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OPTIMAL LARGE-TIME BEHAVIOR OF THE COMPRESSIBLE PHAN-THEIN-TANNER MODEL*

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Abstract. In this paper, we investigate global existence and optimal decay rates of strong solutions to the three dimensional compressible Phan-Thein-Tanner model. We prove the global existence of the solutions by the standard energy method under the small initial data assumptions. Furthermore, if the initial data belong to $L^1(\mathbb{R}^3)$, we establish the optimal time decay rates of the solution as well as its higher-order spatial derivatives. In particular, we also obtain the optimal decay rates of the highest-order spatial derivatives of the velocity. Finally, we derive the lower bound time decay rates for the solution and its spatial derivatives. Our method is based on Hodge decomposition, low-frequency and high-frequency decomposition, delicate spectral analysis, and energy methods.

Keywords. Phan-Thein-Tanner model; Optimal large-time behavior; Global existence.

AMS subject classifications. 35Q30; 76N15; 76P05.

1. Introduction

The theory of Phan-Thein-Tanner model recently gained quite some attention, this model is derived from network theory for the polymeric fluid. This type of fluid is described by the following set of equations

$$\begin{cases} \rho_t+\operatorname{div}(\rho u)=0, \\ \rho(u_t+u\cdot\nabla u)-\mu(\triangle u+\nabla \operatorname{div} u)+\nabla p=\mu_1 \operatorname{div}\tau, \\ \tau_t+u\cdot\nabla\tau+Q(\tau,\nabla u)+(a+b\operatorname{tr}\tau)\tau=\mu_2 D(u), \\ (\rho,u,\tau)|_{t=0}=(\rho_0,u_0,\tau_0),(t,x)\in\mathbb{R}^+\times\mathbb{R}^3. \end{cases} \tag{1.1}$$

The unknowns ρ,u,τ,p are the density, velocity, stress tensor and scalar pressure of fluid, respectively. $D(u)$ is the symmetric part of ∇u, that is

$$D(u)=\frac{1}{2}(\nabla u+(\nabla u)^t).$$

$Q(\tau,\nabla u)$ is a given bilinear form

$$Q(\tau,\nabla u)=\tau\Omega(u)-\Omega(u)\tau+\lambda(D(u)\tau+\tau D(u)),$$

where $\Omega(u)$ is the skew-symmetric part of ∇u, namely

$$\Omega(u)=\frac{1}{2}(\nabla u-(\nabla u)^t).$$

$\mu>0$ is the viscosity coefficient and μ_1 is the elastic coefficient. a and μ_2 are associated to the Debroah number $De=\frac{\mu_2}{a}$, which indicates the relation between the characteristic

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flow time and elastic time [2]. $\lambda\in[-1,1]$ is a physical parameter, in particular, we call the system co-rotational when $\lambda=0$. $b\geq 0$ is a constant related to the rate of creation or destruction for the polymeric network junctions.

To complete the system (1.1), the initial data are given by

$$(\rho,u,\tau)|_{t=0}=(\rho_0,u_0,\tau_0),\quad x\in\mathbb{R}^3. \tag{1.2}$$

In the present paper, we consider the Cauchy problem of (1.1) subject to the initial condition

$$(\rho,u,\tau)(t,x)=(\bar\rho,0,0)\quad\text{as}\quad |x|\to\infty,\in\mathbb{R}^3. \tag{1.3}$$

1.1. History of the problem. Let us review some previous works about the model (1.1) and the related models. If we ignore the stress tensor, (1.1) reduces to the compressible Navier-Stokes (NS) equations. The convergence rate of solution for the compressible Navier-Stokes equations to the steady state has been investigated extensively since the first global existence of small solutions in H^3 was improved by Matsumura and Nishida [19,20]. When the initial perturbation $(\rho_0-1,u_0)\in L^p\cap H^N(N\geq 3)$ with $p\in[1,2]$, the L^2 optimal decay rate of the solution to the NS system is

$$\|(\rho-1,u)(t)\|_{L^2}\leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}.$$

For the small initial perturbation belonging to H^3 only, Matsumura [18] employed the weighted energy method to show the L^2 decay rates. In order to establish optimal decay rates for higher order spatial derivatives of solutions, if the initial perturbation is bounded in $H^{-s}(s\in[0,\frac{3}{2}))$ norm instead of L^1-norm, Guo and Wang [11] developed the time convergence rates, as follows, by using a general energy method,

$$\|\nabla^l(\rho-1,u)(t)\|_{H^{N-l}}\leq C(1+t)^{-\frac{l+s}{2}},$$

for $0\leq l\leq N-1$. In addition, the decay rate of solutions to the NS system was investigated in [3,6,25] and the references therein.

If $b=0$, the system (PTT) reduces to the famous Oldroyd-B model (see [23]) which has been studied widely, most of the results on Oldroyd-B fluids in the literature are about the incompressible model. C. Guillopé and J.C. Saut [10] proved the existence of local strong solutions and the global existence of one dimensional shear flows. Later, the smallness restriction on the coupling constant in [10] was removed by Molinet and Talhouk [21]. In [17], F. Lin, C. Liu and P. Zhang proved that if the initial data are a small perturbation around equilibrium, then the strong solution is global in time. Similar results were obtained in several papers by virtue of different methods, see T. Zhang and D. Fang [29], Y. Zhu [31]. D. Fang and R. Zi [7] proved the global existence of strong solutions with a class of large data. On the other hand, there are relatively few results for the compressible model. Lei [13] proved the local and global existence of classical solutions for a compressible Oldroyd-B system in a torus with small initial data. He also studied the incompressible limit problem and showed that the compressible flows with well-prepared initial data converge to incompressible ones when the Mach number converges to zero. The case of ill prepared initial data was considered by Fang and Zi [8] in the whole space $\mathbb{R}^d$, $d\geq 2$. Recently, the smallness restriction on coupling constant was removed by Zi in [32]. Zhou, Zhu and Zi [30] proved the existence of global strong solution provided the initial data are close to the constant equilibrium state in H^2-framework and obtained the convergence rates of the solutions. Recently, for the compressible Oldroyd type model based on the deformation tensor, the authors proved the decay rate in [16, 28].

In this paper, we focus on the PTT model ($b\neq 0$). To our knowledge, there are a lot of numerical results about the PTT model (see, [1, 9, 24]). Recently, [5] proved that the strong solution in critical Besov spaces exists globally when the initial data are a small perturbation around the equilibrium. [4] proved that the strong solution will blow up in finite time and proved the global existence of strong solution with small initial data. However, there are few results to our knowledge on the compressible PTT model, especially the large-time behavior. Although the authors proved the decay rate in [27], the decay rate of N-th order derivative of solution $(\varrho-1,u)$ coincides with the lower one. Recently, this tricky problem is solved in a series of articles [14, 15] by using the spectrum analysis of the linearized part. On the other hand, compared with the incompressible models, the compressible equations of PTT model are more difficult to deal with because of the strong nonlinearities and interactions among the physical quantities. The main purpose in this paper is to study the global existence and decay rates of smooth solutions for the compressible PTT model. We first establish the global solution of (1.1)-(1.2) in the whole space $\mathbb{R}^3$ near the constant equilibrium state. Then the optimal convergence rates of the higher-order spatial derivatives of the solutions are also obtained. Furthermore, for well-chosen initial data, we also show the lower bounds on the convergence rates. Our method is based on Hodge decomposition, low-frequency and high-frequency decomposition, delicate spectral analysis and energy methods.

Before stating our main result, let us first introduce the notations and conventions used throughout this paper. We use $H^k(\mathbb{R}^3)$ to denote the usual Sobolev space with norm $\|\cdot\|_{H^k}$ and L^p, $1\leq p\leq\infty$ to denote the usual $L^p(\mathbb{R}^3)$ space with norm $\|\cdot\|_{L^p}$. For the sake of conciseness, we do not precise in functional space names when they are concerned with scalar-valued or vector-valued functions, $\|(f,g)\|_X$ denotes $\|f\|_X+\|g\|_X$. We will employ the notation $a\lesssim b$ to mean that $a\leq Cb$ for a universal constant $C>0$ that only depends on the parameters coming from the problem. We denote $\nabla=\partial_x=(\partial_1,\partial_2,\partial_3)$, where $\partial_i=\partial_{x_i}$, $\nabla_i=\partial_i$ and put $\partial_x^\ell f=\nabla^\ell f=\nabla(\nabla^{\ell-1}f)$. Let Λ^s be the pseudo differential operator defined by

$$\Lambda^s f=\mathfrak{F}^{-1}(|\xi|^s\widehat{f}),\ \text{for } s\in\mathbb{R},$$

where $\widehat{f}$ and $\mathfrak{F}(f)$ are the Fourier transforms of f. The homogenous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ with norm given by $\|f\|_{\dot{H}^s}\overset{\triangle}{=}\|\Lambda^s f\|_{L^2}$. For a radial function $\phi\in C_0^\infty(\mathbb{R}^3_\xi)$ such that $\phi(\xi)=1$ when $|\xi|\leq\frac{\eta}{2}$ and $\phi(\xi)=0$ when $|\xi|\geq\eta$, we define the low–frequency part of f by

$$f^l=\mathfrak{F}^{-1}[\phi(\xi)\widehat{f}]$$

and the high–frequency part of f by

$$f^h=\mathfrak{F}^{-1}[(1-\phi(\xi))\widehat{f}].$$

It is direct to check that $f=f^l+f^h$ if Fourier transform of f exists.

