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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS WITH PARAMETERS*

YINBIN DENG† AND YOUJUN WANG‡

Abstract. This paper is devoted to investigating the existence of positive solutions for a class of parameter-dependent quasilinear elliptic equations

$$-\Delta u+V(x)u-\frac{\gamma u}{2\sqrt{1+u^2}}\Delta\sqrt{1+u^2}=\lambda|u|^{p-2}u,\quad u\in H^1(\mathbb{R}^N), \tag{0.1}$$

where γ,λ are positive parameters, $N\geq 3$. For a trapping potential $V(x)$ and $p\in(2,2^*)$, by controlling the range of γ and λ, we establish the existence of positive solutions $u_{\gamma,\lambda}$ for the above problem, where $2^*=\frac{2N}{N-2}$ is critical exponent. For super-critical case, we find a constant $p^*\in[2^*,\min\{\frac{9+2\gamma}{8+2\gamma},\frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}\}2^*)$ such that Equation (0.1) has no positive solution for all $\gamma,\lambda>0$ if $p\geq p^*$ and $\nabla V(x)\cdot x\geq 0$ in $\mathbb{R}^N$. Furthermore, for fixed $\lambda>0$, the asymptotic behavior of positive solutions $u_{\gamma,\lambda}$ is also obtained when $V(x)$ is a positive constant as $\gamma\to 0$.

Keywords. Quasilinear elliptic equations; positive solutions; asymptotic behavior.

AMS subject classifications. 35J20; 35J60.

1. Introduction

In this paper, we study the parameter-dependent quasilinear elliptic equations of the form

$$-\Delta u+V(x)u-\frac{\gamma u}{2\sqrt{1+u^2}}\Delta\sqrt{1+u^2}=f(u),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $V(x)$ is a given potential, $N\geq 3$, γ is a parameter, $f(s)$ is a real function. Equations of this type are related to the solitary wave solutions for the quasilinear Schrödinger equations

$$i\psi_t=-\Delta\psi+W(x)\psi-\rho(|\psi|^2)\psi-\gamma\Delta l(|\psi|^2)l'(|\psi|^2)\psi,\quad x\in\mathbb{R}^N, \tag{1.2}$$

where $\psi(t,x):\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $W(x)$ is a given potential, γ is a parameter, $\rho(s)$ and $l(s)$ are real functions. If $l(s)=\sqrt{1+s}$ and $\rho(s)=1-\frac{1}{\sqrt{1+s}}$, Equation (1.2) is known to describe propagation of high-power ultrashort laser pulse in a medium, see e.g. [5–9]. If $l(s)=\sqrt{1-s}$, Equation (1.2) is the fundamental equation of motion for nonlinear excitations in a classical planar Heisenberg ferromagnetic spin chain in an external field [23,28]. In the case when $l(s)=s$, Equation (1.2) appears in various problems in plasma physics and nonlinear optics, see e.g. [15,22]. We refer the readers to [12,13,16,17,19] and the references therein for more results on the physical background.

In the last decade, a considerable attention has been devoted to the study of solutions to (1.2) when $l(s)=s$, see for example [1,20,24,32] and the references therein. Here, we focus on the case $l(s)=\sqrt{1+s}$. A solution of the form $\psi(t,x)=\exp(-iEt)u(x)$

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†School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China (ybdeng@mail.ccnu.edu.cn).
‡School of Mathematics, South China University of Technology, Guangzhou 510640, China (scyjwang@scut.edu.cn).

is called a solitary wave solution, where $E\in\mathbb{R}$ and $u(x)$ is a real function. Then, we observe that ψ satisfies (1.2) if and only if the function $u(x)$ satisfies (1.1) with $V(x)=W(x)-E$, $f(s)=\rho(|s|^2)s$ and $l(s)=\sqrt{1+s}$.

Setting $\tilde{g}_\gamma(u)=\sqrt{1+\frac{\gamma u^2}{2(1+u^2)}}$, then (1.1) can be reduced to quasilinear elliptic equations

$$-\mathrm{div}(\tilde{g}_\gamma^2(u)\nabla u)+\tilde{g}_\gamma(u)\tilde{g}_\gamma'(u)|\nabla u|^2+V(x)u=f(u),\quad x\in\mathbb{R}^N. \tag{1.3}$$

In the sequel, we always assume that $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ is a trapping potential, that is,

(V) $\quad 0<V_0\le V(x)\le \lim_{|x|\to+\infty}V(x)=V_\infty<+\infty$.

In [24], Shen and Wang proved the existence of nontrivial solutions for problem (1.1) when $\gamma=1$ and the nonlinear term $f(s)$ satisfies the generalized global Ambrosetti–Rabinowitz superlinear condition

$$\exists \mu>2, \textit{such that } 0<\mu\tilde{g}(s)F(s)\le\tilde{G}(s)f(s),\quad \forall s>0, \tag{1.4}$$

where $\tilde{g}(s):=\tilde{g}_1(s)$, $\tilde{G}(s)=\int_0^s\tilde{g}(t)dt$ and $F(s)=\int_0^s f(t)dt$. In view of the definition of $\tilde{g}(s)$, we get that $\frac{s\tilde{g}(s)}{\tilde{G}(s)}\le 6-2\sqrt{6}$ for all $s>0$. So, (1.4) is a consequence of the condition

$$\exists \mu>2, \textit{such that } 0<\mu(6-2\sqrt{6})F(s)\le sf(s),\quad \forall s>0. \tag{1.5}$$

From (1.5), we deduce that there exist constants $C,C_1>0$ such that $F(s)\ge C|s|^{\mu(6-2\sqrt{6})}$ for $s>C_1>0$. Stated in the particular case of (1.5), for $f(s)=|s|^{p-2}s$ with $p\in(12-4\sqrt{6},2^*)$, the existence of a nontrivial solution for (1.1) was proved in [32]. Unfortunately, (1.5) is invalid for $f(s)=|s|^{p-2}s$ if $p\le 12-4\sqrt{6}$ and thus the method used in [32] can not be applied to study this case. Recently, in [10], Deng and Huang proved the existence of positive ground state solutions for (1.1) with $\gamma=1$ and $f(s)=|s|^{p-2}s+|s|^{2^*-2}s$, where $2^*=\frac{2N}{N-2}$, $p\in(2,12-4\sqrt{6}]$ for $N\ge4$ or $p\in(2,4)$ for $N=3$. In their paper, the Pohozaev type identity has been used to find a bounded (PS) sequence and thus conditions on $\nabla V(x)$ were needed. Precisely, they assumed that

(∇V) $\quad$ there exists $C_0\in(0,\frac{(N-2)^2}{2})$ such that $|\nabla V(x)\cdot x|\le\frac{C_0}{|x|^2}$, $\quad\forall x\in\mathbb{R}^N\setminus\{0\}$.

Thus, it is interesting to discuss the existence of positive solutions for (1.1) with general $\gamma>0$ and $f(s)=\lambda|s|^{p-2}s$ when $p\in(2,2^*)$ if the condition (∇V) is abandoned. The present paper is to consider the existence of positive solutions for problem (1.1) for general $\gamma>0$ without assumption (∇V). Precisely, for the following parameter-dependent equation

$$-\Delta u+V(x)u-\frac{\gamma u}{2\sqrt{1+u^2}}\Delta\sqrt{1+u^2}=\lambda|u|^{p-2}u,\quad x\in\mathbb{R}^N, \tag{1.6}$$

where γ and λ are positive parameters, the existence and non-existence of positive solutions are given by the following theorem.

THEOREM 1.1. *Assume that (V) and $p>2$, $N\ge3$. Then, the following statements hold:*

(1) *for all $\lambda>0$ and $p\in(2,2^*)$, Equation (1.6) has a positive classical solution if $\gamma\in(0,\gamma^*)$, where*

$$\gamma^*=\begin{cases}\frac{16(p-2)}{(p-4)^2}, & \textit{if } p<4,\\ +\infty, & \textit{if } p\ge4\end{cases};$$

(2) *for all* $\gamma>0$ *and* $p\in(2,2^*)$*, Equation (1.6) has a positive classical solution if* $\lambda\in(\lambda^*,+\infty)$*, where*

$$\lambda^*=(p-2)^{\frac{2-p}{2}}\left(\frac{2^*-p+2}{2}\right)^{\frac{2(2^*-p+2)(p-2)}{(2^*-p)^2}}2^{\frac{7\cdot 2^*-2-6p}{2(2^*-p)}}S^{-\frac{(2^*-2)(p-2)}{2(2^*-p)}}\cdot(2+\gamma)^{\frac{p(2^*-2)}{2(2^*-p)}}\gamma^{\frac{p-2}{2}}$$

and S *is the best Sobolev constant of inequality* $S\|u\|_{2^*}^2\le\|\nabla u\|_2^2$, $u\in D^{1,2}(\mathbb{R}^N)$.

(3) *for all* $\gamma,\lambda>0$*, there exists a constant* $p^*\in[2^*,\min\{\frac{9+2\gamma}{8+2\gamma},\frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}\}2^*)$ *such that Equation (1.6) has no positive solution if* $p\in[p^*,+\infty)$ *and* $\nabla V(x)\cdot x\ge 0$ *in* $\mathbb{R}^N$.

From the part (1) of Theorem 1.1, for all $\lambda>0$ and $p\in(2,2^*)$, Equation (1.6) has a positive classical solution if $\gamma\in(0,\gamma^*)$. For the case when $V(x)$ is a positive constant and λ is fixed, we have the following delicate result:

THEOREM 1.2. *Suppose* $V(x)=\mu=constant>0$, $p\in(2,2^*)$*, then the corresponding solution* $u_{\gamma,\lambda}$ *of Equation (1.6) obtained in Theorem 1.1 is spherically symmetric and monotone decreasing with respect to* $r=|x|$*. Passing to a subsequence if necessary, we have*

$$u_{\gamma,\lambda}\to u_\lambda \quad in \quad H^2(\mathbb{R}^N)\cap C^2(\mathbb{R}^N) \quad as \quad \gamma\to 0^+,$$

where u_λ *is the ground state of semilinear problem*

$$-\Delta u+\mu u=\lambda|u|^{p-2}u,\quad u\in H^1(\mathbb{R}^N). \tag{1.7}$$

We observe that the natural energy functional corresponding to the Euler-Lagrange Equation (1.6) is:

$$\widetilde{I}_{\gamma,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^N}\tilde{g}_\gamma^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx. \tag{1.8}$$

Notice that although $\widetilde{I}_{\gamma,\lambda}$ is well defined in $H^1(\mathbb{R}^N)$, it is not smooth. It is difficult to find the critical point of $\widetilde{I}_{\gamma,\lambda}(u)$ in $H^1(\mathbb{R}^N)$ by standard variational method. In [24], the authors overcome this difficulty by introducing a change of variables $s=\widetilde{G}_\gamma^{-1}(t)$ for $t\in[0,+\infty)$, where

$$\widetilde{G}_\gamma(s)=\int_0^s\tilde{g}_\gamma(t)dt. \tag{1.9}$$

Then $\widetilde{I}_{\gamma,\lambda}$ was converted to the following C^1 functional:

$$\widetilde{J}_{\gamma,\lambda}(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|\widetilde{G}_\gamma^{-1}(v)|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|\widetilde{G}_\gamma^{-1}(v)|^pdx. \tag{1.10}$$

For $\gamma=1$ and $p\in(12-4\sqrt{6},2^*)$, the existence of positive critical point of $\widetilde{J}_{\gamma,\lambda}$ can be proved via mountain pass theorem, which will lead to the existence of positive critical point of $\widetilde{I}_{\gamma,\lambda}$. It should be pointed out that the condition $p>12-4\sqrt{6}$ plays an important role to prove the boundedness of $(\mathrm{PS})_c$ sequence, see also [32].

