0.$$

For the term I_2, notice that $\mathrm{p}'(r)=A\gamma r^{\gamma-1}\in C([-\delta_0,T+\delta_0];L^{\frac{\gamma}{\gamma-1}}(\widetilde{\Omega}))$, by Lemma A.7, we get, as $\delta\to 0$,

$$\|I_2\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}})}\le C\|\mathrm{div}\,\mathbf{v}\|_{L^1(0,T;L^\infty)}\|[\mathrm{p}'(r)]^{\delta}_{t,x}-\mathrm{p}'(r)\|_{L^\infty(0,T;L^{\frac{\gamma}{\gamma-1}})}\to 0.$$

For the term I_3, it can be rewritten as

$$\begin{aligned}I_3=-\int_{\mathbb{R}}\int_{\mathbb{R}^n}&\Big([\mathrm{p}'(r)](t,x)-[\mathrm{p}'(r)](t-\delta s,x)\Big)\mathrm{div}\,(\mathbf{v}(t,x)\\
&-\mathbf{v}(t-\delta s,x-\delta y))\eta(y)\ \mathrm{d}y\tilde{\eta}(s)\ \mathrm{d}s\end{aligned}$$

$$-\int_{\mathbb{R}}\int_{\mathbb{R}^n}\Big([\mathrm{p}'(r)](t-\delta s,x)-[\mathrm{p}'(r)](t-\delta s,x-\delta y)\Big)\mathrm{div}\,(\mathbf{v}(t,x)\\ -\mathbf{v}(t-\delta s,x-\delta y))\eta(y)\;\mathrm{d}y\tilde{\eta}(s)\;\mathrm{d}s.$$

Thanks to $\mathrm{p}'(r)\in C([-\delta_0,T+\delta_0];L^{\frac{\gamma}{\gamma-1}}(\widetilde{\Omega}))$ and $\mathrm{div}\,\mathbf{v}\in L^1(-\delta_0,T+\delta_0;L^\infty(\widetilde{\Omega}))$, and by Minkowski's integral inequality and Lemma A.3, it follows that, as $\delta\to 0$,

$$\|I_3\|_{L^1(0,T;L^{\frac{\gamma}{\gamma-1}})}\to 0.$$

Therefore, we have, as $\delta\to 0$,

$$\mathrm{p}'(r^\delta_{t,x})\mathrm{div}\,\mathbf{v}^\delta_{t,x}-(\mathrm{p}'(r)\mathrm{div}\,\mathbf{v})^\delta_{t,x}\to 0 \quad \text{strongly in } L^1(0,T;L^{\frac{\gamma}{\gamma-1}}). \tag{3.18}$$

Putting (3.13), (3.14), and (3.16)-(3.18) together, it confirms that

$$\mathrm{P}''(r^\delta_{t,x})E_1(r^\delta_{t,x},\mathbf{v}^\delta_{t,x})\to \mathrm{P}''(r)E_1(r,\mathbf{v}) \quad \text{strongly in } L^1(0,T;L^{\frac{\gamma}{\gamma-1}}), \text{ as } \delta\to 0. \tag{3.19}$$

Using the definition of $E_2(r,\mathbf{v})$ in (1.5), by Lemma A.6, we express $\frac{1}{r^\delta_{t,x}}E_2(r^\delta_{t,x},\mathbf{v}^\delta_{t,x})$ as

$$\begin{aligned}\frac{1}{r^\delta_{t,x}}E_2(r^\delta_{t,x},\mathbf{v}^\delta_{t,x}) =&\partial_t\mathbf{v}^\delta_{t,x}+\mathbf{v}^\delta_{t,x}\cdot\nabla\mathbf{v}^\delta_{t,x}+\frac{1}{r^\delta_{t,x}}\nabla p(r^\delta_{t,x})-\frac{1}{r^\delta_{t,x}}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v}^\delta_{t,x})\\ =&\Big(\frac{1}{r}E_2(r,\mathbf{v})\Big)^\delta_{t,x}+P''(r^\delta_{t,x})\nabla r^\delta_{t,x}-(P''(r)\nabla r)^\delta_{t,x}\\ &+\mathbf{v}^\delta_{t,x}\cdot\nabla\mathbf{v}^\delta_{t,x}-(\mathbf{v}\cdot\nabla\mathbf{v})^\delta_{t,x}-\frac{1}{r^\delta_{t,x}}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v}^\delta_{t,x})+\Big(\frac{1}{r}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v})\Big)^\delta_{t,x}.\end{aligned}$$

Taking a similar argument to (3.19), we have, as $\delta\to 0$,

$$\frac{1}{r^\delta_{t,x}}E_2(r^\delta_{t,x},\mathbf{v}^\delta_{t,x})\to\frac{1}{r}E_2(r,\mathbf{v}) \quad \text{strongly in } L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}). \tag{3.20}$$

Recalling the expression $\Lambda(\cdot)$ in (2.16), noticing that $\|\eta^\delta\|_{L^1(\mathbb{R}^3)}=1$ and $\|\tilde{\eta}^\delta\|_{L^1(\mathbb{R})}=1$, and by Lemmas A.4 and A.6, it yields, as $\delta\to 0$,

$$\Lambda(\mathbf{v}^\delta_{t,x})\le\Lambda(\mathbf{v}). \tag{3.21}$$

By (3.5)-(3.8), and noting the definition of $\mathrm{LS}(\rho,\mathbf{u};\cdot,\cdot)$ in (2.17), we have

$$\mathrm{LS}(\rho,\mathbf{u};r^\delta_{t,x},\mathbf{v}^\delta_{t,x})\to\mathrm{LS}(\rho,\mathbf{u};r,\mathbf{v}) \text{ as } \delta\to 0.$$

In view of (3.7), (3.8), (3.19)-(3.21), and $\mathrm{RS}(\rho,\mathbf{u};\cdot,\cdot)$ defined in (2.17), we have

$$\mathrm{RS}(\rho,\mathbf{u};r^\delta_{t,x},\mathbf{v}^\delta_{t,x})\le\mathrm{RS}(\rho,\mathbf{u};r,\mathbf{v}) \text{ as } \delta\to 0.$$

Thus, we complete the proof. ∎

4. Proof of Theorem 2.1

This section aims to present the proof of Theorem 2.1 with focus on the periodic case. We will point out the difference between the Dirichlet boundary case and the periodic case.

Our proof is starting with the following modified Brenner model:

$$\begin{cases} \partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) = \epsilon \Delta \rho^\epsilon, \\ \partial_t(\rho^\epsilon \mathbf{u}^\epsilon) + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon) + \nabla \mathrm{p}(\rho^\epsilon) + \epsilon^a \nabla (\rho^\epsilon)^\beta = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}^\epsilon) + \epsilon \operatorname{div}(\mathbf{u}^\epsilon \otimes \nabla \rho^\epsilon), \end{cases} \tag{4.1}$$

where $\epsilon \in (0,1]$ is a small parameter, $\beta > \max\{4, \gamma\}$, and a is any positive constant. Here, we put the artificial pressure term $\epsilon^a \nabla (\rho^\epsilon)^\beta$ and the artificial diffusion term $\epsilon \Delta \rho^\epsilon$ at the same level, which differs from the approximation model introduced in [10]. We consider the approximate initial data

$$(\rho^\epsilon, \mathbf{u}^\epsilon)|_{t=0} = (\rho_0^\epsilon, \mathbf{u}_0^\epsilon), \tag{4.2}$$

satisfying

$$\begin{aligned} &\rho_0^\epsilon \in C^3(\Omega), \ \int \rho_0^\epsilon \ \mathrm{d}x \geq C_1 > 0, \ 0 < \epsilon \leq \rho_0^\epsilon \leq \epsilon^{-\frac{a}{2\beta}}, \ \mathbf{u}_0^\epsilon \in C^3(\Omega), \\ &\rho_0^\epsilon \to \rho_0 \text{ strongly in } L^\gamma, \ \sqrt{\rho_0^\epsilon}\mathbf{u}_0^\epsilon \to \frac{\mathbf{m}_0}{\sqrt{\rho_0}} \text{ strongly in } L^2, \text{ as } \epsilon \to 0, \end{aligned} \tag{4.3}$$

where the positive constant $C_1 \leq C_{\rho_0}$ is independent of ϵ, and $(\rho_0, \mathbf{m}_0)$ satisfies (2.21). When we consider the Dirichlet boundary case, the condition $\int \rho_0^\epsilon \ \mathrm{d}x \geq C_1 > 0$ in (4.3) can be removed and $(\rho_0, \mathbf{m}_0)$ satisfies (2.22). In addition, we need to add the boundary conditions $\nabla \rho^\epsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\mathbf{u}^\epsilon|_{\partial\Omega} = 0$.