1.2. Main results. Now, we state our main result about the global existence and decay properties of solution to the system (1.1)-(1.2) in the following theorem.

THEOREM 1.1. *Assume that $(\rho_0-1,\ u_0,\ \tau_0)\in H^\ell(\mathbb{R}^3)$ for an integer $\ell\geq 3$. Then there exists a constant δ_0 such that if*

$$K_0:=\|(\rho_0-1,u_0,\tau_0)\|_{H^\ell}\leq\delta_0, \tag{1.4}$$

then the Cauchy problem (1.1)–(1.2) *admits a unique solution* (ρ,u,τ) *globally in time in the sense that*

$$\|(\rho-1,u,\tau)(t)\|_{H^\ell}^2+\int_0^t(\|\nabla\rho(s)\|_{H^{\ell-1}}^2+\|\nabla u(s)\|_{H^\ell}^2+\|\tau(s)\|_{H^\ell}^2)ds\le CK_0^2. \tag{1.5}$$

Moreover, the following convergence rates hold true.

- **Upper bounds.** *If additionally*

$$N_0:=\|(\rho_0-1,u_0,\tau_0)\|_{L^1}<\infty, \tag{1.6}$$

then for any $t\ge 0$, *it holds that*

$$\left\|\nabla^k(\rho-1,u)(t)\right\|_{H^{\ell-k}}\le C(N_0)(1+t)^{-\frac{3}{4}-\frac{k}{2}},\quad 0\le k\le\ell, \tag{1.7}$$

$$\left\|\nabla^k\tau(t)\right\|_{H^{\ell-k}}\le C(N_0)(1+t)^{-\frac{5}{4}-\frac{k}{2}},\quad 0\le k\le\ell-1. \tag{1.8}$$

- **Lower bounds.** *Let* $(\varrho_0,u_0,\tau_0)=(\rho_0-1,u_0,\tau_0)$ *and assume that Fourier transform of functions* (ϱ_0,u_0,τ_0) *satisfy*

$$\Lambda^{-1}\mathrm{div}\widehat{u_0}(\xi)=\Lambda^{-1}\mathrm{div}\Lambda^{-1}\mathrm{div}\widehat{\tau_0}(\xi)=0,\ \text{and}\ \ |\widehat{\varrho_0}(\xi)|\ge N_0\sqrt{\delta_0}, \tag{1.9}$$

for any $|\xi|\le\eta$. *Then there is a positive constant* c_1 *independent of* t *such that for any large enough* t *and* $0\le k\le\ell$, *it holds that*

$$\min\left\{\|\nabla^k(\rho-1)(t)\|_{H^{\ell-k}},\|\nabla^k u(t)\|_{H^{\ell-k}}\right\}\ge c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{1.10}$$

$$\|\nabla^k\tau(t)\|_{H^{\ell-k}}\ge c_1(1+t)^{-\frac{5}{4}-\frac{k}{2}}. \tag{1.11}$$

REMARK 1.1. Compared with our previous results [27], it is worth noting that the optimal time decay rates of the highest-order spatial derivatives of the velocity are obtained. This is due to the decomposition on the system. On the other hand, to the best knowledge of the authors, there was no result about the lower bounds of decay rates (1.10)-(1.11) for the spatial derivatives of density, velocity, and stress tensor to the compressible Phan-Thein-Tanner Model (1.1) before. That is to say, in this paper, this result was obtained for the first time.

REMARK 1.2. In Theorem 1.1, a smallness assumption on the higher-order Sobolev norms of the initial data is not entirely necessary. Indeed, on the one hand, motivated by the pure energy method the authors [27] proved the global existence result in Theorem 1.1 under the assumption that the H^3 norm of the initial data is small, while the higher-order Sobolev norms can be arbitrarily large. On the other hand, it is clear that in deriving the large-time behavior of solutions, we only need the smallness of H^3 -norm of the initial data. However, this is not our main concern. We will focus our attention on the large-time behavior of the solutions and thus omit the details for the sake of simplicity.

2. Spectral analysis and linear L^2 estimates

2.1. Reformulation. In this subsection, we first reformulate the system (1.1). Denoting $\varrho=\rho-1$, then we can rewrite system (1.1)-(1.2) into the following equivalent form:

$$\begin{cases}\varrho_t+\mathrm{div}u=S_1,\\ u_t+\gamma\nabla\varrho-\mu(\triangle u+\nabla\mathrm{div}u)-\mu_1\mathrm{div}\tau=S_2,\\ \tau_t+a\tau-\mu_2D(u)=S_3,\end{cases} \tag{2.1}$$

where the nonlinear terms $S_i(i=1,2,3)$ are defined by

$$
\begin{aligned}
&S_1=-\mathrm{div}(\varrho u),\\
&S_2=-u\cdot\nabla u-\mu f(\varrho)(\triangle u+\nabla\mathrm{div}u)-g(\varrho)\nabla\varrho-\mu_1 f(\varrho)\mathrm{div}\tau,\\
&S_3=-u\cdot\nabla\tau-Q(\tau,\nabla u)-b\mathrm{tr}\tau\tau,
\end{aligned}
$$

with

$$
(\varrho,u,\tau)(x,0)=(\varrho_0,u_0,\tau_0)\to 0 \text{ as } |x|\to\infty, \tag{2.2}
$$

and here

$$
\gamma=\frac{P'(1)}{1},\quad f(\varrho)=\frac{\varrho}{\varrho+1},\quad g(\varrho)=\frac{P'(\varrho+1)}{\varrho+1}-\frac{P'(1)}{1}.
$$

We define $\tilde U=(\tilde\varrho,\tilde u,\tilde\tau)^t$. In terms of the semigroup theory for evolutionary equation, we can write the corresponding linear system of model (2.1) as follows:

$$
\begin{cases}\tilde U_t=\mathcal{C}\tilde U,\\ \tilde U\big|_{t=0}=U_0.\end{cases} \tag{2.3}
$$

To derive the linear time–decay estimates, if we utilize the method in [19], we need to make a detailed analysis on the properties of the semigroup. Unfortunately, it seems untractable, since the system (2.3) has thirteen equations. To overcome this difficulty, we take Hodge decomposition to system (2.3) such that it can be decoupled into two systems. One has three equations, and the other has two equations. This key observation allows us to derive the optimal linear convergence rates.

Let $v=\Lambda^{-1}\mathrm{div}\tilde\tau$, $\tilde V=(\tilde\varrho,\tilde u,v)^t$. The system (2.3) can be expressed by

$$
\begin{cases}\tilde V_t=\mathcal{B}\tilde V,\\ \tilde V\big|_{t=0}=V_0,\end{cases} \tag{2.4}
$$

where the operator $\mathcal{B}$ is given by

$$
\mathcal{B}=\begin{pmatrix}0 & -\mathrm{div} & 0\\ -\gamma\nabla & \mu(\Delta+\nabla\otimes\nabla) & \mu_1\Lambda\\ 0 & \mu_2\Lambda^{-1}(\Delta+\nabla\otimes\nabla) & -a\end{pmatrix},
$$

here $\nabla^t u:=(\nabla u)^t$ for any vector $u\in(\mathbb{R}^3)$. Applying Fourier transform to the system (2.4), we have

$$
\begin{cases}\widehat{\tilde V}_t=\mathcal{A}(\xi)\widehat{\tilde V},\\ \widehat{\tilde V}\big|_{t=0}=\widehat{V}_0=(\varrho_0,u_0,v_0),\end{cases} \tag{2.5}
$$

where $\widehat{\tilde V}(\xi,t)=\mathfrak{F}(\tilde V(x,t))$, $\xi=(\xi^1,\xi^2,\xi^3)^t$ and $\mathcal{A}(\xi)$ is defined as

$$
\mathcal{A}(\xi)=\begin{pmatrix}0 & -i\xi^t & 0\\ -i\xi & -\mu|\xi|^2\mathrm{I}_{3\times3}-\mu\xi\otimes\xi & -i\mu_1\xi\\ 0 & -\mu|\xi|^{-1}(|\xi|^2\mathrm{I}_{3\times3}+\xi\otimes\xi) & -a\end{pmatrix}.
$$