The underling idea for proving Thereom 1.1-(1) can be processed by a standard way, see [10,25]. The proof of Thereom 1.1–(2) is inspired by the recent work [1,29,30], where some other type of quasilinear elliptic equations were studied. In order to adopt the variational method, we will first modify our problem. Namely, we establish an auxiliary function $g_\gamma(t)$ such that $g_\gamma(t)=\tilde{g}_\gamma(t)$ for $t\in(0,t_1)$, where $t_1>0$ is a proper cut-off point. Then, we consider the modified quasilinear elliptic equation

$$-div(g_\gamma^2(u)\nabla u)+g_\gamma(u)g_\gamma'(u)|\nabla u|^2+V(x)u=\lambda|u|^{p-2}u,\quad x\in\mathbb{R}^N. \tag{1.11}$$

Direct calculations show that if $g_\gamma(t)=\tilde{g}_\gamma(t)$, then Equation (1.11) becomes Equation (1.6). Solutions of (1.11) correspond to critical points of the functional

$$I_{\gamma,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^N}g_\gamma^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx, \tag{1.12}$$

where $I_{\gamma,\lambda}(u)$ is well defined in $H^1(\mathbb{R}^N)$. However, it is nonsmooth. As in [24], we introduce the change of variables $u=G_\gamma^{-1}(v)$ to reformulate functional $I_{\gamma,\lambda}(u)$ by a smooth functional $J_{\gamma,\lambda}(v)$:

$$J_{\gamma,\lambda}(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G_\gamma^{-1}(v)|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^pdx, \tag{1.13}$$

where $G_\gamma(t)=\int_0^t g_\gamma(\tau)d\tau$. Then, we prove that $J_{\gamma,\lambda}(v)$ has a positive critical point and so (1.11) has a positive solution $u_{\gamma,\lambda}=G_\gamma^{-1}(v_{\gamma,\lambda})$. Finally, using elliptic regularity estimate, by choosing proper λ, we show that $|u_{\gamma,\lambda}(x)|\le t_1$ for all $x\in\mathbb{R}^N$. Thus it is indeed a positive solution of (1.6).

The outline of the article is as follows: In Section 2, by establishing an auxiliary function, we modify (1.6). In Section 3, we prove the existence and nonexistence of a positive solution for problem (1.6) by employing the variational technique and a general Pohozaev identity. Finally, we study the asymptotic behavior of solution of (1.6) as $\gamma\to 0^+$ in Section 4.

In this paper, we always make use of the following notations: C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem; The symbol $||u||_p$ is used for the norm of the space $L^p(\mathbb{R}^N)$, $1\le p\le\infty$; By (V), we denote by $H^1(\mathbb{R}^N):=\{u\in L^2(\mathbb{R}^N):|\nabla u|\in L^2(\mathbb{R}^N)\}$ endowed with the norm $||u||:=\sqrt{\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx}$.

2. The modification of Equation (1.6)

To prove our main result, we first introduce an auxiliary function $g_\gamma(t)$ as follows:

$$g_\gamma(t)=\sqrt{\frac{1}{2}\left(1+\frac{\gamma t^2}{1+t^2}\right)\eta(t)+\frac{1}{2}},$$

where $\eta(t)$ is a spatial function satisfying either the following (η_1) or (η_2):

(η_1) $\eta(t)\equiv 1$, for all $t\in\mathbb{R}$;

(η_2) $\eta(t)\in C_0^\infty(\mathbb{R},[0,1])$ is a cut-off function satisfying

$$\eta(t)\begin{cases}=\eta(-t), & \text{if } t\le 0,\\ =1, & \text{if } 0\le t\le\delta_\gamma:=\frac{1}{4}\sqrt{\frac{p-2}{\gamma}},\\ \in(0,1), & \text{if } \frac{1}{4}\sqrt{\frac{p-2}{\gamma}}<t<\frac{1}{2}\sqrt{\frac{p-2}{\gamma}},\\ =0, & \text{if } t\ge\frac{1}{2}\sqrt{\frac{p-2}{\gamma}},\end{cases} \tag{2.1}$$

where $p\in(2,2^*)$. Moreover, it also satisfies

$$-\sigma\sqrt{\eta(t)}\le\eta'(t)t\le 0,\quad \text{for all } t\in\mathbb{R}, \tag{2.2}$$

where σ is a positive constant independent of γ.

For the proper establishment of this kind of spatial function $\eta(t)$, people can refer [30].

Set

$$G_\gamma(t)=\int_0^t g_\gamma(s)ds.$$

Clearly, $G_\gamma(t)$ is an odd C^∞ function and increases in $\mathbb{R}$. Thus, the inverse function $G_\gamma^{-1}(t)$ exists and it is also an odd C^∞ function.

Now we first collect some properties of g_γ and $G_\gamma^{-1}(t)$, which will play important roles in the proof of our main results. By direct calculations, we get the following lemma:

LEMMA 2.1. *The following properties hold:*

(1) $\lim\limits_{t\to 0}\dfrac{G_\gamma^{-1}(t)}{t}=1$;

(2) $\lim\limits_{t\to\infty}\dfrac{G_\gamma^{-1}(t)}{t}=\begin{cases}\sqrt{\frac{2}{2+\gamma}}, & \text{if } (\eta_1) \text{ holds},\\ \sqrt{2}, & \text{if } (\eta_2) \text{ holds},\end{cases}$;

(3) $|G_\gamma^{-1}(t)|\in\begin{cases}\left[\sqrt{\frac{2}{2+\gamma}}|t|,\ |t|\right], & \text{if } (\eta_1) \text{ holds},\\ \left[\sqrt{\frac{2}{2+\gamma}}|t|,\ \sqrt{2}|t|\right], & \text{if } (\eta_2) \text{ holds},\end{cases}$ *for all* $t\in\mathbb{R}$;

(4) $\frac{g_\gamma'(t)t}{g_\gamma(t)}\in\begin{cases}\left[0,\ 1+\frac{4-2\sqrt{4+2\gamma}}{\gamma}\right], & \text{if } (\eta_1) \text{ holds},\\ \left[-\widetilde{C},\ \frac{p-2}{4}\right], & \text{if } (\eta_2) \text{ holds},\end{cases}$ *for some constant* $\widetilde{C}>0$ *and all* $t\in\mathbb{R}$.

Proof. We consider the case (η_2) and the case (η_1) can be treated in exactly the same manner. Since $g_\gamma(t)$ is even and $G_\gamma^{-1}(t)$ is odd, we only consider the case $t\ge 0$. It follows from Hospital's principle that

$$\lim_{t\to 0}\frac{G_\gamma^{-1}(t)}{t}=\lim_{t\to 0}\frac{1}{g_\gamma(G_\gamma^{-1}(t))}=1$$

and

$$\lim_{t\to\infty}\frac{G_\gamma^{-1}(t)}{t}=\lim_{t\to\infty}\frac{1}{g_\gamma(G_\gamma^{-1}(t))}=\sqrt{2}.$$

Thus, the items (1) and (2) are proved.

From the definition of $g_\gamma(t)$, we get $\sqrt{\frac{1}{2}}\le g_\gamma(t)<\sqrt{\frac{2+\gamma}{2}}$ for $t\in\mathbb{R}$. Thus for all $t\ge 0$, we deduce that

$$\sqrt{\frac{1}{2}}t\le G_\gamma(t)=\int_0^t g_\gamma(s)ds\le\sqrt{\frac{2+\gamma}{2}}t,$$

which yields that $\sqrt{\frac{2}{2+\gamma}}t\le G_\gamma^{-1}(t)\le\sqrt{2}t$ for all $t\ge 0$.

Lastly, we prove (4). By (2.2), we get

$$\frac{g'_\gamma(t)t}{g_\gamma(t)}=\frac{2\gamma t^2\eta(t)+(1+t^2)[1+(1+\gamma)t^2]\eta'(t)t}{2(1+t^2)[1+(1+\gamma)t^2]\eta(t)+2(1+t^2)^2}$$
$$\begin{cases}\geq -\frac{\sigma[1+(1+\gamma)t^2]\sqrt{\eta(t)}}{2(1+t^2)}\geq -\frac{1+\gamma t^2}{2}\sigma=\frac{p+2}{8}\sigma=:-\widetilde{C}, & \text{if } 0\leq t<\frac{1}{2}\sqrt{\frac{p-2}{\gamma}},\\ =0, & \text{if } t\geq \frac{1}{2}\sqrt{\frac{p-2}{\gamma}}.\end{cases} \tag{2.3}$$

To prove the second inequality, by (2.3), it suffices to consider the case $0\leq t<\frac{1}{2}\sqrt{\frac{p-2}{\gamma}}$. In fact, we get

$$\begin{aligned}\frac{g'_\gamma(t)t}{g_\gamma(t)}&\leq \frac{2\gamma t^2\eta(t)}{2(1+t^2)[1+(1+\gamma)t^2]\eta(t)+2(1+t^2)^2}\\ &\leq \gamma t^2\eta(t)\\ &\leq \frac{p-2}{4},\quad 0\leq t<\frac{1}{2}\sqrt{\frac{p-2}{\gamma}},\end{aligned}$$

which yields the result. □

REMARK 2.1. We remark that the cut-off point δ_γ in assumption (η_2) is not unique. In fact, as long as the inequality $\frac{g'_\gamma(t)t}{g_\gamma(t)}<\frac{p-2}{2}$ is guaranteed, any $t\in(0,\sqrt{\frac{p-2}{2\gamma}})$ is allowed.

We now consider the modified quasilinear Schrödinger equation of the form:

$$-div(g_\gamma^2(u)\nabla u)+g_\gamma(u)g'_\gamma(u)|\nabla u|^2+V(x)u=\lambda|u|^{p-2}u,\quad x\in\mathbb{R}^N. \tag{2.4}$$

It follows from assumption (η_2) that u must be a positive solution of (1.6), if we can prove the existence of a positive solution u of (2.4) satisfying $0\leq u(x)<\frac{1}{4}\sqrt{\frac{p-2}{\gamma}}$ for all $x\in\mathbb{R}^N$.

The associate variational functional for problem (2.4) is

$$I_{\gamma,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^N}g_\gamma^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx. \tag{2.5}$$

Since $g_\gamma(t)$ is bounded, we can deduce that $I_{\gamma,\lambda}(u)$ is well defined in $H^1(\mathbb{R}^N)$. By introducing the change of variables $u=G_\gamma^{-1}(v)$, we observe that functional $I_{\gamma,\lambda}$ can be written in the following form

$$J_{\gamma,\lambda}(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G_\gamma^{-1}(v)|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^pdx. \tag{2.6}$$

From Lemma 2.1, $J_{\gamma,\lambda}$ is well defined in $H^1(\mathbb{R}^N)$, $J_{\gamma,\lambda}\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and

$$\langle J'_{\gamma,\lambda}(v),\psi\rangle=\int_{\mathbb{R}^N}\left[\nabla v\nabla\psi+V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\psi-\lambda\frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\psi\right]dx, \tag{2.7}$$

for all $v,\psi\in H^1(\mathbb{R}^N)$.

