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ON CONNECTION AMONG QUANTUM-INSPIRED ALGORITHMS OF THE ISING MODEL*

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Abstract. Various combinatorial optimization problems can be reduced to find the minimizer of an Ising model without external fields. This Ising problem is NP-hard and discrete. It is an intellectual challenge to develop algorithms for solving the problem. Over the past decades, many quantum and classical computations have been proposed from physical, mathematical or computational views for finding the minimizer of the Ising model, such as quantum annealing, coherent Ising machine, simulated annealing, adiabatic Hamiltonian systems, etc.. Especially, some of them can be described by differential equations called quantum-inspired algorithms, which use the signum vector of a solution of the differential equations to approach the minimizer of the Ising model. However, the mathematical principle why the quantum-inspired algorithms can work, to the best of our knowledge, is far from being well understood.

In this paper, using Morse's theory we reveal the mathematical principle of these quantum-inspired algorithms for the Ising model. It shows that the Ising problem can be designed by minimizing a smooth function, and those quantum-inspired algorithms are to find a global minimum point of the smooth function. In view of this mathematical principle, it can be proved that several known quantum-inspired algorithms: coherent Ising machine, Kerr-nonlinear parametric oscillators and simulated bifurcation algorithm, can reach the minimizers of the Ising model under suitable conditions. Moreover, we discuss the uniqueness of minimizers for the Ising problem in some senses, and give a sufficient condition to guarantee that the Ising model has a unique minimizer, that is, there is only a pair of minimizers with opposite signs.

Keywords. Ising problem; Quantum-inspired algorithms; Morse theory; Mathematical principle.

AMS subject classifications. 65P99; 90C27; 58E05; 37J06.

1. Introduction

The Ising model without external fields has been extensively studied in combinatorial optimization since many combinatorial problems can be equivalently formulated as finding the ground state of an Ising model [2], for example, the well-known MAX-CUT problem [3, 13] can be described as an Ising problem [2, 7, 14]. As applications, many problems can be equivalently formulated as an Ising problem such as the very-large-scale integrated circuit design [2], drug design [19], traffic flow optimization [16], and financial portfolio management [18]. It is well known that the Ising problem is NP-hard problem for classical computers [2]. Over the past decades, many models and algorithms from the aspects of quantum and classical computations are proposed to find the ground state of the Ising problem such as the coherent Ising machines (CIMs) [22, 23], optical Ising machine [17] and simulated annealing (SA) [11, 12], Kerr-nonlinear parametric oscillators (KPO) [4, 5] and the simulated bifurcation (SB) algorithm [6, 7], etc.. These quantum and classical computations have shown effectiveness in solving large-scale Ising models [6, 7, 10, 14]. A natural question to be asked is why these quantum-inspired algorithms are effective. To the best of our knowledge, little has been well understood or

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justified mathematically about computations.

Some of the above quantum and classical computations, such as CIMs [22, 23], KPO [4,5], SB [7] can be described by some kinds of differential equations. Therefore, we also call them *dynamical system algorithms*. For more dynamical system algorithms (or quantum-inspired algorithms) related optimizations, readers may refer to [20,21] and the references therein. These algorithms use the signum vector of solutions of continuous dynamical systems to approach the minimizers of Ising discrete problem. The aim of the present paper is to understand rigorously why the quantum-inspired algorithms can work in finding the ground state of an Ising model. In particular, we will provide the underlying mathematical principle that the Ising problem can be solved by minimizing some smooth functions.

Let us first recall the Ising problem without external fields, which is

$$\min_{\mathbf{v}} E(\mathbf{v}) := -\frac{1}{2}\mathbf{v}^T S\mathbf{v}, \tag{1.1}$$

where $\mathbf{v}=(v_1,\cdots,v_n)^T$ with $v_i\in\{-1,1\}$ denotes a spin configuration, $S=(s_{i,j})$ is an $n\times n$ symmetric coupling coefficient matrix with $s_{i,i}=0$ for all i. $E(\mathbf{v})$ is called *the Ising energy* and *the candidate set* of the Ising problem is denoted by

$$C(E)=\{-1,1\}^n := \overbrace{\{-1,1\}\times\{-1,1\}\times\cdots\times\{-1,1\}}^{n}.$$

In the other words, the Ising problem is to find some $\mathbf{v}\in C(E)$ such that $E(\mathbf{v})$ takes the minimum value.

We now define a function U on $\mathbf{R}^n$ by

$$U(\mathbf{x})=\sum_{i=1}^{n}\frac{1}{4}x_i^4+\frac{\beta-\alpha^2}{2}\mathbf{x}^T\mathbf{x}-\frac{1}{2}\mathbf{x}^T S\mathbf{x}, \quad \mathbf{x}\in\mathbf{R}^n, \tag{1.2}$$

where $\alpha>1$ is a parameter, β is a given positive constant, and S is the symmetric matrix in (1.1).

Denote the signum vector of $\mathbf{x}$ by

$$\operatorname{sgn}(\mathbf{x}) := (\operatorname{sgn}(x_1),\cdots,\operatorname{sgn}(x_n))^T\in\{-1,0,1\}^n.$$

Our main result is stated as follows.

THEOREM 1.1. *For any a given symmetric matrix S and a positive number β, there exists $\alpha_*>1$ such that for any $\alpha>\alpha_*$, if $\mathbf{x}_0$ is a global minimum point of $U(\mathbf{x})$, then* $\operatorname{sgn}(\mathbf{x}_0)$ *is a minimizer of the Ising problem* (1.1).

This theorem reveals the relationship between minimizers of the Ising problem $\min_{\mathbf{v}} E(\mathbf{v})$ and global minimum points of the smooth function $U(\mathbf{x})$. Local minimum points of $U(\mathbf{x})$ give all the candidates of the Ising model $E(\mathbf{v})$ and the global minimum points of $U(\mathbf{x})$ give minimizers $\min_{\mathbf{v}} E(\mathbf{v})$ of the Ising problem. Therefore, *the quantum-inspired algorithms are to look for global minimum points of some smooth functions.* Various known quantum-inspired algorithms involve finding the global minimum point of various deformation functions of $U(\mathbf{x})$ [4,5,7,22,23]. We leave the proof of Theorem 1.1 in Section 2, and revisit some known quantum-inspired algorithms in Section 3.

Furthermore, we consider if the minimizer of the Ising problem (1.1) is unique in some senses. Note that S is a symmetric matrix in (1.1). Hence, the minimizers of (1.1) appear in pairs with opposite signs. If there is only a pair of minimizers with opposite signs, then we say *the minimizer of the Ising problem* (1.1) *is unique.* To study the uniqueness of the minimizers, we consider the order of the Ising energy. From the order relation we obtain the sufficient condition for the uniqueness of the minimizers. For simplicity, we make the following assumption on the symmetric matrix S in (1.1)

$$(\mathbf{A}) \qquad \sum_{1\le i<j\le n} e_{i,j}s_{i,j} \neq 0,$$

where $s_{i,j}$ is the entry of S, $e_{i,j}$ takes any value in the set $\{-1,0,1\}$ and cannot be zero simultaneously.

Now we give the ascending order of the Ising energy under assumption (A) and obtain the sufficient condition for the uniqueness of the minimizers of the Ising problem (1.1) as follows.

THEOREM 1.2. *Assume that the condition (A) holds for* (1.1). *Then there exists* $\alpha'_* > 1$ *such that for* $\alpha > \alpha'_*$, *all local minimum points* $\mathbf{x}_1,\ldots,\mathbf{x}_N$ *of function* $U(\mathbf{x})$ *in* (1.2) *satisfy that*

$$U(\mathbf{x}_1) \le U(\mathbf{x}_2) < U(\mathbf{x}_3) \le U(\mathbf{x}_4) < \cdots < U(\mathbf{x}_{N-1}) \le U(\mathbf{x}_N), \tag{1.3}$$

where $N = 2^n$. *Let* $\mathbf{v}_i := \mathrm{sgn}(\mathbf{x}_i)$, $i \in \{1,\ldots,N/2\}$. *Then* $\mathbf{v}_{2i-1} = -\mathbf{v}_{2i}$ *and the corresponding Ising energy satisfies that*

$$E(\mathbf{v}_1) = E(\mathbf{v}_2) < E(\mathbf{v}_3) = E(\mathbf{v}_4) < \cdots < E(\mathbf{v}_{N-1}) = E(\mathbf{v}_N).$$

Hence, the Ising problem (1.1) *has a unique minimizer* $\mathbf{v}_1$ *or* $\mathbf{v}_2$ *with* $\mathbf{v}_1 = -\mathbf{v}_2$.

This paper is organized as follows. In Section 2, we provide necessary preliminaries on Morse's theory, and prove Theorem 1.1 and Theorem 1.2. Moreover, we give some examples and a weaker condition on the uniqueness of the minimizers. In Section 3, we apply Theorem 1.1 to revisit some known quantum-inspired algorithms (CIM, KPO and SB algorithm) and prove that the global minimum points found by CIM, KPO and SB algorithm are minimizers of the Ising problem.