We will take Hodge decomposition to analyze the system (2.4), let $\varphi = \Lambda^{-1}\mathrm{div}\tilde{u}, \psi = \Lambda^{-1}\mathrm{div}v$ be the "compressible part" of the velocities $\tilde{u}$ and τ, and denote $\Phi = \Lambda^{-1}\mathrm{curl}\tilde{u}, \Psi = \Lambda^{-1}\mathrm{curl}v$ (with $(\mathrm{curl}z)_i^j = \partial_{x_j} z^i - \partial_{x_i} z^j$) by the "incompressible part" of the velocities $\tilde{u}$ and v. Then we can rewrite (2.4) as

$$\begin{cases} \partial_t \tilde{\varrho} + \Lambda\varphi = 0, \\ \partial_t \varphi - \gamma\Lambda\tilde{\varrho} + 2\mu\Lambda^2\varphi - \mu_1\Lambda\psi = 0, \\ \partial_t \psi + a\psi + 2\mu_2\Lambda\varphi = 0, \\ (\tilde{\varrho}, \varphi, \psi)\big|_{t=0} = (\varrho_0, \Lambda^{-1}\mathrm{div}u_0, \Lambda^{-1}\mathrm{div}v_0)(x), \end{cases} \tag{2.6}$$

and

$$\begin{cases} \partial_t \Phi + \mu\Lambda^2\Phi - \mu_1\Lambda\Psi = 0, \\ \partial_t \Psi + a\Psi + \mu_2\Lambda\Phi = 0, \\ (\Phi, \Psi)\big|_{t=0} = (\Lambda^{-1}\mathrm{curl}u_0, \Lambda^{-1}\mathrm{curl}v_0)(x). \end{cases} \tag{2.7}$$

2.2. Spectral analysis for IVP (2.6). In view of the semigroup theory, we may represent the IVP (2.6) for $\mathcal{U} = (\tilde{\varrho}, \varphi, \psi)^t$ as

$$\begin{cases} \mathcal{U}_t = \mathcal{B}_1 \mathcal{U}, \\ \mathcal{U}\big|_{t=0} = \mathcal{U}_0, \end{cases} \tag{2.8}$$

where the operator $\mathcal{B}_1$ is defined by

$$\mathcal{B}_1 = \begin{pmatrix} 0 & -\Lambda & 0 \\ \gamma\Lambda & -2\mu\Lambda^2 & \mu_1\Lambda \\ 0 & -2\mu_2\Lambda & -a \end{pmatrix}.$$

Taking Fourier transform to system (2.8), we obtain

$$\begin{cases} \widehat{\mathcal{U}}_t = \mathcal{A}_1(\xi)\widehat{\mathcal{U}}, \\ \widehat{\mathcal{U}}\big|_{t=0} = \widehat{\mathcal{U}}_0, \end{cases} \tag{2.9}$$

where $\widehat{\mathcal{U}}(\xi, t) = \mathfrak{F}(\mathcal{U}(x,t))$ and $\mathcal{A}_1(\xi)$ is given by

$$\mathcal{A}_1(\xi) = \begin{pmatrix} 0 & -|\xi| & 0 \\ \gamma|\xi| & -2\mu|\xi|^2 & \mu_1|\xi| \\ 0 & -2\mu_2|\xi| & -a \end{pmatrix}. \tag{2.10}$$

The eigenvalues of the matrix $\mathcal{A}_1(\xi)$ can be computed by

$$\begin{aligned} &\det(\lambda \mathrm{I} - \mathcal{A}_1(\xi)) \\ &= \lambda^3 + (a + 2\mu|\xi|^2)\lambda^2 + (2a\mu + 2\mu_1\mu_2 + \gamma)|\xi|^2\lambda + a\gamma|\xi|^2 \\ &= 0, \end{aligned}$$

which implies that matrix $\mathcal{A}_1(\xi)$ possesses three eigenvalues:

$$\begin{cases} \lambda_1 = -a + \frac{2\mu_1\mu_2}{a}|\xi|^2 + \mathcal{O}(|\xi|^3), \\ \lambda_2 = \sqrt{\gamma}i|\xi| - \frac{a\mu + \mu_1\mu_2}{a}|\xi|^2 + \mathcal{O}(|\xi|^3), \\ \lambda_3 = -\sqrt{\gamma}i|\xi| - \frac{a\mu + \mu_1\mu_2}{a}|\xi|^2 + \mathcal{O}(|\xi|^3). \end{cases} \tag{2.11}$$

Consequently, we can represent the solution of IVP (2.6) as

$$\widehat{\mathcal{U}}(\xi,t) = \mathrm{e}^{t\mathcal{A}_1(\xi)}\widehat{\mathcal{U}}_0(\xi) = \left(\sum_{i=1}^{3} \mathrm{e}^{\lambda_i t} P_i(\xi)\right)\widehat{\mathcal{U}}_0(\xi), \tag{2.12}$$

where the projection operators P_i can be computed as

$$P_1(\xi) = \frac{1}{a^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a\mu_1|\xi| \\ 0 & 2a\mu_2|\xi| & a^2 \end{pmatrix} + \mathcal{O}(|\xi|^2), \tag{2.13}$$

$$P_2(\xi) = \frac{1}{2a\sqrt{\gamma}i}\begin{pmatrix} a\sqrt{\gamma}i + (a\mu+\mu_1\mu_2-\gamma)|\xi| & -a-\sqrt{\gamma}i|\xi| & -\mu_1|\xi| \\ a\gamma+\gamma\sqrt{\gamma}i|\xi| & a\sqrt{\gamma}i - (\gamma+a\mu+\mu_1\mu_2)|\xi| & \mu_1\sqrt{\gamma}i|\xi| \\ -2\mu_2\gamma|\xi| & -2\mu_2\sqrt{\gamma}i|\xi| & 0 \end{pmatrix}$$
$$+\mathcal{O}(|\xi|^2), \tag{2.14}$$

$$P_3(\xi) = -\frac{1}{2a\sqrt{\gamma}i}\begin{pmatrix} -a\sqrt{\gamma}i + (a\mu+\mu_1\mu_2-\gamma)|\xi| & -a+\sqrt{\gamma}i|\xi| & -\mu_1|\xi| \\ a\gamma-\gamma\sqrt{\gamma}i|\xi| & -a\sqrt{\gamma}i - (\gamma+a\mu+\mu_1\mu_2)|\xi| & -\mu_1\sqrt{\gamma}i|\xi| \\ -2\mu_2\gamma|\xi| & 2\mu_2\sqrt{\gamma}i|\xi| & 0 \end{pmatrix}$$
$$+\mathcal{O}(|\xi|^2). \tag{2.15}$$

By virtue of (2.12)-(2.15), we can establish the following estimates on low-frequency part of the solution $\widehat{\mathcal{U}}(\xi,t)$ to the IVP (2.6).

LEMMA 2.1. *Let $\nu_1 = \frac{a\mu+\mu_1\mu_2}{a}$, then there exists a sufficiently small positive constant η_1, such that the following estimates hold*

$$\left|\widehat{\tilde{\varrho}}\right|, |\widehat{\varphi}|, \left|\widehat{\psi}\right| \lesssim e^{-\nu_1|\xi|^2 t}\left(|\widehat{\varrho}_0| + |\widehat{\varphi}_0| + \left|\widehat{\psi}_0\right|\right) \tag{2.16}$$

for any $|\xi| \le \eta_1$.

With the key estimate (2.16) in hand, we are able to establish the optimal L^2-convergence rate on the low-frequency part of the solution, which is stated in the following proposition.

PROPOSITION 2.1 (L^2–theory). *For any $k > -\frac{3}{2}$, there exists a positive constant C independent of t, such that*

$$\left\|\nabla^k\left(\tilde{\varrho}^l, \varphi^l, \psi^l\right)(t)\right\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left\|\widehat{\mathcal{U}}^l(0)\right\|_{L^\infty}, \tag{2.17}$$

for any $t \ge 0$.

