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HOMOGENIZATION OF A TRANSMISSION PROBLEM WITH SIGN-CHANGING COEFFICIENTS AND INTERFACIAL FLUX JUMP*

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Abstract. We study the homogenization of a scalar problem posed in a composite medium made up of two materials, a positive and a negative one. An important feature is the presence of a flux jump across their oscillating interface. The main difficulties of this study are due to the sign-changing coefficients and the appearance of an unsigned surface integral term in the variational formulation. A proof by contradiction (nonstandard in this context) and T−coercivity technics are used in order to cope with these difficulties.

Keywords. Negative materials; transmission problem; homogenization; imperfect interfaces; flux jump.

AMS subject classifications. 35B27; 80M35; 80M40; 35Q60; 78M35.

1. Introduction

In this paper, we study the homogenization of a class of scalar problems stated in a composite medium made up of two ε-periodically distributed materials, a positive and a negative one. Here, ε is a small real parameter, related to the characteristic dimension of our domain. There are two important features of the problem under study: the sign-changing coefficients of the two materials and the presence of a flux jump across their interface. Negative materials are encountered in optics in the framework of metamaterials, which are composite structures displaying unusual properties (see, e.g., [27,28]). Such materials can exhibit, over some frequency ranges, negative dielectric permittivity and magnetic permeability, leading to a negative index of refraction.

Our goal in this paper is to analyze the limit behaviour, as $\varepsilon \to 0$, of problem (2.3), in which we notice the continuity of the solution and the presence of a flux jump on the oscillating interface between the two constituents. More precisely, the flux jump, given in $(2.3)_{(iv)}$, depends on the solution u^ε, on a jump function r^ε and on a scaling factor ε^γ, with γ a real parameter. To the best of our knowledge, this is the first mathematical study of a scalar problem involving flux jump at the interface between a positive material and a negative metamaterial. Let us emphasize that, as pointed out in [22, Chapter 4], there are applications for which the use of such more involved transmission conditions is relevant in the positive-negative case. For scalar problems posed in composites with two positive materials, the continuity of the solution and the flux jump across the interface correspond to physical and chemical phenomena (we refer the reader to [2–4, 21, 23]).

In the non-elliptic case studied here, the main difficulties consist in proving the well-posedness of both the microscopic and macroscopic problems, as well as of the local ones. This is due to the sign-changing coefficients and to the presence, in the variational formulations, of an unsigned surface integral term, coming from the special form of the imposed flux jump. To cope with the lack of coercivity due to the

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sign-changing coefficients, we apply T−coercivity technics, which can be seen as a variation of Lax-Milgram lemma. The T−coercivity was first used in [9] for the study of problems with sign-changing coefficients. Since then, such problems have been studied in several contexts. For instance, we can refer the reader interested in the analysis of indefinite problems to [6] for scalar transmission problems, [18] for Helmholtz type problems, [7] for Maxwell's system, [24] for a non linear sign-changing problem, [16] for the study of scattering resonances and [8, 17] for numerical aspects. In the context of the homogenization for perfect transmission problems with sign-changing coefficients, the T−coercivity has been used in [12–14] for scalar diffusion problems and in [11] for Maxwell's system.

In order to overcome the difficulties generated by the presence of the unsigned term, we use, for the well-posedness and the homogenization result, a proof by contradiction similar to the one in [11]. More precisely, two cases which are relevant are studied, namely $\gamma=1$ and $\gamma=0$ in problem (2.3). It is worth mentioning that some of our proofs highly use results from [11] concerning the homogenization of scalar problems with sign-changing coefficients, in the case of perfect interfaces (without jump fluxes). Nevertheless, due to the presence of jumps at the interfaces, here the proof of convergence and the obtained limit problems are quite different from those in [11]. In particular, an extra zero-order term appears in the homogenized problem (see (4.3) and (5.3)), to keep memory of the integral term on the imperfect interface between the two materials. For each case, we first establish the homogenization result, by assuming a uniform energy estimate condition. Next, we prove by contradiction that this uniform energy estimate condition is indeed satisfied, under some assumptions on the data. These assumptions are related to the contrast κ between the two conductivities (see (2.2)), that should be large or small enough, and to the special properties of the jump function r. For elliptic problems, such a function r was already encountered in perforated domains in [5] and in two-component domains in [15]. Following the value of γ, its effect at the limit is seen in the homogenized problems and in the special form of the corrector (see Theorems 4.1 and 5.1).

The paper is organized as follows. In Section 2, we introduce the main notation and set the problem. In Section 3, we recall some results concerning the homogenization of non-elliptic perfect transmission problems that will be used in this paper. Sections 4 and 5 are devoted to the proof of the homogenization results in the case of weak coupling ($\gamma=1$ in problem (2.3)) and, respectively, strong coupling ($\gamma=0$ in problem (2.3)). We give in Section 6 some concluding remarks and perspectives. Finally, we collect in the Appendix the definitions of the periodic unfolding operators and their main properties used throughout the paper (see [19]).

2. Setting of the problem

Let $\Omega\subset\mathbb{R}^N$ $(N\geqslant 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and denote by $Y=(0,1)^N$ the reference cell in $\mathbb{R}^N$. We suppose that Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2\subset Y$ and $Y=Y_1\cup\overline{Y}_2$. We also assume that $\Gamma=\partial Y_2$ is Lipschitz continuous. For each $\mathbf{k}\in\mathbb{Z}^N$, we set $Y^{\mathbf{k}}=\mathbf{k}+Y$, $Y_p^{\mathbf{k}}=\mathbf{k}+Y_p$, for $p\in\{1,2\}$, and $\Gamma^{\mathbf{k}}=\mathbf{k}+\Gamma$.

Let $\varepsilon\in(0,1)$ be a sequence of strictly real positive numbers, such that $\varepsilon^{-1}\in\mathbb{N}^*$. The small parameter ε is related to the characteristic dimension of the spatial variations in our domain. We define, for each $p\in\{1,2\}$, the following sets (see Figure 2.1):

$$\Omega_p^\varepsilon=\{x\in\Omega\,|\,x\in\varepsilon Y_p^{\mathbf{k}},\mathbf{k}\in\mathbb{Z}^N\},\quad \Gamma^\varepsilon=\{x\in\Omega\,|\,x\in\varepsilon\Gamma^{\mathbf{k}},\mathbf{k}\in\mathbb{Z}^N\}.$$

We denote by n^ε the unit outward normal to Ω_2^ε. This choice of the domain Ω is made

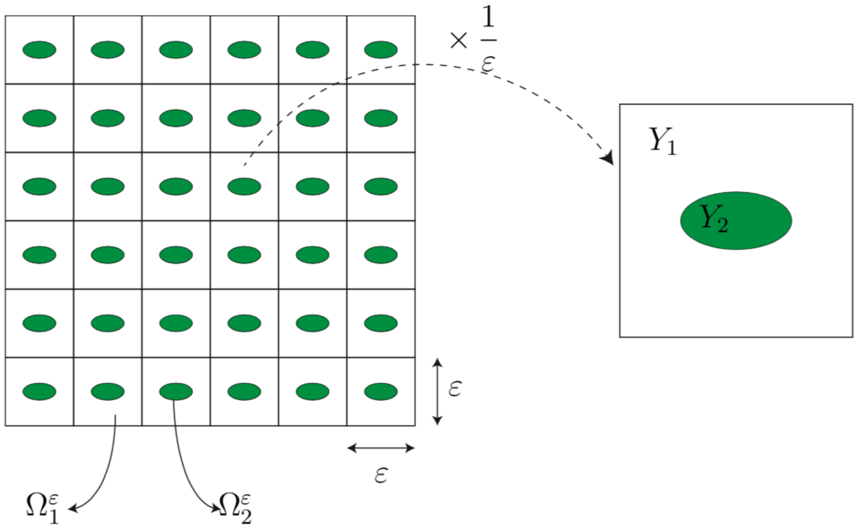

FIG. 2.1. *The composite periodic material and the corresponding reference cell Y.*

without loss of generality. Indeed, all the results contained in this paper hold true for any bounded open set Ω with Lipschitz continuous boundary $\partial\Omega$.