For any fixed $\epsilon > 0$ and any $T \in (0, \infty)$, by the Faedo-Galerkin approximation adopted by Feireisl et al. ([10], Proposition 2.1), the system (4.1)-(4.2) has a global weak solution $(\rho^\epsilon, \mathbf{u}^\epsilon)$, which satisfies the energy inequality

$$\begin{aligned} &\int \frac{1}{2}\rho^\epsilon |\mathbf{u}^\epsilon|^2 + \mathrm{P}(\rho^\epsilon) + \epsilon^a \mathrm{Q}(\rho^\epsilon) \ \mathrm{d}x + \int_0^t \int \mathbb{S}(\nabla \mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^t \int \epsilon \mathrm{P}''(\rho^\epsilon) |\nabla \rho^\epsilon|^2 \\ &+ \epsilon^{1+a} \mathrm{Q}''(\rho^\epsilon) |\nabla \rho^\epsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq \int \frac{1}{2}\rho_0^\epsilon |\mathbf{u}_0^\epsilon|^2 + \mathrm{P}(\rho_0^\epsilon) + \epsilon^a \mathrm{Q}(\rho_0^\epsilon) \ \mathrm{d}x, \end{aligned} \tag{4.4}$$

for any $t \in [0,T]$, where $\mathrm{P}(\rho^\epsilon) = \frac{A}{\gamma - 1}(\rho^\epsilon)^\gamma$ and $\mathrm{Q}(\rho^\epsilon) = \frac{1}{\beta - 1}(\rho^\epsilon)^\beta$.

By (4.3) and Lemma A.2, it follows from (4.4) that

$$\sup_{t \in [0,T]} \|\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon\|_{L^2} \leq C, \tag{4.5}$$

$$\sup_{t \in [0,T]} \|\rho^\epsilon\|_{L^\gamma} \leq C, \tag{4.6}$$

$$\int_0^T \|\mathbf{u}^\epsilon\|_{H^1} \ \mathrm{d}t \leq C, \tag{4.7}$$

$$\epsilon \int_0^T \int |\rho^\epsilon|^{\gamma-2} |\nabla \rho^\epsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq C. \tag{4.8}$$

Here, the estimate (4.7) requires $\gamma \geq \frac{6}{5}$ from Lemma A.2. (For the Dirichlet boundary case, the estimate (4.7) holds for $\gamma > 1$ by Poincaré's inequality.)

The conclusion ([10], Proposition 2.1) tells us that

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) = \epsilon \Delta \rho^\epsilon \quad \text{a.e. in } (0,T) \times \Omega.$$

Multiplying $B'(\rho^\epsilon)$ on the both sides of above equation, we arrive at

$$\partial_t B(\rho^\epsilon) + \operatorname{div}(B(\rho^\epsilon)\mathbf{u}^\epsilon) + (B'(\rho^\epsilon)\rho^\epsilon - B(\rho^\epsilon)) \operatorname{div} \mathbf{u}^\epsilon = \epsilon \operatorname{div}(B'(\rho^\epsilon)\nabla\rho^\epsilon) - \epsilon B''(\rho^\epsilon)|\nabla\rho^\epsilon|^2, \tag{4.9}$$

where $B\in C([0,\infty))\cap C^2((0,\infty))$ with $B'(z)=0$ for large $z\in\mathbb{R}^+$.

On the one hand, taking $B(z)=z\ln z$ for $z\in[0,1]$ in (4.9), and integrating the result over $(0,T)\times\{x\in\Omega:0<\rho^\epsilon\le 1\}$, it follows from (4.6) and (4.7) that

$$\begin{aligned}&\epsilon\int_0^T\int_{\{x:0<\rho^\epsilon\le 1\}}(\rho^\epsilon)^{-1}|\nabla\rho^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_{\{x:\rho^\epsilon\le 1\}}\rho^\epsilon\ln\rho^\epsilon\,\mathrm{d}x\Big|_0^T-\int_0^T\int_{\{x:\rho^\epsilon\le 1\}}\rho^\epsilon\mathrm{div}\,\mathbf{u}^\epsilon\,\mathrm{d}x\,\mathrm{d}t\le C.\end{aligned}$$

On the other hand, by (4.8), it infers that

$$\epsilon\int_0^T\int_{\{x:\rho^\epsilon\ge 1\}}(\rho^\epsilon)^{-1}|\nabla\rho^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}t\le\epsilon\int_0^T\int(\rho^\epsilon)^{\gamma-1}\Big|\frac{\nabla\rho^\epsilon}{(\rho^\epsilon)^{\frac12}}\Big|^2\,\mathrm{d}x\,\mathrm{d}t\le C.$$

Then, it implies that

$$\epsilon^{\frac12}(\rho^\epsilon)^{-\frac12}\nabla\rho^\epsilon\in L^2(0,T;L^2).\tag{4.10}$$

Similar to (4.10), if we take $B(z)=-4\sqrt{z}$ for $z\in[0,1]$, then we have

$$\epsilon^{\frac12}(\rho^\epsilon)^{-\frac34}\nabla\rho^\epsilon\in L^2(0,T;L^2).\tag{4.11}$$

By the estimates (4.6) and (4.10), and Hölder's inequality, one has

$$\epsilon^{\frac12}\nabla\rho^\epsilon\in L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}).\tag{4.12}$$

Using the energy estimates (4.6) and (4.7), up to a subsequence $(\rho^\epsilon,\mathbf{u}^\epsilon)$ without relabeling, there exists a weak limit $(\rho,\mathbf{u})$ such that

$$\rho^\epsilon\rightharpoonup\rho\quad\text{weakly-}*\text{ in }L^\infty(0,T;L^\gamma),\tag{4.13}$$
$$\mathbf{u}^\epsilon\rightharpoonup\mathbf{u}\quad\text{weakly in }L^2(0,T;H^1).\tag{4.14}$$

Notice that $\sqrt{\rho^\epsilon}\in L^\infty(0,T;L^{2\gamma})$ from (4.6), there exists a function $\tilde\rho$ such that

$$\sqrt{\rho^\epsilon}\rightharpoonup\sqrt{\tilde\rho}\quad\text{weakly-}*\text{ in }L^\infty(0,T;L^{2\gamma}).\tag{4.15}$$

Taking $B(\rho^\epsilon)=\sqrt{\rho^\epsilon}$ in (4.9), and by the estimates (4.5)-(4.7), (4.10) and (4.11), then we have

$$\begin{aligned}\partial_t(\sqrt{\rho^\epsilon})&=-\,\mathrm{div}\,(\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon)+\frac12\sqrt{\rho^\epsilon}\mathrm{div}\,\mathbf{u}^\epsilon+\epsilon\mathrm{div}\Big(\frac{1}{2\sqrt{\rho^\epsilon}}\nabla\rho^\epsilon\Big)+\frac14\epsilon(\rho^\epsilon)^{-\frac32}|\nabla\rho^\epsilon|^2\\&\in L^\infty(0,T;W^{-1,2})+L^2(0,T;L^{\frac{2\gamma}{\gamma+1}})+L^2(0,T;W^{-1,2})+L^1(0,T;L^1)\\&\subset L^2(0,T;W^{-1,\frac{2\gamma}{\gamma+1}})\cap L^1(0,T;L^1)\subset L^1(0,T;W^{-1,1}).\end{aligned}\tag{4.16}$$

With the help of Lemma A.10, we have

$$\rho^\epsilon=\sqrt{\rho^\epsilon}\sqrt{\rho^\epsilon}\to\tilde\rho\quad\text{in }\mathcal{D}'((0,T)\times\Omega).\tag{4.17}$$