Proof. By virtue of (2.16) and the Plancherel theorem, one has

$$\begin{aligned}\left\|\nabla^k \mathrm{e}^{t\mathcal{A}_1} * \mathcal{U}^l(0)\right\|_{L^2}^2 &= \left\||\xi|^k \mathrm{e}^{t\mathcal{A}_1(\xi)}\widehat{\mathcal{U}}^l(0)\right\|_{L^2}^2 \\ &\lesssim \int_{|\xi|\le\eta} \mathrm{e}^{-2\nu_1|\xi|^2 t}|\xi|^{2k}|\widehat{\mathcal{U}}^l(0)|^2 d\xi \\ &\le C(1+t)^{-\frac{3}{2}-k}\left\|\widehat{\mathcal{U}}^l(0)\right\|_{L^\infty}^2,\end{aligned}$$

which implies (2.17). Therefore, we have completed the proof of Proposition 2.1. ∎

In order to derive the optimal linear convergence rates, we need to establish the lower bounds on the convergence rates which are stated in the following proposition.

PROPOSITION 2.2. *Assume that* $(\varrho_0,\varphi_0,\psi_0,)\in L^1$ *satisfies*

$$\widehat{\varphi_0}(\xi)=\widehat{\psi_0}(\xi)=0, \quad \textit{and} \quad |\widehat{\varrho_0}(\xi)|\geq c_0 \tag{2.18}$$

for any $|\xi|\leq\eta$, *where* c_0 *is a positive constant. Then, there exists a positive constant* C_1 *independent of* t, *such that it holds that*

$$\begin{aligned}&\min\{\|\tilde{\varrho}^l(t)\|_{L^2},\|\varphi^l(t)\|_{L^2}\}\geq C_1c_0(1+t)^{-\frac{3}{4}},\\&\|\psi^l(t)\|_{L^2}\geq C_1c_0(1+t)^{-\frac{5}{4}},\end{aligned} \tag{2.19}$$

for sufficiently large t.

Proof. Due to (2.18), it follows from (2.12)-(2.15) that

$$\begin{aligned}\widehat{\tilde{\varrho}}^l=&\left(\frac{1}{2}+\frac{a\mu+\mu_1\mu_2-\gamma}{2a\sqrt{\gamma}i}|\xi|+\mathcal{O}(|\xi|^2)\right)\mathrm{e}^{\lambda_2(|\xi|)t}\widehat{\varrho}_0^l+\left(\frac{1}{2}-\frac{a\mu+\mu_1\mu_2-\gamma}{2a\sqrt{\gamma}i}|\xi|+\mathcal{O}(|\xi|^2)\right)\mathrm{e}^{\lambda_3(|\xi|)t}\widehat{\varrho}_0^l\\&\sim\mathrm{e}^{-\nu_1|\xi|^2t}\left(\frac{a\mu+\mu_1\mu_2-\gamma}{a\sqrt{\gamma}}|\xi|\sin(\sqrt{\gamma}|\xi|t)+\cos(\sqrt{\gamma}|\xi|t)\right)\widehat{\varrho}_0^l,\end{aligned}$$

which together with Plancherel theorem and the double angle formula implies

$$\begin{aligned}\|\tilde{\varrho}^l(t)\|_{L^2}^2=&\|\widehat{\tilde{\varrho}}^l(t)\|_{L^2}^2\\\geq&\frac{c_0^2}{2}\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_1|\xi|^2t}\cos^2(\sqrt{\gamma}|\xi|t)\mathrm{d}\xi\\&-(\frac{a\mu+\mu_1\mu_2-\gamma}{a\sqrt{\gamma}})^2\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_1|\xi|^2t}|\xi|^2\sin^2(\sqrt{\gamma}|\xi|t)|\widehat{\varrho}_0^l|^2\mathrm{d}\xi\\\geq&\frac{c_0^2}{4}\left(\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_1|\xi|^2t}\mathrm{d}\xi+\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_1|\xi|^2t}\cos(2\sqrt{\gamma}|\xi|t)\mathrm{d}\xi\right)\\&-(\frac{a\mu+\mu_1\mu_2-\gamma}{a\sqrt{\gamma}})^2\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_1|\xi|^2t}|\xi|^2\sin^2(\sqrt{\gamma}|\xi|t)|\widehat{\varrho}_0^l|^2\mathrm{d}\xi\\\geq&C_1c_0(1+t)^{-\frac{3}{2}},\end{aligned}$$

if t large enough. Similarly, for the terms φ^l,ψ^l, we have

$$\|\varphi^l(t)\|_{L^2}\geq C_1c_0(1+t)^{-\frac{3}{2}}.$$

$$\begin{aligned}\widehat{\psi}^l=&\left(\frac{\mu_2\sqrt{\gamma}}{ai}|\xi|+\mathcal{O}(|\xi|^2)\right)\left(\mathrm{e}^{\lambda_3(|\xi|)t}-\mathrm{e}^{\lambda_2(|\xi|)t}\right)\widehat{\varrho}_0^l\\&\sim\frac{2\mu_2\sqrt{\gamma}}{a}|\xi|\mathrm{e}^{-\nu_1|\xi|^2t}\sin(\sqrt{\gamma}|\xi|t)\widehat{\varrho}_0^l,\end{aligned}$$

which together with Plancherel theorem leads to

$$\|\psi^l(t)\|_{L^2}^2=\left\|\widehat{\psi}^l\right\|_{L^2}^2\geq C_2c_0(1+t)^{-\frac{5}{2}}.$$

Therefore, we have completed the proof of Proposition 2.2. $\square$

2.3. Spectral analysis for IVP (2.7). Similar to the proof of (2.8), we may express the IVP (2.7) for $\mathcal{V}=(\Phi,\Psi)^t$ as

$$\begin{cases} \mathcal{V}_t=\mathcal{B}_2\mathcal{V}, \\ \mathcal{V}\big|_{t=0}=\mathcal{V}_0, \end{cases} \tag{2.20}$$

where the operator $\mathcal{B}_2$ is defined by

$$\mathcal{B}_2=\begin{pmatrix} -\mu\Lambda^2 & \mu_1\Lambda \\ -\mu_2\Lambda & -a \end{pmatrix}.$$

Taking Fourier transform to system (2.20), we obtain

$$\begin{cases} \widehat{\mathcal{V}}_t=\mathcal{A}_2(\xi)\widehat{\mathcal{V}}, \\ \widehat{\mathcal{V}}\big|_{t=0}=\widehat{\mathcal{V}}_0, \end{cases} \tag{2.21}$$

where $\widehat{\mathcal{V}}(\xi,t)=\mathfrak{F}(\mathcal{V}(x,t))$ and $\mathcal{A}_2(\xi)$ is given by

$$\mathcal{A}_2(\xi)=\begin{pmatrix} -\mu|\xi|^2 & \mu_1|\xi| \\ -\mu_2|\xi| & -a \end{pmatrix}. \tag{2.22}$$

We compute the eigenvalues of matrix $\mathcal{A}_2(\xi)$ from the determinant

$$\det(\lambda \mathrm{I}-\mathcal{A}_2(\xi))=\lambda^2+(a+\mu|\xi|^2)\lambda+(a\mu+\mu_1\mu_2)|\xi|^2=0,$$

which implies that matrix $\mathcal{A}_2(\xi)$ possesses two eigenvalues:

$$\begin{cases} \lambda_1=-a+\frac{\mu_1\mu_2}{a}|\xi|^2+\mathcal{O}(|\xi|^3), \\ \lambda_2=-\frac{a\mu+\mu_1\mu_2}{a}|\xi|^2+\mathcal{O}(|\xi|^3). \end{cases} \tag{2.23}$$

Consequently, we can represent the solution of IVP (2.7) as

$$\widehat{\mathcal{V}}(\xi,t)=\mathrm{e}^{t\mathcal{A}_2(\xi)}\widehat{\mathcal{V}}_0(\xi)=\left(\sum_{i=1}^{2}\mathrm{e}^{\lambda_i t}Q_i(\xi)\right)\widehat{\mathcal{V}}_0(\xi), \tag{2.24}$$

where

$$Q_1(\xi)=-\frac{1}{a}\begin{pmatrix} 0 & \mu_1|\xi| \\ -\mu_2|\xi| & -a \end{pmatrix}+\mathcal{O}(|\xi|^2), \tag{2.25}$$

$$Q_2(\xi)=\frac{1}{a}\begin{pmatrix} a & \mu_1|\xi| \\ -\mu_2|\xi| & 0 \end{pmatrix}+\mathcal{O}(|\xi|^2). \tag{2.26}$$

By virtue of (2.24)-(2.26), we can establish the following estimates on low-frequency part of the solution $\widehat{\mathcal{V}}(\xi,t)$ to the IVP (2.7).

LEMMA 2.2. *There exists a sufficiently small positive constant η_2, such that the following estimates hold*

$$\left|\widehat{\Phi}\right|,\left|\widehat{\Psi}\right|\lesssim e^{-\nu_1|\xi|^2t}\left(\left|\widehat{\Phi}_0\right|+\left|\widehat{\Psi}_0\right|\right) \tag{2.27}$$

for any $|\xi|\le\eta_2$.