We consider two real constants a_1 and a_2 such that $a_1 a_2 < 0$. Let $a \in L^\infty(\mathbb{R}^N)$ denote the function given by

$$a(y) = a_1 \mathbf{1}_{Y_1}(y) + a_2 \mathbf{1}_{Y_2}(y)$$

and extended by $Y-$periodicity to the whole of $\mathbb{R}^N$ ($\mathbf{1}_{\mathcal{O}}$ denotes the characteristic function of a set $\mathcal{O}$). Without loss of generality, we can suppose that

$$a_1 > 0, \quad a_2 < 0 \tag{2.1}$$

and we define the contrast κ as being the positive number

$$\kappa = \left| \frac{a_1}{a_2} \right| = \frac{a_1}{|a_2|}. \tag{2.2}$$

We set

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \qquad \text{a.e. in } \Omega.$$

For every function v defined on Ω, we set

$$v_1 := v|_{\Omega_1^\varepsilon}, \qquad v_2 := v|_{\Omega_2^\varepsilon}.$$

Similarly, for every function v defined on the reference cell Y, we set

$$v_1 := v|_{Y_1}, \qquad v_2 := v|_{Y_2}.$$

Let

$$r^\varepsilon(x) = r\left(\frac{x}{\varepsilon}\right), \qquad \text{a.e. on } \Gamma^\varepsilon,$$

where r is a $Y-$periodic function in $L^\infty(\Gamma)$. Given $f\in L^2(\Omega)$ and a real number γ, our goal is to analyze the asymptotic behavior, as $\varepsilon\to 0$, of the solution u^ε of the problem:

$$\begin{cases} -a_1\Delta u_1^\varepsilon = f & \text{in } \Omega_1^\varepsilon,\\ -a_2\Delta u_2^\varepsilon = f & \text{in } \Omega_2^\varepsilon,\\ u_1^\varepsilon - u_2^\varepsilon = 0 & \text{on } \Gamma^\varepsilon,\\ a_1\nabla u_1^\varepsilon\cdot n^\varepsilon - a_2\nabla u_2^\varepsilon\cdot n^\varepsilon = \varepsilon^\gamma r^\varepsilon u^\varepsilon & \text{on } \Gamma^\varepsilon,\\ u_1^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here, the positive and the negative materials occupy the domains Ω_1^ε and Ω_2^ε, respectively. Across their common boundary Γ^ε, we notice the continuity of the solution and the presence of a flux jump. The flux jump, given in $(2.3)_{(iv)}$, depends on the solution u^ε, on the jump function r^ε and on a scaling factor ε^γ.

Our goal is to study problem (2.3) for two relevant values of the real parameter γ, namely $\gamma=1$ and $\gamma=0$. Indeed, in these two cases one keeps track of the flux jump in the limit.

The variational formulation of problem (2.3) is the following one: find $u^\varepsilon\in H_0^1(\Omega)$ such that

$$\mathcal{A}^\varepsilon(u^\varepsilon,v) = \ell(v), \quad \forall v\in H_0^1(\Omega), \tag{2.4}$$

where the bilinear form $\mathcal{A}^\varepsilon : H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$ and the linear form $\ell : H_0^1(\Omega)\to\mathbb{R}$ are given by

$$\mathcal{A}^\varepsilon(u,v) = \int_\Omega a^\varepsilon \nabla u\cdot\nabla v\,\mathrm{d}x + \varepsilon^\gamma\int_{\Gamma^\varepsilon} r^\varepsilon uv\,\mathrm{d}\sigma_x \tag{2.5}$$

and

$$\ell(v) = \int_\Omega fv\,\mathrm{d}x,$$

respectively. Here, the space $H_0^1(\Omega)$ is classically endowed with the norm $\|\nabla u\|_{L^2(\Omega)}$. The variational formulation (2.4) can be equivalently written in the abstract operator form

$$\mathbb{A}^\varepsilon u^\varepsilon = F,$$

where $\mathbb{A}^\varepsilon\in\mathcal{L}(H_0^1(\Omega))$ is given by

$$\mathbb{A}^\varepsilon := \mathbb{A}_0^\varepsilon + \mathbb{K}^\varepsilon,$$

in which the operators $\mathbb{A}_0^\varepsilon\in\mathcal{L}(H_0^1(\Omega))$ and $\mathbb{K}^\varepsilon\in\mathcal{L}(H_0^1(\Omega))$ are defined by

$$(\nabla(\mathbb{A}_0^\varepsilon u),\nabla v)_{L^2(\Omega)} := \int_\Omega a^\varepsilon\nabla u\cdot\nabla v\,\mathrm{d}x, \qquad \forall u,v\in H_0^1(\Omega), \tag{2.6}$$

$$(\nabla(\mathbb{K}^\varepsilon u),\nabla v)_{L^2(\Omega)} := \varepsilon^\gamma\int_{\Gamma^\varepsilon} r^\varepsilon uv\,\mathrm{d}\sigma_x, \qquad \forall u,v\in H_0^1(\Omega), \tag{2.7}$$

and the right-hand side F is such that

$$(\nabla F,\nabla v)_{L^2(\Omega)} = \ell(v), \qquad \forall v\in H_0^1(\Omega).$$

It is worth noticing that the operator $\mathbb{A}_0^\varepsilon$ corresponds to the case without jump in the flux (the normal trace $a\nabla u\cdot n$ being continuous through Γ^ε). The well-posedness of the problem corresponding to the operator $\mathbb{A}_0^\varepsilon$ has been first studied in [12] and then in [10] and [11]. For reader's convenience, we recall in Section 3 some results corresponding to this case that will be used in what follows.

For any bounded open set D of $\mathbb{R}^N$, we denote by $H^1_{\rm per}(D)$ the closure of $\mathscr{C}^\infty_{\rm per}(\overline{D})$ for the norm of $H^1(D)$, where $\mathscr{C}^\infty_{\rm per}(\overline{D})$ is the subset of functions of $\mathscr{C}^\infty(\overline{D})$ satisfying periodic boundary conditions on ∂D. We introduce the space $H^1_{{\rm per},\diamond}$ defined by

$$H^1_{{\rm per},\diamond}(D):=\left\{\varphi\in H^1_{\rm per}(D)\,\Big|\,\int_D\varphi\,{\rm d}y=0\right\}.$$

We also define

$$\mathcal{M}_D v(\cdot)=\frac{1}{|D|}\int_D v(\cdot,y)\,{\rm d}y,\quad \forall v\in L^1(\Omega\times D)$$

and we set

$$y_{\mathcal{M}}=y-\mathcal{M}_Y(y).$$

Throughout the paper, C denotes a positive constant, independent of ε, whose value can change from line to line.

3. Background on the homogenization of non-elliptic perfect transmission problems

In this section, we collect some results concerning the homogenization of the operator $\mathbb{A}_0^\varepsilon$ defined by (2.6), which corresponds to the case of a perfect transmission condition across the interface Γ^ε. In other words, for a given f in $L^2(\Omega)$, we are interested in the asymptotic behavior of solutions $v^\varepsilon\in H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases}-{\rm div}\,(a^\varepsilon\nabla v^\varepsilon)=f & \text{in } \Omega,\\ v^\varepsilon=0 & \text{on } \partial\Omega.\end{cases}\tag{3.1}$$

In the elliptic case (i.e. when $a_1>0$ and $a_2>0$), it is well-known (see, for instance, [25] or [1, Section 2]) that (3.1) admits a unique solution $v^\varepsilon\in H^1_0(\Omega)$ that weakly converges in $H^1_0(\Omega)$ to the unique solution $v\in H^1_0(\Omega)$ of the homogenized problem

$$\begin{cases}-{\rm div}\left(a^{\rm hom}\nabla v\right)=f & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\end{cases}\tag{3.2}$$

where the entries of the homogenized matrix $a^{\rm hom}=(a^{\rm hom}_{ij})_{1\leqslant i,j\leqslant N}$ are given by

$$a^{\rm hom}_{ij}=\int_Y a(y)\nabla\left(\chi^i+y_i\right)\cdot\nabla\left(\chi^j+y_j\right)\,{\rm d}y.\tag{3.3}$$

Here, the cell functions $\chi^j\in H^1_{{\rm per},\diamond}(Y)$, $1\leqslant j\leqslant N$, solve the variational problem

$$\int_Y a(y)\nabla\chi^j\cdot\nabla\chi'\,{\rm d}y=(a_1-a_2)\int_\Gamma(e_j\cdot n)\chi'\,{\rm d}\sigma_y\qquad\forall\chi'\in H^1_{{\rm per},\diamond}(Y).\tag{3.4}$$

Note that χ^j is the unique solution in $H^1_{{\rm per},\diamond}(Y)$ of the second order equation

$$\begin{cases}-{\rm div}_y\left(a(y)(\nabla_y\chi^j+e_j)\right)=0 & \text{in } Y,\\ \mathcal{M}_Y(\chi^j)=0,\end{cases}$$

or equivalently, of the following transmission problem:

$$\begin{cases} -\Delta\chi_1^j = 0 & \text{in } Y_1, \\ -\Delta\chi_2^j = 0 & \text{in } Y_2, \\ \chi_1^j - \chi_2^j = 0 & \text{on } \Gamma, \\ a_1\partial_n\chi_1^j - a_2\partial_n\chi_2^j = -(a_1-a_2)(e_j\cdot n) & \text{on } \Gamma, \\ \mathcal{M}_Y(\chi^j) = 0, \end{cases} \tag{3.5}$$

where χ_1^j and χ_2^j denote the restrictions of χ^j to Y_1 and Y_2, e_j stands for the j^{th} vector of the canonical basis of $\mathbb{R}^N$ and n is the outgoing unit normal to ∂Y_2.