2. Mathematical principle on the quantum-inspired algorithms

In this section, we first recall some preliminaries on Morse's theory. We then classify critical points of $U(\mathbf{x})$ according to their Morse indices in Proposition 2.2. At last, we transform the Ising problem into looking for global minimum points of the smooth function $U(\mathbf{x})$ and thus obtain Theorem 1.1 and Theorem 1.2.

2.1. Preliminary results.

The notion of *Morse index* of a smooth real-valued function f at a critical point is given as follows.

DEFINITION 2.1. *Suppose that* f *is a smooth real-valued function on* $\mathbf{R}^n$ *and* $\mathbf{x}$ *is a critical point of* f, *i.e.,* $\nabla f(\mathbf{x}) = 0$. *The Morse index* $i(\mathbf{x})$ *of* f *at* $\mathbf{x}$ *is defined as the number of negative eigenvalues of the Hessian* $D^2 f(\mathbf{x})$ *counted with multiplicity and the nullity* $\nu(\mathbf{x})$ *is the dimension of kernel at* $\mathbf{x}$. *Namely,*

$$i_f(\mathbf{x}) := \max\left\{\dim V \mid V \subset \mathbf{R}^n \text{ is a subspace with } \mathbf{e}^T D^2 f(\mathbf{x})\mathbf{e} < 0, \forall \mathbf{e} \in V\setminus\{0\}\right\},$$
$$\nu_f(\mathbf{x}) := \dim\ker D^2 f(\mathbf{x}).$$

If $\nu_f(\mathbf{x}) = 0$, *then* f *is called non-degenerate at* $\mathbf{x}$.

Using the Morse index, critical points of f can be classified into *local maximum points* whose Morse indices are n, *local minimum points* whose Morse indices are 0 and *saddle points* whose Morse indices are between 0 and n. The sets of all local maximum points, all local minimum points, all saddle points, are denoted by $\mathcal{C}_n(f)$, $\mathcal{C}_0(f)$ and $\mathcal{C}_s(f)$, respectively. We refer readers to [1] and [15] for more details on the Morse theory.

Let

$$\bar{U}(\mathbf{x}) = \sum_{i=1}^{n} \frac{1}{4} x_i^4 - \frac{\alpha^2}{2} \mathbf{x}^T \mathbf{x}, \quad \alpha > 1.$$

Then $U(\mathbf{x}) = \bar{U}(\mathbf{x}) + \frac{\beta}{2}\mathbf{x}^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T S\mathbf{x}$ by (1.2). Denote by $\mathcal{C}(U)$ and $\mathcal{C}(\bar{U})$ the sets of all critical points of $U(\mathbf{x})$ and $\bar{U}(\mathbf{x})$ respectively. It is straightforward to obtain that

$$\mathcal{C}(U) = \{\mathbf{x} \in \mathbf{R}^n |\ (x_1^3 + (\beta - \alpha^2)x_1, \ldots, x_n^3 + (\beta - \alpha^2)x_n)^T - S\mathbf{x} = 0\},$$

and

$$\mathcal{C}(\bar{U}) = \{\mathbf{x} \in \mathbf{R}^n |\ (x_1^3 - \alpha^2 x_1, \ldots, x_n^3 - \alpha^2 x_n)^T = 0\}.$$

Both $\mathcal{C}(\bar{U})$ and $\mathcal{C}(U)$ are non-empty since $\mathbf{0} \in \mathcal{C}(U) \cap \mathcal{C}(\bar{U})$. Define the set $\mathbf{A} = \{-\alpha, \alpha\}$ and $\mathbf{A}_0 = \{-\alpha, 0, \alpha\}$. Then we define

$$\mathbf{A}_0^n = \{-\alpha, 0, \alpha\}^n, \quad \mathbf{A}^n = \{-\alpha, \alpha\}^n.$$

The critical points of $\bar{U}(\mathbf{x})$ can be classified as follows.

LEMMA 2.1. *$\mathcal{C}(\bar{U}) = \mathbf{A}_0^n$. Moreover, for any a $\mathbf{x} \in \mathcal{C}(\bar{U})$, there holds*

$$i_{\bar{U}}(\mathbf{x}) = \#\{j \mid x_j = 0, x_j \textit{ is the } j\textit{-th component of } \mathbf{x}\},$$

where $\#\{\cdot\}$ represents the number of all elements of set $\{\cdot\}$. Then $\mathcal{C}(\bar{U})$ can be classified as follows.

(i) *The unique local maximum point is the origin, i.e., $\mathcal{C}_n(\bar{U}) = \{0\}^n$;*

(ii) *the set of local minimum points $\mathcal{C}_0(\bar{U}) = \mathbf{A}^n$;*

(iii) *the set of saddle points $\mathcal{C}_s(\bar{U}) = \mathcal{C}(\bar{U}) \setminus \big(\mathcal{C}_n(\bar{U}) \cup \mathcal{C}_0(\bar{U})\big)$.*

Proof. Solving $\nabla \bar{U}(\mathbf{x}) = 0$ directly, we obtain the solutions $\mathbf{x} = (x_1, \ldots, x_n)^T$ with $x_i \in \mathbf{A}_0$. Therefore, the number of the critical points of $\bar{U}(\mathbf{x})$ is 3^n.

The Hessian of $\bar{U}$ is given by $D^2\bar{U}(\mathbf{x}) = \mathrm{diag}\{3x_1^2 - \alpha^2, \ldots, 3x_n^2 - \alpha^2\}$. If $x_i = 0$, then $3x_i^2 - \alpha^2 < 0$; if $x_i = \pm\alpha$, then $3x_i^2 - \alpha^2 > 0$. Its Morse index is given by $\#\{j | x_j = 0, x_j \text{ is the } j\text{-th component of } \mathbf{x}\}$. Therefore, the origin is the unique local maximum point, and each $\mathbf{x} \in \mathbf{A}^n$ is a local minimum point. Each $\mathbf{x} \in \mathcal{C}(\bar{U}) \setminus (\mathbf{A}^n \cup \{0\})$ is a saddle point, namely at least one $x_i = 0$ and at least one $x_j \in \mathbf{A}$. ∎

Note that $\mathcal{C}_0(\bar{U}) = \mathbf{A}^n$ where $\mathbf{A} = \{-\alpha, \alpha\}$. Recall the candidate set $C(E) = \{-1, 1\}^n$. Via the signum map, the following result holds.

COROLLARY 2.1. *$\{\mathrm{sgn}(\mathbf{x}) | \mathbf{x} \in \mathcal{C}_0(\bar{U})\} = C(E)$.*

For critical points of U, we have an a priori estimate as follows.

When α is large enough, each critical point of $U(\mathbf{x})$ can be approximated by a unique critical point of $\bar{U}(\mathbf{x})$ as follows.

PROPOSITION 2.1. *For every $\mathbf{x}\in\mathcal{C}(U)$, there exist constants B_1, large enough α_1 and one unique $\bar{\mathbf{x}}\in\mathcal{C}(\bar{U})$, such that for any $\alpha>\alpha_1$,*

$$|\mathbf{x}-\bar{\mathbf{x}}|<\frac{B_1}{\alpha}. \tag{2.1}$$

Furthermore, $i_U(\mathbf{x})=i_{\bar{U}}(\bar{\mathbf{x}})$.

Proof. Each $\mathbf{x}\in\mathcal{C}(U)$ satisfies that $\left(x_1^3,\dots,x_n^3\right)^T=\alpha^2\mathbf{x}+(S-\beta I)\mathbf{x}$. Denote

$$(f_1,\dots,f_n)^T\equiv\left(x_1^3,\dots,x_n^3\right)^T-\alpha^2\mathbf{x}-(S-\beta I)\mathbf{x}.$$

Note that $\mathbf{x}\in\mathcal{C}(U)$ if and only if $f_i(\mathbf{x})=0$. Without loss of generality, for any given α, assume $|x_1|=\max\{|x_1|,\dots,|x_n|\}$ and denote $s=\max\{s_{i,j}\}$. Then the i-th component of $(S-\beta I)\mathbf{x}$ satisfies

$$|((S-\beta I)\mathbf{x})_i|=\left|\sum_{j=1}^{n}s_{i,j}x_j-\beta x_i\right|\le(ns+\beta)|x_1|<(ns+\beta+1)|x_1|.$$