Combining (4.13) and (4.17), and the uniqueness of limit implies, for a.e. $(t,x)\in(0,T)\times\Omega$,

$$\rho=\tilde\rho.\tag{4.18}$$

By (4.15), (4.16) and (4.18), and Lemma A.9 gives

$$\sqrt{\rho^\epsilon} \to \sqrt{\rho} \quad \text{in } C([0,T];L^{2\gamma}-w).$$

Since $2\gamma > 2 > \frac{6}{5}$, by the interpolation relation $L^{2\gamma} \hookrightarrow\hookrightarrow H^{-1}$, Aubin-Lions lemma gives

$$\sqrt{\rho^\epsilon} \to \sqrt{\rho} \quad \text{in } C([0,T];H^{-1}). \tag{4.19}$$

It follows from (4.14) and (4.19) that

$$\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon \to \sqrt{\rho}\mathbf{u} \quad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Since $\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon \in L^\infty(0,T;L^2)$ (recall (4.5)), one has

$$\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon \rightharpoonup \sqrt{\rho}\mathbf{u} \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2). \tag{4.20}$$

By (4.13), (4.16) and (4.20), in view of Lemma A.10, we have

$$\rho^\epsilon\mathbf{u}^\epsilon = \sqrt{\rho^\epsilon}\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon \to \rho\mathbf{u} \quad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Since $\rho^\epsilon\mathbf{u}^\epsilon \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})$ (recall (4.5) and (4.6)), it implies

$$\rho^\epsilon\mathbf{u}^\epsilon \rightharpoonup \rho\mathbf{u} \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}). \tag{4.21}$$

From the discussion in Section 3, we can choose smooth functions $0 < r \in C^\infty([0,T]\times\Omega)$ and $\mathbf{v} \in C^\infty([0,T]\times\Omega)$. Taking the inner product with $(4.1)_2$ by $-\mathbf{v}$, multiplying $(4.1)_1$ by $\frac{1}{2}|\mathbf{v}|^2 - \mathrm{P}'(r)$, and integrating the result over $(0,t)\times\Omega$, it deduces that

$$\begin{aligned}
&\int_0^t \frac{\mathrm{d}}{\mathrm{dt}} \int -\rho^\epsilon\mathbf{u}^\epsilon\cdot\mathbf{v} + \frac{1}{2}\rho^\epsilon|\mathbf{v}|^2 - \mathrm{P}'(r)\rho^\epsilon \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^t \int \rho^\epsilon\partial_t\mathbf{v}\cdot(\mathbf{v}-\mathbf{u}^\epsilon) + \rho^\epsilon(\mathbf{u}^\epsilon\cdot\nabla)\mathbf{v}\cdot(\mathbf{v}-\mathbf{u}^\epsilon) + \mathbb{S}(\nabla\mathbf{u}^\epsilon):\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^t \int -\mathrm{p}(\rho^\epsilon)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x - \epsilon^a(\rho^\epsilon)^\beta \mathrm{div}\,\mathbf{v} + \epsilon(\nabla\rho^\epsilon\cdot\nabla)\mathbf{v}\cdot(\mathbf{u}^\epsilon-\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^t \int -\rho^\epsilon\partial_t\mathrm{P}'(r) - \rho^\epsilon\mathbf{u}^\epsilon\cdot\nabla\mathrm{P}'(r) + \epsilon\nabla\rho^\epsilon\cdot\nabla\mathrm{P}'(r)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.22}$$

Since both $r > 0$ and $\mathbf{v}$ are smooth functions, we integrate (2.2) over $(0,t)$ to give

$$\int_0^t \frac{\mathrm{d}}{\mathrm{dt}} \int \mathrm{P}'(r)r - \mathrm{P}(r)\,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int r\partial_t\mathrm{P}'(r) + r\mathbf{v}\cdot\nabla\mathrm{P}'(r) + \mathrm{p}(r)\mathrm{div}\,\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.23}$$

Adding up (4.4), (4.22) and (4.23), and by means of the definitions of $E_1(r,\mathbf{v})$ and $E_2(r,\mathbf{v})$ in (1.5), we arrive at

$$\begin{aligned}
&\int \frac{1}{2}\rho^\epsilon|\mathbf{u}^\epsilon-\mathbf{v}|^2 + \mathrm{P}(\rho^\epsilon) - \mathrm{P}'(r)(\rho^\epsilon - r) - \mathrm{P}(r) + \epsilon^a\mathrm{Q}(\rho^\epsilon)\,\mathrm{d}x \\
&\quad + \int_0^t \int \mathbb{S}(\nabla\mathbf{u}^\epsilon - \nabla\mathbf{v}):\nabla(\mathbf{u}^\epsilon-\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int \frac{1}{2}\rho_0^\epsilon|\mathbf{u}_0^\epsilon - \mathbf{v}_0|^2 + \mathrm{P}(\rho_0^\epsilon) - \mathrm{P}'(r_0)(\rho_0^\epsilon - r_0) - \mathrm{P}(r_0) + \epsilon^a\mathrm{Q}(\rho_0^\epsilon)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int -\rho(\mathbf{u}^\epsilon-\mathbf{v})\cdot\mathbb{D}(\mathbf{v})\cdot(\mathbf{u}^\epsilon-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t+\int_0^t\int \frac{\rho^\epsilon-r}{r}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u}^\epsilon)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^t\int \big(\mathrm{p}(\rho^\epsilon)-\mathrm{p}'(r)(\rho^\epsilon-r)-\mathrm{p}(r)\big)\mathrm{div}\,\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t-\int_0^t\int \epsilon^a(\rho^\epsilon)^\beta\mathrm{div}\,\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^t\int (r-\rho^\epsilon)\mathrm{P}''(r)E_1(r,\mathbf{v})+\frac{\rho^\epsilon}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u}^\epsilon)\ \mathrm{d}x\ \mathrm{d}t\\
&+\epsilon\int_0^t\int (\nabla\rho^\epsilon\cdot\nabla)\mathbf{v}\cdot(\mathbf{u}^\epsilon-\mathbf{v})+\nabla\rho^\epsilon\cdot\nabla\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}
$$

Using the same way as (2.11) to deal with the third term on the right-hand side of the above equality, then applying Grönwall's inequality (integral form) to the result gives

$$
\begin{aligned}
&\int \frac12\rho^\epsilon|\mathbf{u}^\epsilon-\mathbf{v}|^2+\mathrm{P}(\rho^\epsilon)-\mathrm{P}'(r)(\rho^\epsilon-r)-\mathrm{P}(r)+\epsilon^a\mathrm{Q}(\rho^\epsilon)\ \mathrm{d}x\\
&\quad+\frac12\int_0^t\int \mathbb{S}(\nabla\mathbf{u}^\epsilon-\nabla\mathbf{v}):\nabla(\mathbf{u}^\epsilon-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s\\
&\le\Big(\int_0^t C_0\Lambda(\mathbf{v})\exp\Big(\int_s^t C_0\Lambda(\mathbf{v})\ \mathrm{d}s\Big)\ \mathrm{d}s+1\Big)\Big(\mathcal{R}_0+\sum_{i=1}^3\mathcal{R}_i\Big),
\end{aligned}
\tag{4.24}
$$

where

$$
\begin{aligned}
&\mathcal{R}_0=\int \frac12\rho_0^\epsilon|\mathbf{u}_0^\epsilon-\mathbf{v}_0|^2+\mathrm{P}(\rho_0^\epsilon)-\mathrm{P}'(r_0)(\rho_0^\epsilon-r_0)-\mathrm{P}(r_0)+\epsilon^a\mathrm{Q}(\rho_0^\epsilon)\ \mathrm{d}x,\\
&\mathcal{R}_1=\int_0^t\int (r-\rho^\epsilon)\mathrm{P}''(r)E_1(r,\mathbf{v})+\frac{\rho^\epsilon}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u}^\epsilon)\ \mathrm{d}x\ \mathrm{d}s,\\
&\mathcal{R}_2=\epsilon\int_0^t\int (\nabla\rho^\epsilon\cdot\nabla)\mathbf{v}\cdot(\mathbf{u}^\epsilon-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s,\\
&\mathcal{R}_3=\epsilon\int_0^t\int \nabla\rho^\epsilon\cdot\nabla\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}s.
\end{aligned}
$$

The next step is to recover a dissipative solution by passing to the limits in (4.24) as ε tends to zero.