In the case of sign-changing coefficients studied here (see (2.1)), the bilinear forms associated with the cell problems and the microscopic problem become indefinite and the homogenization requires the development of new techniques. It has been shown in [10–12] that the above homogenization results are still valid provided that the contrast κ between the two materials (see (2.2)) is small or large enough. For reader's convenience, we collect these results in the following proposition.

PROPOSITION 3.1. *Assume that $a_1>0$ and $a_2<0$ and let $\kappa = \frac{a_1}{|a_2|}$. Then, there exist two positive constants κ_Y and κ'_Y (depending only on the geometry of Y_2) such that, for all*

$$0<\kappa<1/\kappa'_Y \qquad \textit{or} \qquad \kappa>\kappa_Y, \tag{3.6}$$

the following assertions hold true:

(1) *The operator $\mathbb{A}_0^\varepsilon$ is uniformly $\mathbb{T}-$coercive as ε tends to zero, i.e. there exists ε^* such that for $\varepsilon\in(0,\varepsilon^*)$ and a family of invertible operators $\mathbb{T}^\varepsilon$ of $\mathcal{L}(H_0^1(\Omega))$, there exists $\alpha>0$ independent of ε such that:*

$$\left|(\nabla(\mathbb{A}_0^\varepsilon(\mathbb{T}^\varepsilon u)),\nabla u)_{L^2(\Omega)}\right| = \left|\int_\Omega a^\varepsilon\nabla(\mathbb{T}^\varepsilon u)\cdot\nabla u\,\mathrm{d}x\right| \geqslant \alpha\|\nabla u\|_{L^2(\Omega)}, \tag{3.7}$$

for all $u\in H_0^1(\Omega)$. In particular, $\mathbb{A}_0^\varepsilon$ is uniformly invertible as ε tends to zero, or equivalently, there exists ε^* such that, for $\varepsilon\in(0,\varepsilon^*)$, problem (3.1) admits a unique solution $v^\varepsilon\in H_0^1(\Omega)$ and

$$\|\nabla v^\varepsilon\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)},$$

for some constant C independent of ε.

(2) *For every $g\in L^2(\Gamma)$, the variational cell problem*

$$\int_Y a(y)\nabla\chi^j\cdot\nabla\chi'\,\mathrm{d}y = \int_\Gamma g\chi'\,\mathrm{d}\sigma_y, \qquad \forall\chi'\in H^1_{\mathrm{per},\diamond}(Y), \tag{3.8}$$

is well-posed. In particular, the cell problems (3.4), corresponding to the right-hand sides $g=(a_1-a_2)(e_j\cdot n)$, are well-posed and the homogenized matrix (3.3) is thus well-defined.

(3) *The homogenized matrix (3.3) is positive definite and, hence, the homogenized problem (3.2) admits a unique solution $v\in H_0^1(\Omega)$.*

(4) *The sequence (v^ε) converges to v weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$.*

Obviously, the Lax-Milgram lemma cannot be applied to prove existence results for problems with sign-changing coefficients. The T−coercivity method is a slight variation of Lax-Milgram lemma, in which the main idea is to check the coercivity of the bilinear form $a(T\cdot,\cdot)$ instead of $a(\cdot,\cdot)$ for some isomorphism T. This method turns out to be a quite efficient alternative to handle sign-changing problems, by choosing an isomorphism T that changes the sign of the solution in the subdomains where the coefficients are negative (and leaving it unchanged in the "positive subdomains"). Of course, this cannot be always done in such a rough way. In order to also ensure the continuity of the traces at the interfaces and to remain in H^1, we may be led to introduce some correcting terms.

The results of Proposition 3.1 have been first obtained in [12] using the T−coercivity method, in the case of large contrasts only. Then, they have been proved for small and large contrasts in [10] through the analysis of the spectrum of the Neumann-Poincaré operator. Studying a similar homogenization problem for Maxwell's system, the authors of [11] investigated the above scalar problem and obtained the precise values of the bounds κ_Y and κ'_Y through a variational approach (these bounds are related to the constants m and M of Theorem 3.14 in [11]). Let us emphasize that these results have been also obtained for small and large contrasts using the T−coercivity method when studying the homogenization of a scalar problem in thin periodic domains (see [13]).

More precisely, assertions 1 and 2 are proved in [13] using two distinct T−coercivity operators depending on the value (large or small) of the contrast (see Proposition 3.8 and Proposition 3.10 for assertion 1 and Theorem 3.6 for assertion 2). They can also be found in Theorem 3.14 and Theorem 4.1 of [11] where the proof is based on variational arguments and different T−coercivity operators. The proof of assertion 3 can be found in [10, Corollary 5.1] or [11, Proposition 4.2]. Finally, assertion 4 is proved in [10, Proposition 5.5.].

It is worth noticing that the third assertion of Proposition 3.1 implies, in particular, that under assumption (3.6), the operator $A^{\rm hom}$ defined by

$$A^{\rm hom}u := -\operatorname{div}(a^{\rm hom}\nabla u), \qquad \forall u\in D(A^{\rm hom}),$$

where $D(A^{\rm hom}) := \{u\in H^1_0(\Omega)\,|\,\operatorname{div}(a^{\rm hom}\nabla u)\in L^2(\Omega)\}$, is a positive and selfadjoint operator with compact resolvent. Thus, it is diagonalizable with an orthonormal basis of eigenfunctions associated with a sequence of strictly positive eigenvalues $(\lambda_n^{\rm hom})_{n\geqslant 1}$ tending to $+\infty$. In the sequel, we set

$$\Lambda^{\rm hom} := \{\lambda_n^{\rm hom},\ n\geqslant 1\}. \tag{3.9}$$

It is also worth mentioning that the first assertion of Proposition 3.1 implies that, for extreme contrasts (i.e. if condition (3.6) is satisfied), $\mathbb{A}^\varepsilon = \mathbb{A}_0^\varepsilon + \mathbb{K}^\varepsilon$ is a Fredholm operator.

Corollary 3.1. *Assume that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6). Then, there exists ε^* such that for every (fixed) $\varepsilon\in(0,\varepsilon^*)$, $\mathbb{A}^\varepsilon = \mathbb{A}_0^\varepsilon + \mathbb{K}^\varepsilon$ is a Fredholm operator of index 0.*

Proof. The result follows immediately from the invertibility of $\mathbb{A}_0^\varepsilon$, according to the first assertion in Proposition 3.1, and the fact that $\mathbb{K}^\varepsilon\in\mathcal{L}(H^1_0(\Omega))$ is a compact operator. The last assertion on $\mathbb{K}^\varepsilon$ is a consequence of its definition (2.7), due to the compactness of the trace operator for fixed ε. ∎

We conclude this section with some comments concerning the case of "moderate" contrasts (i.e. for $\kappa\in(1/\kappa'_Y,\kappa_Y)$, $\kappa\neq 1$). The analysis turns out to be much more complicated and leads to several open questions, even in the case of perfect transmission conditions (continuity of the traces and fluxes). First of all, the homogenized matrix is probably not anymore elliptic, as it can be expected from the numerical experiments performed in [11] (see Figures 4.1 and 4.2 therein). Hence, the well-posedness of the homogenized problem probably fails. Secondly, for every *fixed value* ε, the microscopic problem is well-posed except for a countable set of critical values of the contrast converging to 1 (see [11, Section 3.4.]). However, this countable set of critical values changes with ε, and it is not clear at all if there exists a segment of $(1/\kappa'_Y, \kappa_Y)$ which is uniformly free of such critical values as ε tends to 0.