Note that any critical points of $J_{\gamma,\lambda}$ correspond to the solutions of the equation

$$-\Delta v+V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}=\lambda\frac{|G_\gamma^{-1}(v)|^{p-2}G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))},\quad x\in\mathbb{R}^N. \tag{2.8}$$

In order to find positive solutions of (2.4), it suffices to study the existence of positive solutions of Equation (2.8).

REMARK 2.2. It is easy to verify that $u=G_\gamma^{-1}(v)\in C^2(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$ must be a classical solution for (2.4) if $v\in C^2(\mathbb{R}^N)\cap H^1(\mathbb{R}^N)$ is a critical point of $J_{\gamma,\lambda}$.

REMARK 2.3. Because we look for positive solutions, we can rewrite the functional $J_{\gamma,\lambda}$ in the following

$$J_{\gamma,\lambda}(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G_\gamma^{-1}(v)|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v^+)|^pdx,$$

where $v^+=\max\{v,0\}$. Standard regularity arguments show that any critical points v belong to C^2 and $v(x)>0$ from the strong maximum principle if v is nontrival.

3. Proof of Theorem 1.1

Thanks to Lemma 2.1$-$(3), it is easy to prove that the functional $J_{\gamma,\lambda}$ exhibits the mountain pass geometry.

LEMMA 3.1. *(i) $J_{\gamma,\lambda}(v)\geq C||v||^2+o(||v||^2)$ as $v\to 0$ in $H^1(\mathbb{R}^N)$;*
(ii) there exists a $e\in H^1(\mathbb{R}^N)$, $e\neq 0$ satisfying $J_{\gamma,\lambda}(e)\leq 0$.

In view of Lemma 3.1, applying the mountain pass theorem [31], it follows that there exists a $(PS)_{c_{\gamma,\lambda}}$ sequence $\{v_n\}\subset H^1(\mathbb{R}^N)$, i.e., a sequence such that $J_{\gamma,\lambda}(v_n)\to c_{\gamma,\lambda}$ and $J'_{\gamma,\lambda}(v_n)\to 0$, where $c_{\gamma,\lambda}$ is the mountain pass level of $J_{\gamma,\lambda}$ characterized by

$$c_{\gamma,\lambda}=\inf_{\xi\in\Gamma_{\gamma,\lambda}}\sup_{t\in[0,1]}J_{\gamma,\lambda}(\xi(t)) \tag{3.1}$$

and $\Gamma_{\gamma,\lambda}=\{\xi(t)\in C([0,1],H^1(\mathbb{R}^N)):\xi(0)=0,\xi(1)\neq 0,J_{\gamma,\lambda}(\xi(1))<0\}$. Moreover, from Lemma 3.1, we get $c_{\gamma,\lambda}>0$.

We next claim that the $(PS)_{c_{\gamma,\lambda}}$ sequence for $J_{\gamma,\lambda}$ is bounded. To this end, we assert that the item (4) in Lemma 2.1 plays an important role. Indeed, let $\{v_n\}$ be a $(PS)_{c_{\gamma,\lambda}}$ sequence for $J_{\gamma,\lambda}$, namely,

$$J_{\gamma,\lambda}(v_n)=c_{\gamma,\lambda}+o_n(1),\quad \langle J'_{\gamma,\lambda}(v_n),\psi\rangle=o_n(1)||\psi||,\quad \forall\psi\in H^1(\mathbb{R}^N), \tag{3.2}$$

where $o_n(1)\to 0$ as $n\to\infty$. Let $\psi_n=G_\gamma^{-1}(v_n)g_\gamma(G_\gamma^{-1}(v_n))$. From Lemma 2.1$-$(3) and (4),

$$|\nabla\psi_n|=\left|\left(1+\frac{G_\gamma^{-1}(v_n)g'_\gamma(G_\gamma^{-1}(v_n))}{g_\gamma(G_\gamma^{-1}(v_n))}\right)\nabla v_n\right|\leq C|\nabla v_n|,\quad |\psi_n|\leq C|v_n|.$$

Thus $\psi_n\in H^1(\mathbb{R}^N)$. By choosing $\psi=\psi_n$ as a test function and from Lemma 2.1–(3), (4), we get

$$\begin{aligned}pc_{\gamma,\lambda}+o_n(1)+o_n(1)||v_n||=&pJ_{\gamma,\lambda}(v_n)-\langle J_{\gamma,\lambda}(v_n),\psi_n\rangle\\ &=\int_{\mathbb{R}^N}\left(\frac{p-2}{2}-\frac{G_\gamma^{-1}(v_n)g'_\gamma(G_\gamma^{-1}(v_n))}{g_\gamma(G_\gamma^{-1}(v_n))}\right)|\nabla v_n|^2dx\\ &+\frac{p-2}{2}\int_{\mathbb{R}^N}V(x)|G_\gamma^{-1}(v_n)|^2dx.\end{aligned}$$

By Lemma 2.1–(4), if (η_1) occurs, we get $\frac{p-2}{2}-\frac{G_\gamma^{-1}(t)g_\gamma'(G_\gamma^{-1}(t))}{g_\gamma(G_\gamma^{-1}(t))}>\frac{p-2}{2}-\frac{4+\gamma-2\sqrt{4+2\gamma}}{\gamma}>0$ if $p\in(2,2^*)$ and $\gamma\in(0,\gamma^*)$. On the other hand, if (η_2) occurs, we get $\frac{p-2}{2}-\frac{G_\gamma^{-1}(t)g_\gamma'(G_\gamma^{-1}(t))}{g_\gamma(G_\gamma^{-1}(t))}>\frac{p-2}{4}$. This together with Lemma 2.1–(3) imply that $\|v_n\|$ is bounded.

Thus, up to subsequence, we may assume that there is $v_{\gamma,\lambda}\in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}&v_n\rightharpoonup v_{\gamma,\lambda} \quad in \quad H^1(\mathbb{R}^N),\\ &v_n\to v_{\gamma,\lambda} \quad in \quad L^q_{loc}(\mathbb{R}^N),\quad q\in[1,2^*),\\ &v_n\to v_{\gamma,\lambda} \quad a.e. \quad in \quad \mathcal{O}:=supp\psi\end{aligned}$$

and there exists $w_q(x)\in L^q(\mathcal{O})$, such that for any n, $|v_n(x)|\le|w_q(x)|$ a.e. in $\mathcal{O}$. Now we are going to prove that $v_{\gamma,\lambda}$ is a positive solution of (2.8).

LEMMA 3.2. *Suppose $g_\gamma(t)$ satisfy either (η_1) or (η_2), then $v_{\gamma,\lambda}$ obtained above is a positive solution for modified problem (2.8).*

Proof. We first show that $\langle J'_{\gamma,\lambda}(v_{\gamma,\lambda}),\psi\rangle=0$ for any $\psi\in C_0^\infty(\mathbb{R}^N)$, i.e., $v_{\gamma,\lambda}$ is a critical point of $J_{\gamma,\lambda}$. Note that as $n\to\infty$, we get

$$\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\to\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))},\quad \text{a.e. in } \mathcal{O}, \tag{3.3}$$

$$\frac{|G_\gamma^{-1}(v_n)|^{p-2}G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\to\frac{|G_\gamma^{-1}(v_{\gamma,\lambda})|^{p-2}G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))},\quad \text{a.e. in } \mathcal{O}. \tag{3.4}$$

Furthermore, by Lemma 2.1−(3), we have

$$\left|\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\psi\right|\le C_1|v_n||\psi|\le C_1|w_2||\psi|,\quad \text{a.e. in } \mathcal{O}, \tag{3.5}$$

$$\left|\frac{|G_\gamma^{-1}(v_n)|^{p-2}G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\psi\right|\le C_2|v_n|^{p-1}|\psi|\le C_2|w_p|^{p-1}|\psi|,\quad \text{a.e. in } \mathcal{O}. \tag{3.6}$$

Now, combining (3.3)–(3.6), the Lebesgue dominated convergence theorem and the weak convergence $v_n\rightharpoonup v_{\gamma,\lambda}$ in $H^1(\mathbb{R}^N)$, we have $\langle J'_{\gamma,\lambda}(v_n),\psi\rangle\to\langle J'_{\gamma,\lambda}(v_{\gamma,\lambda}),\psi\rangle$ as $n\to\infty$. Because $J'_{\gamma,\lambda}(v_n)\to0$ as $n\to\infty$, we conclude that $J'_{\gamma,\lambda}(v_{\gamma,\lambda})=0$. By Remark 2.3, we may assume $v_{\gamma,\lambda}\ge0$. If $v_{\gamma,\lambda}\not\equiv0$, by the strong maximum principle, we get $v_{\gamma,\lambda}>0$. Otherwise, assuming $v_{\gamma,\lambda}\equiv0$, then, as in [24], $\{v_n\}$ is also a $(\mathrm{PS})_{c_{\gamma,\lambda}}$ for the function $J^\infty_{\gamma,\lambda}:H^1(\mathbb{R}^N)\to\mathbb{R}$:

$$J^\infty_{\gamma,\lambda}(v)=\frac12\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{V_\infty}{2}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^pdx. \tag{3.7}$$

Next, we claim that there exist α, $R>0$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_R(y_n)}v_n^2dx\ge\alpha>0. \tag{3.8}$$

Suppose by contradiction that for all $R>0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}v_n^2dx=0. \tag{3.9}$$

Then, by Lions compactness lemma [18], we deduce that $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2,2^*)$. So by Lemma 2.1–(1) and (2), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |G_\gamma^{-1}(v_n)|^p dx = 0 \tag{3.10}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{|G_\gamma^{-1}(v_n)|^{p-2}G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))} v_n dx = 0. \tag{3.11}$$

Thanks to Lemma 2.1−(1) , for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|v_n(x)| < \delta$, we have

$$\int_{\{x\in\mathbb{R}^N:|v_n(x)|\le\delta\}} V(x)\left|\frac{v_n}{g_\gamma(G_\gamma^{-1}(v_n))G_\gamma^{-1}(v_n)} - 1\right| \left|G_\gamma^{-1}(v_n)\right|^2 dx \le V_\infty \varepsilon \int_{\mathbb{R}^N} v_n^2 dx \le C\varepsilon. \tag{3.12}$$

On the other hand, by Lemma 2.1−(2) and (3), we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|v_n(x)|\ge\delta\}} V(x)\left|\frac{v_n}{g_\gamma(G_\gamma^{-1}(v_n))G_\gamma^{-1}(v_n)} - 1\right| \left|G_\gamma^{-1}(v_n)\right|^2 dx\\ &\le C V_\infty \delta^{2-p} \lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p dx = 0.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), since ε is arbitrary, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\left|G_\gamma^{-1}(v_n)\right|^2 dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))} v_n dx. \tag{3.14}$$

Thus, by (3.11) and (3.14), we deduce that

$$\begin{aligned}0 &= \lim_{n\to\infty}\langle J'_{\gamma,\lambda}(v_n), v_n\rangle\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n - \lambda\frac{|G_\gamma^{-1}(v_n)|^{p-2}G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n\right)dx\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)|G_\gamma^{-1}(v_n)|^2\right)dx.\end{aligned} \tag{3.15}$$

Then combining (3.10) and (3.15), we get $J_{\gamma,\lambda}(v_n) \to 0$ as $n \to \infty$, which is a contradiction since $J_{\gamma,\lambda}(v_n) \to c_{\gamma,\lambda} > 0$ as $n \to \infty$. The claim is proved, i.e., (3.8) holds.