We first deal with the left-hand side of (4.24). By the weak convergences (4.13), (4.14), (4.20) and (4.21), and in view of the low semi-continuous of L^2-norm and the convexity of $\mathrm{P}(\cdot)$, we get, as $\epsilon\to 0$,

$$
\begin{aligned}
&\int \frac12\rho^\epsilon|\mathbf{u}^\epsilon-\mathbf{v}|^2+\mathrm{P}(\rho^\epsilon)-\mathrm{P}'(r)(\rho^\epsilon-r)-\mathrm{P}(r)+\epsilon^a\mathrm{Q}(\rho^\epsilon)\ \mathrm{d}x\\
&\quad+\frac12\int_0^t\int \mathbb{S}(\nabla\mathbf{u}^\epsilon-\nabla\mathbf{v}):\nabla(\mathbf{u}^\epsilon-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s\\
&\ge\int \frac12\rho|\mathbf{u}-\mathbf{v}|^2+\mathrm{P}(\rho)-\mathrm{P}'(r)(\rho-r)-\mathrm{P}(r)\ \mathrm{d}x\\
&\quad+\frac12\int_0^t\int \mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s.
\end{aligned}
\tag{4.25}
$$

Next, we tackle with the reminder terms $\mathcal{R}_i\,(i=0,1,2,3)$ on the right-hand side of (4.24). For the term $\mathcal{R}_0$, by the initial conditions (4.3), we have, as $\epsilon\to 0$,

$$
\mathcal{R}_0\to\int \frac12\rho_0\Big|\frac{\mathbf{m}_0}{\rho_0}-\mathbf{v}_0\Big|^2+\mathrm{P}(\rho_0)-\mathrm{P}'(r_0)(\rho_0-r_0)-\mathrm{P}(r_0)\ \mathrm{d}x. \tag{4.26}
$$

For the term $\mathcal{R}_1$, it follows from (4.13) and (4.21) that, as $\epsilon\to 0$,

$$\mathcal{R}_1 \to \int_0^t \int (r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v})+\frac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ dx\ ds. \tag{4.27}$$

We turn to the term $\mathcal{R}_2$. By (4.10), (4.12) and (4.20), it implies that

$$\epsilon(\sqrt{\rho^\epsilon})^{-1}\nabla\rho^\epsilon\cdot(\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon)\to 0 \quad \text{in } L^2(0,T;L^1), \tag{4.28}$$

$$\epsilon\nabla\rho^\epsilon\to 0 \quad \text{in } L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}). \tag{4.29}$$

In view of (4.28) and (4.29), we have, as $\epsilon\to 0$,

$$\mathcal{R}_2 = \epsilon\int_0^t\int(\sqrt{\rho^\epsilon})^{-1}\nabla\rho^\epsilon\cdot\nabla\mathbf{v}\cdot\sqrt{\rho^\epsilon}\mathbf{u}^\epsilon\ dx\ ds - \epsilon\int_0^t\int(\nabla\rho^\epsilon\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\ dx\ ds\to 0. \tag{4.30}$$

Finally, for the term $\mathcal{R}_3$, by Hölder's inequality, it follows from (4.29) that, as $\epsilon\to 0$,

$$\mathcal{R}_3 \le C\|\epsilon\nabla\rho^\epsilon\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+1}})}\|\nabla r\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}})}\to 0. \tag{4.31}$$

Making use of Proposition 3.1, we conclude that (2.17) holds for the functions r and $\mathbf{v}$ satisfying (2.18) and (2.20). Thus, we complete the proof of Theorem 2.1.

5. Proof of Theorem 2.2

The goal of this section is to show that the weak solution of the compressible isentropic Navier-Stokes equations is also a dissipative solution in the sense of Definition 2.1.

Let $(\rho,\mathbf{u})$ be the weak solution of the problem (1.1)-(1.3) or (1.1), (1.2) and (1.4). Let $\chi_n\in C_c^\infty((0,T))$ and $\phi_m\in C_c^\infty(\Omega)$. We can also let $(r,\mathbf{v})$ be the smooth functions by the arguments in Section 3. Taking the test function $\Phi=\chi_n\phi_m\mathbf{v}$ in (2.28) of Definition 2.2 gives

$$\begin{aligned}
0=&\int_0^T\int\rho\mathbf{u}\cdot\partial_t(\chi_n\phi_m\mathbf{v})\ dx\ dt+\int_0^T\int\rho\mathbf{u}\otimes\mathbf{u}:\nabla(\chi_n\phi_m\mathbf{v})\ dx\ dt\\
&+\int_0^T\int\mathrm{p}(\rho)\mathrm{div}\,(\chi_n\phi_m\mathbf{v})\ dx\ dt-\int_0^T\int\mathbb{S}(\nabla\mathbf{u}):\nabla(\chi_n\phi_m\mathbf{v})\ dx\ dt\\
=&\int_0^T\partial_t\chi_n\int\phi_m\rho\mathbf{u}\cdot\mathbf{v}\ dx\ dt+\int_0^T\chi_n\int\phi_m\rho\mathbf{u}\cdot\partial_t\mathbf{v}\ dx\ dt\\
&+\int_0^T\chi_n\int\phi_m\rho\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{u}\ dx\ dt+\int_0^T\chi_n\int\phi_m\mathrm{p}(\rho)\mathrm{div}\,\mathbf{v}\ dx\ dt\\
&-\int_0^T\chi_n\int\phi_m\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{v}\ dx\ dt+\int_0^T\chi_n\int\rho\mathbf{u}\otimes\mathbf{u}:(\nabla\phi_m\otimes\mathbf{v})\ dx\ dt\\
&+\int_0^T\chi_n\int\mathrm{p}(\rho)\nabla\phi_m\cdot\mathbf{v}\ dx\ dt-\int_0^T\chi_n\int\mathbb{S}(\nabla\mathbf{u}):(\nabla\phi_m\otimes\mathbf{v})\ dx\ dt.
\end{aligned}\tag{5.1}$$

Choosing $\Psi=\frac{1}{2}|\mathbf{v}|^2\chi_n\phi_m$ and $\Psi=\mathrm{P}'(r)\chi_n\phi_m$ in (2.27) of Definition 2.2, respectively, one sees that

$$0=\int_0^T\int\rho\partial_t\Big(\frac{1}{2}|\mathbf{v}|^2\chi_n\phi_m\Big)\ dx\ dt+\int_0^T\int\rho\mathbf{u}\cdot\nabla\Big(\frac{1}{2}|\mathbf{v}|^2\chi_n\phi_m\Big)\ dx\ dt$$

$$
\begin{aligned}
&=\int_0^T \partial_t\chi_n \int \phi_m\rho\Big(\frac{1}{2}|\mathbf{v}|^2\Big)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m\rho\mathbf{v}\cdot\partial_t\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T \chi_n\int \phi_m\rho\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \frac{1}{2}|\mathbf{v}|^2\rho\mathbf{u}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
0&=\int_0^T\int \rho\partial_t(\mathrm{P}'(r)\chi_n\phi_m)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T\int \rho\mathbf{u}\cdot\nabla(\mathrm{P}'(r)\chi_n\phi_m)\ \mathrm{d}x\ \mathrm{d}t\\
&=\int_0^T \partial_t\chi_n\int \phi_m\rho\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m\rho\partial_t\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T \chi_n\int \phi_m\rho\mathbf{u}\cdot\nabla\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \rho\mathrm{P}'(r)\mathbf{u}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}
\tag{5.3}
$$