Similar to the proof of Proposition 2.1, we can also get the following lemma which is concerned with L^2-convergence rate on the low-frequency part of the solution $\mathcal{V}$.

PROPOSITION 2.3 (L^2–theory). *For any $k>-\frac{3}{2}$, there exists a positive constant C independent of t, such that*

$$\left\|\nabla^k\left(\Phi^l,\Psi^l\right)(t)\right\|_{L^2}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left\|\widehat{\mathcal{V}}^l(0)\right\|_{L^\infty},\tag{2.28}$$

for any $t\geq 0$.

Proof. By virtue of (2.27) and the Plancherel theorem, one has

$$\begin{aligned}\left\|\nabla^k \mathrm{e}^{t\mathcal{A}_2}*\mathcal{V}^l(0)\right\|_{L^2}^2&=\left\||\xi|^k\mathrm{e}^{t\mathcal{A}_2(\xi)}\widehat{\mathcal{V}^l}(0)\right\|_{L^2}^2\\&\lesssim\int_{|\xi|\leq\eta}\mathrm{e}^{-2\nu_2|\xi|^2t}|\xi|^{2k}|\widehat{\mathcal{V}^l}(0)|^2d\xi\\&\leq C(1+t)^{-\frac{3}{2}-k}\left\|\widehat{\mathcal{U}}^l(0)\right\|_{L^\infty}^2,\end{aligned}$$

for any $t\geq 0$. ∎

Noticing the definition of φ,ψ,Φ and Ψ, and the fact that the relations

$$u=-\wedge^{-1}\nabla\varphi-\wedge^{-1}\mathrm{div}\Phi, v=-\wedge^{-1}\nabla\psi-\wedge^{-1}\mathrm{div}\Psi,\tag{2.29}$$

involve pseudodifferential operators of degree zero, the estimates in space H^k for the original function (u,v) will be the same as for (φ,ψ,Φ,Ψ). Combining Propositions 2.1, 2.2 and 2.3, we have the following result concerning long-time properties for the solution semigroup $e^{-t\mathcal{A}}$.

PROPOSITION 2.4. *Let $k>-\frac{3}{2}$ and assume that the initial data $U_0\in L^1(\mathbb{R}^3)$, then for any $t\geq 0$, the global solution $\tilde{V}=(\tilde{\varrho},\tilde{u},v)^t$ of the IVP (2.4) satisfies*

$$\left\|\nabla^k\left(\tilde{\varrho}^l,\tilde{u}^l,v^l\right)\right\|_{L^2}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|V(0)\|_{L^1}.\tag{2.30}$$

If in addition, the initial data satisfies (2.18), the following lower-bounds on convergence rate hold

$$\begin{aligned}&\min\left\{\left\|\tilde{\varrho}^l(t)\right\|_{L^2},\left\|\tilde{u}^l(t)\right\|_{L^2}\right\}\geq C_1c_0(1+t)^{-\frac{3}{4}},\\&\left\|v^l(t)\right\|_{L^2}\geq C_1c_0(1+t)^{-\frac{5}{4}},\end{aligned}\tag{2.31}$$

if t large enough.

LEMMA 2.3. *Let $k>-\frac{3}{2}$ and assume that the initial data $U_0\in L^1(\mathbb{R}^3)$, then for any $t\geq 0$, it holds that*

$$\left\|\nabla^k\tilde{\tau}^l\right\|_{L^2}\leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}}.\tag{2.32}$$

Proof. By delicate calculation to the system $(2.3)_3$, we can deduce that

$$\tilde{\tau}^l=\mathrm{e}^{-at}\tau_0^l+\mu_2\int_0^t\mathrm{e}^{-a(t-s)}D(u)^lds,$$

then integrating the equality over $\mathbb{R}^3$, and by virtue of (2.30) and the Plancherel theorem, one has

$$\begin{aligned}\left\|\nabla^k\tilde{\tau}^l\right\|_{L^2} &\lesssim e^{-at}\left\|\nabla^k\tau_0^l\right\|_{L^2}+\int_0^t e^{-a(t-s)}\left\|\nabla^k D(u)^l\right\|_{L^2}ds\\ &\lesssim e^{-at}\left\|\nabla^k\tau_0^l\right\|_{L^2}+\int_0^t e^{-a(t-s)}(1+t)^{-\frac{5}{4}-\frac{k}{2}}ds\\ &\le C(1+t)^{-\frac{5}{4}-\frac{k}{2}},\end{aligned}$$

for any $t\ge 0$. □

Noticing the definition of v, combining Proposition 2.4 and Lemma 2.3, we finally deduce that the solution $\tilde{U}$ to (2.3) has the following decay rates in time.

Proposition 2.5. *Under the assumption of Proposition 2.4, the global solution $\tilde{U}=(\tilde{\varrho},\tilde{u},\tilde{\tau})^t$ of the IVP (2.3) satisfies*

$$\left\|\nabla^k\left(\tilde{\varrho}^l,\tilde{u}^l\right)(t)\right\|_{L^2}\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U(0)\|_{L^1}, \tag{2.33}$$

$$\left\|\nabla^k\tilde{\tau}^l(t)\right\|_{L^2}\le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}\|U(0)\|_{L^1}, \tag{2.34}$$

$$\min\left\{\left\|\tilde{\varrho}^l(t)\right\|_{L^2},\left\|\tilde{u}^l(t)\right\|_{L^2}\right\}\ge C_1c_0(1+t)^{-\frac{3}{4}}, \tag{2.35}$$

$$\left\|\tilde{\tau}^l(t)\right\|_{L^2}\ge C_1c_0(1+t)^{-\frac{5}{4}}, \tag{2.36}$$

for any $t\ge 0$.

3. Optimal convergence rate

In this section, we devote ourselves to deriving the a priori energy estimates for the nonlinear system (2.1). To see this, we assume a priori that for sufficiently small $\delta>0$,

$$\|(\varrho,u,\tau)(t)\|_{H^\ell}\le\delta, \tag{3.1}$$

by (3.1) and Sobolev's inequality, we obtain

$$\frac{1}{2}\le\varrho+1\le 2.$$

Hence, we immediately have

$$|f(\varrho)|,|g(\varrho)|\le C|\varrho|,\quad |f^{(k)}(\varrho)|,|g^{(k)}(\varrho)|\le C\quad\text{for any}\quad k\ge 1. \tag{3.2}$$

In this section, we shall show the optimal convergence rate of the solution stated in Theorem 1.1. The global existence and uniqueness of the solution to the Cauchy problem (2.1) have been proven in [27] based on the classical energy method, therefore we omit the details for the sake of simplicity.

Theorem 3.1. *Assume that $(\rho_0-1,\ u_0,\ \tau_0)\in H^\ell(\mathbb{R}^3)$ for an integer $\ell\ge 3$. Then there exists a constant δ_0 such that if*

$$\|(\rho_0-1,u_0,\tau_0)\|_{H^\ell}\le\delta_0, \tag{3.3}$$

then the problem (1.1)-(1.2) has a unique global solution (ρ,u,τ) satisfying that for all $t\geq 0$,

$$\|(\rho-1,u,\tau)(t)\|_{H^\ell}^2+\int_0^t(\|\nabla\rho(s)\|_{H^{\ell-1}}^2+\|\nabla u(s)\|_{H^\ell}^2+\|\tau(s)\|_{H^\ell}^2)ds\leq C\|(\rho_0-1,u_0,\tau_0)\|_{H^\ell}^2. \tag{3.4}$$

In what follows, we devote ourselves to proving the optimal convergence rate of the solution stated in Theorem 1.1. We first prove the upper bound on the optimal convergence rate of the solution stated in (1.7)–(1.8). To do this, for any $0\leq k\leq \ell$, we define the time-weighted energy functional

$$E_k^\ell(t)=\sup_{0\leq s\leq t}\left\{(1+s)^{\frac{3}{4}+\frac{k}{2}}\left(\|\nabla^k(\varrho,u,\tau)(s)\|_{H^{\ell-k}}\right)\right\}. \tag{3.5}$$

Therefore, it suffices to prove that for any $0\leq k\leq \ell, E_k^\ell(t)$ has a uniform time-independent bound. We will take advantage of the low-frequency and high-frequency decomposition and use the key linear convergence estimates to achieve this goal by induction.

THEOREM 3.2. *Assume that the hypotheses of Theorem 1.1 and (1.6) are in force. Then there exists a positive constant C independent of t, such that*

$$E_k^\ell(t)\leq C(N_0),$$

for $0\leq k\leq \ell$.