4. Case 1: Weak coupling

This section deals with problem (2.3) in the case where $\gamma = 1$. The main result is stated in Theorem 4.1. We notice in the homogenized problem (4.3) the presence of an extra-term of order zero, coming from the properly scaled flux jump function r. The proof of Theorem 4.1 is done by a nonstandard argument. More precisely, we start by establishing in Proposition 4.1 the homogenization result, assuming the uniform energy estimate (4.8). Next, we prove by contradiction in Proposition 4.2 that this uniform energy estimate is indeed satisfied, under suitable assumptions on the data.

4.1. Main result. In the Appendix, we recall, following [19], the definitions and the main properties for the periodic unfolding operators. These operators will be used in the homogenization process. According to Chapter 1 of [19], one has the following classical convergence results.

LEMMA 4.1. *Let (u^ε) be a sequence in $H^1_0(\Omega)$ satisfying the uniform energy estimate*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C.$$

Then, there exist $u \in H^1_0(\Omega)$ and $\widehat{u} \in L^2(\Omega, H^1_{per}(Y))$, with $\mathcal{M}_Y(\widehat{u}) = 0$, such that, up to a subsequence, we have the following convergence results as $\varepsilon \to 0$:

$$\begin{cases} \mathcal{T}^\varepsilon(u^\varepsilon) \to u \quad \text{strongly in } L^2(\Omega, H^1(Y)), \\ \mathcal{T}^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y), \\ \dfrac{\mathcal{T}^\varepsilon(u^\varepsilon) - \mathcal{M}_Y(\mathcal{T}^\varepsilon(u^\varepsilon))}{\varepsilon} \rightharpoonup y_{\mathcal{M}} \cdot \nabla u + \widehat{u} \quad \text{weakly in } L^2(\Omega, H^1(Y)). \end{cases} \tag{4.1}$$

We can now state the main result of this section.

THEOREM 4.1. *Assume that $\gamma = 1$ in (2.3) and that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6). Assume also that*

$$-(r, \mathbb{1})_{L^2(\Gamma)} \notin \Lambda^{\text{hom}}, \tag{4.2}$$

where Λ^{hom} is the spectrum defined in (3.9).

Then, there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, problem (2.3) (or, equivalently, the weak formulation (2.4)) admits a unique solution u^ε. Moreover, u^ε converges, in the sense of (4.1), to $(u, \widehat{u}) \in H^1_0(\Omega) \times L^2(\Omega, H^1_{\text{per},\diamond}(Y))$, where u is the unique solution of the well-posed homogenized problem

$$\begin{cases} -\text{div}(a^{\text{hom}} \nabla u) + (r, \mathbb{1})_{L^2(\Gamma)} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

and

$$\widehat{u}(x,y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi^j(y). \tag{4.4}$$

Here, a^{hom} is the homogenized matrix, given, for $i,j=1,\dots,N$, by

$$a_{ij}^{\text{hom}} = \int_Y a(y)\nabla\left(\chi^i + y_i\right)\cdot\nabla\left(\chi^j + y_j\right)\,\mathrm{d}y, \tag{4.5}$$

in terms of the cell functions $\chi^j \in H^1_{\text{per}}(Y)$ $(j=1,\dots,N)$, solving the following well-posed cell problems:

$$\begin{cases} -\mathrm{div}_y\left(a(y)(\nabla_y\chi^j + e_j)\right) = 0, & \text{in } Y, \\ \mathcal{M}_Y(\chi^j) = 0, \end{cases} \tag{4.6}$$

which read equivalently

$$\begin{cases} -\Delta\chi_1^j = 0 & \text{in } Y_1, \\ -\Delta\chi_2^j = 0 & \text{in } Y_2, \\ \chi_1^j - \chi_2^j = 0 & \text{on } \Gamma, \\ a_1\partial_n\chi_1^j - a_2\partial_n\chi_2^j = -(a_1 - a_2)(e_j\cdot n) & \text{on } \Gamma, \\ \mathcal{M}_Y(\chi^j) = 0. \end{cases} \tag{4.7}$$

REMARK 4.1. Let us note that if $(r,\mathbb{1})_{L^2(\Gamma)} \geqslant 0$, then condition (4.2) is automatically satisfied, since $\Lambda^{hom} \subset (0,+\infty)$.

REMARK 4.2. Note that the cell problems (4.7) coincide with those obtained for $r \equiv 0$, namely problems (3.5).

Theorem 4.1 follows from the results given in Proposition 4.1 and Proposition 4.2, respectively, which are proved in the next two subsections.

4.2. Proof of the homogenization result under uniform energy estimate condition.

PROPOSITION 4.1. *Assume that $\gamma = 1$ in (2.3) and that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6). Let (u^ε) be a sequence of solutions of problem (2.3) (or, equivalently, of the weak formulation (2.4)) satisfying the uniform estimate*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C. \tag{4.8}$$

Then, the following properties hold true:

(1) *The cell problems (4.7) are well-posed and the homogenized coefficients (4.5) are well-defined.*

(2) *The sequence (u^ε) converges, in the sense of (4.1), to $(u,\widehat{u}) \in H^1_0(\Omega)\times L^2(\Omega, H^1_{\text{per},\diamond})$, where u solves (4.3) and $\widehat{u}$ is given by (4.4).*

Proof.

(1) This statement follows from the second assertion in Proposition 3.1.

(2) The passage to the limit in the variational formulation (2.4) is done by using the periodic unfolding method (see, for instance, [19] and the references therein). We recall in the Appendix the results from [19] that we use in the paper.

In order to get the limit problem (4.3), we unfold the variational formulation (2.4) and, then, we take in the unfolded problem the admissible test function

$$v = \varphi(x) + \varepsilon\omega(x)\psi\left(\frac{x}{\varepsilon}\right), \tag{4.9}$$

with φ, $\omega \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\mathrm{per}}(Y)$. The following convergences hold true:

$$\mathcal{T}^\varepsilon(v) \to \varphi \quad \text{strongly in } L^2(\Omega\times Y), \tag{4.10}$$

$$\mathcal{T}^\varepsilon(\nabla v) \to \nabla\varphi + \nabla_y \Phi \quad \text{strongly in } L^2(\Omega\times Y), \tag{4.11}$$

where $\Phi(x,y) = \omega(x)\psi(y)$. The passage to the limit with $\varepsilon \to 0$ in the unfolded formulation of problem (2.4) is standard, by using convergences (4.1) and (4.10)-(4.11). In this way, we are led to

$$\int_{\Omega\times Y} a(y)(\nabla u + \nabla_y \widehat{u})\cdot(\nabla\varphi + \nabla_y\Phi)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} r(y)u\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega\times Y} f\varphi\,\mathrm{d}x\,\mathrm{d}y, \tag{4.12}$$

which holds, by standard density arguments, for all $\varphi \in H^1_0(\Omega)$ and $\Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$.

By taking successively in (4.12) $\varphi = 0$ and, respectively, $\Phi = 0$, we get

$$\int_{\Omega\times Y} a(y)(\nabla u + \nabla_y \widehat{u})\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{4.13}$$

and

$$\int_{\Omega\times Y} a(y)(\nabla u + \nabla_y \widehat{u})\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} r(y)u\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega\times Y} f\varphi\,\mathrm{d}x\,\mathrm{d}y. \tag{4.14}$$

Using now the factorization (4.4) and (4.6), we obtain immediately the homogenized problem (4.3), with the homogenized coefficients given by (4.5). □

REMARK 4.3. The result of Proposition 4.1 can be easily generalized to the case where the right-hand side in (2.3) is a sequence (f_n) strongly converging to some limit f in $L^2(\Omega)$.

4.3. Proof of the uniform energy estimate.

PROPOSITION 4.2. *Assume that* $\gamma = 1$ *and that the contrast* $\kappa = \dfrac{a_1}{|a_2|}$ *satisfies* (3.6). *Assume also that*

$$-(r,\mathbb{1})_{L^2(\Gamma)} \notin \Lambda^{\mathrm{hom}},$$

where Λ^{hom} *is the spectrum defined in* (3.9).

Then, there exists ε^* *such that for all* $\varepsilon \in (0,\varepsilon^*)$, *problem* (2.3) *(or, equivalently, the weak formulation (2.4)) admits a unique solution* u^ε. *Moreover, the uniform estimate*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C \tag{4.15}$$

holds for all $\varepsilon \in (0,\varepsilon^*)$.