Noticing that the relations $\mathrm{P}'(r)r-\mathrm{P}(r)=\mathrm{p}(r)$ and $\mathrm{P}''(r)r=\mathrm{p}'(r)$, we have

$$
\begin{aligned}
0&=\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int [\mathrm{P}'(r)r-\mathrm{P}(r)]\chi_n\phi_m\ \mathrm{d}x\ \mathrm{d}t\\
&=\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{p}(r)\chi_n\phi_m\ \mathrm{d}x\ \mathrm{d}t+\int_0^T\int \mathrm{div}(\mathrm{p}(r)\mathbf{v}\chi_n\phi_m)\ \mathrm{d}x\ \mathrm{d}t\\
&=\int_0^T \partial_t\chi_n\int \phi_m[\mathrm{P}'(r)r-\mathrm{P}(r)]\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m r\partial_t\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T \chi_n\int \phi_m r\mathbf{v}\cdot\nabla\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m\mathrm{p}(r)\mathrm{div}\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T \chi_n\int \mathrm{p}(r)\mathbf{v}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}
\tag{5.4}
$$

Collecting (5.1)-(5.4) together, we obtain that

$$
\begin{aligned}
&-\int_0^T \partial_t\chi_n\int \phi_m\Big[-\rho\mathbf{u}\cdot\mathbf{v}+\frac{1}{2}\rho|\mathbf{v}|^2-\mathrm{P}'(r)\rho+\mathrm{P}'(r)r-\mathrm{P}(r)\Big]\ \mathrm{d}x\ \mathrm{d}t\\
&=-\int_0^T \chi_n\int \phi_m\rho(\mathbf{u}-\mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{u}-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T \chi_n\int \phi_m(r-\rho)\partial_t\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m(r\mathbf{v}-\rho\mathbf{u})\cdot\nabla\mathrm{P}'(r)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_0^T \chi_n\int \phi_m(\mathrm{p}(\rho)-\mathrm{p}(r))\mathrm{div}\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t+\int_0^T \chi_n\int \phi_m\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t+\mathcal{R}_{\chi_n\phi_m},
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
\mathcal{R}_{\chi_n\phi_m}=&\underbrace{-\int_0^T \partial_t\chi_n\int (1-\phi_m)\Big[\frac{1}{2}\rho|\mathbf{u}|^2+\mathrm{P}(\rho)\Big]\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_1}\\
&\underbrace{-\int_0^T \chi_n\int \rho\mathbf{u}\otimes\mathbf{u}:(\nabla\phi_m\otimes\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t-\int_0^T \chi_n\int \mathrm{p}(\rho)\mathbf{v}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_2}
\end{aligned}
$$

$$
+\underbrace{\int_0^T \chi_n \int \mathbb{S}(\nabla \mathbf{u}):(\nabla \phi_m \otimes \mathbf{v})\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_3}+\underbrace{\int_0^T \chi_n \int \frac{1}{2}|\mathbf{v}|^2 \rho \mathbf{u}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_4}
$$

$$
\underbrace{-\int_0^T \chi_n \int \rho \mathrm{P}'(r)\mathbf{u}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_5}+\underbrace{\int_0^T \chi_n \int \mathrm{p}(r)\mathbf{v}\cdot\nabla\phi_m\ \mathrm{d}x\ \mathrm{d}t}_{\mathfrak{X}_6}.
$$

We endow a sequence $\phi_m \in C_c^\infty(\Omega)$ with

$$
\begin{aligned}
&0\le\phi_m\le 1,\ \phi_m=1 \text{ for } x\in\Omega,\ \mathrm{dist}(x,\partial\Omega)\ge\frac{1}{m},\\
&\phi_m\to 1,\ |\nabla\phi_m|\le 2m \ \text{ for } x\in\Omega.
\end{aligned}
$$

We first tackle with the term $\mathcal{R}_{\chi_n\phi_m}$. For the term $\mathfrak{X}_1$ in $\mathcal{R}_{\chi_n\phi_m}$, in view of (2.26), and using the Lebesgue's dominated convergence theorem, one has, as $m\to\infty$,

$$
\mathfrak{X}_1\to 0. \tag{5.6}
$$

For the term $\mathfrak{X}_2$, by Hölder's inequality and (2.26), we get, as $m\to\infty$,

$$
\begin{aligned}
\mathfrak{X}_2 &\le \int_0^T \chi_n \int (\rho|\mathbf{u}|^2+\mathrm{p}(\rho))|\nabla\phi_m \mathrm{dist}(x,\partial\Omega)||\mathbf{v}[\mathrm{dist}(x,\partial\Omega)]^{-1}|\ \mathrm{d}x\ \mathrm{d}t\\
&\le C(n)\int_0^T \|\mathbf{v}[\mathrm{dist}(x,\partial\Omega)]^{-1}\|_{L^\infty}\int_{\{x:\mathrm{dist}(x,\partial\Omega)\le\frac{1}{m}\}}\rho|\mathbf{u}|^2+\mathrm{p}(\rho)\ \mathrm{d}x\ \mathrm{d}t\\
&\le C(n)\sup_{0\le t\le T}\int_{\{x:\mathrm{dist}(x,\partial\Omega)\le\frac{1}{m}\}}\rho|\mathbf{u}|^2+\mathrm{p}(\rho)\ \mathrm{d}x\times\|\mathbf{v}[\mathrm{dist}(x,\partial\Omega)]^{-1}\|_{L^1(0,T;L^\infty)}\\
&\to 0.
\end{aligned} \tag{5.7}
$$

Here, we have employed the fact that $\mathbf{v}[\mathrm{dist}(x,\partial\Omega)]^{-1}\in L^1(0,T;L^\infty)$. Indeed, by Hardy's inequality, we have

$$
\|\mathbf{v}[\mathrm{dist}(x,\partial\Omega)]^{-1}\|_{L^p}\le C\|\nabla\mathbf{v}\|_{L^p}\le C\|\nabla\mathbf{v}\|_{L^\infty},\ \text{for } 1<p<\infty.
$$

Since $\nabla\mathbf{v}\in L^\infty(0,T;L^q)$ for $q>1$ and $\nabla\mathbf{v}\in L^1(0,T;L^\infty)$ (see Remark 2.2), by Lemma A.5, the conclusion holds.

With a similar argument to the term $\mathfrak{X}_2$, by Hölder's inequality and Hardy's inequality, along with (2.26) and the regularities of r and $\mathbf{v}$ in Theorem 2.2, it infers that, as $m\to\infty$,

$$
\mathfrak{X}_i\to 0,\ \text{for } i=3,4,5,6. \tag{5.8}
$$

Then, we have, as $m\to\infty$,

$$
\mathcal{R}_{\chi_n\phi_m}\to 0.
$$

Taking

$$
\chi(t)=\begin{cases}
0, & t<\tilde{s},\ t\ge\tilde{t}\\
\frac{1}{h}(t-\tilde{s}), & \tilde{s}\le t<\tilde{s}+h,\\
1, & \tilde{s}+h\le t<\tilde{t}-h,\\
-\frac{1}{h}(t-\tilde{t}), & \tilde{t}-h\le t<\tilde{t},
\end{cases}
$$

for any $0<\tilde{s}<\tilde{t}<T$ and $0<h<\frac{\tilde{t}-\tilde{s}}{2}$, and $\chi_n(t)=[\chi*\frac{1}{n}\tilde{\eta}(\frac{\cdot}{n})](t)\in C_c^\infty((0,T))$ for large n, where $\tilde{\eta}(t)$ is the mollifier in (3.2), it follows from Lemmas A.6 and A.7 that, as $n\to\infty$,

$$\chi_n\to\chi,\quad \partial_t\chi_n\to\chi_w',$$

for a.e. $t\in(0,T)$, where

$$\chi_w'(t)=\begin{cases}0, & t<\tilde{s},\ t\geq\tilde{t}\\ \frac{1}{h}, & \tilde{s}\leq t<\tilde{s}+h,\\ 0, & \tilde{s}+h\leq t<\tilde{t}-h,\\ -\frac{1}{h}, & \tilde{t}-h\leq t<\tilde{t},\end{cases}$$

which is the weak derivative of $\chi(t)$ with respect to time.