Proof. We will employ mathematical induction method to prove Theorem 3.2. Therefore, by noticing (3.5), it suffices to prove the following Lemmas 3.1 and 3.2. Thus, the proof of Theorem 3.2 is completed. ∎

The first lemma is concerned with the estimate on $E_0^\ell(t)$.

LEMMA 3.1. *Assume that the hypotheses of Theorem 3.1 and (1.6) are in force. Then there exists a positive constant C, which is independent of t, such that*

$$E_0^\ell(t)\leq C(N_0). \tag{3.6}$$

Proof. Under the assumption of Theorem 1.1, we derive the following energy estimates on (ϱ,u,τ) which have been proven in [27]:

$$\frac{d}{dt}\mathcal{E}_0^\ell(t)+C(\|\nabla\varrho\|_{H^{\ell-1}}^2+\|\nabla u\|_{H^\ell}^2+\|\tau\|_{H^\ell}^2)\leq 0, \tag{3.7}$$

where $\mathcal{E}_0^\ell(t)$ is equivalent to $\|(\varrho,u,\tau)(t)\|_{H^\ell}^2$, using the fact that $\|(\varrho^h,u^h,\tau^h)(t)\|_{L^2}\leq C\|\nabla(\varrho,u,\tau)(t)\|_{L^2}$, (3.7) implies that there exists a positive constant D_1 such that

$$\frac{d}{dt}\mathcal{E}_0^\ell(t)+D_1\mathcal{E}_0^\ell\leq C\|(\varrho^l,u^l,\tau^l)(t)\|_{L^2}^2. \tag{3.8}$$

Defining $\mathcal{S}=(S^1,S^2,S^3)^t$, it follows from Duhamel's principle that

$$U^l=\mathrm{e}^{t\mathcal{C}}U^l(0)+\int_0^t\mathrm{e}^{(t-s)\mathcal{C}}\mathcal{S}^l(s)\mathrm{d}s, \tag{3.9}$$

which together with Plancherel theorem, integration by parts, Proposition 2.5, Lemma A.4 and (3.4) implies

$$
\begin{aligned}
&\|(\varrho^l,u^l,\tau^l)(t)\|_{L^2}=\|(\widehat{\varrho}^l,\widehat{u}^l,\widehat{\tau}^l)(t)\|_{L^2}\\
\lesssim&(1+t)^{-\frac34}\|U(0)\|_{L^1}+\int_0^t(1+t-s)^{-\frac34}\|\mathcal{S}^l(s)\|_{L^1}\mathrm{d}s\\
\lesssim&N_0(1+t)^{-\frac34}+\int_0^t(1+t-s)^{-\frac34}\|(\varrho,u,\tau)(s)\|_{L^2}\|\nabla(\varrho,u,\tau)(s)\|_{H^1}\mathrm{d}s\\
\lesssim&N_0(1+t)^{-\frac34}+\int_0^{\frac{t}{2}}(1+t-s)^{-\frac34}(1+s)^{-\frac34}E_0^\ell(t)\|\nabla(\varrho,u,\tau)(s)\|_{H^1}\mathrm{d}s\\
&+\int_{\frac{t}{2}}^t(1+t-s)^{-\frac34}(1+s)^{-\frac34}E_0^\ell(t)\|\nabla(\varrho,u,\tau)(s)\|_{H^1}\mathrm{d}s\\
\lesssim&N_0(1+t)^{-\frac34}+E_0^\ell(t)(1+t)^{-\frac34}\left(\int_0^{\frac{t}{2}}(1+s)^{-\frac32}ds\right)^{\frac12}\left(\int_0^{\frac{t}{2}}\|\nabla(\varrho,u,\tau)(s)\|_{H^1}^2ds\right)^{\frac12}\\
&+E_0^\ell(t)(1+t)^{-\frac34}\left(\int_{\frac{t}{2}}^t(1+t-s)^{-\frac32}ds\right)^{\frac12}\left(\int_{\frac{t}{2}}^t\|\nabla(\varrho,u,\tau)(s)\|_{H^1}^2ds\right)^{\frac12}\\
\lesssim&(1+t)^{-\frac34}\left(N_0+\delta_0E_0^\ell(t)\right).
\end{aligned}\tag{3.10}
$$

Substituting (3.10) into (3.8), one has

$$
\frac{d}{dt}\mathcal{E}_0^\ell(t)+D_1\mathcal{E}_0^\ell\le C(1+t)^{-\frac32}\left(N_0+\delta_0E_0^\ell(t)\right)^2.\tag{3.11}
$$

Applying Gronwall's inequality to the above inequality, we can infer that

$$
\begin{aligned}
\mathcal{E}_0^\ell(t)\le&\ \mathrm{e}^{-D_1t}\mathcal{E}_0^\ell(0)+C\int_0^t\mathrm{e}^{-D_1(t-s)}(1+s)^{-\frac32}\left[N_0+\delta_0E_0^\ell(s)\right]^2\mathrm{d}s\\
\le&\ C(1+t)^{-\frac32}\left[N_0+\delta_0E_0^\ell(t)\right]^2,
\end{aligned}
$$

which together with (3.5) implies that

$$
\begin{aligned}
&(1+t)^{\frac32}\|(\varrho,u,\tau)(t)\|_{H^\ell}^2\lesssim N_0^2+\delta_0^2(E_0^\ell(t))^2,\\
&E_0^\ell(t)\le C(N_0),
\end{aligned}\tag{3.12}
$$

if δ_0 is small enough. Therefore, we have completed the proof of Lemma 3.1. □

The next lemma is devoted to closing the estimates $E_k^\ell(t)$ for $1\le k\le\ell$.

LEMMA 3.2. *Assume that the hypotheses of Theorem 3.1 and (1.6) are in force. If additionally*

$$
E_{k-1}^\ell(t)\le C(N_0).\tag{3.13}
$$

Then it holds that

$$
E_k^\ell(t)\le C(N_0),\tag{3.14}
$$

for $1\le k\le\ell$.

Proof. We will combine the key linear estimates with delicate nonlinear energy estimates based on good properties of the low–frequency and high–frequency decomposition to prove Lemma 3.2, and the process involves the following six steps.

Step 1. L^2 estimate of $(\nabla^j\varrho^l,\nabla^j u^l,\nabla^j\tau^l)$ with $1\leq j\leq k$. First, similar to the proof of (3.10), we also have

$$\begin{aligned}\|\nabla^j(\varrho^l,u^l,\tau^l)(t)\|_{L^2}\lesssim&(1+t)^{-\frac{3}{4}-\frac{j}{2}}+\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{3}{4}-\frac{j}{2}}\|U(0)\|_{L^1}\mathrm{d}s\\&+\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{5}{4}}\||\xi|^{j-1}\widehat{\mathcal{S}}^l(s)\|_{L^\infty}\mathrm{d}s\\\lesssim&C(N_0)(1+t)^{-\frac{3}{4}-\frac{j}{2}}+\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{5}{4}}\||\xi|^{j-1}\widehat{\mathcal{S}}^l(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.15}$$

Next, we shall estimate the second term on the right-hand side of (3.15). The main idea of our approach is to make full use of the benefit of the low-frequency and high-frequency decomposition. To see this, by virtue of (3.13), Lemma A.2, and Lemma A.3, we can bound the term $\||\xi|^{j-1}\widehat{\mathcal{S}}^l(s)\|_{L^\infty}$ by

$$\begin{aligned}&\||\xi|^{j-1}\widehat{\mathcal{S}}^l(s)\|_{L^\infty}\\\lesssim&\left\|\nabla^{j-1}\Big(\mathrm{div}(\varrho \mathrm{u}),\mathrm{u}\cdot\nabla \mathrm{u},\mathrm{g}(\varrho)\nabla\varrho,\mu_1\mathrm{f}(\varrho)\mathrm{div}\tau,\mathrm{u}\cdot\nabla\tau,\mathrm{Q}(\tau,\nabla \mathrm{u}),\mathrm{btr}\tau\tau\Big)(\mathrm{t})\right\|_{\mathrm{L}^1}\\&+\|\nabla^{\max\{0,j-2\}}\mu f(\varrho)(\triangle u+\nabla\mathrm{div}u)(t)\|_{L^1}\\\lesssim&\|(\varrho,u,\tau)(t)\|_{L^2}\|\nabla^j(\varrho,u,\tau)(t)\|_{L^2}+\|\nabla(\varrho,u,\tau)(t)\|_{L^2}\|\nabla^{j-1}(\varrho,u,\tau)(t)\|_{L^2}\\&+\|\tau(t)\|_{L^2}\|\nabla^{j-1}\tau(t)\|_{L^2}+\|\nabla^{\max\{0,j-2\}}\varrho(t)\|_{L^2}\|\nabla^2u(t)\|_{L^2}\\\lesssim&(1+t)^{-\frac{3}{4}}E_0^\ell(t)(1+t)^{-\frac{3}{4}-\frac{j-1}{2}}E_{j-1}^\ell(t)\\\lesssim&C(N_0)(1+t)^{-1-\frac{j}{2}}.\end{aligned}\tag{3.16}$$

Substituting (3.16) into (3.15) yields that

$$\|\nabla^j(\varrho^l,u^l,\tau^l)(t)\|_{L^2}\leq C(N_0)(1+t)^{-\frac{3}{4}-\frac{j}{2}}.\tag{3.17}$$

It should be mentioned that the low-frequency convergence estimate (3.17) plays a critical role in proving the optimal convergence rates of the highest-order derivatives of solutions.