Let $c_1=ns+\beta+1$, and then each component of $(S-\beta I)\mathbf{x}$ belongs to $(-c_1|x_1|,c_1|x_1|)$. We have

$$x_1^3-\alpha^2x_1-c_1|x_1|<f_1(x_1)<x_1^3-\alpha^2x_1+c_1|x_1|.$$

If $x_1=0$, clearly $\mathbf{x}=0$. If $x_1\neq0$, then

$$x_1^2-\alpha^2-c_1<\frac{f_1(x_1)}{x_1}<x_1^2-\alpha^2+c_1.$$

Thus solutions of $f_1(x_1)=0$ must satisfy $|x_1|\in\left(\sqrt{\alpha^2-c_1},\sqrt{\alpha^2+c_1}\right)$. Now considering $f_i(\mathbf{x})$ for $i>1$, we have

$$x_i^3-\alpha^2x_i-c_1\sqrt{\alpha^2+c_1}<f_i(x_i)<x_i^3-\alpha^2x_i+c_1\sqrt{\alpha^2+c_1}.$$

Choosing $\alpha_2>2c_1$ large enough such that $2\alpha>\sqrt{\alpha^2+c_1}+1$ as $\alpha>\alpha_2$, and

$$\begin{aligned}f_i\left(\frac{2c_1}{\alpha}\right)&<\left(\frac{2c_1}{\alpha}\right)^3-\alpha^2\left(\frac{2c_1}{\alpha}\right)+c_1\sqrt{\alpha^2+c_1}\\&<1-2c_1\alpha+c_1\sqrt{\alpha^2+c_1}\\&<c_1\left(1+\sqrt{\alpha^2+c_1}-2\alpha\right)\\&<0.\end{aligned}$$

Similarly, one can prove $f_i\left(-\frac{2c_1}{\alpha}\right)>0$, thus there is a solution of $f_i(x_i)=0$ in $\left(-\frac{2c_1}{\alpha},\frac{2c_1}{\alpha}\right)$. By the same arguments, one can prove there are solutions of $f_i(x_i)$ belonging to one of the following three intervals

$$\left(-\alpha-\frac{2c_1}{\alpha},-\alpha+\frac{2c_1}{\alpha}\right),\quad\left(-\frac{2c_1}{\alpha},\frac{2c_1}{\alpha}\right)\quad\left(\alpha-\frac{2c_1}{\alpha},\alpha+\frac{2c_1}{\alpha}\right).$$

Moreover, all solutions of $f_i(x_i)$ should belong to these intervals since $f_i(x_i)$ are negative or positive as $x_i\in\mathbb{R}\setminus\left(\left(-\alpha-\frac{2c_1}{\alpha},-\alpha+\frac{2c_1}{\alpha}\right)\cup\left(-\frac{2c_1}{\alpha},\frac{2c_1}{\alpha}\right)\cup\left(\alpha-\frac{2c_1}{\alpha},\alpha+\frac{2c_1}{\alpha}\right)\right)$.

Note that each component of $\mathbf{x}$, x_i, satisfies one of following inequalities for all $\alpha > \alpha_2$,

$$|x_i - \alpha| < \frac{B_1}{\alpha}, \quad |x_i - 0| < \frac{B_1}{\alpha}, \quad |x_i + \alpha| < \frac{B_1}{\alpha}. \tag{2.2}$$

By (2.2), we know that $\lim_{\alpha\to\infty} \mathbf{x}/\alpha \in \{-1,0,1\}^n$ and denoted by $\bar{\mathbf{v}}$.

Let $\bar{\mathbf{x}} := \alpha\bar{\mathbf{v}}$ with $\bar{\mathbf{x}} \in \mathbf{A}_0^n$ which is uniquely determined by $\mathbf{x}$. Without loss of generality, assume $i_{\bar{U}}(\bar{\mathbf{x}}) = n - i_0$. By (2.2), we know that the number of x_i satisfying $|x_i - \alpha| < \frac{B_1}{\alpha}$ or $|x_i + \alpha| < \frac{B_1}{\alpha}$ is i_0 and the number of x_i satisfying $|x_i| < \frac{B_1}{\alpha}$ is $n - i_0$ for $\alpha > \alpha_2$.

The Hessian of $U(\mathbf{x})$ is given by

$$D^2U = \mathrm{diag}\{3x_1^2 + \beta - \alpha^2, \ldots, 3x_n^2 + \beta - \alpha^2\} - S.$$

Decompose D^2U as the sum of $\mathrm{diag}\{3x_1^2 - \alpha^2, \ldots, 3x_n^2 - \alpha^2\}$ and $\beta I_n - S$. Suppose that the eigenvalues of $\frac{1}{\alpha^2}D^2U$ are $\sigma(\frac{1}{\alpha^2}D^2U) = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1 \geq \cdots \geq \lambda_n$. Furthermore, suppose that

$$\sigma(\frac{1}{\alpha^2}\mathrm{diag}\{3x_1^2 - \alpha^2, \ldots, 3x_n^2 - \alpha^2\}) = \{\bar{\lambda}_1, \ldots, \bar{\lambda}_n\},$$

$$\sigma((\beta I_n - S)/\alpha^2) = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\},$$

where $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n$ and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n$. Since $(\beta I_n - S)$ is a constant matrix and $i_{\bar{U}}(\bar{\mathbf{x}}) = n - i_0$, there exists $\alpha_1 > \alpha_2$ such that $\max\{|\tilde{\lambda}_1|, |\tilde{\lambda}_n|\} < 1/3$, $\bar{\lambda}_i > 5/3$ for $1 \leq i \leq i_0$ and $\bar{\lambda}_i < -2/3$ for $i_0 + 1 \leq i \leq n$. According to Weyl's inequality (cf. [9, Theorem 4.3.1]), λ_i satisfies that $\bar{\lambda}_i + \tilde{\lambda}_n < \lambda_i < \bar{\lambda}_i + \tilde{\lambda}_1$. Therefore, λ_i possesses the same sign as $\bar{\lambda}_i$.

$$\lambda_i \geq 1, \text{ for } 1 \leq i \leq i_0, \text{ and } \lambda_i \leq -\frac{1}{3}, \text{ for } i_0 + 1 \leq i \leq n.$$

Then $i_U(\mathbf{x}) = n - i_0$ and all critical points of $U(\mathbf{x})$ are non-degenerate. Then the proposition follows. ∎

As α tends to positive infinity, every critical point $\mathbf{x}$ of $U(\mathbf{x})$ satisfies $|\mathbf{x} - \bar{\mathbf{x}}| \to 0$ by (2.1). According to Proposition 2.1, when $\alpha > \alpha_1$, $\mathbf{x}$ can be written as $\mathbf{x} = \bar{\mathbf{x}} + \delta$ where $\bar{\mathbf{x}} \in \mathcal{C}(\bar{U})$ and $|\delta| \sim \mathcal{O}(1/\alpha)$.

COROLLARY 2.2. *Suppose $\mathbf{x} = (x_1, \ldots, x_n)^T$ is one local minimum point of $U(\mathbf{x})$. There exists $\alpha_3 > \alpha_1$ such that for all $\alpha > \alpha_3$ the following statements hold.*

(i) *$\mathbf{x} = \bar{\mathbf{x}} + \delta$ where $\bar{\mathbf{x}} \in \mathbf{A}^n$ and $\delta = (\delta_1, \ldots, \delta_n)^T$ with $\delta_i \sim \mathcal{O}(1/\alpha)$ as $\alpha \to \infty$;*

(ii) *$x_i \neq 0$ for all $i \in \{1, \ldots, n\}$;*

(iii) *$\mathrm{sgn}(\mathbf{x}) = \mathrm{sgn}(\bar{\mathbf{x}})$.*

Proof. Since $\mathbf{x}$ is a local minimum point, then $i_U(\mathbf{x}) = 0$ and $i_{\bar{U}}(\bar{\mathbf{x}}) = 0$. By Lemma 2.1, for each i, $\bar{x}_i$ satisfies $|\bar{x}_i| = \alpha$. By (2.1), one can choose a proper $\alpha_3 > \alpha_2$ such that $\bar{x}_i + \delta_i \neq 0$ and $\mathrm{sgn}(x_i) = \mathrm{sgn}(\bar{x}_i)$ when $\alpha > \alpha_3$ for all i. ∎

PROPOSITION 2.2. *For any given β and S, there exists a sufficiently large constant $\alpha_0 > \alpha_3$ such that for all $\alpha > \alpha_0$,*

(i) *$U(\mathbf{x})$ possesses 2^n local minimum points;*

(ii) *$\{\mathrm{sgn}(\mathbf{x}) | \mathbf{x} \in \mathcal{C}(U)\} = C(E)$.*

Proof. Denote the unit open ball with the center $\mathbf{x}$ by $\mathcal{B}(\mathbf{x},1)$. Suppose that $\bar{\mathbf{x}}\in\mathcal{C}_0(\bar{U})$ with $\bar{\mathbf{x}}=(\bar{x}_1,\ldots,\bar{x}_n)$. For $\mathbf{x}\in\partial\mathcal{B}(\bar{\mathbf{x}},1)$, there exists u such that $\mathbf{x}=\bar{\mathbf{x}}+u$ where $u=(u_1,\ldots,u_n)^T$ with $|u|=1$. Note that for $\mathbf{x}\in\partial\mathcal{B}(\bar{\mathbf{x}},1)$,

$$\begin{aligned}
&U(\mathbf{x})-U(\bar{\mathbf{x}})\\
&=\sum_{i=1}^{n}(\bar{x}_i^3u_i+\frac{3}{2}\bar{x}_i^2u_i^2+\bar{x}_iu_i^3+\frac{1}{4}u_i^4)+\sum_{i=1}^{n}\frac{\beta-\alpha^2}{2}(2\bar{x}_iu_i+u_i^2)-u^TS\bar{x}-\frac{1}{2}u^TSu\\
&=\sum_{j=1}^{n}(\alpha^2u_i^2+\beta u_i\bar{x}_i+\bar{x}_iu_i^3+\frac{1}{4}u_i^4+\frac{\beta}{2}u_i^2)-u^TS\bar{x}-\frac{1}{2}u^TSu.
\end{aligned}$$