Proof. Assume by contradiction that there exists a sequence $\varepsilon_n \to 0$ such that problem (2.3) is not well-posed for $\varepsilon = \varepsilon_n$. According to Corollary 3.1 and the Fredholm alternative, the well-posedness of (2.3) is equivalent to the question of the uniqueness of its solution. Hence, there exists a sequence denoted (u_n) of non-zero solutions of (2.3) for $\varepsilon = \varepsilon_n$ with $f = 0$. We can suppose without loss of generality that $\|\nabla u_n\|_{L^2(\Omega)} = 1$. Using the equivalent weak formulation (2.4) of (2.3), we have thus

$$\|\nabla u_n\|_{L^2(\Omega)} = 1, \qquad \mathcal{A}^{\varepsilon_n}(u_n, v) = 0, \quad \forall v \in H_0^1(\Omega). \tag{4.16}$$

Similarly, if we assume by contradiction that (4.15) is not satisfied, we obtain the existence of a sequence (u_n) in $H_0^1(\Omega)$ and a sequence of right-hand sides (f_n) in $L^2(\Omega)$ such that

$$\begin{aligned} &\|\nabla u_n\|_{L^2(\Omega)} = 1, \qquad \|f_n\|_{L^2(\Omega)} \to 0, \\ &\mathcal{A}^{\varepsilon_n}(u_n, v) = \int_\Omega f_n v \,\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{4.17}$$

Since (4.16) can be seen as a particular case of (4.17) (corresponding to a null sequence (f_n)), the proposition will be proved if we can derive a contradiction when assuming (4.17). Hence, from now on, let us assume that (u_n) and (f_n) satisfy (4.17). According to Proposition 4.1 and Remark 4.3, we can pass to the limit in (4.17) and obtain that u_n weakly converges in $H_0^1(\Omega)$ (and strongly in $L^2(\Omega)$) to the solution u of the following problem

$$\begin{cases} -\mathrm{div}\,(a^{\mathrm{hom}} \nabla u) + (r, \mathbb{1})_{L^2(\Gamma)}\, u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

Since $-(r, \mathbb{1})_{L^2(\Gamma)} \notin \Lambda^{\mathrm{hom}}$, we have $u = 0$.

On the other hand, using the definition (2.5) of the bilinear form $\mathcal{A}^{\varepsilon_n}(\cdot,\cdot)$, (4.17), we get

$$\int_\Omega a^{\varepsilon_n} \nabla u_n \cdot \nabla v \,\mathrm{d}x = \int_\Omega f_n v \,\mathrm{d}x - \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n v \,\mathrm{d}\sigma_x.$$

Choosing $v = \mathbb{T}^{\varepsilon_n} u_n$ in the above relation, where $\mathbb{T}^\varepsilon$ is the T$-$coercivity operator from Proposition 3.1, we obtain from (3.7) that:

$$\alpha \|\nabla u_n\|^2_{L^2(\Omega)} \leqslant \left| \int_\Omega f_n \mathbb{T}^{\varepsilon_n} u_n \,\mathrm{d}x \right| + \left| \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n \mathbb{T}^{\varepsilon_n} u_n \,\mathrm{d}\sigma_x \right|. \tag{4.19}$$

Below, we need to deal separately with the cases of large and small contrasts.

The case of large contrasts: $\kappa > \kappa_Y$.

In this case, the operator $\mathbb{T}^{\varepsilon_n}$ is the one defined in Proposition 3.8, [13], and it verifies

$$(\mathbb{T}^{\varepsilon_n} u_n)|_{\Gamma^{\varepsilon_n}} = u_n.$$

Using its value in the last term of (4.19), we obtain that

$$\alpha \|\nabla u_n\|^2_{L^2(\Omega)} \leqslant \left| \int_\Omega f_n \mathbb{T}^{\varepsilon_n} u_n \,\mathrm{d}x \right| + \left| \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} |u_n|^2 \,\mathrm{d}\sigma_x \right|. \tag{4.20}$$

Let us prove that the two terms of the right-hand side of (4.20) tend to zero, which will provide the desired contradiction, since $\|\nabla u_n\|_{L^2(\Omega)} = 1$ and $\alpha > 0$.

Concerning the first term $\int_\Omega f_n \mathbb{T}^{\varepsilon_n} u_n \, dx$, it is enough to notice that (f_n) converges to 0 strongly in $L^2(\Omega)$ and that $(\mathbb{T}^{\varepsilon_n} u_n)$ is bounded in $H_0^1(\Omega)$, since $\mathbb{T}^{\varepsilon_n}$ is uniformly bounded in $\mathcal{L}(H_0^1(\Omega))$.

Concerning the second term, we note that, by unfolding, according with Proposition A.1$_{(iii)}$, it can be written as follows:

$$\varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}(x) |u_n(x)|^2 \, \mathrm{d}\sigma_x = \int_{\Omega\times\Gamma} r(y) |\mathcal{T}^{\varepsilon_n}(u_n)(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma_y.$$

Using in the above equality the continuity of the trace operator on Γ, the identity $\nabla_y(\mathcal{T}^\varepsilon u) = \varepsilon \mathcal{T}^\varepsilon(\nabla u)$ (see Proposition A.1$_{(iv)}$), and the boundedness of the function r, we get

$$\left| \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}(x) |u_n(x)|^2 \, \mathrm{d}\sigma_x \right| \leqslant C \left\{ \|\mathcal{T}^{\varepsilon_n}(u_n)\|^2_{L^2(\Omega\times Y)} + \varepsilon_n^2 \|\mathcal{T}^{\varepsilon_n}(\nabla u_n)\|^2_{L^2(\Omega\times Y)} \right\}.$$

Since $\|u_n\|_{L^2(\Omega)} \to 0$, it follows from Proposition A.2$_{(ii)}$ that $\|\mathcal{T}^{\varepsilon_n}(u_n)\|_{L^2(\Omega\times Y)}$ tends to zero. On the other hand, since $\|\nabla u_n\|_{L^2(\Omega)} = 1$, it follows from Proposition A.1$_{(i)}$ and A.1$_{(ii)}$ that $\|\mathcal{T}^{\varepsilon_n}(\nabla u_n)\|_{L^2(\Omega\times Y)} = 1$. Hence,

$$\lim_{n\to+\infty} \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}(x) |u_n(x)|^2 \, \mathrm{d}\sigma_x = 0 \tag{4.21}$$

and the proof is complete for large contrasts.

The case of small contrasts: $0 < \kappa < 1/\kappa'_Y$. In this case, the operator $\mathbb{T}^{\varepsilon_n}$ is the one defined in Proposition 3.10, [13], and it verifies

$$(\mathbb{T}^{\varepsilon_n} u_n)|_{\Gamma^{\varepsilon_n}} = -u_n + 2\mathcal{M}_2^{\varepsilon_n}(u_n),$$

where $\mathcal{M}_2^{\varepsilon_n}(u_n)$ is the function, piecewise constant on each cell $\varepsilon Y_2^{\mathbf{k}}$, defined on Ω_2^ε by:

$$\mathcal{M}_2^{\varepsilon_n}(u_n)(x) := \frac{1}{|\varepsilon_n Y_2^{\mathbf{k}}|} \int_{\varepsilon_n Y_2^{\mathbf{k}}} u_n \, \mathrm{d}x, \qquad \forall x \in \varepsilon Y_2^{\mathbf{k}}.$$

Using its value, the last term of (4.19) reads as follows

$$\begin{aligned} \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n \mathbb{T}^{\varepsilon_n} u_n \, \mathrm{d}\sigma_x &= \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n \left(-u_n + 2\mathcal{M}_2^{\varepsilon_n}(u_n)\right) \mathrm{d}\sigma_x \\ &= \varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} |u_n|^2 \, \mathrm{d}\sigma_x - 2\varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n \left(u_n - \mathcal{M}_2^{\varepsilon_n}(u_n)\right) \mathrm{d}\sigma_x. \end{aligned}$$

Since the first term of the right-hand side in the above equation tends to zero according to (4.21), it suffices to prove that the second one also tends to zero. Using Proposition A.1$_{(iii)}$, we get that:

$$\begin{aligned} &\varepsilon_n \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n \left(u_n - \mathcal{M}_2^{\varepsilon_n}(u_n)\right) \mathrm{d}\sigma_x \\ &= \int_{\Omega\times\Gamma} r(y) \mathcal{T}^{\varepsilon_n}(u_n) \left(\mathcal{T}^{\varepsilon_n}(u_n) - \mathcal{M}_{Y_2}(\mathcal{T}^{\varepsilon_n}(u_n))\right) \mathrm{d}x \, \mathrm{d}\sigma_y. \end{aligned} \tag{4.22}$$

One can prove as above (in the case of large contrasts) that

$$\mathcal{T}^{\varepsilon_n}(u_n) \longrightarrow 0 \qquad \text{in } L^2(\Omega\times\Gamma). \tag{4.23}$$