Define $\tilde{v}_n(x) = v_n(x + y_n)$. Since $\{v_n\}$ is a $(\mathrm{PS})_{c_{\gamma,\lambda}}$ sequence for $J^\infty_{\gamma,\lambda}$, $\{\tilde{v}_n\}$ is also a $(\mathrm{PS})_{c_{\gamma,\lambda}}$ sequence for $J^\infty_{\gamma,\lambda}$. Arguing as in the case of $\{v_n\}$, we get $\{\tilde{v}_n\}$ is bounded. So, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}_\gamma$ in $H^1(\mathbb{R}^N)$ with $(J^\infty_{\gamma,\lambda})'(\tilde{v}_\gamma) = 0$. By (3.8), we have $\tilde{v}_\gamma \ne 0$.

Let

$$E(v) = \int_{\mathbb{R}^N}\left(\frac{p-2}{2} - \frac{g'_\gamma(G_\gamma^{-1}(v))G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\right)|\nabla v|^2 dx.$$

By Theorem 1.6 in [27], $E(v)$ is weakly lower semi-continuous. Then according to

Fatou's lemma, we have

$$\begin{aligned}
pc_{\gamma,\lambda} &= \lim_{n\to\infty}(pJ^{\infty}_{\gamma,\lambda}(\tilde{v}_n)-\langle (J^{\infty}_{\gamma,\lambda})'(\tilde{v}_n), G^{-1}_{\gamma}(\tilde{v}_n)g_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_n))\rangle)\\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{p-2}{2}-\frac{g'_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_n))G^{-1}_{\gamma}(\tilde{v}_n)}{g_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_n))}\right)|\nabla\tilde{v}_n|^2dx\\
&\quad +\frac{p-2}{2}V_{\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}|G^{-1}_{\gamma}(\tilde{v}_n)|^2dx\\
&\geq \int_{\mathbb{R}^N}\left(\frac{p-2}{2}-\frac{g'_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_\gamma))G^{-1}_{\gamma}(\tilde{v}_\gamma)}{g_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_\gamma))}\right)|\nabla\tilde{v}_\gamma|^2dx+\frac{p-2}{2}V_{\infty}\int_{\mathbb{R}^N}|G^{-1}_{\gamma}(\tilde{v}_\gamma)|^2dx\\
&= pJ^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma)-\langle (J^{\infty}_{\gamma,\lambda})'(\tilde{v}_\gamma), G^{-1}_{\gamma}(\tilde{v}_\gamma)g_{\gamma}(G^{-1}_{\gamma}(\tilde{v}_\gamma))\rangle\\
&= pJ^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma),
\end{aligned} \tag{3.16}$$

which yields that $J^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma)\leq c_{\gamma,\lambda}$.

Analogous to the arguments used in [14], we can get a path $\chi(t):[0,L]\to H^1(\mathbb{R}^N)$ such that

$$\begin{cases}\chi(0)=0, J^{\infty}_{\gamma,\lambda}(\chi(L))<0, \tilde{v}_\gamma\in\chi([0,L]),\\ \chi(t)(x)>0, \forall x\in\mathbb{R}^N, t\in[0,L],\\ \max\limits_{t\in[0,L]}J^{\infty}_{\gamma,\lambda}(\chi(t))=J^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma).\end{cases} \tag{3.17}$$

Define the set

$$\Gamma^{\infty}_{\gamma,\lambda}=\{\chi\in C([0,1],H^1(\mathbb{R}^N)):\chi(0)=0,\chi(1)\neq 0, J^{\infty}_{\gamma,\lambda}(\chi(1))<0\}.$$

After a suitable scale change in t, we can assume $\chi(t)\in\Gamma^{\infty}_{\gamma,\lambda}$. Particularly,

$$\max_{t\in[0,1]}J^{\infty}_{\gamma,\lambda}(\chi(t))=J^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma)\leq c_{\gamma,\lambda}.$$

With restriction we can assume that $V(x)\leq V_{\infty}$ but $V(x)\not\equiv V_{\infty}$ (otherwise there is nothing to prove). Thus, $\chi(t)\in\Gamma^{\infty}_{\gamma,\lambda}\subset\Gamma_{\gamma}$, and hence

$$c_{\gamma,\lambda}\leq \max_{t\in[0,1]}J_{\gamma,\lambda}(\chi(t)):=J_{\gamma,\lambda}(\chi(\bar{t}))<J^{\infty}_{\gamma}(\chi(\bar{t}))\leq\max_{t\in[0,1]}J^{\infty}_{\gamma,\lambda}(\chi(t))=J^{\infty}_{\gamma,\lambda}(\tilde{v}_\gamma)\leq c_{\gamma,\lambda}$$

which is a contradiction. It follows from Remark 2.2 that $v_{\gamma,\lambda}>0$ is a critical point of $J_{\gamma,\lambda}$ and hence $v_{\gamma,\lambda}$ is a positive solution of (2.8). ◻

For all $\gamma>0$, if $p\in(2,2^*)$ and $\gamma\in(0,\gamma^*)$, we take $\eta(t)$ satisfying (η_1). In this case, $\tilde{g}_\gamma(t)=g_\gamma(t)$ in (2.4) and hence (2.4) turns into (1.6). According to the above arguments, we get $u_{\gamma,\lambda}=G^{-1}_{\gamma}(v_{\gamma,\lambda})>0$ is a solution of (1.6).

However, if (η_2) occurs, (2.4) can not be transformed into (1.6) unless $v_{\gamma,\lambda}$ obtained above satisfies $0\leq u_{\gamma,\lambda}(x)=G^{-1}_{\gamma}(v_{\gamma,\lambda}(x))<\frac{1}{4}\sqrt{\frac{p-2}{\gamma}}$ for all $x\in\mathbb{R}^N$. To this end, we next establish the L^{∞} estimate for $v_{\gamma,\lambda}$. First we give the boundedness of its gradient.

LEMMA 3.3. *The solution $v_{\gamma,\lambda}$ of (2.8) satisfies* $||\nabla v_{\gamma,\lambda}||_2\leq\sqrt{2}(\frac{1}{2+\gamma})^{\frac{p}{2(2-p)}}\lambda^{\frac{1}{2-p}}$.

Proof. Since $v_{\gamma,\lambda}$ is a critical point of $J_{\gamma,\lambda}$, then

$$\begin{aligned}
pc_{\gamma,\lambda}&=pJ_{\gamma,\lambda}(v_{\gamma,\lambda})-\langle J'_{\gamma,\lambda}(v_{\gamma,\lambda}),G^{-1}_{\gamma}(v_{\gamma,\lambda})g_{\gamma}(G^{-1}_{\gamma}(v_{\gamma,\lambda}))\rangle\\
&\geq\frac{p-2}{4}\int_{\mathbb{R}^N}|\nabla v_{\gamma,\lambda}|^2dx+\frac{p-2}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}_{\gamma}(v_{\gamma,\lambda})|^2dx.
\end{aligned}$$

It follows that,

$$||\nabla v_{\gamma,\lambda}||_2^2 \le \frac{4p}{p-2} c_{\gamma,\lambda}. \tag{3.18}$$

On the other hand, by Lemma 2.1−(3), we conclude that

$$J_{\gamma,\lambda}(v) \le P_{\gamma,\lambda}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + 2V_\infty \int_{\mathbb{R}^N} |v|^2 dx - \frac{\lambda}{p}\left(\frac{2}{2+\gamma}\right)^{\frac{p}{2}} \int_{\mathbb{R}^N} |v|^p dx.$$

Denote

$$\Sigma_{\gamma,\lambda} = \{\xi \in C([0,1], H^1(\mathbb{R}^N)) : \xi(0) = 0, \xi(1) \ne 0, P_{\gamma,\lambda}(\xi(1)) < 0\}$$

and note that $\Sigma_{\gamma,\lambda} \subset \Gamma_{\gamma,\lambda}$, we have

$$c_{\gamma,\lambda} = \inf_{\xi \in \Gamma_{\gamma,\lambda}} \sup_{t\in[0,1]} J_{\gamma,\lambda}(\xi(t)) \le \inf_{\xi \in \Sigma_{\gamma,\lambda}} \sup_{t\in[0,1]} J_{\gamma,\lambda}(\xi(t)) \le \inf_{\xi \in \Sigma_{\gamma,\lambda}} \sup_{t\in[0,1]} P_{\gamma,\lambda}(\xi(t)). \tag{3.19}$$

Let us set

$$S_p = \inf\left\{ \int_{\mathbb{R}^N} (|\nabla v|^2 + 4V_\infty |v|^2) dx : v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |v|^p dx = 1 \right\}.$$

It is well known that $S_p > 0$ and it is achieved at some v^*, see e.g. [2].

Now, we take

$$\phi(x) = \begin{cases} v^*(x), & \text{if } S_p \le 1, \\ v^*(S_p^{\frac{p}{(N-2)p-2N}} x), & \text{if } S_p > 1. \end{cases}$$

Then, we have

$$\begin{aligned} \max_{t\in\mathbb{R}} P_{\gamma,\lambda}(t\phi) &= P_{\gamma,\lambda}(t_{max}\phi) \\ &= \frac{p-2}{2p}\left(\frac{2}{2+\gamma}\right)^{\frac{p}{2-p}} \lambda^{\frac{2}{2-p}} \left(\int_{\mathbb{R}^N} (|\nabla\phi|^2 + 4V_\infty|\phi|^2)dx\right)^{\frac{p}{p-2}} \left(\int_{\mathbb{R}^N} |\phi|^p dx\right)^{\frac{2}{2-p}} \\ &\le \frac{p-2}{2p}\left(\frac{2}{2+\gamma}\right)^{\frac{p}{2-p}} \lambda^{\frac{2}{2-p}}. \end{aligned} \tag{3.20}$$

Note that we can choose large $T > t_{max}$ such that $P_{\gamma,\lambda}(T\phi) < 0$. Thus for $t \in [0,1]$, we get $\xi(t) := tT\phi \in \Sigma_{\gamma,\lambda}$ such that $P_{\gamma,\lambda}(\xi(t)) \le \frac{p-2}{2p}(\frac{1}{2+\gamma})^{\frac{p}{2-p}} \lambda^{\frac{2}{2-p}}$. It follows from (3.19) that

$$c_{\gamma,\lambda} \le \frac{p-2}{2p}\left(\frac{1}{2+\gamma}\right)^{\frac{p}{2-p}} \lambda^{\frac{2}{2-p}},$$

which yields the result. □

Remark 3.1. Note that equation

$$-\Delta v + 4V_\infty v = \lambda\left(\frac{2}{2+\gamma}\right)^{\frac{p}{2}} |v|^{p-2} v, \quad x \in \mathbb{R}^N \tag{3.21}$$

is the Euler–Lagrange equation associated to the energy functional $P(v)$. In [21], Pohozaev showed that (3.21) possesses a solution if and only if $p \in (2, 2^*)$, $N \geq 3$ (see also [3]).

REMARK 3.2. From Lemma 3.3 and Sobolev inequality, we have

$$\|v_{\gamma,\lambda}\|_{2^*} \leq S^{-\frac{1}{2}} \|\nabla v_{\gamma,\lambda}\|_2 \leq \sqrt{2}\left(\frac{1}{2+\gamma}\right)^{\frac{p}{2(2-p)}} S^{-\frac{1}{2}} \lambda^{\frac{1}{2-p}},$$

where S is the best Sobolev constant.