Now, letting $m\to\infty$ and then $n\to\infty$ in (5.5), by the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&\frac{1}{h}\int_{\tilde{t}-h}^{\tilde{t}}\int\Big[-\rho\mathbf{u}\cdot\mathbf{v}+\frac{1}{2}\rho|\mathbf{v}|^2-\mathrm{P}'(r)\rho+\mathrm{P}'(r)r-\mathrm{P}(r)\Big]\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\frac{1}{h}\int_{\tilde{s}}^{\tilde{s}+h}\int\Big[-\rho\mathbf{u}\cdot\mathbf{v}+\frac{1}{2}\rho|\mathbf{v}|^2-\mathrm{P}'(r)\rho+\mathrm{P}'(r)r-\mathrm{P}(r)\Big]\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_{\tilde{s}}^{\tilde{t}}\chi\int\rho(\mathbf{u}-\mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t+\int_{\tilde{s}}^{\tilde{t}}\chi\int\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_{\tilde{s}}^{\tilde{t}}\chi\int(r-\rho)\partial_t\mathrm{P}'(r)\,\mathrm{d}x\,\mathrm{d}t+\int_{\tilde{s}}^{\tilde{t}}\chi\int(r\mathbf{v}-\rho\mathbf{u})\cdot\nabla\mathrm{P}'(r)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_{\tilde{s}}^{\tilde{t}}\chi\int(\mathrm{p}(\rho)-\mathrm{p}(r))\mathrm{div}\,\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t+\int_{\tilde{s}}^{\tilde{t}}\chi\int\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Letting $h\to0$ above, by the Lebesgue's differentiation theorem and Lebesgue's dominated convergence theorem, it deduces that

$$\begin{aligned}&\mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})\big|_{t=\tilde{t}}-\mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})\big|_{t=\tilde{s}}\\&=-\int_{\tilde{s}}^{\tilde{t}}\int\rho(\mathbf{u}-\mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x+\int_{\tilde{s}}^{\tilde{t}}\int\rho(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\\&\quad+\int_{\tilde{s}}^{\tilde{t}}\int(r-\rho)\partial_t\mathrm{P}'(r)\,\mathrm{d}x\,\mathrm{d}t+\int_{\tilde{s}}^{\tilde{t}}\int(r\mathbf{v}-\rho\mathbf{u})\cdot\nabla\mathrm{P}'(r)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_{\tilde{s}}^{\tilde{t}}\int(\mathrm{p}(\rho)-\mathrm{p}(r))\mathrm{div}\,\mathbf{v}\,\mathrm{d}x+\int_{\tilde{s}}^{\tilde{t}}\int\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{5.9}$$

where

$$\mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})=\int\Big[-\rho\mathbf{u}\cdot\mathbf{v}+\frac{1}{2}\rho|\mathbf{v}|^2-\mathrm{P}'(r)\rho+\mathrm{P}'(r)r-\mathrm{P}(r)\Big]\,\mathrm{d}x.$$

Adding $\int\frac{1}{2}\rho|\mathbf{u}|^2+\mathrm{P}(\rho)\,\mathrm{d}x\big|_{t=\tilde{t}}+\int_{\tilde{s}}^{t}\int\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t$ to both sides of (5.9), using the energy inequality (2.29) in Definition 2.2, and by means of the definitions $E_1(r,\mathbf{v})$ and $E_2(r,\mathbf{v})$ in (1.5), we have, for a.e. $\tilde{t}\in(0,T)$,

$$\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})\big|_{t=\tilde{t}}+\int_{\tilde{s}}^{\tilde{t}}\int\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \int \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \mathrm{P}(\rho_0)\ \mathrm{d}x + \mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})|_{t=\tilde{s}} \\
&\quad + \int_{\tilde{s}}^{\tilde{t}}\int -\rho(\mathbf{u}-\mathbf{v})\cdot\mathbb{D}(\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) - \big(\mathrm{p}(\rho)-\mathrm{p}'(r)(\rho-r)-\mathrm{p}(r)\big)\mathrm{div}\,\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_{\tilde{s}}^{\tilde{t}}\int (r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v}) + \frac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_{\tilde{s}}^{\tilde{t}}\int \frac{\rho-r}{r}\mathrm{div}\,\mathbb{S}(\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t. \qquad (5.10)
\end{aligned}
$$

With the same way as (2.11) to deal with the last term on the right-hand side of (5.10), applying Grönwall's inequality (integral form), it yields, for any $0<\tilde{s}<\tilde{t}<T$, that

$$
\begin{aligned}
&\mathcal{E}(\rho,\mathbf{u};r,\mathbf{v})\big|_{t=\tilde{t}} + \frac{1}{2}\int_{\tilde{s}}^{\tilde{t}}\int \mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{v}):\nabla(\mathbf{u}-\mathbf{v})\ \mathrm{d}x\ \mathrm{d}s \\
&\leq \Big(\int_0^{\tilde{t}} C_0\Lambda(\mathbf{v})\exp\Big(\int_s^{\tilde{t}} C_0\Lambda(\mathbf{v})\ \mathrm{d}s\Big)\ \mathrm{d}s + 1\Big)\Big\{\int \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \mathrm{P}(\rho_0)\ \mathrm{d}x + \mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})|_{t=\tilde{s}} \\
&\quad + \int_0^{\tilde{t}}\int \big|(r-\rho)\mathrm{P}''(r)E_1(r,\mathbf{v})\big| + \Big|\frac{\rho}{r}E_2(r,\mathbf{v})\cdot(\mathbf{v}-\mathbf{u})\Big|\ \mathrm{d}x\ \mathrm{d}s\Big\}.
\end{aligned}
$$

Using (2.26), (2.27) and (2.28) in Definition 2.2, and making the density arguments, the weak solution $(\rho,\mathbf{u})$ belongs to the regularity class

$$
\rho\in C([0,T];L^{\gamma}-w), \quad \rho\mathbf{u}\in C([0,T];L^{\frac{2\gamma}{\gamma+1}}-w). \qquad (5.11)
$$

Choosing $\tilde{s}:=s_n$ with $s_n\to 0\,(n\to\infty)$, and taking advantage of (5.11), it implies that

$$
\begin{aligned}
&\int \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \mathrm{P}(\rho_0)\ \mathrm{d}x + \mathcal{G}(\rho,\mathbf{u};r,\mathbf{v})|_{t=\tilde{s}} \\
&= \int \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \mathrm{P}(\rho_0)\ \mathrm{d}x - \int \rho\mathbf{u}\cdot\mathbf{v}\ \mathrm{d}x\Big|_{t=s_n} + \int \frac{1}{2}\rho|\mathbf{v}|^2\ \mathrm{d}x\Big|_{t=s_n} \\
&\qquad - \int \rho\mathrm{P}'(r)\ \mathrm{d}x\Big|_{t=s_n} + \int \mathrm{P}'(r)r - \mathrm{P}(r)\ \mathrm{d}x\Big|_{t=s_n} \\
&\to \int \frac{1}{2}\rho_0\Big|\frac{\mathbf{m}_0}{\rho_0}-\mathbf{v}_0\Big|^2 + \mathrm{P}(\rho_0) - \mathrm{P}'(r_0)(\rho_0-r_0) - \mathrm{P}(r_0)\ \mathrm{d}x \\
&=: \mathcal{E}_0(\rho_0,\mathbf{m}_0;r_0,\mathbf{v}_0) \ \text{ as } n\to\infty.
\end{aligned}
$$

Thus, we complete the proof of Theorem 2.2.

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Appendix. In this appendix, we first collect a few classical inequalities, some properties of L^p space, and two weak convergence results. Then, we add two supplementary lemmas (Lemmas A.2 and A.8) and give their proofs. These facts are frequently used in the proof of our main results. We point out that the Lemma A.8 subjects to the density arguments to deduce (3.7), (3.8) and (3.18) in Section 3.

First, we list the Poincaré inequalities.

LEMMA A.1 ([3], Chapter 9, Poincaré-Wirtinger's inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary $\partial\Omega$. Then,*

$$\left\| f - \frac{1}{|\Omega|}\int_\Omega f \, \mathrm{d}x \right\|_{L^q(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}, \ \forall f \in W^{1,p},$$

where $1 \le q \le \frac{pn}{n-p}$ for $n > p$.