Step 2. L^2 estimate of $\nabla^j\varrho$. Applying the operator ∇^j to $(2.1)_1$, multiplying the resulting equation by $\nabla^j\varrho$, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j\varrho|^2\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^j\mathrm{div}u\nabla^j\varrho\mathrm{d}x\\=&-\int_{\mathbb{R}^3}\nabla^j\mathrm{div}(\varrho u)\nabla^j\varrho\mathrm{d}x\\=&-\int_{\mathbb{R}^3}\nabla^j(\varrho\mathrm{div}u)\cdot\nabla^j\varrho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^j(u\cdot\nabla^j\varrho)\cdot\nabla^j\varrho\mathrm{d}x.\end{aligned}\tag{3.18}$$

We shall estimate each term on the right-hand side of (3.18). By using integration by parts, (3.4), Lemma A.1 and Hölder's inequality, one has

$$\left|\int_{\mathbb{R}^3}\nabla^j(\varrho\mathrm{div}u)\cdot\nabla^j\varrho\mathrm{d}x\right|$$

$$
\begin{aligned}
&\lesssim \|\varrho\|_{L^\infty}\|\nabla^{j+1}u\|_{L^2}\|\nabla^j\varrho\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}^2\\
&\quad +\mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m\varrho\|_{L^4}\|\nabla^{j-m+1}u\|_{L^4}\|\nabla^j\varrho\|_{L^2}\\
&\lesssim \|\varrho\|_{L^\infty}\|\nabla^{j+1}u\|_{L^2}\|\nabla^j\varrho\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}^2\\
&\quad +\mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^{\frac{5}{2}}u\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^{j+1}u\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}\\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2+\|\nabla^{j+1}u\|_{L^2}^2),
\end{aligned}\tag{3.19}
$$

and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\nabla^j(u\cdot\nabla\varrho)\cdot\nabla^j\varrho \mathrm{d}x\right|\\
&\lesssim \left|\int_{\mathbb{R}^3}\mathrm{div}u|\nabla^j\varrho|^2\mathrm{d}x\right|+\|\nabla u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}^2+\|\nabla\varrho\|_{L^\infty}\|\nabla^j u\|_{L^2}\|\nabla^j\varrho\|_{L^2}\\
&\qquad +\mathcal{H}(j-2)\sum_{m=2}^{j-1}\|\nabla^m u\|_{L^4}\|\nabla^{j-m+1}\varrho\|_{L^4}\|\nabla^j\varrho\|_{L^2}\\
&\lesssim \|\nabla u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}^2+\|\nabla\varrho\|_{L^\infty}\|\nabla^j u\|_{L^2}\|\nabla^j\varrho\|_{L^2}\\
&\qquad +\mathcal{H}(j-2)\sum_{m=2}^{j-1}\|\nabla^{\frac{5}{2}}u\|_{L^2}^{\frac{4(j-m)-3}{4j-10}}\|\nabla^j u\|_{L^2}^{\frac{4m-7}{4j-10}}\|\nabla^{\frac{5}{2}}\varrho\|_{L^2}^{\frac{4m-7}{4j-10}}\|\nabla^j\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-10}}\|\nabla^j\varrho\|_{L^2}\\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2+\|\nabla^j u\|_{L^2}^2),
\end{aligned}\tag{3.20}
$$

where $\mathcal{H}=\mathcal{X}(0,\infty)$ is the Heaviside function

Combining (3.18) – (3.20), by Cauchy's inequality, we deduce

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j\varrho|^2\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^j\mathrm{div}u\nabla^j\varrho\mathrm{d}x\lesssim\delta(\|\nabla^j\varrho\|_{L^2}^2+\|\nabla^j u\|_{L^2}^2+\|\nabla^{j+1}u\|_{L^2}^2). \tag{3.21}
$$

Step 3. L^2 estimate of $\nabla^j u$. Applying the operator ∇^j to $(2.1)_2$, multiplying the resulting equation by $\nabla^j u$, and integrating over $\mathbb{R}^3$, we can deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j u|^2\mathrm{d}x-\gamma\int_{\mathbb{R}^3}\nabla^j\mathrm{div}u\nabla^j\varrho\mathrm{d}x+\mu\int_{\mathbb{R}^3}|\nabla^{j+1}u|^2\mathrm{d}x\\
&\qquad+\mu\int_{\mathbb{R}^3}|\nabla^j\mathrm{div}u|^2\mathrm{d}x-\mu_1\int_{\mathbb{R}^3}\nabla^j\mathrm{div}\tau\nabla^j u\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}\nabla^j(u\cdot\nabla u)\cdot\nabla^j u\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^j[g(\varrho)\nabla\varrho]\cdot\nabla^j u\mathrm{d}x\\
&\qquad-\mu\int_{\mathbb{R}^3}\nabla^j[f(\varrho)(\triangle u+\nabla\mathrm{div}u)]\cdot\nabla^j u\mathrm{d}x-\mu_1\int_{\mathbb{R}^3}\nabla^j[f(\varrho)\mathrm{div}\tau]\cdot\nabla^j u\mathrm{d}x\\
&:=\sum_{i=1}^4 I_i.
\end{aligned}\tag{3.22}
$$

Next, we shall estimate the terms on the right-hand side of (3.22) one by one. For the term I_1, employing a similar argument used in proof of (3.20), we have

$$
|I_1|\lesssim\delta\|\nabla^j u\|_{L^2}^2. \tag{3.23}
$$

For the term I_2, due to (3.2),(3.4), we have from Lemma A.1 that

$$\begin{aligned}
|I_2| &= \left|\int_{\mathbb{R}^3} \nabla^{j-1}[g(\varrho)\nabla\varrho]\cdot\nabla^{j+1}u\mathrm{d}x\right| \\
&\lesssim \|g(\varrho)\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}\|\nabla^{j+1}u\|_{L^2} + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m g(\varrho)\|_{L^4}\|\nabla^{j-m}\varrho\|_{L^4}\|\nabla^{j+1}u\|_{L^2} \\
&\lesssim \|\varrho\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}\|\nabla^{j+1}u\|_{L^2} \\
&\quad + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^{j+1}u\|_{L^2} \\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2 + \|\nabla^{j+1}u\|_{L^2}^2).
\end{aligned} \tag{3.24}$$

For the term I_3, applying a similar argument used in (3.24), we obtain

$$\begin{aligned}
|I_3| &= \left|-\mu\int_{\mathbb{R}^3}\nabla^j[f(\varrho)(\triangle u+\nabla\mathrm{div}u)]\cdot\nabla^j u\mathrm{d}x\right| \\
&\approx \left|\mu\int_{\mathbb{R}^3}\nabla^{j-1}(f(\varrho)\nabla^2 u)\cdot\nabla^{j+1}u\mathrm{d}x\right| \\
&\lesssim \|g(\varrho)\|_{L^\infty}\|\nabla^{j+1}u\|_{L^2}^2 + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m f(\varrho)\|_{L^4}\|\nabla^{j-m+1}u\|_{L^4}\|\nabla^{j+1}u\|_{L^2} \\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2 + \|\nabla^{j+1}u\|_{L^2}^2).
\end{aligned} \tag{3.25}$$

And, for the term I_4, one has

$$\begin{aligned}
|I_4| &= \left|\mu_1\int_{\mathbb{R}^3}\nabla^{j-1}[f(\varrho)\mathrm{div}\tau]\cdot\nabla^{j+1}u\mathrm{d}x\right| \\
&\lesssim \|f(\varrho)\|_{L^\infty}\|\nabla^j\tau\|_{L^2}\|\nabla^{j+1}u\|_{L^2} \\
&\quad + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m f(\varrho)\|_{L^4}\|\nabla^{j-m-1}\mathrm{div}\tau\|_{L^4}\|\nabla^{j+1}u\|_{L^2} \\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2 + \|\nabla^{j+1}u\|_{L^2}^2 + \|\nabla^j\tau\|_{L^2}^2).
\end{aligned} \tag{3.26}$$