Note that $\alpha^2u_i^2$ possesses the order of α^2; the terms of $\beta u_i\bar{x}_i$, $\bar{x}_iu_i^3$ and $u^TS\bar{x}$ possess the order of α; and the remaining terms are bounded when α is sufficiently large. Then there exists $\alpha_4>\alpha_3$ such that $U(\mathbf{x})-U(\bar{\mathbf{x}})>0$ for $\alpha>\alpha_4$. Since $\overline{\mathcal{B}(\bar{\mathbf{x}},1)}$ is compact and $U(\bar{\mathbf{x}}+u)>U(\bar{\mathbf{x}})$ for any $|u|=1$, U possesses at least one local minimum point in $B(\bar{\mathbf{x}},1)$. Since $\#\mathcal{C}_0(\bar{U})=2^n$ and $\mathcal{B}(\bar{\mathbf{x}}_i,1)\cap\mathcal{B}(\bar{\mathbf{x}}_j,1)=\emptyset$ for any $\bar{\mathbf{x}}_i,\bar{\mathbf{x}}_j\in\mathcal{C}_0(U)$, we have that U possesses at least 2^n local minimum points.

For any $\mathbf{x}\in\mathcal{B}(\bar{\mathbf{x}},1)$, there exists $u=(u_1,\ldots,u_n)$ satisfying $|u|=1$ such that $\mathbf{x}=\bar{\mathbf{x}}+u=(\bar{x}_1+u_1,\ldots,\bar{x}_n+u_n)$. The Hessian of U at $\bar{\mathbf{x}}$ is given by

$$\begin{aligned}
D^2U(\mathbf{x})=&\mathrm{diag}\{3(\bar{x}_1+u_1)^2-\alpha^2,\ldots,3(\bar{x}_n+u_n)^2-\alpha^2\}+\beta I_n-S\\
=&\mathrm{diag}\{2\alpha^2+3(2\bar{x}_1u_1+u_1^2),\ldots,2\alpha^2+3(2\bar{x}_1u_1+u_1^2)\}+\beta I_n-S.
\end{aligned}$$

It yields that there exists $\alpha_0>\alpha_4$ such that if $\alpha>\alpha_0$, $D^2U(\mathbf{x})$ is positive definite. Therefore, $U(x)$ is strictly convex in $\mathcal{B}(\bar{\mathbf{x}},1)$. Hence there exists the unique local minimum $\mathbf{x}_0$ of $U(\mathbf{x})$ in $\mathcal{B}(\bar{\mathbf{x}},1)$.

Therefore, if $\alpha\ge\alpha_0$, $U(\mathbf{x})$ possesses 2^n local minimum points. Then the conclusion (i) of this proposition follows.

By the conclusion (iii) of Corollary 2.2, the signum vectors of local minimum points of $U(\mathbf{x})$ are the same as the ones of local minimum of $\bar{U}(\mathbf{x})$. By Corollary 2.1, the conclusion (ii) of this proposition holds. ◻

Now we can establish the bridge between the Ising energy E and the function U through function $\bar{U}$ and prove Theorem 1.1 and Theorem 1.2.

For any given Ising model $E(\mathbf{v})$, we can write the Ising energy in ascending order. Without loss of generality, we label the candidates according to their Ising energy as follows.

$$E(\mathbf{v}_1)=E(\mathbf{v}_2)\le E(\mathbf{v}_3)=E(\mathbf{v}_4)\le\cdots\le E(\mathbf{v}_{N-1})=E(\mathbf{v}_N),$$

where $\mathbf{v}_i\in C(E)=\{-1,1\}^n$ and $\mathbf{v}_{2i-1}=-\mathbf{v}_{2i}$. Let

$$d_i=E(\mathbf{v}_{2i})-E(\mathbf{v}_{2i-2})\ge 0,\quad \text{where } i\in\{2,\ldots,N/2\}. \tag{2.3}$$

We can label the local minimum points of $\bar{U}(\mathbf{x})$ by $\bar{\mathbf{x}}_i$ with $\bar{\mathbf{x}}_i=\alpha\mathbf{v}_i$. Since $|\bar{\mathbf{x}}_i|=|\bar{\mathbf{x}}_j|$ for any $\bar{\mathbf{x}}_i,\bar{\mathbf{x}}_j\in\mathcal{C}_0(\bar{U})$, we have that $\bar{U}(\bar{\mathbf{x}}_i)=\bar{U}(\bar{\mathbf{x}}_j)$. For $i\in\{2,\ldots,N/2\}$, we have that

$$\begin{aligned}
U(\bar{\mathbf{x}}_{2i})-U(\bar{\mathbf{x}}_{2i-2})=&\left(\bar{U}(\bar{\mathbf{x}}_{2i})+\frac{\beta}{2}|\bar{\mathbf{x}}_{2i}|^2-\frac{1}{2}\bar{\mathbf{x}}_{2i}^TS\bar{\mathbf{x}}_{2i}\right)\\
&-\left(\bar{U}(\bar{\mathbf{x}}_{2i-2})+\frac{\beta}{2}|\bar{\mathbf{x}}_{2i-2}|^2-\frac{1}{2}\bar{\mathbf{x}}_{2i-2}^TS\bar{\mathbf{x}}_{2i-2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\bar{\mathbf{x}}_{2i-2}^T S \bar{\mathbf{x}}_{2i-2} - \bar{\mathbf{x}}_{2i}^T S \bar{\mathbf{x}}_{2i}\right) \\
&= \alpha^2 d_i,
\end{aligned}
$$

where the last equality holds by $\bar{\mathbf{x}}_{2i} = \alpha \mathbf{v}_{2i}$. It follows that

$$
U(\bar{\mathbf{x}}_1) = U(\bar{\mathbf{x}}_2) \le U(\bar{\mathbf{x}}_3) = U(\bar{\mathbf{x}}_4) \le \cdots \le U(\bar{\mathbf{x}}_{N-1}) = U(\bar{\mathbf{x}}_N). \tag{2.4}
$$

By Proposition 2.1 and the finiteness of $\#\mathcal{C}(U)$, there exists $\alpha_5 > \alpha_0$, such that if $\alpha > \alpha_5$, we can also label the local minimum points of $U(\mathbf{x})$ by $\mathbf{x}_i$ for $1 \le i \le N$ and $\mathbf{x}_i$ satisfies

$$
|\mathbf{x}_i - \bar{\mathbf{x}}_i| < B/\alpha. \tag{2.5}
$$

LEMMA 2.2. *For any $\mathbf{x}_i \in \mathcal{C}_0(U)$, there exist $M > 0$ and $A_{\max} > \alpha_5$ such that for $\alpha > A_{\max}$,*

$$
|U(\mathbf{x}_i) - U(\bar{\mathbf{x}}_i)| < M.
$$

Proof. By Corollary 2.2 and Proposition 2.2, we have that $\mathbf{x}_i = \bar{\mathbf{x}}_i + \delta_i$ where $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n})^T$, $\bar{\mathbf{x}}_i = (\bar{x}_{i,1}, \ldots, \bar{x}_{i,n})^T \in \mathcal{C}_0(\bar{U})$, and $\delta_i = (\delta_{i,1}, \ldots, \delta_{i,n})^T$. It follows that

$$
\begin{aligned}
&U(\mathbf{x}_i) - U(\bar{\mathbf{x}}_i) \\
&= \sum_{j=1}^{n} \left(\bar{x}_{i,j}^3 \delta_{i,j} + \frac{3}{2}\bar{x}_{i,j}^2 \delta_{i,j}^2 + \bar{x}_{i,j}\delta_{i,j}^3 + \frac{1}{4}\delta_{i,j}^4\right) + \sum_{j=1}^{n} \frac{\beta - \alpha^2}{2}\left(2\bar{x}_{i,j}\delta_{i,j} + \delta_{i,j}^2\right) - \delta^T S \bar{x} - \frac{1}{2}\delta^T S \delta \\
&= \sum_{j=1}^{n} \left(\alpha^2 \delta_{i,j}^2 + \beta \delta_{i,j}\bar{x}_{i,j} + \bar{x}_{i,j}\delta_{i,j}^3 + \frac{1}{4}\delta_{i,j}^4 + \frac{\beta}{2}\delta_{i,j}^2\right) - \delta^T S \bar{x} - \frac{1}{2}\delta^T S \delta,
\end{aligned}
$$