LEMMA A.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary $\partial\Omega$, and a nonnegative function $\tilde{r}$ satisfies*

$$0 < M_0 \le \int_\Omega \tilde{r} \, \mathrm{d}x, \quad \int_\Omega \tilde{r}^\gamma \, \mathrm{d}x \le M_1,$$

for some positive constants M_0 and M_1, where it assumes $\gamma \ge \frac{2n}{n+2}$ for $n \ge 3$. Then, there exists a constant $\tilde{c}_\gamma := C(n,\gamma,\Omega,M_0,M_1)$ such that

$$\|f\|_{H^1(\Omega)} \le \tilde{c}_\gamma\big(\|\nabla f\|_{L^2(\Omega)} + \|\sqrt{\tilde{r}} f\|_{L^2(\Omega)}\big).$$

REMARK A.1. Lemmas A.1 and A.2 still hold when $\Omega = \mathbb{T}^n$. Lemma A.2 can be directly deduced by Theorem 11.23 called generalized Korn-Poincaré inequality in [9]. For reader's convenience, we give a brief proof by means of Lemma A.1 as follows.

Proof. By Minkowski's inequality and Poincaré's inequality in Lemma A.1, it follows that

$$\begin{aligned}\|f\|_{L^2(\Omega)} &\le \left\| f - \frac{1}{|\Omega|}\int_\Omega f \, \mathrm{d}x \right\|_{L^2(\Omega)} + \left\| \frac{1}{|\Omega|}\int_\Omega f \, \mathrm{d}x \right\|_{L^2(\Omega)} \\ &\le C\|\nabla f\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Then, we know

$$\|f\|_{H^1(\Omega)} \le C\|\nabla f\|_{L^2(\Omega)} + C\int_\Omega |f| \, \mathrm{d}x.$$

Using Hölder's inequality, one has

$$\begin{aligned}\int_\Omega r \, \mathrm{d}x \frac{1}{|\Omega|}\int_\Omega |f| \, \mathrm{d}x &\le \int_\Omega r\left| f - \frac{1}{|\Omega|}\int f \, \mathrm{d}x \right| \mathrm{d}x + \int_\Omega r|f| \, \mathrm{d}x \\ &\le \|r\|_{L^{\frac{p}{p-1}}(\Omega)} \left\| f - \frac{1}{|\Omega|}\int_\Omega f \, \mathrm{d}x \right\|_{L^p(\Omega)} + \|\sqrt{r}\|_{L^2(\Omega)}\|\sqrt{r} f\|_{L^2(\Omega)}.\end{aligned}$$

Taking $p = \frac{2n}{n-2}$, it sees that $\frac{p}{p-1} = \frac{2n}{n+2}$. By the Poincaré-Wirtinger's inequality stated in Lemma A.1 and the restrictions on γ, the conclusion follows. ∎

Next, we recall some basic properties of L^p space.

Lemma A.3 ([3], Lemma 4.3 in Chapter 4). *Let $\Omega\subset\mathbb{R}^n$ be an open set, $K\subset\subset\Omega$, $K=\Omega$ if $\Omega=\mathbb{T}^n$ or $\mathbb{R}^n$. Then, for $f\in L^p(\Omega)$, $1\le p<\infty$, as $\xi\to 0$,*

$$\|f(x+\xi)-f(x)\|_{L^p(K)}\to 0.$$

Lemma A.4 ([3], Theorem 4.15 in Chapter 4). *Let $f\in L^1(\mathbb{R}^n)$ and $g\in L^p(\mathbb{R}^n)$ with $1\le p\le\infty$. Define*

$$(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)\ \mathrm{d}y.$$

Then,

$$\|f*g\|_{L^p(\mathbb{R}^n)}\le\|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}.$$

Lemma A.5. *Let $\Omega\subset\mathbb{R}^n$, $0<T<\infty$. If $f\in L^\infty(0,T;L^{p_1}(\Omega))$ for some $p_1\ge 1$, and $\|f\|_{L^{p_0}(0,T;L^p(\Omega))}\le M<\infty$ for fixed $1\le p_0\le\infty$ and all $1\le p<\infty$, where the constant M is independent of p. Then, for $1\le p_0<\infty$, $f\in L^{p_0}(0,T;L^\infty(\Omega))$; for $1<p_0\le\infty$, as $p\to\infty$,*

$$\|f\|_{L^{p_1}(0,T;L^p(\Omega))}\to\|f\|_{L^{p_1}(0,T;L^\infty(\Omega))},\ 1\le p_1<p_0.$$

Proof. Since $f\in L^\infty(0,T;L^{p_1})$ for some $p_1\ge 1$, by Rudin ([17], Exercises 4(e), Chapter 3), it gives

$$\|f\|_{L^p(\Omega)}\to\|f\|_{L^\infty(\Omega)}\ \text{a.e. in}\ t\in(0,T).$$

For $1\le p_0<\infty$,

$$\|f\|^{p_0}_{L^p(\Omega)}\in L^1(0,T),$$

Fatou's lemma gives

$$\|f\|^{p_0}_{L^{p_0}(0,T;L^\infty)}=\int_0^T\liminf_{p\to\infty}\|f\|^{p_0}_{L^p(\Omega)}\ \mathrm{d}t\le\liminf_{p\to\infty}\int_0^T\|f\|^{p_0}_{L^p(\Omega)}\ \mathrm{d}t\le M^{p_0}.$$

For $1<p_0\le\infty$, Egoroff's theorem implies

$$\|f\|_{L^{p_1}(0,T;L^p(\Omega))}\to\|f\|_{L^{p_1}(0,T;L^\infty(\Omega))},$$

as $p\to\infty$, where $1\le p_1<p_0$. ◻

The following three lemmas are about mollifier. Let f be a locally integrable function. Recall the definitions of mollifiers $\eta(x)$ in (3.1) and $\tilde\eta(t)$ in (3.2). We use f^δ_x to denote the mollification of f with respect to x, and $f^\delta_{t,x}$ to denote the mollification of f with respect to both x and t, that is,

$$f^\delta_x(t,x)=(f*\eta^\delta)(t,x)=\int_{\mathbb{R}^n} f(t,x-y)\eta^\delta(y)\ \mathrm{d}y,$$
$$f^\delta_{t,x}(t,x)=(f^\delta*\tilde\eta^\delta)(t,x)=\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(s,x-y)\eta^\delta(y)\ \mathrm{d}y\tilde\eta^\delta(t-s)\ \mathrm{d}s.$$

Lemma A.6. *Let $f\in W^{k,p}$ with $1\le p<\infty$, then, for $|\alpha|\le k$,*

$$D^\alpha f^\delta_x=(D^\alpha f)^\delta_x.$$

REMARK A.2. This lemma is one conclusion during the proof of Theorem 1 in Section 5.3 of [6], which means that the α^{th} order partial derivative of the smooth function f_x^δ is the mollification of the α^{th} order weak partial derivative of f. Similarly, the regularization in t has the same property. We directly write it as follows, if $\partial_t f \in L^q(0,T;L^p)$, $1 \le p,q < \infty$, then $\partial_t f_{t,x}^\delta = (\partial_t f)_{t,x}^\delta$.

LEMMA A.7 ([3], Theorem 4.22 in Chapter 4). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $K \subset\subset \Omega$, $K = \Omega$ if $\Omega = \mathbb{T}^n$ or $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ is a locally integrable function. Then, $f_x^\delta \to f$ on a.e. $x \in K$, and for $f \in L^p(\Omega)$, $1 \le p < \infty$,*

$$\|f_x^\delta - f\|_{L^p(K)} \to 0 \ \text{ as } \delta \to 0.$$

LEMMA A.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $K \subset\subset \Omega$, $K = \Omega$ if $\Omega = \mathbb{T}^n$, and $(t_1,t_2) \subset\subset (0,T)$ with any $T \in (0,\infty)$. Then*

(1) *For $\partial_t f \in L^1(0,T;L^1(\Omega))$, $\nabla f \in L^\infty(0,T;L^1(\Omega))$, we have*

$$\|f_{t,x}^\delta - f\|_{L^\infty(t_1,t_2;L^\infty(K))} \to 0 \ \text{ as } \delta \to 0.$$

(2) *For $\partial_t f \in L^1(0,T;L^1(\Omega))$, $f \in L^\infty(0,T;L^p(\Omega))$ with $1 \le p < \infty$, we have*

$$\|f_{t,x}^\delta - f\|_{L^\infty(t_1,t_2;L^p(K))} \to 0 \ \text{ as } \delta \to 0.$$

Proof.