Setting as in [20]

$$U_n := \frac{\mathcal{T}^{\varepsilon_n}(u_n) - \mathcal{M}_{Y_2}(\mathcal{T}^{\varepsilon_n}(u_n))}{\varepsilon_n},$$

we note that $U_n \in L^2(\Omega, H^1(Y_2))$, since $u_n \in H^1_0(\Omega)$. By the Poincaré-Wirtinger inequality applied in the domain Y_2, we have

$$\begin{aligned} \|U_n\|_{L^2(\Omega\times Y_2)} &= \left\| \frac{\mathcal{T}^{\varepsilon_n}(u_n) - \mathcal{M}_{Y_2}(\mathcal{T}^{\varepsilon_n}(u_n))}{\varepsilon_n} \right\|_{L^2(\Omega\times Y_2)} \\ &\leqslant C \left\| \frac{\nabla_y \mathcal{T}^{\varepsilon_n}(u_n)}{\varepsilon_n} \right\|_{L^2(\Omega\times Y_2)} \\ &= C \|\mathcal{T}^{\varepsilon_n}(\nabla u_n)\|_{L^2(\Omega\times Y_2)} \\ &\leqslant C. \end{aligned}$$

Moreover, using the definition of U_n and Proposition A.1$_{(iv)}$, we get

$$\|\nabla U_n\|_{L^2(\Omega\times Y_2)} = \left\| \frac{1}{\varepsilon_n} \nabla_y \mathcal{T}^{\varepsilon_n}(u_n) \right\|_{L^2(\Omega\times Y_2)} = \|\mathcal{T}^{\varepsilon_n}(\nabla u_n)\|_{L^2(\Omega\times Y_2)} \leqslant C.$$

Consequently, by the trace inequality,

$$\mathcal{T}^{\varepsilon_n}(u_n) - \mathcal{M}_{Y_2}(\mathcal{T}^{\varepsilon_n}(u_n)) = \varepsilon_n U_n \longrightarrow 0 \qquad \text{in } L^2(\Omega\times\Gamma). \tag{4.24}$$

Plugging (4.23) and (4.24) in (4.22), we obtain the desired result. ∎

Remark 4.4. Let us remark that Theorem 4.1 also holds true in the elliptic case, namely when both a_1 and a_2 are positive. In this case, the result holds true for any value of the contrast κ.

5. Case 2: Strong coupling

This section deals with problem (2.3) in the case $\gamma = 0$, under the additional assumption that $\mathcal{M}_\Gamma(r) = 0$. The main result is stated in Theorem 5.1. We first remark the presence of ξ, a scalar function which is the unique solution of the additional cell problem (5.1). This function ξ appears, as expected, in the corrector (5.4) and gives rise in the homogenized problem (5.3) to an extra-term of order zero, namely $(r,\xi)_{L^2(\Gamma)} u$. The proof of Theorem 5.1 is done by using the same type of nonstandard argument as in Section 4.

5.1. Main result.

Theorem 5.1. *Assume that $\gamma = 0$ in (2.3) and that $\mathcal{M}_\Gamma(r) = 0$. Assume also that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6). Then, there exists a unique Y-periodic scalar function ξ solution of the following well-posed cell problem:*

$$\begin{cases} -\Delta \xi_1 = 0 & \text{in } Y_1, \\ -\Delta \xi_2 = 0 & \text{in } Y_2, \\ \xi_1 - \xi_2 = 0 & \text{on } \Gamma, \\ a_1 \partial_n \xi_1 - a_2 \partial_n \xi_2 = r(y) & \text{on } \Gamma, \\ \mathcal{M}_Y(\xi) = 0. \end{cases} \tag{5.1}$$

If we assume in addition that

$$-(r,\xi)_{L^2(\Gamma)} \notin \Lambda^{\text{hom}}, \tag{5.2}$$

where Λ^{hom} is the spectrum defined in (3.9), then there exists ε^* such that for all $\varepsilon \in (0,\varepsilon^*)$, problem (2.3) (or, equivalently, the weak formulation (2.4)) admits a unique solution u^ε. Furthermore, (u^ε) converges, in the sense of (4.1), to $(u,\widehat{u}) \in H_0^1(\Omega)\times L^2(\Omega, H^1_{per,\diamond}(Y))$, where u is the unique solution of the well-posed homogenized problem

$$\begin{cases} -\mathrm{div}\,(a^{\mathrm{hom}}\nabla u) + (r,\xi)_{L^2(\Gamma)}\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.3}$$

and

$$\widehat{u}(x,y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\chi^j(y) + \xi(y)u(x). \tag{5.4}$$

Here, a^{hom} is the homogenized matrix defined in (4.5)-(4.6).

REMARK 5.1. According to (5.1), one has

$$\int_Y a(y)\nabla\xi\cdot\nabla\xi'\,\mathrm{d}y = -\int_\Gamma r\xi'\,\mathrm{d}\sigma_y \qquad \forall \xi' \in H^1_{\mathrm{per},\diamond}(Y).$$

We first remark that in the elliptic case (i.e. when both a_1 and a_2 are positive), by taking $\xi' = \xi$ in the above relation, we get $(r,\xi)_{L^2(\Gamma)} < 0$. In the non-elliptic case studied here, the same choice of test function leads to a term with indefinite sign. Thus, we need to use a T$-$coercivity argument in the cell Y. More precisely, we use the results proved in [13] (see Proposition 3.3 for $\kappa > \kappa_Y$ and Proposition 3.5 for $\kappa < 1/\kappa'_Y$ therein). First of all, for $\kappa > \kappa_Y$, there exists an operator $\mathbf{T}_Y \in \mathcal{L}(H^1_{\mathrm{per},\diamond}(Y))$ such that:

$$\alpha\|\nabla\xi\|^2_{L^2(Y)} \leqslant \int_Y a(y)\nabla\xi\cdot\nabla(\mathbf{T}_Y\xi)\,\mathrm{d}y = -\int_\Gamma r\,\mathbf{T}_Y\xi\,\mathrm{d}\sigma_y.$$

Using the fact that, on Γ, $\mathbf{T}_Y\xi$ is equal to ξ up to an additive constant, we obtain that $(r,\xi)_{L^2(\Gamma)} < 0$, since $\mathcal{M}_\Gamma(r) = 0$. Hence, assumption (5.2) is needed to ensure the well-posedness of the homogenized problem (5.3) for large contrasts. On the contrary, condition (5.2) is always satisfied for small contrasts since, in this case, $\mathbf{T}_Y\xi$ is equal on Γ to $-\xi$ up to an additive constant, and, hence, one has $(r,\xi)_{L^2(\Gamma)} > 0$.

5.2. Proof of the homogenization result under uniform energy estimate condition.

PROPOSITION 5.1. *Assume that $\gamma = 0$ in (2.3) and that $\mathcal{M}_\Gamma(r) = 0$. Assume that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6) and let (u^ε) be a sequence of solutions of problem (2.3) (or, equivalently, of the weak formulation (2.4)) satisfying the uniform estimate*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C. \tag{5.5}$$

Then, the following properties hold true:

(1) *The cell problems (4.7) and (5.1) are well-posed.*

(2) *(u^ε) converges, in the sense of (4.1), to $(u,\widehat{u}) \in H_0^1(\Omega)\times L^2(\Omega, H^1_{\mathrm{per},\diamond}(Y))$ defined as in Theorem 5.1.*

Proof.

(1) The well-posedness of the cell problems (4.7) and (5.1) for contrasts satisfying (3.6) follows from the second assertion in Proposition 3.1, applied for $g=(a_1-a_2)(e_j\cdot n)$ and for $g=-r$, respectively.

(2) The passage to the limit in the variational formulation (2.4) is done by using the periodic unfolding method. We start by unfolding this variational formulation. Using Proposition A.1, we obtain the unfolded problem

$$\int_{\Omega\times Y}\mathcal{T}^\varepsilon(A^\varepsilon)\mathcal{T}^\varepsilon(\nabla u^\varepsilon)\cdot\mathcal{T}^\varepsilon(\nabla v)\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{\varepsilon}\int_{\Omega\times\Gamma}r(y)\mathcal{T}_b^\varepsilon(u^\varepsilon)\mathcal{T}_b^\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}\sigma_y$$
$$=\int_{\Omega\times Y}\mathcal{T}^\varepsilon(f)\mathcal{T}^\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}y,$$

for all $v\in H_0^1(\Omega)$.