where the second equality holds by $|\bar{x}_{i,j}| = \alpha$. For each j, the terms $\alpha^2\delta_{i,j}^2$, $\beta\delta_{i,j}\bar{x}_{i,j}$ and $\delta^T S \bar{\mathbf{x}}$ are all bounded because $|\bar{x}_{i,j}| = \alpha$ and $|\delta_{i,j}| < B_2/\alpha$. The terms $\bar{x}_{i,j}\delta_{i,j}^3$, $\frac{1}{4}\delta_{i,j}^4$, $\frac{\beta}{2}\delta^2$ and $\frac{1}{2}\delta^T S \delta$ tend to zero as α tends to infinity. Therefore, there exist $A_i > \alpha_5$ and $M_i > 0$ such that for all $\alpha \ge A_i$, $|U(\mathbf{x}_i) - U(\bar{\mathbf{x}}_i)| < M_i$. Note that A_i and M_i only depend on β and S. By the finiteness of the $\mathcal{C}_0(U)$, we define that

$$
A_{\max} := \max\{A_i | 1 \le i \le N\}, \; M := \max\{M_i | 1 \le i \le N\}.
$$

The proof is complete. ∎

2.2. Proofs of main results. Note that d_i in (2.3) only depend on β and S. By the finiteness of the $\mathcal{C}_0(U)$, we define that

$$
d_{\min} := \min\{d_i | d_i \ne 0, 2 \le i \le N/2\}.
$$

We first prove Theorem 1.1 as follows.

Proof. Suppose $\mathbf{x}_0$ is a global minimum point of $U(\mathbf{x})$. Let $\mathbf{v}_0 = \mathrm{sgn}(\mathbf{x}_0) \in C(E)$. By Lemma 2.2, when $\alpha > A_{\max}$,

$$
|U(\mathbf{x}_i) - U(\bar{\mathbf{x}}_i)| < M, \quad \forall 1 \le i \le N. \tag{2.6}
$$

Let

$$
\alpha_* := \max\left\{A_{\max}, \sqrt{\frac{3M}{d_{min}}}\right\}.
$$

Note that $d_{\min} \neq 0$. Assume by contradiction that there is $\mathbf{v}' \in C(E)$ such that $E(\mathbf{v}') < E(\mathbf{v}_0)$. Then $E(\mathbf{v}_0) - E(\mathbf{v}') > d_{\min} > 0$. Let $\bar{\mathbf{x}}_0 = \alpha \mathbf{v}_0$ and $\bar{\mathbf{x}}' = \alpha \mathbf{v}'$. When $\alpha > \alpha_*$, we have that $U(\bar{\mathbf{x}}_0) - U(\bar{\mathbf{x}}') > \alpha_*^2 d_{\min} \geq 3M$. Together with (2.6), we have that $|U(\mathbf{x}_0) - U(\bar{\mathbf{x}}_0)| < M$ and $|U(\mathbf{x}') - U(\bar{\mathbf{x}}')| < M$. It follows that when $\alpha > \alpha_*$,

$$U(\mathbf{x}_0) - U(\mathbf{x}') > M > 0,$$

which contradicts that $\mathbf{x}_0$ is a global minimum point of $U(\mathbf{x})$. Then $\mathbf{v}_0$ is a minimizer of $E(\mathbf{v})$. The proof of Theorem 1.1 is finished. ∎

Next we prove Theorem 1.2.

Proof. In view of assumption (A), we assert that $E(\mathbf{v}') \neq E(\mathbf{v}'')$ if $\mathbf{v}' \neq \pm \mathbf{v}''$.

If not, there exist $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,n})^T$ and $\mathbf{v}_j = (v_{j,1}, \ldots, v_{j,n})^T$ with $\mathbf{v}_i \neq \pm \mathbf{v}_j$ such that $E(\mathbf{v}_i) - E(\mathbf{v}_j) = 0$. Note that there exists some $\mathbf{e} \in \{-1, 0, 1\}^n$,

$$E(\mathbf{v}_i) - E(\mathbf{v}_j) = \frac{1}{2} \sum_{1 \leq k,l \leq n} (-v_{i,k} v_{i,l} + v_{j,k} v_{j,l}) s_{kl} = 2 \left(\sum_{1 \leq k,l \leq n} e_{k,l} s_{kl} \right) = 0, \tag{2.7}$$

where the second equality holds by $(v_{i,k} v_{i,l} - v_{j,k} v_{j,l}) \in \{0, \pm 2\}$. Therefore, we have that (2.7) contradicts to the Assumption (A).

It follows that the Ising energy of candidates can be arranged in ascending order. Without loss of generality, we label all the candidates according to their Ising energy as follows.

$$E(\mathbf{v}_1) = E(\mathbf{v}_2) < E(\mathbf{v}_3) = E(\mathbf{v}_4) < \cdots < E(\mathbf{v}_{N-1}) = E(\mathbf{v}_N), \tag{2.8}$$

where $\mathbf{v}_{2i-1} = -\mathbf{v}_{2i}$ for all $1 \leq i \leq N/2$. Following a similar argument as (2.4), we obtain that

$$U(\bar{\mathbf{x}}_1) = U(\bar{\mathbf{x}}_2) < U(\bar{\mathbf{x}}_3) = U(\bar{\mathbf{x}}_4) < \cdots < U(\bar{\mathbf{x}}_{N-1}) = U(\bar{\mathbf{x}}_N), \tag{2.9}$$

where $\bar{\mathbf{x}}_i = \alpha \mathbf{v}_i$. See Figure 2.1 which is the schematic diagram of the ascending order of Ising energy at candidates and the ascending order of U at its local minimum points. And Figure 2.1 gives an intuitive explanation of the following proof. Also $d_i > 0$ in (2.3) for $1 \leq i \leq N/2$. Then we label local minimum points of $U(\mathbf{x})$ as $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ satisfying (2.5). Without loss of generality, we assume that $U(\mathbf{x}_{2i-1}) \leq U(\mathbf{x}_{2i})$ for all $1 \leq i \leq N/2$. Otherwise, we can interchange the $\mathbf{v}_{2i-1}$ and $\mathbf{v}_{2i}$ in (2.8) such that $U(\mathbf{x}_{2i-1}) \leq U(\mathbf{x}_{2i})$, (2.8) and (2.9) hold simultaneously.

Following a similar argument used in the proof of Theorem 1.1, there exists $\alpha'_* > \alpha_*$ such that for $\alpha > \alpha'_*$, $|U(\bar{\mathbf{x}}_i) - U(\mathbf{x}_i)| < M_i < M$, $U(\bar{\mathbf{x}}_{2i}) - U(\bar{\mathbf{x}}_{2i-2}) = \alpha^2 d_i > 3M$, and $U(\mathbf{x}_{2i-1}) \leq U(\mathbf{x}_{2i})$ for all $1 \leq i \leq N/2$. Then (1.3) holds. By (3) of Corollary 2.2 and $\mathbf{v}_i = \operatorname{sgn}(\bar{\mathbf{x}}_i)$, it follows that $\mathbf{v}_i = \operatorname{sgn}(\mathbf{x}_i)$. Then the proof of Theorem 1.2 is completed. ∎

For any given Ising model, there always exists an ascending order of Ising energy as

$$E(\mathbf{v}_1) = E(\mathbf{v}_2) \leq E(\mathbf{v}_3) = E(\mathbf{v}_4) \leq \cdots \leq E(\mathbf{v}_{N-1}) = E(\mathbf{v}_N),$$

but it is unclear whether there exist only two minimizers of Ising problem and how to label the candidates according to their Ising energy. In fact, Theorem 1.2 still holds under a weaker assumption than (A):

$$E(\mathbf{v}_i) \neq E(\mathbf{v}_j), \quad \forall \mathbf{v}_i \neq \pm \mathbf{v}_j.$$

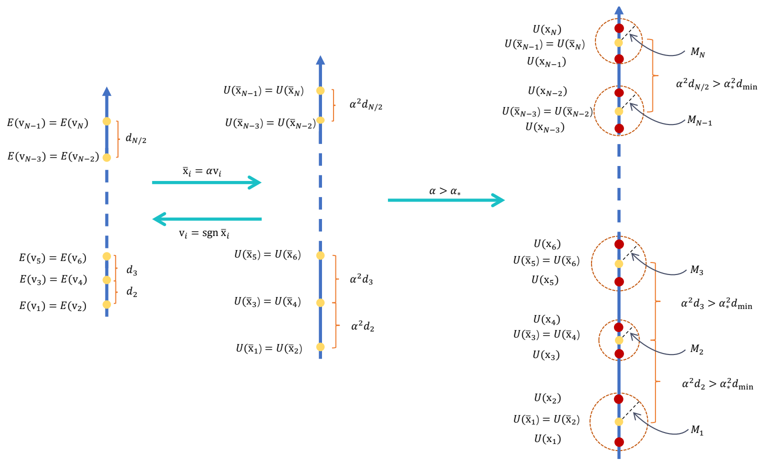


FIG. 2.1. *We use this picture to illustrate the proof of Theorem 1.2. For $\alpha \geq \alpha_*$, the ascending order of Ising energy at candidates is the same as the ascending order of U at its local minimum points.*

One can compute and list the values of U at the local minimum points as in (1.3). Via the signum map, one can label the candidates and obtain the global minimizers of the Ising problem.