PROPOSITION 3.1. *The solution $v_{\gamma,\lambda}$ of (2.8) satisfies*

$$||v_{\gamma,\lambda}||_\infty \leq \left(\frac{2^*-p+2}{2}\right)^{\frac{2(2^*-p+2)}{(2^*-p)^2}} 2^{\frac{2\cdot 2^*-2-p}{2(2^*-p)}} S^{-\frac{2^*-2}{2(2^*-p)}} \left(\frac{1}{2+\gamma}\right)^{\frac{p(2^*-2)}{2(2-p)(2^*-p)}} \lambda^{\frac{1}{2-p}}.$$

Proof. The result can be proved in a similar way as Proposition 3.1 in [1], we give the outline of the proof here. In what follows, for convenience, we denote $v_{\gamma,\lambda}$ by v. For each $m \in N$ and $\beta > 1$, let $A_m = \{x \in \mathbb{R}^N : |v|^{\beta-1} \leq m\}$ and $B_m = \mathbb{R}^N \setminus A_m$. Define

$$v_m = \begin{cases} v|v|^{2(\beta-1)}, & \text{in } A_m, \\ m^2 v, & \text{in } B_m. \end{cases}$$

Note that $v_m \in H^1(\mathbb{R}^N)$. Using v_m as a test function in (2.7), we deduce that

$$\int_{\mathbb{R}^N} \left[\nabla v \nabla v_m + V(x) \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} v_m\right] dx = \lambda \int_{\mathbb{R}^N} \frac{|G_\gamma^{-1}(v)|^{p-2} G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} v_m dx. \tag{3.22}$$

Besides, we have

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m dx = (2\beta - 1) \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 dx + m^2 \int_{B_m} |\nabla v|^2 dx. \tag{3.23}$$

Let

$$w_m = \begin{cases} v|v|^{\beta-1}, & \text{in } A_m, \\ mv, & \text{in } B_m. \end{cases}$$

Then

$$\int_{\mathbb{R}^N} |\nabla w_m|^2 dx = \beta^2 \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 dx + m^2 \int_{B_m} |\nabla v|^2 dx. \tag{3.24}$$

Thus from (3.23) and (3.24), we get

$$\int_{\mathbb{R}^N} (|\nabla w_m|^2 - \nabla v \nabla v_m) dx = (\beta - 1)^2 \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 dx. \tag{3.25}$$

Combining Lemma 2.1–(3), (3.22), (3.23) and (3.25), since $\beta > 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx &\leq \left[\frac{(\beta-1)^2}{2\beta-1} + 1\right] \int_{\mathbb{R}^N} \nabla v \nabla v_m dx \\ &\leq \beta^2 \int_{\mathbb{R}^N} \left[\nabla v \nabla v_m + V(x) \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} v_m\right] dx \\ &= \beta^2 \lambda \int_{\mathbb{R}^N} \frac{|G_\gamma^{-1}(v)|^{p-2} G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} v_m dx \\ &\leq \sqrt{2} \beta^2 \lambda \int_{\mathbb{R}^N} |v|^{p-2} w_m^2 dx. \end{aligned}$$

By Hölder inequality, and since $|w_m|\leq|v|^\beta$ in $\mathbb{R}^N$ and $|w_m|=|v|^\beta$ in A_m, we get

$$\left(\int_{A_m}|v|^{\beta 2^*}dx\right)^{\frac{N-2}{N}}\leq\sqrt{2}\lambda\beta^2||v||_{2^*}^{p-2}\left(\int_{\mathbb{R}^N}|v|^{2\beta q_1}dx\right)^{\frac{1}{q_1}},$$

where $q_1=\frac{2^*}{2^*-p+2}$. By Monotone Convergence Theorem, letting $m\to\infty$, we have

$$||v||_{\beta 2^*}\leq\beta^{\frac{1}{\beta}}(\sqrt{2}\lambda||v||_{2^*}^{p-2})^{\frac{1}{2\beta}}||v||_{2\beta q_1}. \tag{3.26}$$

Setting $\sigma=\frac{2^*}{2q_1}=\frac{2^*-p+2}{2}$ and $\beta=\sigma$ in (3.26), we obtain $2q_1\beta=2^*$ and

$$||v||_{\sigma 2^*}\leq\sigma^{\frac{1}{\sigma}}(\sqrt{2}\lambda||v||_{2^*}^{p-2})^{\frac{1}{2\sigma}}||v||_{2^*}. \tag{3.27}$$

Taking $\beta=\sigma^2$ in (3.26), we have

$$||v||_{\sigma^2 2^*}\leq\sigma^{\frac{2}{\sigma^2}}(\sqrt{2}\lambda||v||_{2^*}^{p-2})^{\frac{1}{2\sigma}}||v||_{\sigma 2^*}. \tag{3.28}$$

From (3.27) and (3.28),

$$||v||_{\sigma^2 2^*}\leq\sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}}(\sqrt{2}\lambda||v||_{2^*}^{p-2})^{\frac{1}{2\sigma}+\frac{1}{2\sigma^2}}||v||_{2^*}.$$

Taking $\beta=\sigma^i$ $(i=1,2,\cdots)$ and iterating (3.26), we get

$$||v||_{\sigma^j 2^*}\leq\sigma^{\sum_{i=1}^j\frac{i}{\sigma^i}}(\sqrt{2}\lambda||v||_{2^*}^{p-2})^{\frac{1}{2}\sum_{i=1}^j\frac{1}{\sigma^i}}||v||_{2^*}.$$

Therefore, by (3.20), using Sobolev inequality, taking the limit of $j\to+\infty$, we get

$$\begin{aligned}||v||_\infty&\leq\sigma^{\frac{\sigma}{(\sigma-1)^2}}2^{\frac{1}{4(\sigma-1)}}\lambda^{\frac{1}{2(\sigma-1)}}||v||_{2^*}^{\frac{2^*-2}{2^*-p}}\\&=\left(\frac{2^*-p+2}{2}\right)^{\frac{2(2^*-p+2)}{(2^*-p)^2}}2^{\frac{2\cdot 2^*-2-p}{2(2^*-p)}}S^{-\frac{2^*-2}{2(2^*-p)}}\left(\frac{1}{2+\gamma}\right)^{\frac{p(2^*-2)}{2(2-p)(2^*-p)}}\lambda^{\frac{1}{2-p}}.\end{aligned}$$

This ends the proof. ∎

Proof of Theorem 1.1–(1). For all $\gamma>0$, if $p\in(2,2^*)$ and $\gamma\in(0,\gamma^*)$, we take $\eta(t)$ satisfying (η_1). In this case, $\tilde{g}_\gamma(t)=g_\gamma(t)$ in (2.4) and hence (2.4) turns into (1.6). It follows from Lemma 3.2 and Remark 2.2 that $u_{\gamma,\lambda}=G_\gamma^{-1}(v_{\gamma,\lambda})>0$ is a solution of (1.6).

Proof of Theorem 1.1–(2). From Proposition 3.1, for any $\gamma>0$, we set $K=\left(\frac{2^*-p+2}{2}\right)^{\frac{2(2^*-p+2)}{(2^*-p)^2}}2^{\frac{2\cdot 2^*-2-p}{2(2^*-p)}}S^{-\frac{2^*-2}{2(2^*-p)}}\left(\frac{1}{2+\gamma}\right)^{\frac{p(2^*-2)}{2(2-p)(2^*-p)}}$ and choose $\lambda^*=d\gamma^{\frac{p-2}{2}}$ with $d=(\frac{\sqrt{p-2}}{4\sqrt{2}K})^{2-p}$ such that

$$\begin{aligned}||u_{\gamma,\lambda}||_\infty&=||G_\gamma^{-1}(v_{\gamma,\lambda})||_\infty\\&\leq\sqrt{2}||v_{\gamma,\lambda}||_\infty\leq\sqrt{2}K\lambda^{\frac{1}{2-p}}\leq\frac{1}{4}\sqrt{\frac{p-2}{\gamma}},\quad\forall\lambda\in(\lambda^*,+\infty).\end{aligned}$$

In this case, we take $\eta(t)$ satisfying (η_2). It follows from above estimate that $\tilde{g}_\gamma(t)=g_\gamma(t)$ in (2.4) and hence (2.4) turns into (1.6) if $\lambda\in(\lambda^*,\ +\infty)$. Again using Lemma 3.2 and Remark 2.2 we obtain that $u_{\gamma,\lambda}=G_\gamma^{-1}(v_{\gamma,\lambda})>0$ is a solution of (1.6).

Proof of Theorem 1.1–(3). We are going to find a constant

$$p^* \in \left[2^*, \min\left\{\frac{9+2\gamma}{8+2\gamma}, \quad \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}\right\} 2^*\right)$$

such that problem (1.6) has no positive solution $u \in H^1(\mathbb{R}^N)$ for $p \geq p^*$ if $x \cdot \nabla V(x) \geq 0$ in $\mathbb{R}^N$. It suffices to prove that problem (2.8) has no positive solution.

Suppose by contradiction that $v \in H^1(\mathbb{R}^N)$ is a positive solution of (2.8), it follows from the Pohozaev identity that

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))|G_\gamma^{-1}(v)|^2dx &= \int_{\mathbb{R}^N} K(G_\gamma^{-1}(v))dx \\ &=: \int_{\{x\in\mathbb{R}^N: 0\leq u < \frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx + \int_{\{x\in\mathbb{R}^N: u\geq \frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx,\end{aligned} \tag{3.29}$$

where $u = G_\gamma^{-1}(v)$ and

$$K(u) = \frac{(N-2)\lambda}{2}\frac{G_\gamma(u)u^{p-1}}{g_\gamma(u)} - \frac{N\lambda}{p}u^p + \frac{N}{2}u^2 - \frac{N-2}{2}\frac{G_\gamma(u)u}{g_\gamma(u)}.$$

The assumption $x \cdot \nabla V(x) \geq 0$ implies that

$$-\frac{1}{2}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))|G_\gamma^{-1}(v)|^2dx < 0.$$

Therefore, to complete the proof of our Theorem 1.1–(3), it suffices to verify that the right-hand side of (3.29) is nonnegative.

Using Lemma 2.1–(4), we get $K(u) > 0$ if $p \geq \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}2^* > 2^*$. Noting that $\frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma} \to 1$ as $\gamma \to 0$. Hence, we only need to consider the case $p \in [2^*, \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}2^*)$.

Noting that

$$\begin{aligned}K(u) &\geq \frac{(N-2)\lambda}{2}\frac{G_\gamma(u)u^{p-1}}{g_\gamma(u)} - \frac{N\lambda}{2^*}u^p + \frac{N}{2}u^2 - \frac{N-2}{2}\frac{G_\gamma(u)u}{g_\gamma(u)} \\ &= \frac{N-2}{2}\frac{u}{g_\gamma(u)}\big(ug_\gamma(u) - G_\gamma(u)\big)\big(1-\lambda u^{p-2}\big) + u^2,\end{aligned} \tag{3.30}$$

we see

$$\int_{\{x\in\mathbb{R}^N: 0\leq u < \frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx > 0. \tag{3.31}$$

Observing (3.30), we can choose $\bar{t} > \frac{1}{\lambda^{\frac{1}{p-2}}}$ (which can be independent of p) such that $K(t) \geq 0$, $\forall t \in [\frac{1}{\lambda^{\frac{1}{p-2}}}, \bar{t}]$. Now, by direct calculation, we see

$$\begin{aligned}\frac{tg_\gamma'(t)}{g_\gamma(t)} &= \frac{1}{2t^{-2} + (4+\gamma) + (2+\gamma)t^2} \\ &\leq \frac{1}{2\bar{t}^{-2} + (4+\gamma) + (2+\gamma)\bar{t}^2} =: \eta(\bar{t}) \leq \frac{1}{8+2\gamma}, \ \forall t \geq \bar{t}.\end{aligned}$$

Hence, if we choose $p \geq (1+\eta(\bar{t}))2^* =: p^*$, we find

$$K(u) = \frac{N\lambda u^{p-1}}{pg_\gamma(u)}\Big(\frac{p}{2^*}G_\gamma(u) - ug_\gamma(u)\Big) + \frac{N-2}{2}\big(ug_\gamma(u) - G_\gamma(u)\big) + u^2$$
$$> \frac{N\lambda u^{p-1}}{pg_\gamma(u)}\Big[(1+\eta(\bar{t}))G_\gamma(u) - ug_\gamma(u)\Big] \geq 0,$$

which combined with (3.31) implies that the right-hand side of (3.29) is positive.