Choosing in the unfolded problem the test function (4.9) and using convergences (4.1) and (4.10)-(4.11), one can pass to the limit with $\varepsilon\to 0$. The only term which needs more attention is the one involving the function r. One has the following convergence result:

$$\frac{1}{\varepsilon}\int_{\Omega\times\Gamma}r(y)\mathcal{T}_b^\varepsilon(u^\varepsilon)\mathcal{T}_b^\varepsilon(v)\,\mathrm{d}x\,\mathrm{d}\sigma_y\longrightarrow\int_{\Omega\times\Gamma}r(y)\widehat{u}(x,y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\sigma_y$$
$$+\int_{\Omega\times\Gamma}r(y)u(x)\omega(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}\sigma_y,\quad(5.6)$$

whose proof uses in an essential way the fact that the mean value of r over Γ is zero (see [26] and [15]).

Passing to the limit in the unfolded problem and using standard density arguments, we obtain that for all $\varphi\in H_0^1(\Omega)$ and $\Phi\in L^2(\Omega,H^1_{\mathrm{per}}(Y))$ one has:

$$\int_{\Omega\times Y}a(y)(\nabla u+\nabla_y\widehat{u})\cdot(\nabla\varphi+\nabla_y\Phi)\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega\times\Gamma}r(y)\widehat{u}\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y+\int_{\Omega\times\Gamma}r(y)u\Phi\,\mathrm{d}x\,\mathrm{d}\sigma_y$$
$$=\int_{\Omega\times Y}f\varphi\,\mathrm{d}x\,\mathrm{d}y.$$

By taking successively in the above relation $\varphi=0$ and $\Phi=0$, we get

$$\int_{\Omega\times Y}a(y)(\nabla u+\nabla_y\widehat{u})\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega\times\Gamma}r(y)u\Phi\,\mathrm{d}x\,\mathrm{d}\sigma_y=0\quad(5.7)$$

and

$$\int_{\Omega\times Y}a(y)(\nabla u+\nabla_y\widehat{u})\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega\times\Gamma}r(y)\widehat{u}\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y=\int_{\Omega\times Y}f\varphi\,\mathrm{d}x\,\mathrm{d}y.\quad(5.8)$$

Using now the factorization (5.4), the cell problems (4.6) and (5.1) and the compatibility condition $(i=1,\dots,N)$

$$\int_Y a(y)\frac{\partial\xi}{\partial y_i}(y)\,\mathrm{d}y+\int_\Gamma r(y)\chi^i(y)\,\mathrm{d}\sigma_y=0,$$

we obtain as in [5] and [15] the homogenized problem (5.3), with the homogenized coefficients given by (4.5). ∎

Remark 5.2. The result of Proposition 5.1 can be easily generalized to the case where the right-hand side is a sequence (f_n) strongly converging to some limit f in $L^2(\Omega)$.

5.3. Proof of the uniform energy estimate.

PROPOSITION 5.2. *Assume that* $\gamma = 0$ *in* (2.3) *and that* $\mathcal{M}_\Gamma(r) = 0$. *Assume also that the contrast* $\kappa = \dfrac{a_1}{|a_2|}$ *satisfies* (3.6) *and that*

$$-(r,\xi)_{L^2(\Gamma)} \notin \Lambda^{\mathrm{hom}},$$

where ξ *solves the cell problem* (5.1) *and* Λ^{hom} *is the spectrum defined in* (3.9).

Then, there exists ε^* *such that, for all* $\varepsilon \in (0,\varepsilon^*)$, *problem* (2.3) *(or, equivalently, the weak formulation (2.4)) admits a unique solution* u^ε. *Moreover, the uniform estimate*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leqslant C \tag{5.9}$$

holds for all $\varepsilon \in (0,\varepsilon^*)$.

Proof. The proof goes along the same lines as the one of Proposition 4.2, but it is slightly more technical. Let us point out the main differences. Proceeding by contradiction, as in Proposition 4.2, we obtain the existence of a sequence $\varepsilon_n \to 0$ and of two corresponding sequences (u_n) in $H^1_0(\Omega)$ and (f_n) in $L^2(\Omega)$ such that:

$$\begin{gathered} \|\nabla u_n\|_{L^2(\Omega)} = 1, \qquad \|f_n\|_{L^2(\Omega)} \to 0, \\ \int_\Omega a^{\varepsilon_n} \nabla u_n \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} u_n v \,\mathrm{d}\sigma_x = \int_\Omega f_n v \,\mathrm{d}x, \quad \forall v \in H^1_0(\Omega). \end{gathered} \tag{5.10}$$

Moreover, by assumption $-(r,\xi)_{L^2(\Gamma)} \notin \Lambda^{\mathrm{hom}}$ and, according to the homogenization result of Proposition 5.1 and Remark 5.2, we can prove that u_n converges to 0 weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. In addition, the two-scale convergence results (4.1) and the trace theorem imply that

$$\mathcal{T}^{\varepsilon_n}(u_n) \to 0 \quad \text{strongly in } L^2(\Omega, H^1(Y)) \tag{5.11}$$

and

$$\frac{\mathcal{T}^{\varepsilon_n}(u_n) - M_n}{\varepsilon} \ \text{ is bounded in } L^2(\Omega \times \Gamma), \tag{5.12}$$

where M_n is the function of the macroscopic variable defined by:

$$M_n(x) := \mathcal{M}_Y(\mathcal{T}^{\varepsilon_n}(u_n)(x,y)) = \frac{1}{|Y|}\int_Y \mathcal{T}^{\varepsilon_n}(u_n)(x,y)\,\mathrm{d}y.$$

As in the proof of Proposition 4.2, we choose in (5.10) the test function $v = \mathbb{T}^{\varepsilon_n} u_n$, and we consider separately the cases of large and small contrasts.

The case of large contrasts: $\kappa > \kappa_Y$. In this case, we have $\mathbb{T}^{\varepsilon_n} u_n = u_n$ on Γ^{ε_n} and the use of the coercivity result (3.7) in (5.10) yields

$$\alpha \|\nabla u_n\|^2_{L^2(\Omega)} \leqslant \left| \int_\Omega f_n \mathbb{T}^{\varepsilon_n} u_n \,\mathrm{d}x \right| + \left| \int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n} |u_n|^2 \,\mathrm{d}\sigma_x \right|.$$

Since u_n is normalized in $H^1_0(\Omega)$ and since the first term of the right-hand side of the above relation tends to zero (as in the proof of Proposition 4.2), the desired contradiction will be obtained if we show that the term $\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}|u_n|^2\,\mathrm{d}\sigma_x$ tends to zero. To do so, we rewrite this term as follows:

$$\begin{aligned}\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}|u_n|^2\,\mathrm{d}\sigma_x &= \frac{1}{\varepsilon_n}\int_{\Omega\times\Gamma} r(y)|\mathcal{T}^{\varepsilon_n}(u_n)|^2\,\mathrm{d}x\,\mathrm{d}\sigma_y\\ &= \frac{1}{\varepsilon_n}\int_{\Omega\times\Gamma} r(y)\,[\mathcal{T}^{\varepsilon_n}(u_n)-M_n]\,\mathcal{T}^{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\ &\quad+\frac{1}{\varepsilon_n}\int_{\Omega\times\Gamma} r(y)M_n\mathcal{T}^{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\ &= \int_{\Omega\times\Gamma} r(y)\,\frac{\mathcal{T}^{\varepsilon_n}(u_n)-M_n}{\varepsilon_n}\,\mathcal{T}^{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\ &\quad+\int_{\Omega\times\Gamma} r(y)\,\frac{\mathcal{T}^{\varepsilon_n}(u_n)-M_n}{\varepsilon_n}\,M_n\,\mathrm{d}x\,\mathrm{d}\sigma_y+\frac{1}{\varepsilon_n}\int_{\Omega\times\Gamma} r(y)M_n^2\,\mathrm{d}x\,\mathrm{d}\sigma_y.\end{aligned}$$

Since r has zero mean value on Γ and M_n depends only on x, the last term in the above relation vanishes. Moreover, due to (5.12), we have

$$\left|\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}|u_n|^2\,\mathrm{d}\sigma_x\right|\leqslant C\|r\|_{L^\infty(\Gamma)}\left(\|\mathcal{T}^{\varepsilon_n}(u_n)\|_{L^2(\Omega\times\Gamma)}+\|M_n\|_{L^2(\Omega\times\Gamma)}\right).$$

The left-hand side tends to zero when ε tends to zero. Indeed, by using the same arguments as in Proposition 4.2, we have

$$\|\mathcal{T}^{\varepsilon_n}(u_n)\|_{L^2(\Omega\times\Gamma)}\leqslant C\left\{\|\mathcal{T}^{\varepsilon_n}(u_n)\|^2_{L^2(\Omega\times Y)}+\varepsilon_n^2\|\mathcal{T}^{\varepsilon_n}(\nabla u_n)\|^2_{L^2(\Omega\times Y)}\right\}.$$

Finally, the last term $\|M_n\|_{L^2(\Omega\times\Gamma)}=|\Gamma|^{\frac{1}{2}}\|M_n\|_{L^2(\Omega)}$ also tends to zero by using statement (i) of Proposition 1.25 in [19], applied to $v_\varepsilon=u_n$.