2.3. Examples. As an application of Theorem 1.1, we take a look at 2- and 3-spin cases.

EXAMPLE 2.1. *In 2-spin case, the Ising model can be reduced to $E_2(\mathbf{v}) = -\frac{1}{2}\mathbf{v}^T S_2 \mathbf{v}$ with $\mathbf{v} \in \{-1,1\}^2$ and $S_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For the any given $\beta > 0$, when $\alpha > \alpha_* = \sqrt{\beta+2}$, $U(\mathbf{x})$ possesses 4 local minimum points as shown in Table 2.1, where $(\lambda_1, \lambda_1)^T$ and $(-\lambda_1, -\lambda_1)^T$ with $\lambda_1 = \sqrt{\alpha^2 - \beta + 1}$ are global minimum points. Taking $\beta = 2$, we draw the contour plots of $U(x_1, x_2)$ depending on α, which shows that the number of local minimum points varies with the parameter α, see Figure 2.2. Note that $(1,1)^T$ and $(-1,-1)^T$ are the minimizers of E_2. In 3-spin case, take for example $S_3 = \left(\begin{smallmatrix} 0 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 0 \end{smallmatrix}\right)$. The minimizers of $E_3(\mathbf{v}) = -\frac{1}{2}\mathbf{v}^T S_3 \mathbf{v}$ are $(-1,1,1)^T$ and $(1,-1,-1)^T$. It is shown that when $\alpha > 4.6$, the sigum vectors of global minimum points of $U(\mathbf{x})$ are $(-1,1,1)^T$ and $(1,-1,-1)^T$.*

EXAMPLE 2.2. *We provide two examples here to explain Theorem 1.2.*

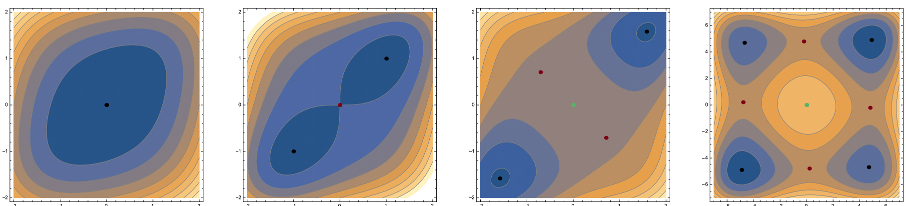
(i) *In the n-spin case, let*

$$S_n = (s_{i,j})_{n\times n} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2^2} & \cdots & -\frac{1}{2^{n-1}} \\ -\frac{1}{2} & 0 & -\frac{1}{2^n} & \cdots & \frac{1}{2^{2n-3}} \\ \frac{1}{2^2} & -\frac{1}{2^n} & 0 & \cdots & \frac{1}{2^{3n-6}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2^{n-1}} & \frac{1}{2^{2n-3}} & \frac{1}{2^{3n-6}} & \cdots & 0 \end{pmatrix}. \tag{2.10}$$

Via direct computations, one can check that the assumption (A) holds for S_n. Then the Ising problem $E(\mathbf{v}) = -\frac{1}{2}\mathbf{v}^T S_n \mathbf{v}$ possesses only a pair of minimizers and the ascending order of the candidates can be given by computing and comparing the corresponding U at its local minimum points. One can further

α	Min	Saddle	Max
$\alpha^2<\beta-1$	$(0,0)^T$	NA	NA
$\alpha^2\in(\beta-1,\beta+1)$	$(\lambda_1,\lambda_1)^T$, $(-\lambda_1,-\lambda_1)^T$	$(0,0)^T$	NA
$\alpha^2\in(\beta+1,\beta+2)$	$(\lambda_1,\lambda_1)^T$, $(-\lambda_1,-\lambda_1)^T$	$(\lambda_2,-\lambda_2)^T$, $(-\lambda_2,\lambda_2)^T$	$(0,0)^T$
$\alpha^2>\beta+2$	$(\lambda_1,\lambda_1)^T$, $(-\lambda_1,-\lambda_1)^T$ $(\lambda_2,-\lambda_2)^T$, $(-\lambda_2,\lambda_2)^T$	$(\lambda_3,-\lambda_4)^T$, $(-\lambda_3,\lambda_4)^T$ $(\lambda_4,-\lambda_3)^T$, $(-\lambda_4,\lambda_3)^T$	$(0,0)^T$

TABLE 2.1. *The critical points of $U(\mathbf{x})$ when $S=S_2$ are given here. If $\alpha>\sqrt{\beta+2}$, both $(\lambda_1,\lambda_1)^T$ and $(-\lambda_1,-\lambda_1)^T$ are the global minimum points and their signum vectors $(-1,-1)^T$ and $(1,1)^T$ minimize the Ising energy E_2. The local properties of the critical points of $U(\mathbf{x})$ change with the increasing of α where $\lambda_1=\sqrt{\alpha^2-\beta+1}$, $\lambda_2=\sqrt{\alpha^2-\beta-1}$, $\lambda_3=\sqrt{(\alpha^2-\beta+\sqrt{(\alpha^2-\beta)^2-4})/2}$, and $\lambda_4=\sqrt{(\alpha^2-\beta-\sqrt{(\alpha^2-\beta)^2-4})/2}$.*



(a) $\alpha=0$ (b) $\alpha=\sqrt{2}$ (c) $\alpha=\sqrt{7/2}$ (d) $\alpha=5$

FIG. 2.2. *The contour plots of $U(\mathbf{x})=\frac{1}{4}(x_1^4+x_2^4)+\frac{2-\alpha^2}{2}(x_1^2+x_2^2)-x_1x_2$ depend on α where $\beta=2$. The black dots are local minimum points; the red dots are saddles; and the green dots are local maximum points. In (a), $(0,0)^T$ is the unique local minimum point. In (b), there are one saddle and two local minimum points. In (c), there are only two saddles, two local minimum points, and a unique local maximum point. In (d), there are four saddles, four local minimum points, and a unique local maximum point. When $\alpha^2>1$, $(\lambda_1,\lambda_1)^T$ and $(-\lambda_1,-\lambda_1)^T$ are the global minimum points as in (b), (c) and (d).*

consider $S_n=(s_{i,j})_{n\times n}$, where $s_{i,j}$ satisfy that

$$\{s_{1,2},s_{1,3},\ldots,s_{1,n},\ldots,s_{n-2,n-1},s_{n-2,n},s_{n-1,n}\}\subset\left\{\pm\frac{1}{2},\pm\frac{1}{2^2},\ldots,\pm\frac{1}{2^{\frac{n(n-1)}{2}}}\right\},$$

and $|s_{i,j}|\neq|s_{i',j'}|$ for all $(i,j)\neq(i',j')$. Namely, if one permutes $s_{i,j}$, the same results still hold.

(ii) *In the 3-spin case, let $S_3'=\begin{pmatrix}0&-1&-1\\-1&0&-1\\-1&-1&0\end{pmatrix}$. One can prove that the assumption (A) does not hold. Via direct computations, $(1,1,-1)$, $(-1,-1,1)$, $(-1,1,-1)$ and $(1,-1,1)$ are four minimizers of the Ising problem. Hence, the minimizers of the Ising problem $E(\mathbf{v})=-\frac{1}{2}\mathbf{v}^TS_3'\mathbf{v}$ are not unique.*

We extend S_n in (2.10) to more general cases as follows.

COROLLARY 2.3. *If $S=(s_{i,j})_{n\times n}$ satisfies that there exists $\rho\in\{k,\frac{1}{k}|k\in\mathbf{Z}\setminus\{0,\pm1\}\}$, such that*

$$\{s_{1,2},s_{1,3},\ldots,s_{1,n},s_{2,3},\ldots,s_{2,n},\ldots,s_{n-2,n-1},s_{n-2,n},s_{n-1,n}\}\subset\left\{\pm\frac{1}{\rho},\pm\frac{1}{\rho^2},\ldots,\pm\frac{1}{\rho^{\frac{n(n-1)}{2}}}\right\},$$

and $|s_{i,j}| \neq |s_{i',j'}|$ *for all* $(i,j) \neq (i',j')$*, then the Ising problem satisfies Assumption (A) and the results of Theorem 1.2 hold.*

3. Revisit some known quantum-inspired algorithms

In this section, we study some known quantum-inspired algorithms of the Ising problem. Using the mathematical principle found in last section, we can prove that these algorithms are valid under some conditions such as CIM proposed in [22, 23], KPO proposed in [5] and SB algorithm proposed in [7]. In the following, we use the original notations in cited papers.