As a result, we complete the proof of Theorem 1.1–(3).

Remark 3.3. Since we can not find the explicit form of $G_\gamma(t)$, it is difficult for us to give the exact value of $\bar{t}$, below which $K(u)$ in (3.30) is nonnegative. However, we guess that $\bar{t}$ there should be $+\infty$, which implies that p^* is exactly 2^*, the critical exponent.

4. Asymptotic behavior of positive solution $u_{\gamma,\lambda}$

In what follows, we assume that $V(x) = \mu > 0$. For fixed $\lambda > 0$, we study the asymptotic behavior of $u_{\gamma,\lambda}$ as $\gamma \to 0^+$.

Define

$$m_{\gamma,\lambda} = \inf\{J_{\gamma,\lambda}(v); v \in H^1(\mathbb{R}^N)\setminus\{0\} \text{ is a solution of (2.8)}\}.$$

Following the arguments of Berestycki and Lions in [3], we can prove that $m_{\gamma,\lambda} > 0$ and $m_{\gamma,\lambda}$ is attained by $v_{\gamma,\lambda}$ satisfying

(1) $v_{\gamma,\lambda} > 0$ is spherically symmetric and $v_{\gamma,\lambda}$ decreases with respect to $|x|$;

(2) $v_{\gamma,\lambda} \in C^2(\mathbb{R}^N)$;

(3) $v_{\gamma,\lambda}$ together with its derivatives up to order 2 have exponential decay at infinity:

$$|D^\alpha v_{\gamma,\lambda}| \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N,$$

for some C, $\delta > 0$ and $|\alpha| \leq 2$.

In [14], Jeanjean and Tanaka proved that $m_{\gamma,\lambda} = c_{\gamma,\lambda}$, where $c_{\gamma,\lambda}$ is defined in (3.1) with $V(x)$ being replaced by μ. Moreover, we choose $\gamma_1 \in (0, \gamma^*)$ such that $u_{\gamma,\lambda} = G_\gamma^{-1}(v_{\gamma,\lambda})$ is indeed of a solution of (1.6) with $V(x) = \mu$ for $\gamma \in (0, \gamma_1]$. Similar to the proof of Proposition 3.1, we can prove $v_{\gamma,\lambda}$ is uniformly bounded with respect to γ.

We introduce the set $\widetilde{\mathcal{P}}$ of non-trivial solutions of (2.8) satisfying Pohozaev identity as follows:

$$\widetilde{\mathcal{P}} = \Big\{v \in H^1(\mathbb{R}^N)\setminus\{0\} : \widetilde{P}(v) := \frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla v|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^N}|G_\gamma^{-1}(v)|^p dx = 0\Big\}.$$

Then, similar to the proof of Lemma 3.1 in [14], we deduce that

$$m_{\gamma,\lambda} = \inf_{v \in \widetilde{\mathcal{P}}} J_{\gamma,\lambda}(v)$$

From Lemma 2.1–(3), Lemma 3.3 and the definition of $g_\gamma(t)$, we get

$$||u_{\gamma,\lambda}|| \leq C||v_{\gamma,\lambda}|| \leq C, \tag{4.1}$$

which implies that $u_{\gamma,\lambda}$ is uniformly bounded with respect to γ in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we may assume that as $\gamma \to 0^+$,

$$\begin{aligned} &u_{\gamma,\lambda} \rightharpoonup u_\lambda \ \text{ in } \ H^1(\mathbb{R}^N), \\ &u_{\gamma,\lambda} \to u_\lambda \ \text{ in } \ L^q_{loc}(\mathbb{R}^N), \quad q \in [1,2^*), \\ &u_{\gamma,\lambda} \to u_\lambda \ \text{ a.e. in } \ \mathcal{K} := supp\varphi, \quad \varphi(x) \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{4.2}$$

Moreover, there exists a function $\phi(x) \in L^q(\mathcal{K})$ such that $|u_{\gamma,\lambda}| \le \phi(x)$ a.e. in $\mathcal{K}$ for all γ.

We claim that u_λ is a solution of problem (1.7), namely, $\langle I'_\lambda(u_\lambda), \varphi\rangle = 0$, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$, where $I_\lambda(u)$ is defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \mu u^2)dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

In fact, by (4.2), we have

$$\begin{aligned} 0 &= \langle I'_\gamma(u_{\gamma,\lambda}), \varphi\rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_{\gamma,\lambda} \nabla\varphi + \mu u_{\gamma,\lambda}\varphi)dx \\ &\quad - \gamma\int_{\mathbb{R}^N} \left[\frac{u_{\gamma,\lambda}}{2(1+u_{\gamma,\lambda}^2)^2}|\nabla u_{\gamma,\lambda}|^2\varphi + \frac{u_{\gamma,\lambda}^2}{2(1+u_{\gamma,\lambda}^2)}\nabla u_{\gamma,\lambda}\nabla\varphi\right]dx - \lambda\int_{\mathbb{R}^N} |u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\varphi dx \\ &= \int_{\mathbb{R}^N} (\nabla u_{\gamma,\lambda}\nabla\varphi + \mu u_{\gamma,\lambda}\varphi)dx - \lambda\int_{\mathbb{R}^N} |u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\varphi dx + o(1) \\ &= \int_{\mathbb{R}^N} (\nabla u\nabla\varphi + \mu u\varphi)dx - \lambda\int_{\mathbb{R}^N} |u|^{p-2}u\varphi dx + o(1). \end{aligned} \tag{4.3}$$

Thus, we obtain

$$\int_{\mathbb{R}^N} (\nabla u_\lambda + \mu u_\lambda - \lambda|u_\lambda|^{p-2}u_\lambda)\varphi dx = 0, \tag{4.4}$$

which yields u_λ is a solution of problem (1.7). Since $u_{\gamma,\lambda}(x) > 0$ and $u_{\gamma,\lambda}(x) \in C^2$, we have $u_\lambda(x) \ge 0$.

Note that at this stage, we do not know whether $u_\lambda(x) \not\equiv 0$ or not. Next we prove $u_\lambda(x) \not\equiv 0$ and thus $u_\lambda(x) > 0$.

To this end, set

$$\widetilde{m}_{\gamma,\lambda} = \inf\{I_{\gamma,\lambda}(u); u \in H^1(\mathbb{R}^N)\setminus\{0\} \text{ is a solution of } (2.4)\}.$$

By Lemma 2.1, for $v \in H^1(\mathbb{R}^N)$, $u = G_\gamma^{-1}(v) \in H^1(\mathbb{R}^N)$, while for $u \in H^1(\mathbb{R}^N)$, $v = G_\gamma(u) \in H^1(\mathbb{R}^N)$. Moreover, since

$$I_{\gamma,\lambda}(u) = J_{\gamma,\lambda}(v),$$

$$\langle I'_{\gamma,\lambda}(u), \varphi\rangle = \langle J'_{\gamma,\lambda}(v), g_\gamma(G_\gamma^{-1}(v))\varphi\rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N),$$

we have $\widetilde{m}_{\gamma,\lambda} = m_{\gamma,\lambda}$.

Next, we set

$$\mathcal{P}=\left\{u\in H^1(\mathbb{R}^N)\setminus\{0\}:P(u):=\frac{N-2}{2N}\int_{\mathbb{R}^N}\left[1+\frac{\gamma u^2}{2(1+u^2)}\right]|\nabla u|^2dx+\frac{\mu}{2}\int_{\mathbb{R}^N}|u|^2dx\right.$$
$$\left.-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx=0\right\}.$$

Then, since $P(u)=\widetilde{P}(v)$ for $u=G_\gamma^{-1}(v)$, we get that

$$m_{\gamma,\lambda}=\inf_{v\in\widetilde{\mathcal{P}}}J_{\gamma,\lambda}(v)=\inf_{u\in\mathcal{P}}I_{\gamma,\lambda}(u).$$

Lemma 4.1.

$$\limsup_{\gamma\to 0^+} m_{\gamma,\lambda}\le m_\lambda.$$

where m_λ is the ground state level of (1.7) defined by

$$m_\lambda=\inf\{I_\lambda(u):u\in H^1(\mathbb{R}^N)\setminus\{0\},I_\lambda'(u)=0\}.$$

Proof. Let u be a ground state of (1.7) such that $I_\lambda(u)=m_\lambda$. By [3], $u\in L^\infty$. Moreover, u statisfies the Pohozaev identity:

$$\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{\mu}{2}\int_{\mathbb{R}^N}u^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx=0. \tag{4.5}$$

For $\tau>0$, we let

$$P\Big(u\Big(\frac{x}{\tau}\Big)\Big):=\frac{N-2}{2N}\tau^{N-2}\int_{\mathbb{R}^N}\left[1+\frac{\gamma u^2}{2(1+u^2)}\right]|\nabla u|^2dx$$
$$+\frac{\mu}{2}\tau^N\int_{\mathbb{R}^N}u^2dx-\frac{\lambda}{p}\tau^N\int_{\mathbb{R}^N}|u|^pdx. \tag{4.6}$$

It follows from (4.6) and (4.5) that

$$P\Big(u\Big(\frac{x}{\tau}\Big)\Big):=\frac{N-2}{2N}\tau^{N-2}\left[(1-\tau^2)\int_{\mathbb{R}^N}|\nabla u|^2dx+\gamma\int_{\mathbb{R}^N}\frac{u^2}{2(1+u^2)}|\nabla u|^2dx\right].$$

Let

$$\tau_{\gamma,\lambda}=\sqrt{\frac{\int_{\mathbb{R}^N}\left[1+\frac{\gamma u^2}{2(1+u^2)}\right]|\nabla u|^2dx}{\int_{\mathbb{R}^N}|\nabla u|^2dx}},$$

we get $P(u(\frac{x}{\tau}))=0$ and $\tau_{\gamma,\lambda}\to 1$ as $\gamma\to 0^+$. Clearly, $u(\frac{x}{\tau_{\gamma,\lambda}})\in\mathcal{P}$.