The case of small contrasts: $0<\kappa<1/\kappa'_Y$. In this case, we have on Γ^{ε_n}

$$\mathbb{T}^{\varepsilon_n}u_n=-u_n+2\mathcal{M}_2^{\varepsilon_n}(u_n).$$

The use of the T$-$coercivity result (3.7) in (5.10) implies that

$$\begin{aligned}\alpha\|\nabla u_n\|^2_{L^2(\Omega)} &\leqslant \left|\int_\Omega f_n\mathbb{T}^{\varepsilon_n}u_n\,\mathrm{d}x\right|+\left|\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}u_n(-u_n+2\mathcal{M}_2^{\varepsilon_n}(u_n))\,\mathrm{d}\sigma_x\right|\\ &\leqslant \left|\int_\Omega f_n\mathbb{T}^{\varepsilon_n}u_n\,\mathrm{d}x\right|+\left|\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}|u_n|^2\,\mathrm{d}\sigma_x\right|\\ &\qquad+2\left|\int_{\Gamma^{\varepsilon_n}} r^{\varepsilon_n}u_n\,(u_n-\mathcal{M}_2^{\varepsilon_n}(u_n))\,\mathrm{d}\sigma_x\right|.\end{aligned}$$

The two first terms above have been already proved to converge to zero, the computations being the same as those detailed above for large contrasts. Following exactly the same arguments as in the proof of Proposition 4.2 for the case of small contrasts, we show that the last term also converges to zero and the proof is completed. ∎

6. Conclusion

In this paper, we studied the homogenization of a class of scalar problems stated in a composite medium made up of two ε-periodically distributed materials with different signs, in the presence of a flux jump at the interface. Two typical limit cases were considered: the case of weak coupling ($\gamma=1$) and the one of strong coupling ($\gamma=0$). The proof is achieved by contradiction, taking advantage of the fact that the homogenized problems are of Fredholm type.

The same technique used here might also work for the class of problems

$$\begin{cases} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) + \beta u^\varepsilon = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where a^ε is as in the present paper and β is a given real number. Indeed, proceeding by contradiction like we did in this paper, one can prove that the solution u^ε of problem (6.1) converges weakly in $H_0^1(\Omega)$ to the unique solution u of the homogenized problem

$$\begin{cases} -\operatorname{div}\left(a^{\mathrm{hom}} \nabla u\right) + \beta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This convergence holds true provided that the contrast $\kappa = \dfrac{a_1}{|a_2|}$ satisfies (3.6) and that $-\beta \notin \Lambda^{\mathrm{hom}}$, where Λ^{hom} is the spectrum defined in (3.9).

In [14], we treated a non-elliptic double-porosity problem of the form (6.1) (with $a_1 = 1$ and $a_2 = -\varepsilon^2$), only for $\beta = 0$. The case $\beta > 0$, which was left open there (see Remark 3.4, problem (3.10)), as well as the case $\beta < 0$, might also be solved by applying the approach used in this paper.

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Appendix. We recall here the definitions of the periodic unfolding operators and some of their properties. We refer the reader to the monograph of Ciorănescu, Damlamian and Griso [19].

Without loss of generality, we suppose that the domain Ω is exactly covered by an entire number of cells $\varepsilon Y^{\mathbf{k}} = \varepsilon(\mathbf{k} + Y)$ ($\mathbf{k} \in \mathbb{Z}^N$). Moreover, we omit writing the measure of Y, since it equals 1 if $Y = (0,1)^N$.

For $x \in \mathbb{R}^N$, we denote by $[x]$ its integer part in $\mathbb{Z}^N$, such that $x - [x] \in Y$, where $Y = (0,1)^N$. We set $\{x\} = x - [x]$ for $x \in \mathbb{R}^N$. In particular, for a. e. $x \in \mathbb{R}^N$ and every $\varepsilon > 0$, one has

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon \left\{\frac{x}{\varepsilon}\right\}. \tag{A.1}$$

Definition A.1 (Unfolding Operators).

(i) *For every function φ Lebesgue measurable in Ω, the periodic unfolding operator $\mathcal{T}^\varepsilon$ is defined by*

$$\mathcal{T}^\varepsilon(\varphi)(x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \textit{for } (x,y) \in \Omega \times Y.$$

(ii) *For every function φ Lebesgue measurable on Γ^ε, the boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by*

$$\mathcal{T}_b^\varepsilon(\varphi)(x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \textit{for } (x,y) \in \Omega \times \Gamma.$$

We remark that the operators $\mathcal{T}^\varepsilon$ and $\mathcal{T}_b^\varepsilon$ are linear and continuous. We list below some of their properties, that we use throughout the paper.

PROPOSITION A.1.

(i) *If* $v \in L^1(\Omega)$, *then*

$$\int_\Omega v(x)\,\mathrm{d}x = \int_{\Omega\times Y} \mathcal{T}^\varepsilon(v)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

(ii) *If* v,w *are Lebesgue measurable functions, then*

$$\mathcal{T}^\varepsilon(vw) = \mathcal{T}^\varepsilon(v)\mathcal{T}^\varepsilon(w).$$

(iii) *If* $v \in L^1(\Gamma^\varepsilon)$, *then*

$$\int_{\Gamma^\varepsilon} v(x)\,\mathrm{d}\sigma_x = \frac{1}{\varepsilon}\int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(v)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

(iv) *If* $v \in H^1(\Omega)$, *then* $\mathcal{T}^\varepsilon(\nabla v) \in L^2(\Omega, H^1(Y))$ *and*

$$\mathcal{T}^\varepsilon(\nabla v) = \frac{1}{\varepsilon}\nabla_y(\mathcal{T}^\varepsilon(\nabla v)).$$

(v) *Let* w *be Lebesgue measurable in* Y, *extended by* Y*-periodicity to the whole space* $\mathbb{R}^N$ *and define the sequence* $w^\varepsilon(x) = w\left(\frac{x}{\varepsilon}\right)$. *Then,*

$$\mathcal{T}^\varepsilon(w^\varepsilon|_\Omega)(x,y) = w(y).$$

(vi) *If* $v \in H^1(\Omega)$, *then*

$$\mathcal{T}_b^\varepsilon(v) = \mathcal{T}^\varepsilon(v)|_{\Omega\times\Gamma}.$$

(vii) *If* $w \in L^p(\Gamma)$, $p \geqslant 1$, *is* Y*-periodic and* $w^\varepsilon(x) = w\left(\frac{x}{\varepsilon}\right)$, *then*

$$\mathcal{T}_b^\varepsilon(w^\epsilon)(x) = \mathcal{T}_b^\varepsilon(w)\left(\frac{x}{\varepsilon}\right) = w(y).$$

The next proposition recalls the main general convergence results used in this paper.

PROPOSITION A.2.

(i) *If* $v \in L^2(\Omega)$, *then* $\mathcal{T}^\varepsilon(v) \to v$ *strongly in* $L^2(\Omega\times Y)$.

(ii) *If* v^ε *strongly converges to* v *in* $L^2(\Omega)$, *then*

$$\mathcal{T}^\varepsilon(v^\varepsilon) \to v \text{ strongly in } L^2(\Omega\times Y).$$

(iii) *If* v^ε *weakly converges to* v *in* $H^1(\Omega)$, *then*

$$\mathcal{T}^\varepsilon(v^\varepsilon) \rightharpoonup v \text{ weakly in } L^2(\Omega; H^1(Y))$$

and there exists $\widehat{v} \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$ *with* $\mathcal{M}_Y(\widehat{v}) = 0$ *such that, up to a subsequence,*

$$\begin{aligned} \mathcal{T}^\varepsilon(\nabla v^\varepsilon) &\rightharpoonup \nabla v + \nabla_y \widehat{v} && \text{weakly in } L^2(\Omega\times Y),\\ \mathcal{T}_b^\varepsilon(v^\varepsilon) &\rightharpoonup v && \text{weakly in } L^2(\Omega\times\Gamma). \end{aligned}$$

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