3.1. Coherent Ising machines. The CIM was proposed [22, 23] to find minimizers of Ising problem

$$\min_{\mathbf{v}} E := -\frac{1}{2}\mathbf{v}^T \Xi \mathbf{v}, \tag{3.1}$$

where $\mathbf{v} \in C(E) = \{-1,1\}^n$, $\Xi = (\xi_{ij})_{n\times n}$ is symmetric and $\xi_{ii} = 0$. Via the quantum computation, CIM uses the network of optical parametric oscillators. Each optical parametric oscillator possesses two stable oscillating states above the threshold. In view of this property, it is suitable to represent -1 and 1 in the Ising model. In contrast to the decreasing in annealing, the CIMs obtain the configuration with the lowest the ground state by increasing the gain gradually [12, 22, 23].

The CIM in [22] can be described as

$$\begin{cases} \dot{c}_j = \left(-1 + p - \left(c_j^2 + s_j^2\right)\right) c_j + \sum_{l=1, l\neq j}^n \xi_{jl} c_l, \\ \dot{s}_j = \left(-1 - p - \left(c_j^2 + s_j^2\right)\right) s_j + \sum_{l=1, l\neq j}^n \xi_{jl} s_l, \end{cases} \tag{3.2}$$

where c_j is the normalized in-phase component, s_j is the normalized quadrature component, $p > 1$ is the pump rate and $(\xi_{ij})_{n\times n}$ is as in (3.1). Let p be a constant. If $(\mathbf{c}, \mathbf{s})$ is a classical solution of (3.2), then $(\mathbf{c}, \mathbf{s}) \in C^2(\mathbf{R}, \mathbf{R}^{2n})$. We define the function $U_d \in C^2(\mathbf{R}^{2n}, \mathbf{R})$ by

$$U_d(\mathbf{c}, \mathbf{s}) := \sum_{j=1}^n \left(\frac{1}{4}(c_j^2 + s_j^2)^2 - \frac{p}{2}(c_j^2 - s_j^2) + \frac{1}{2}(c_j^2 + s_j^2) \right) - \frac{1}{2}\mathbf{c}^T \Xi \mathbf{c} - \frac{1}{2}\mathbf{s}^T \Xi \mathbf{s}.$$

Via direct computations, (3.2) can be rewritten as

$$\begin{cases} \dot{c}_j = -\frac{\partial U_d}{\partial c_j}, \\ \dot{s}_j = -\frac{\partial U_d}{\partial s_j}. \end{cases}$$

We further define a function $\tilde{U}_d(\mathbf{c}) \in C^2(\mathbf{R}^n, \mathbf{R})$ by

$$\tilde{U}_d(\mathbf{c}) := U_d(\mathbf{c}, 0) = \sum_{i=1}^n \left(\frac{1}{4} c_i^4 + \frac{1-p}{2} c_i^2 \right) - \frac{1}{2}\mathbf{c}^T \Xi \mathbf{c}.$$

Denote the set of critical points of $U_d(\mathbf{c}, \mathbf{s})$ and the set of critical points of $\tilde{U}_d(\mathbf{c})$ by $\mathcal{C}(U_d)$ and $\mathcal{C}(\tilde{U}_d)$ respectively. Furthermore, define the projection map π_d by

$$\pi_d : \mathcal{C}(U_d) \to \mathcal{C}(\tilde{U}_d), \quad (\mathbf{c}, \mathbf{s}) \mapsto \mathbf{c}.$$

Then we have the following result.

LEMMA 3.1. *For $p>\lambda_\Xi$ where λ_Ξ is the largest eigenvalue of Ξ, if $(\mathbf{c},\mathbf{s})\in\mathcal{C}(U_d)$, then $\mathbf{s}=0$. Moreover, the map π_d is bijective and $i_{U_d}((\mathbf{c},0))=i_{\tilde{U}_d}(\mathbf{c})$.*

Proof. Note that $\nabla U_d=0$ is equivalent to

$$\begin{cases}\left(1-p+\left(c_j^2+s_j^2\right)\right)c_j-\sum_{l=1,l\neq j}^n \xi_{jl}c_l=0,\\ \left(1+p+\left(c_j^2+s_j^2\right)\right)s_j-\sum_{l=1,l\neq j}^n \xi_{jl}s_l=0.\end{cases}$$

Since $p>\lambda_\Xi$, $\nabla U_d=0$ implies $s_j=0$. Therefore, $\nabla U_d=0$ can be reduced to

$$\left(1-p+c_j^2\right)c_j-\sum_{l=1,l\neq j}^n \xi_{jl}c_l=0,\ 1\le j\le n. \tag{3.3}$$

Note that (3.3) is equivalent to $\nabla\tilde{U}_d=0$. It yields that $(\mathbf{c},0)\in\mathcal{C}(U_d)$ if and only if $\mathbf{c}\in\mathcal{C}(\tilde{U}_d)$. Therefore, the map π_d is a bijection between $\mathcal{C}(U_d)$ and $\mathcal{C}(\tilde{U}_d)$.

The Hessian of U_d at the critical point $(\mathbf{c},0)$ is given by

$$\begin{aligned}D^2U_d(\mathbf{c},0)=&\operatorname{diag}\{3C,O_n\}+\operatorname{diag}\{(1-p)I_n,(1+p)I_n\}-\operatorname{diag}\{\Xi,\Xi\}\\ =&\operatorname{diag}\{3C+(1-p)I_n-\Xi,(1+p)I_n-\Xi\},\end{aligned}$$

where $C=\operatorname{diag}\{c_1^2,\dots,c_n^2\}^T$, O_n is an $n\times n$ matrix with all elements being 0 and $\mathbf{c}=(c_1,\dots,c_n)$. By direct computations, $D^2\tilde{U}_d(\mathbf{c})=3C+(1-p)I_n-\Xi$. Note that $(1+p)I_n-\Xi$ is positive definite since $p>\lambda_\Xi$. It follows that $i_{U_d}((\mathbf{c},0))=i_{\tilde{U}_d}(\mathbf{c})$. The proof is complete. ◻

Via Theorem 1.1, we have following result.

PROPOSITION 3.1. *When $p>\max\{\alpha_*^2,\lambda_\Xi\}$, if $(\mathbf{c},0)$ is a global minimum point of U_d, then $\operatorname{sgn}(\mathbf{c})$ minimizes Ising model $E=-\frac{1}{2}\mathbf{v}^T\Xi\mathbf{v}$.*

Proof. By Lemma 3.1 and $U_d(\mathbf{c},0)=\tilde{U}_d(\mathbf{c})$, if $(\mathbf{c},0)$ minimizes $U_d(\mathbf{c},\mathbf{s})$ globally, then $\mathbf{c}$ minimizes $\tilde{U}_d$ globally for $p>\lambda_{\max}$. Assume that the parameters of $U(\mathbf{x})$ in (1.2) satisfy $\alpha=\sqrt{p}$, $\beta=1$ and $S=\Xi$. Then function $U=\tilde{U}_d$. Via Theorem 1.1, it follows that if $(\mathbf{c},0)$ minimizes U_d globally, the signum vector $\operatorname{sgn}(\mathbf{c})$ is a minimizer of Ising model (3.1) for $p>\max\{\alpha_*^2,\lambda_\Xi\}$. ◻

Another CIM, proposed in [23], can also be written in classical variables. Define that

$$U_c(\mathbf{x})=\sum_{i=1}^n\frac{1}{4}x_i^4+\frac{1-p}{2}x_i^2-\epsilon\mathbf{x}^TS_c\mathbf{x},$$

where x_i is the in-phase amplitude. The dynamics of $\mathbf{x}$ is given as follows.

$$\dot{\mathbf{x}}=-\nabla U_c, \tag{3.4}$$

where $p>0$ is the normalized pump rate, ϵ is a small constant with $0<\epsilon\ll 1$ and $S_c=(s_{ij})_{n\times n}$ is symmetric coupling constant matrix with $s_{ii}=0$.

Let $\beta=1$, $\alpha=\sqrt{p}$, and $S=2\epsilon S_c$. The function U_c is the same as $U(\mathbf{x})$ given by (1.2). So minimizing $E=-\frac{1}{2}\mathbf{v}^TS\mathbf{v}$ is equivalent to minimizing $E=-\frac{\epsilon}{2}\mathbf{v}^TS_c\mathbf{v}$. Then we apply Theorem 1.1 directly and obtain the following result.

PROPOSITION 3.2. *For $p>\alpha_*^2$, if $\mathbf{x}$ is a global minimum point of $U_c(\mathbf{x})$, then $\operatorname{sgn}(\mathbf{x})$ is a minimizer of Ising model $E=-\frac{1}{2}\mathbf{x}^TS_c\mathbf{x}$.*

REMARK 3.1. From the mathematical point of view, CIM in (3.2) and (3.4) are both designed to minimize U_d globally via the gradient descent flow. A global minimum point yields a minimizer of the Ising model.