Therefore, we have

$$\begin{aligned}m_{\gamma,\lambda}&\le I_{\gamma,\lambda}\Big(u\Big(\frac{x}{\tau_{\gamma,\lambda}}\Big)\Big)\\&=\frac{1}{2}\tau_\gamma^{N-2}\int_{\mathbb{R}^N}\left[1+\frac{\gamma u^2}{2(1+u^2)}\right]|\nabla u|^2dx+\frac{1}{2}\mu\tau_\gamma^N\int_{\mathbb{R}^N}u^2dx-\frac{\lambda}{p}\tau_\gamma^N\int_{\mathbb{R}^N}|u|^pdx,\end{aligned}$$

which yields

$$\limsup_{\gamma\to 0^+} m_{\gamma,\lambda} \le \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+\mu u^2)dx - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p dx$$
$$= I_\lambda(u) = m_\lambda.$$

□

LEMMA 4.2. *For any given $\tilde{\gamma}>0$, there exists some positive constant $c_{\tilde{\gamma},\lambda}$ such that $m_{\gamma,\lambda} > c_{\tilde{\gamma},\lambda}$ for all $\gamma\in(0,\tilde{\gamma})$.*

Proof. For $\tilde{\gamma}>0$, we define the functional

$$Q_{\tilde{\gamma},\lambda}(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{\mu}{2+\tilde{\gamma}}\int_{\mathbb{R}^N} v^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^N}|v|^p dx$$

and the set

$$\Sigma_{\tilde{\gamma},\lambda} = \{\xi\in C([0,1],H^1(\mathbb{R}^N)) : \xi(0)=0, \xi(1)\neq 0, Q_{\tilde{\gamma},\lambda}(v)(\xi(1))<0\}.$$

By Lemma 2.1–(3), we have

$$Q_{\tilde{\gamma},\lambda}(v) \le J_{\gamma,\lambda}(v)$$

and thus $\Gamma_{\gamma,\lambda}\subset\Sigma_{\tilde{\gamma},\lambda}$. So, we obtain

$$0 < c_{\tilde{\gamma},\lambda} = \inf_{\xi\in\Sigma_{\tilde{\gamma},\lambda}}\sup_{t\in[0,1]} Q_{\tilde{\gamma},\lambda}(\xi(t))$$
$$\le \inf_{\xi\in\Gamma_{\gamma,\lambda}}\sup_{t\in[0,1]} Q_{\tilde{\gamma},\lambda}(\xi(t)) \le \inf_{\xi\in\Gamma_{\gamma,\lambda}}\sup_{t\in[0,1]} J_{\gamma,\lambda}(\xi(t)) = c_{\gamma,\lambda} = m_{\gamma,\lambda}.$$

The proof is finished. □

LEMMA 4.3. *Assume that $u_{\gamma,\lambda}$ is a solution of (2.4), then there exist $\ell\in\mathbb{N}\cup\{0\}$, $\{y_\gamma^j\}\subset\mathbb{R}^N$, $j=1,2,\cdots,\ell$ and $u_\lambda^j\in H^1(\mathbb{R}^N)\setminus\{0\}$ such that as $\gamma\to 0^+$,*

(1) $I_{\gamma,\lambda}(u_{\gamma,\lambda}) \to I_\lambda(u_\lambda) + \sum_{j=1}^{\ell} I_\lambda(u_\lambda^j)$;

(2) $||u_{\gamma,\lambda} - u_\lambda - \sum_{j=1}^{\ell} u_\lambda^j(\cdot - y_\gamma^j)|| \to 0$;

(3) $I'_\lambda(u_\lambda^j) = 0$, $|y_\gamma^j|\to\infty$, $|y_\gamma^i - y_\gamma^j|\to 0$, $\quad i\neq j$.

Proof. We follow the arguments developed by Benci and Cerami, see [4]. Let $u_{\gamma,\lambda}^1 := u_{\gamma,\lambda} - u_\lambda$, then $u_{\gamma,\lambda}^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and thus

$$||u_{\gamma,\lambda}^1||^2 = ||u_{\gamma,\lambda}||^2 - ||u_\lambda||^2 + o(1), \tag{4.7}$$

where $o(1)\to 0$ as $\gamma\to 0^+$.

By Brezis-Lieb lemma [31], we get

$$||u_{\gamma,\lambda}^1||_q^q = ||u_{\gamma,\lambda}||_q^q - ||u_\lambda||_q^q + o(1), \quad q\in[2,2^*). \tag{4.8}$$

Since $||u_{\gamma,\lambda}||_\infty \le C$ and $||u_{\gamma,\lambda}||\le C$, by (4.7) and (4.8), we have

$$m_{\gamma,\lambda} = I_{\gamma,\lambda}(u_{\gamma,\lambda})$$
$$= \frac{1}{2}\int_{\mathbb{R}^N}\left[1+\frac{\gamma u_{\gamma,\lambda}^2}{2(1+u_{\gamma,\lambda}^2)}\right]|\nabla u_{\gamma,\lambda}|^2dx + \frac{1}{2}\mu\int_{\mathbb{R}^N} u_{\gamma,\lambda}^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u_{\gamma,\lambda}|^p dx$$

$$=I_\lambda(u^1_{\gamma,\lambda})+I_\lambda(u_\lambda)+o(1) \tag{4.9}$$

and in a similar way that

$$\begin{aligned}
0=&\langle I'_\gamma(u_{\gamma,\lambda}),\varphi\rangle\\
=&\int_{\mathbb{R}^N}\Big[1+\frac{\gamma u^2_{\gamma,\lambda}}{2(1+u^2_{\gamma,\lambda})}\Big]\nabla(u^1_{\gamma,\lambda}+u_\lambda)\nabla\varphi dx-\gamma\int_{\mathbb{R}^N}\frac{u_{\gamma,\lambda}}{(1+u^2_{\gamma,\lambda})^2}|\nabla(u^1_{\gamma,\lambda}+u_\lambda)|^2\varphi dx\\
&+\mu\int_{\mathbb{R}^N}(u^1_{\gamma,\lambda}+u_\lambda)\varphi dx-\int_{\mathbb{R}^N}|u^1_{\gamma,\lambda}+u_\lambda|^{p-2}(u^1_{\gamma,\lambda}+u_\lambda)\varphi dx\\
=&\langle I'_\lambda(u^1_{\gamma,\lambda}),\varphi\rangle+\langle I'_\lambda(u_\lambda),\varphi\rangle+o(1)\\
=&\langle I'_\lambda(u^1_{\gamma,\lambda}),\varphi\rangle+o(1),\quad \forall\varphi\in H^1(\mathbb{R}^N).
\end{aligned} \tag{4.10}$$

Define

$$\delta=\lim_{n\to\infty}\sup_{x\in\mathbb{R}^N}\int_{B_1(y)}|u^1_{\gamma,\lambda}|^2dx.$$

If $\delta=0$, then using Lions lemma [18], $u^1_{\gamma,\lambda}\to0$ in $L^p(\mathbb{R}^N)$, $p\in(2,2^*)$. Since by (4.10), $\langle I'_\lambda(u^1_{\gamma,\lambda}),u^1_{\gamma,\lambda}\rangle\to0$, we have $u^1_{\gamma,\lambda}\to0$ in $H^1(\mathbb{R}^N)$, namely, $u_{\gamma,\lambda}\to u_\lambda$ in $H^1(\mathbb{R}^N)$ and the proof is complete. If $\delta>0$, we may assume that there exists $\{y^1_\gamma\}\subset\mathbb{R}^N$ such that

$$\int_{B_1(y^1_\gamma)}|u^1_{\gamma,\lambda}|^2dx>\frac{\delta}{2},$$

that is,

$$\int_{B_1(0)}|u^1_{\gamma,\lambda}(x+y^1_\gamma)|^2dx>\frac{\delta}{2}. \tag{4.11}$$

We may assume that $u^1_\lambda(x+y^1_\gamma)\rightharpoonup u^1_\lambda$ in $H^1(\mathbb{R}^N)$. By (4.11), $u^1_\lambda\not\equiv0$ and since $u^1_\lambda\rightharpoonup0$ in $H^1(\mathbb{R}^N)$, we have $|y^1_\gamma|\to\infty$. Let $u^2_{\gamma,\lambda}=u^1_{\gamma,\lambda}-u^1_\lambda(\cdot-y^1_\gamma)$, we get

$$||u^2_{\gamma,\lambda}||^2=||u_{\gamma,\lambda}||^2-||u_\lambda||^2-||u^1_\lambda||^2+o(1),$$

$$||u^2_{\gamma,\lambda}||^p_p=||u_{\gamma,\lambda}||^p_p-||u_\lambda||^p_p-||u^1_\lambda||^p_p+o(1)$$

and in a similar way that

$$m_{\gamma,\lambda}=I_\lambda(u_\lambda)+I_\lambda(u^1_\lambda)+I_\lambda(u^2_{\gamma,\lambda})+o(1),$$

$$\langle I'_\lambda(u^1),\varphi\rangle=0$$

and

$$\langle I'_\lambda(u^2_{\gamma,\lambda}),\varphi\rangle=o(1),\quad\forall\varphi\in H^1(\mathbb{R}^N).$$

Iterating the above procedure, since $I_\lambda(u^j)>0$ for every j, the iteration must terminate at some finite index, we get the result. □

We now prove $u_\lambda\not\equiv0$. In fact, we have the following result:

Lemma 4.4. *There exists $y_\gamma \in \mathbb{R}^N$ such that $u_{\gamma,\lambda}(\cdot - y_\gamma) \to u_\lambda(\cdot) > 0$ in $H^1(\mathbb{R}^N)$ as $\gamma \to 0^+$.*

Proof. In view of Lemma 4.3, if $u_\lambda \not\equiv 0$, we have

$$\lim_{\gamma\to 0^+} m_{\gamma,\lambda} = I_\lambda(u_\lambda) + \sum_{j=1}^{\ell} I_\lambda(u_\lambda^j) \geq (\ell+1) m_\lambda.$$

However, by Lemma 4.1, we get $\limsup_{\gamma\to 0^+} m_{\gamma,\lambda} \leq m_\lambda$. Thus $\ell = 0$ and the proof is complete provided $y_\gamma = 0$.

If $u_\lambda \equiv 0$, then by Lemma 4.3 again, $\ell = 1$. Thus we have $u_{\gamma,\lambda} \to u_\lambda^1(\cdot - y_\gamma^1)$ in $H^1(\mathbb{R}^N)$ and $I'_\lambda(u_\lambda^1) = 0$. Since the ground state of (1.7) is unique up to translation, it follows that $u_\lambda^1(x) = u_\lambda(x+\tilde{y})$ for some $\tilde{y} \in \mathbb{R}^N$, where u_λ is the ground state of (1.7). So, $u_{\gamma,\lambda} \to u_\lambda(\cdot - y_\gamma^1 + \tilde{y})$ in $H^1(\mathbb{R}^N)$. ∎

Lemma 4.5. *$||\nabla u_{\gamma,\lambda}||_\infty \leq C$.*

Proof. Recalling that $v_{\gamma,\lambda}$ satisfies

$$-\Delta v_{\gamma,\lambda} = -\mu \frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))} + \lambda \frac{|G_\gamma^{-1}(v_{\gamma,\lambda})|^{p-2} G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}.$$

By Lemma 2.1, we get

$$|\Delta v_{\gamma,\lambda}| \leq C(|v_{\gamma,\lambda}| + |v_{\gamma,\lambda}|^{p-1}).$$

For any $q > 2^*$, we have

$$\begin{aligned} ||\Delta v_{\gamma,\lambda}||_q &\leq C||v_{\gamma,\lambda}||_q + C||v_\gamma^{p-1}||_q \\ &\leq C\left[||v_{\gamma,\lambda}||_\infty^{\frac{q-2^*}{q}} + ||v_{\gamma,\lambda}||_\infty^{\frac{q(p-1)-2^*}{q}} \right] ||v_{\gamma,\lambda}||_{2^*}^{\frac{2^*}{q}} \\ &\leq C. \end{aligned} \tag{4.12}$$