(1) By the definition of mollifiers in (3.1) and (3.2), and with some direct computations, it has

$$\begin{aligned}
f_{t,x}^\delta(t,x) - f(t,x) &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} [f(t-s,x-y) - f(t,x)]\eta^\delta(y)\, \mathrm{d}y \tilde{\eta}^\delta(s)\, \mathrm{d}s \\
= &\int_{\mathbb{R}} \int_{\mathbb{R}^n} [f(t-s,x-y) - f(t,x-y)]\eta^\delta(y)\, \mathrm{d}y \tilde{\eta}^\delta(s)\, \mathrm{d}s \\
&+ \int_{\mathbb{R}^n} [f(t,x-y) - f(t,x)]\eta^\delta(y)\, \mathrm{d}y \\
= &\underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}^n} [f(t-\delta s,x-\delta y) - f(t,x-\delta y)]\eta(y)\, \mathrm{d}y \tilde{\eta}(s)\, \mathrm{d}s}_{y:=y'=\delta y,\ s:=s'=\delta s} \\
&+ \underbrace{\int_{\mathbb{R}^n} [f(t,x-\delta y) - f(t,x)]\eta(y)\, \mathrm{d}y}_{y:=y'=\delta y} \\
= &\int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(t-\tau\delta s,x-\delta y)\, \mathrm{d}\tau \eta(y)\, \mathrm{d}y \tilde{\eta}(s)\, \mathrm{d}s \\
&+ \int_{\mathbb{R}^n} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(t,x-\tau\delta y)\, \mathrm{d}\tau \eta(y)\, \mathrm{d}y \\
= &\underbrace{\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^n} \partial_t f(t-\tau\delta s,x-\delta y) \times (-\delta s)\eta(y)\, \mathrm{d}y \tilde{\eta}(s)\, \mathrm{d}s\, \mathrm{d}\tau}_{\text{Fubini's theorem}} \\
&+ \underbrace{\int_0^1 \int_{\mathbb{R}^n} \nabla f(t,x-\tau\delta y) \cdot (-\delta y)\eta(y)\, \mathrm{d}y\, \mathrm{d}\tau}_{\text{Fubini's theorem}}
\end{aligned}$$

$$=:I_1+I_2. \tag{A.1}$$

For the first part I_1, noticing that supp $\eta(y)=\{y\in\mathbb{R}^n:|y|\le 1\}$ and supp $\tilde{\eta}(s)=\{s\in\mathbb{R}:|s|\le 1\}$, we have

$$I_1\le C\delta\int_0^1\int_{\{s:|s|\le 1\}}\int_{\{y:|y|\le 1\}}|\partial_t f(t-\tau\delta s,x-\delta y)|\ \mathrm{d}y\ \mathrm{d}s\ \mathrm{d}\tau.$$

For $(t,x)\in(t_1,t_2)\times K\subset\subset(0,T)\times\Omega$, since $\tau\in[0,1]$, $|s|\le 1$ and $|y|\le 1$, it follows that

$$(t-\tau\delta s,x-\delta y)\in\big([t_1,t_2]-\{s':|s'|\le\delta\}\big)\times\big(K-\{y':|y'|\le\delta\}\big),$$

where $[t_1,t_2]-\{s':|s'|\le\delta\}=\{t-s':t\in[t_1,t_2],s'\in\mathbb{R},|s'|\le\delta\}$ and $K-\{y':|y'|\le\delta\}=\{x-y':x\in K,y'\in\mathbb{R}^n,|y'|\le\delta\}$. There exists $\delta_0>0$ such that $[t_1,t_2]-\{s':|s'|\le\delta\}\subset(0,T)$ and $K-\{y':|y'|\le\delta\}\subset\Omega$ for $0<\delta<\delta_0$. Then,

$$\begin{aligned}I_1\le& C\delta\int_0^1\int_{[t_1,t_2]\cup([t_1,t_2]-\{s':|s'|\le\delta\})}\int_{K\cup(K-\{y':|y'|\le\delta\})}|\partial_t f(t,x)|\ \mathrm{d}x\ \mathrm{d}t\ \mathrm{d}\tau\\ \le& C\delta\|\partial_t f\|_{L^1(0,T;L^1(\Omega))}.\end{aligned} \tag{A.2}$$

Similarly, for the second part I_2, it holds

$$\begin{aligned}I_2\le& C\delta\int_0^1\int_{\{y:|y|\le 1\}}|\nabla f(t,x-\tau\delta y)|\ \mathrm{d}y\ \mathrm{d}\tau\\ \le& C\delta\int_0^1\int_{K\cup(K-\{y':|y'|\le\delta\})}|\nabla f(t,x)|\ \mathrm{d}x\ \mathrm{d}\tau\\ \le& C\delta\|\nabla f(t,\cdot)\|_{L^1(\Omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|f^\delta_{t,x}-f\|_{L^\infty(t_1,t_2;L^\infty(K))}\\ \le& C\delta\|\partial_t f\|_{L^1(0,T;L^1(\Omega))}+C\delta\|\nabla f\|_{L^\infty(0,T;L^1(\Omega))}\to 0,\end{aligned}$$

as $\delta\to 0$.

(2) Taking the similar arguments to (A.1) and (A.2), we have

$$\begin{aligned}&f^\delta_{t,x}(t,x)-f(t,x)\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}^n}[f(t-s,x-y)-f(t,x)]\eta^\delta(y)\ \mathrm{d}y\tilde{\eta}^\delta(s)\ \mathrm{d}s\\ =&\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}^n}\partial_t f(t-\tau\delta s,x-\delta y)\times(-\delta s)\eta(y)\ \mathrm{d}y\tilde{\eta}(s)\ \mathrm{d}s\ \mathrm{d}\tau\\ &+\int_{\mathbb{R}^n}[f(t,x-\delta y)-f(t,x)]\eta(y)\ \mathrm{d}y\\ \le& C\delta\|\partial_t f\|_{L^1(0,T;L^1(\Omega))}+C\int_{\{y:|y|\le 1\}}|f(t,x-\delta y)-f(t,x)|\ \mathrm{d}y.\end{aligned}$$

Using Minkowski's integral inequality, then we arrive at

$$\|f^\delta_{t,x}(t,x)-f(t,x)\|_{L^\infty(t_1,t_2;L^p(K))}$$

$$\leq C\delta\|\partial_t f\|_{L^1(0,T;L^1(\Omega))} + C\sup_{0\leq t\leq T}\int_{\{y:|y|\leq 1\}}\|f(t,x-\delta y)-f(t,x)\|_{L^p(K)}\,\mathrm{d}y.$$

By Lemma A.3, it gives

$$\|f^\delta_{t,x}-f\|_{L^\infty(t_1,t_2;L^p(K))}\to 0,$$

as $\delta\to 0$. □

Finally, we recall two important lemmas to deal with the product of two weak convergence sequences.

Lemma A.9 ([15], Lemma C.1). *Let X be a reflexive Banach space, Y be a Banach space, $X\hookrightarrow Y$, Y' is separable and dense in X'. Assume a sequence $\{f_n\}$ satisfies $f_n\in L^\infty(0,T;X)$ and $\partial_t f_n\in L^p(0,T;Y)$ with $1<p\leq\infty$. Then, f_n is relatively compact in $C([0,T];X-w)$.*

Lemma A.10 ([16], Lemma 5.1). *Assume $g_n\rightharpoonup g$ weakly in $L^{p_1}(0,T;L^{p_2})$, $h_n\rightharpoonup h$ weakly in $L^{q_1}(0,T;L^{q_2})$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=1$, $1\leq p_1,p_2\leq\infty$, $q_1>1$. In addition, $\partial_t g_n$ is uniformly bounded in $L^1(0,T;W^{-k,1})$ for some $k\geq 0$, and $\|h_n(t,x)-h_n(t,x+\xi)\|_{L^{q_1}(0,T;L^{q_2})}\to 0$ as $|\xi|\to 0$ for any n. Then, $g_nh_n\to gh$ in $\mathcal{D}'((0,T)\times\Omega)$.*

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