Readers may refer [10, 14] for the large-scale of CIMs. With the help of the Brownian motion in CIMs, these computations show their power on solving the large-scale combinatorial problems.

3.2. Adiabatic Hamiltonian systems. Suppose the Ising problem

$$\min_{\mathbf{v}} E := -\frac{1}{2}\mathbf{v}^T J\mathbf{v}, \tag{3.5}$$

where $\mathbf{v} \in C(E) = \{-1,1\}^n$ and $J = (J_{i,j})_{n\times n}$ is a symmetric matrix with $J_{i,i} = 0$. Another quantum computation Ising machine based on Kerr-nonlinear parametric oscillators (KPO) is first proposed theoretically in [4] and be realized in experiment by superconducting circuits [8]. Instead of threshold and stable oscillations in CIM, KPO mainly uses the detuning and the Kerr effect. It follows that KPO is nondissipative. The nondissipative linear couplings between KPOs is introduced by the coupling coefficients $J_{i,j}$ of the Ising model. This network can be used to find the ground state of the Ising model via quantum adiabatic evolution by increasing the pumping rate gradually. The system can be formulated as the following classical Hamiltonian system

$$\begin{cases} \dot{x}_i = \frac{\partial H_k}{\partial y_i} = \left(p(t) + \Delta + K\left(x_i^2 + y_i^2\right)\right) y_i - \xi_0 \sum_{j=1}^n J_{i,j} y_j, \\ \dot{y}_i = -\frac{\partial H_k}{\partial x_i} = \left(p(t) - \Delta - K\left(x_i^2 + y_i^2\right)\right) x_i + \xi_0 \sum_{j=1}^n J_{i,j} x_j \end{cases}$$

where K is a positive Kerr coefficient, Δ is the detuning, $\xi_0 > 0$ is a constant, $p(t) > 0$ is the parametric pumping rate with $\dot{p} > 0$ and $J_{i,j}$ are the coupling coefficients in (3.5). The corresponding Hamiltonian is

$$\begin{aligned} H_k(\mathbf{x},\mathbf{y},t) = &\sum_{i=1}^n \left(\frac{K}{4}\left(x_i^2 + y_i^2\right)^2 - \frac{p(t)}{2}\left(x_i^2 - y_i^2\right) + \frac{\Delta}{2}\left(x_i^2 + y_i^2\right)\right) \\ &- \frac{\xi_0}{2}\sum_{i=1}^n \sum_{j=1}^n J_{i,j}\left(x_i x_j + y_i y_j\right). \end{aligned}$$

For the problem (3.5), a heuristic algorithm called simulated bifurcation (SB) algorithm is proposed in [7]. It is a nonlinear oscillator network of classical nonlinear Hamiltonian systems. It uses two bifurcations in each non-linear oscillator to represent the two states of the Ising spin. It is a mechanical Hamiltonian system given as follows

$$\begin{cases} \dot{x}_i = \frac{\partial H_s}{\partial y_i} = \Delta y_i, \\ \dot{y}_i = -\frac{\partial H_s}{\partial x_i} = -\left(K x_i^2 + \Delta - p(t)\right) x_i + \xi_0 \sum_{j=1}^n J_{i,j} x_j. \end{cases}$$

The corresponding Hamiltonian is

$$H_s(\mathbf{x},\mathbf{y},t) = \sum_{i=1}^n \frac{\Delta}{2} y_i^2 + \sum_{i=1}^n \left(\frac{K}{4} x_i^4 + \frac{\Delta - p(t)}{2} x_i^2\right) - \frac{\xi_0}{2}\mathbf{x}^T J\mathbf{x},$$

where $K, \Delta, \xi_0 > 0$ are constants, and $p(t) > 0$ is a function with $\dot{p}(t) > 0$. The SB algorithm can be simulated in digitial computers because it is a classical Hamiltonian system.

To apply Theorem 1.1, we define the function $U_h(\mathbf{x})$ which is the potential of H_s as follows

$$U_h(\mathbf{x}) = \sum_{i=i}^{n} \frac{K}{4} x_i^4 + \left(\frac{\Delta - p}{2}\right) x_i^2 - \frac{\xi_0}{2} \mathbf{x}^T J \mathbf{x},$$

where p is a parameter. Define the project maps $\pi_k : C(H_k) \to \mathcal{C}(U_h), (\mathbf{x},\mathbf{y}) \mapsto \mathbf{x}$ and $\pi_s : C(H_s) \to \mathcal{C}(U_h), (\mathbf{x},\mathbf{y}) \mapsto \mathbf{x}$. We first discuss the relationship between $\mathcal{C}(H_k)$ (resp. $\mathcal{C}(H_s)$) of H_k (resp. H_s) and $\mathcal{C}(U_h)$.

LEMMA 3.2. *For $p > \lambda_J$ where λ_J is the largest eigenvalue of $\xi_0 J$, if $(\mathbf{x},\mathbf{y}) \in \mathcal{C}(H_k)$, then $\mathbf{y} = 0$. Furthermore, π_k is a bijection and $i_{H_k}((\mathbf{x},\mathbf{y})) = i_{U_h}(\mathbf{x})$.*

Since the proof of this lemma is similar to the one of Lemma 3.1, we only sketch the proof.

Proof. **(The sketch of proof of Lemma 3.2.)** For $p > \lambda_J$, the critical point $(\mathbf{x},\mathbf{y})$ of $\nabla H_k = 0$ satisfies that $\mathbf{y} = 0$ and $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the root of

$$-\left(p - \Delta - K x_i^2\right) x_i + \xi_0 \sum_{j=1}^{n} J_{i,j} x_j = 0, \quad 1 \le i \le n.$$

Therefore, if $(\mathbf{x},\mathbf{y}) \in \mathcal{C}(H_k)$, we have that $\dot{x} = \left(p + \Delta + K\left(x_i^2 + y_i^2\right)\right) y_i - \xi_0 \sum_{j=1}^{n} J_{i,j} y_j = 0$. Since $p > \xi_0 J$, $pI - \xi_0 J$ is positive definite. Moreover, the diagonal matrix $\mathrm{diag}(\Delta + K(x_1^2 + y_1^2), \ldots, \Delta + K(x_n^2 + y_n^2))$ is always a positive definite matrix. It follows that if $(\mathbf{x},\mathbf{y}) \in \mathcal{C}(H_k)$, then $\mathbf{y} = 0$ and $\mathbf{x} \in \mathcal{C}(U_h)$, and vice versa. It follows that π_h is a bijection.

Note that the Hessian of H_k at the critical point $(\mathbf{x}, 0)$ is given by

$$D^2 H_k(\mathbf{x}, 0) = \mathrm{diag}\{3KX + (\Delta - p) I_n - \xi_0 J, (\Delta + p) I_n - \xi_0 J\},$$

where $X = \mathrm{diag}\{x_1^2, \ldots, x_n^2\}$. Note that $(\Delta + p) I_n - \xi_0 J$ is positive definite for $p > \lambda_J$. It follows that $i_{H_k}((\mathbf{x},\mathbf{y})) = i_{U_h}(\mathbf{x})$. This lemma follows. ∎

PROPOSITION 3.3. *For $p > \max\{\alpha_*^2, \lambda_J\}$, if $(\mathbf{x},\mathbf{y})$ is a global minimum point of $H_k(\mathbf{x},\mathbf{y})$, then $\mathrm{sgn}(\mathbf{x})$ minimizes the Ising model (3.5).*

Proof. By Lemma 3.1, if $(\mathbf{x}, 0) \in \mathcal{C}(H_k)$, then $H_k(\mathbf{x}, 0) = U_h(\mathbf{x})$. Therefore, if $(\mathbf{x}, 0)$ minimizes H_k globally, then x minimizes U_h globally. Via re-scaling of x and Theorem 1.1, the signum vector of the global minimum of U_h is a minimizer of the Ising model $E(\mathbf{v}) = -\frac{1}{2}\mathbf{v}^T J \mathbf{v}$. It follows that if $\mathbf{x}$ minimizes H_k globally, $\mathrm{sgn}(\mathbf{x})$ is a minimizer of Ising model. ∎

Similary, the following results for SB algorithm hold.

PROPOSITION 3.4. *For $p > \alpha_*^2$, if $(\mathbf{x},\mathbf{y}) \in \mathcal{C}(H_s)$, then $\mathbf{y} = 0$. Moreover, the map π_s is a well-defined bijection and $i_{H_s}((\mathbf{x},\mathbf{y})) = i_{U_h}(\mathbf{x})$. If $(\mathbf{x},\mathbf{y})$ is a global minimum point of $H_s(\mathbf{x},\mathbf{y})$, then $\mathrm{sgn}(\mathbf{x})$ minimizes the Ising model (3.5).*

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