By Corollary 9.10 in [11], $||D^2 u_{\gamma,\lambda}||_q \leq C||\Delta u_{\gamma,\lambda}||_q$ for $C = C(n,p) > 0$. Then, by the interpolation, we have $||v_{\gamma,\lambda}||_{W^{2,q}(\mathbb{R}^N)} \leq C$. Since $q > 2^*$, by Sobolev inequalities $W^{2,q}(\mathbb{R}^N) \hookrightarrow C^{1,\beta}(\mathbb{R}^N)$, we get $||v_{\gamma,\lambda}||_{C^{1,\beta}(\mathbb{R}^N)} \leq C$, where the constant C depends only on β and q. The result follows from the fact $||\nabla u_{\gamma,\lambda}||_\infty \leq C||\nabla u_{\gamma,\lambda}||_\infty$. ∎

Lemma 4.6. *$u_{\gamma,\lambda} \to u_\lambda$ in $H^2(\mathbb{R}^N)$.*

Proof. We claim that there exists $C > 0$ independent of $\gamma \in (0, \gamma_0)$ such that $||\Delta u_{\gamma,\lambda}||_2 \leq C$. Indeed, we observe that

$$-\left(1 + \frac{\gamma u_{\gamma,\lambda}^2}{\sqrt{1+u_{\gamma,\lambda}^2}}\right) \Delta u_{\gamma,\lambda} = -\mu u_{\gamma,\lambda} + \frac{\gamma u_{\gamma,\lambda}}{\sqrt{(1+u_{\gamma,\lambda}^2)^3}} |\nabla u_{\gamma,\lambda}|^2 + \lambda |u_{\gamma,\lambda}|^{p-2} u_{\gamma,\lambda}.$$

Thus, by Lemma 4.5 and (4.1), we have

$$\begin{aligned} ||\Delta u_{\gamma,\lambda}||_2 &= \left\| \frac{\sqrt{1+u_{\gamma,\lambda}^2}}{\sqrt{1+u_{\gamma,\lambda}^2} + \gamma u_{\gamma,\lambda}^2} \left[-\mu u_{\gamma,\lambda} + \frac{\gamma u_{\gamma,\lambda}}{\sqrt{(1+u_\gamma^2)^3}} |\nabla u_{\gamma,\lambda}|^2 + \lambda |u_{\gamma,\lambda}|^{p-2} u_{\gamma,\lambda} \right] \right\|_2 \\ &\leq C. \end{aligned} \tag{4.13}$$

Let $L=-\Delta+\mu I$, then L^{-1} is a bounded operator from $L^2(\mathbb{R}^N)$ to $H^2(\mathbb{R}^N)$,

$$u_{\gamma,\lambda}=L^{-1}\left[\frac{\gamma u_{\gamma,\lambda}^2}{\sqrt{1+u_{\gamma,\lambda}^2}}\Delta u_{\gamma,\lambda}+\frac{\gamma u_{\gamma,\lambda}}{\sqrt{(1+u_{\gamma,\lambda}^2)^3}}|\nabla u_{\gamma,\lambda}|^2+\lambda|u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\right]$$

and

$$u_\lambda=L^{-1}(\lambda|u_\lambda|^{p-2}u_\lambda).$$

Thus, we get

$$||u_{\gamma,\lambda}-u_\lambda||_{H^2}\le C(\gamma||\Delta u_{\gamma,\lambda}||_2+\gamma||u_{\gamma,\lambda}|\nabla u_{\gamma,\lambda}|^2||_2)+\lambda|||u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}-|u_\lambda|^{p-2}u_\lambda||_2. \tag{4.14}$$

By Lemma 4.5 and (4.13), we get

$$\gamma||\Delta u_{\gamma,\lambda}||_2+\gamma||u_{\gamma,\lambda}|\nabla u_{\gamma,\lambda}|^2||_2\to 0. \tag{4.15}$$

Since $u_{\gamma,\lambda}$ is radial, by the radial lemma [26], we have

$$|u_{\gamma,\lambda}|\le\frac{C}{|x|}||u_{\gamma,\lambda}||\le\frac{C}{|x|},\quad |x|\ge 1.$$

Thus, for any $\varepsilon>0$, there exists $R>0$ such that

$$|||u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}-|u|^{p-2}u||_{L^2(\mathbb{R}^N\setminus B_R(0))}<\varepsilon. \tag{4.16}$$

Since $u_{\gamma,\lambda}\rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$, it follows that there exists $\phi(x)\in L^1(B_R(0))$ such that

$$|u_{\gamma,\lambda}|^{p-1}\le C|\phi|\in L^1(B_R(0)).$$

Moreover

$$|u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\to|u_\lambda|^{p-2}u_\lambda,\ \text{a.e. in}\ \ B_R(0).$$

Thus, by Lebesgue dominated convergence theorem, we have

$$|||u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}-|u_\lambda|^{p-2}u_\lambda||_{L^2(B_R(0))}\to 0. \tag{4.17}$$

Finally, combining (4.14) – (4.17), we obtain

$$\lim_{\gamma\to 0^+}||u_{\gamma,\lambda}-u_\lambda||_{H^2}=0.$$

□

LEMMA 4.7. *$u_{\gamma,\lambda}\to u_\lambda$ in $C^2(\mathbb{R}^N)$.*

Proof. First, we show that $v_{\gamma,\lambda}\to u_\lambda$ in $C^2(\mathbb{R}^N)$. Since

$$|v_{\gamma,\lambda}|\le\frac{C}{|x|}||v_{\gamma,\lambda}||\le\frac{C}{|x|},\quad |x|\ge 1,$$

for any $q>2$ and $\varepsilon>0$, there exists $R>0$ independent of γ such that

$$\left\|-\mu\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}+\frac{|G_\gamma^{-1}(v_{\gamma,\lambda})|^{p-2}G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}\right\|_{L^q(\mathbb{R}^N\setminus B_R(0))}<\varepsilon$$

and

$$||\mu u_\lambda||_{L^q(\mathbb{R}^N\setminus B_R(0))}+|||u_\lambda|^{p-1}||_{L^q(\mathbb{R}^N\setminus B_R(0))}<\varepsilon.$$

On the other hand, since

$$||u_{\gamma,\lambda}||_\infty=||G_\gamma^{-1}(v_{\gamma,\lambda})||_\infty\le C,$$

we have

$$G_\gamma^{-1}(v_{\gamma,\lambda})\to u_\lambda, \quad \text{a.e. in } \mathbb{R}^N,$$

$$-\mu\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{\sqrt{1+G_\gamma^{-1}(v_{\gamma,\lambda})^2}}\to-\mu u_\lambda, \quad \text{a.e. in } \mathbb{R}^N.$$

By Lebesgue dominated convergence theorem, we get

$$\left\|-\mu\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{\sqrt{1+G_\gamma^{-1}(v_{\gamma,\lambda})^2}}+\mu u\right\|_{L^q(B_R(0))}+\lambda\left\||u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}-|u_\lambda|^{p-2}u_\lambda\right\|_{L^q(B_R(0))}\to 0. \tag{4.18}$$

Thus we have $\limsup_{\gamma\to0^+}||\Delta(v_{\gamma,\lambda}-u_\lambda)||_{L^q}\le 2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $v_{\gamma,\lambda}\to u_\lambda$ in $W^{2,q}(\mathbb{R}^N)$ for any $q>2$ as $\gamma\to0^+$. By Sobolev embedding, we have $v_{\gamma,\lambda}\to u_\lambda$ in $C^{1,\beta}(\mathbb{R}^N)$. By the bootstrap arguments, we have $v_{\gamma,\lambda}\to u_\lambda$ in $C^2(\mathbb{R}^N)$.

Next, we prove $v_{\gamma,\lambda}-u_{\gamma,\lambda}\to 0$ in $C^2(\mathbb{R}^N)$. Clearly, we have

$$\begin{aligned}|v_{\gamma,\lambda}-u_{\gamma,\lambda}|&=\left|\int_0^{u_{\gamma,\lambda}}\left[\sqrt{1+\frac{\gamma t^2}{2(1+t^2)}}-1\right]dt\right|\\&\le\frac{1}{2}\sqrt{\gamma}u_\gamma^2.\end{aligned} \tag{4.19}$$

Thus, from Proposition 3.1 that

$$\sup_{x\in\mathbb{R}^N}|v_{\gamma,\lambda}(x)-u_{\gamma,\lambda}(x)|\le C\sqrt{\gamma}\to 0.$$

From Lemma 4.5 and $\nabla u_{\gamma,\lambda}=g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))\nabla v_{\gamma,\lambda}$, we get

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}|\nabla v_{\gamma,\lambda}(x)-\nabla u_{\gamma,\lambda}(x)|&=\sup_{x\in\mathbb{R}^N}\left|\frac{\gamma u_\gamma^2\nabla u_{\gamma,\lambda}}{\sqrt{2+(2+\gamma)u_\gamma^2}\left[\sqrt{2(1+u_\gamma^2)}+\sqrt{2+(2+\gamma)u_\gamma^2}\right]}\right|\\&\le C\gamma\to 0.\end{aligned}$$

Similarly, we get

$$\sup_{x\in\mathbb{R}^N}\left|-\mu\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}+\frac{|G_\gamma^{-1}(v_{\gamma,\lambda})|^{p-2}G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}+\mu u_{\gamma,\lambda}-\lambda|u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\right|\to 0.$$

From

$$\begin{aligned}|\Delta u_{\gamma,\lambda}|&=\left|\frac{\sqrt{1+u_\gamma^2}}{\sqrt{1+u_\gamma^2+\gamma u_\gamma^2}}\left[-\mu u_{\gamma,\lambda}+\frac{\gamma u_{\gamma,\lambda}}{\sqrt{(1+u_\gamma^2)^3}}|\nabla u_{\gamma,\lambda}|^2+\lambda|u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\right]\right|\\&\le C,\end{aligned}$$

we obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}^N}|\Delta(v_{\gamma,\lambda}-u_{\gamma,\lambda})|\le&\gamma\sup_{x\in\mathbb{R}^N}\left|\frac{u_{\gamma,\lambda}}{\sqrt{(1+u_\gamma^2)^3}}|\nabla u_{\gamma,\lambda}|^2\right|+\gamma\sup_{x\in\mathbb{R}^N}\left|\frac{u_\gamma^2}{\sqrt{1+u_\gamma^2}}\Delta u_{\gamma,\lambda}\right|\\&+\sup_{x\in\mathbb{R}^N}\left|-\mu\frac{G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}+\lambda\frac{|G_\gamma^{-1}(v_{\gamma,\lambda})|^{p-2}G_\gamma^{-1}(v_{\gamma,\lambda})}{g_\gamma(G_\gamma^{-1}(v_{\gamma,\lambda}))}\right.\\&\left.+\mu u_{\gamma,\lambda}-\lambda|u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda}\right|\\\to&0.\end{aligned}\tag{4.20}$$

In a similar way, using (4.20), together with Sobolev interpolation inequality, we can show

$$\sup_{x\in\mathbb{R}^N}|D^j(v_{\gamma,\lambda}-u_{\gamma,\lambda})|\to0,\quad |j|\le2,$$

and this completes the proof of Lemma 4.7. □

Proof. **(Proof of Theorem 1.2.)** Since $u_{\gamma,\lambda}(x)=G_\gamma^{-1}(v_{\gamma,\lambda}(x))$, $G_\gamma^{-1}(t)$ is an odd C^∞ function and increases in $\mathbb{R}$, $v_{\gamma,\lambda}(x)$ is spherically symmetric and monotone decreasing with respect to $r=|x|$, we deduce that $u_{\gamma,\lambda}(x)$ is also spherically symmetric and monotone decreasing with respect to $r=|x|$. Finally, the asymptotic behavior of $u_{\gamma,\lambda}$ follows from Lemmas 4.6 and 4.7. □

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