Combining (3.23)-(3.26), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j u|^2\mathrm{d}x - \gamma\int_{\mathbb{R}^3}\nabla^j\mathrm{div}u\nabla^j\varrho\mathrm{d}x + \mu\int_{\mathbb{R}^3}|\nabla^{j+1}u|^2\mathrm{d}x \\
&\quad + \mu\int_{\mathbb{R}^3}|\nabla^j\mathrm{div}u|^2\mathrm{d}x - \mu_1\int_{\mathbb{R}^3}\nabla^j\mathrm{div}\tau\nabla^j u\mathrm{d}x \\
&\lesssim \delta(\|\nabla^j\varrho\|_{L^2}^2 + \|\nabla^j u\|_{L^2}^2 + \|\nabla^{j+1}u\|_{L^2}^2 + \|\nabla^j\tau\|_{L^2}^2).
\end{aligned} \tag{3.27}$$

Step 4. L^2 estimate of $\nabla^j\tau$. Applying the operator ∇^j to $(2.1)_3$, multiplying the resulting equation by $\nabla^j\tau$, and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j\tau|^2\mathrm{d}x + a\int_{\mathbb{R}^3}|\nabla^j\tau|^2\mathrm{d}x - \mu_2\int_{\mathbb{R}^3}\nabla^j D(u)\nabla^j\tau\mathrm{d}x \\
&= -\int_{\mathbb{R}^3}\nabla^j(u\cdot\nabla\tau)\cdot\nabla^j\tau\mathrm{d}x - \int_{\mathbb{R}^3}\nabla^j Q(\tau,\nabla u)\cdot\nabla^j\tau\mathrm{d}x - b\int_{\mathbb{R}^3}\nabla^j(\mathrm{tr}\tau\tau)\cdot\nabla^j\tau\mathrm{d}x := \sum_{i=1}^3 J_i.
\end{aligned} \tag{3.28}$$

Next, we shall estimate the terms on the right-hand side of (3.28) one by one. For the term J_1, employing a similar argument used in proof of (3.20), we have

$$|J_1| \lesssim \delta(\|\nabla^j u\|_{L^2}^2 + \|\nabla^j \tau\|_{L^2}^2). \tag{3.29}$$

Similarly,

$$|J_2| \lesssim \delta(\|\nabla^j u\|_{L^2}^2 + \|\nabla^j \tau\|_{L^2}^2), \tag{3.30}$$

and

$$|J_3| \lesssim \delta\|\nabla^j \tau\|_{L^2}^2. \tag{3.31}$$

Combining (3.29) – (3.31), we deduce from (3.28) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^j\tau|^2\mathrm{d}x + a\int_{\mathbb{R}^3}|\nabla^j\tau|^2\mathrm{d}x - \mu_2\int_{\mathbb{R}^3}\nabla^j D(u)\nabla^j\tau\mathrm{d}x \lesssim \delta(\|\nabla^j u\|_{L^2}^2 + \|\nabla^j\tau\|_{L^2}^2). \tag{3.32}$$

Step 5. Dissipation of $\nabla^j\varrho^h$. Applying the operator $\nabla^{j-1}\mathfrak{F}^{-1}(1-\phi(\xi))$ to $(2.1)_2$, multiplying the resulting equation by $\nabla^{j-1}\nabla\varrho^h$, and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned}
\gamma\int_{\mathbb{R}^3}|\nabla^j\varrho^h|^2\mathrm{d}x &= -\int_{\mathbb{R}^3}\nabla^{j-1}u_t^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x + \mu\int_{\mathbb{R}^3}\nabla^{j-1}(\triangle u^h + \nabla \mathrm{div} u^h)\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x\\
&\quad + \mu_1\int_{\mathbb{R}^3}\nabla^{j-1}\mathrm{div}\tau^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x + \int_{\mathbb{R}^3}\nabla^{j-1}S_2^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x\\
&:= \sum_{i=1}^4 K_i.
\end{aligned} \tag{3.33}$$

For first term on the right-hand side of (3.33), by $(2.1)_1$, Lemma A.1, and Lemma A.4, we can use integration by parts, Hölder's inequality and Young's inequality to deduce that

$$\begin{aligned}
|K_1| &= -\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x - \int_{\mathbb{R}^3}\nabla^{j-1}\mathrm{div}u^h\cdot\nabla^{j-1}\varrho_t^h\mathrm{d}x\\
&= -\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla^{j-1}\mathrm{div}u^h|^2\mathrm{d}x + \int_{\mathbb{R}^3}\nabla^{j-1}\mathrm{div}u^h\nabla^{j-1}\mathrm{div}(\varrho u)^h\mathrm{d}x\\
&\lesssim -\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x + \|\nabla^{j-1}\mathrm{div}u\|_{L^2}^2 + \|\varrho\|_{L^\infty}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}^2\\
&\quad + \|u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}\|\nabla^{j-1}\mathrm{div}u\|_{L^2} + \|\varrho\|_{L^\infty}\|\nabla^j u\|_{L^2}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}\\
&\quad + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m\varrho\|_{L^4}\|\nabla^{j-m-1}\mathrm{div}u\|_{L^4}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}\\
&\quad + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^m u\|_{L^4}\|\nabla^{j-m}\varrho\|_{L^4}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}\\
&\lesssim -\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x + \|\nabla^{j-1}\mathrm{div}u\|_{L^2}^2\\
&\quad + \|\varrho\|_{L^\infty}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}^2 + \|u\|_{L^\infty}\|\nabla^j\varrho\|_{L^2}\|\nabla^{j-1}\mathrm{div}u\|_{L^2} + \|\varrho\|_{L^\infty}\|\nabla^j u\|_{L^2}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}\\
&\quad + \mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^{\frac{3}{2}}u\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^j u\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{H}(j-1)\sum_{m=1}^{j-1}\|\nabla^{\frac{3}{2}}u\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^j u\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^{\frac{3}{2}}\varrho\|_{L^2}^{\frac{4m-3}{4j-6}}\|\nabla^j\varrho\|_{L^2}^{\frac{4(j-m)-3}{4j-6}}\|\nabla^{j-1}\mathrm{div}u\|_{L^2}\\
&\lesssim -\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x+\|\nabla^{j-1}\mathrm{div}u\|_{L^2}^2+\delta(\|\nabla^j\varrho\|_{L^2}^2+\|\nabla^j u\|_{L^2}^2). \qquad (3.34)
\end{aligned}
$$

Employing Hölder's inequality and Young's inequality, the terms K_2 and K_3 can be bounded by

$$
|K_2|+|K_3|\lesssim\frac{\gamma}{4}\int_{\mathbb{R}^3}|\nabla^j\varrho^h|^2\mathrm{d}x+C(\|\nabla^{j+1}u\|_{L^2}^2+\|\nabla^j\tau\|_{L^2}^2). \qquad (3.35)
$$

For the last term K_4, we rewrite it as

$$
\begin{aligned}
K_4=&-\int_{\mathbb{R}^3}\nabla^{j-1}(u\cdot\nabla u)^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^{j-1}[g(\varrho)\nabla\varrho]^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x\\
&-\mu\int_{\mathbb{R}^3}\nabla^{j-1}[f(\varrho)(\triangle u+\nabla\mathrm{div}u)]^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x-\mu_1\int_{\mathbb{R}^3}\nabla^{j-1}[f(\varrho)\mathrm{div}\tau]^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x\\
:=&\sum_{i=1}^4 K_{4i}. \qquad (3.36)
\end{aligned}
$$

Now we shall estimate the four terms on the right-hand side of (3.36) as follows. For the first term, employing a similar argument used in the proof of (3.19), it holds that

$$
|K_{41}|\lesssim\delta(\|\nabla^j\varrho^h\|_{L^2}^2+\|\nabla^j u\|_{L^2}^2). \qquad (3.37)
$$

Similarly, for the terms $K_{42}-K_{44}$, we also have

$$
|K_{42}|+|K_{43}|+|K_{44}|\lesssim\delta(\|\nabla^j\varrho^h\|_{L^2}^2+\|\nabla^j u\|_{L^2}^2+\|\nabla^{j+1}u\|_{L^2}^2+\|\nabla^j\tau\|_{L^2}^2). \qquad (3.38)
$$

Plugging the estimates (3.34) – (3.38) into (3.33), by Cauchy's inequality, since δ is small, we then obtain

$$
\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x+\frac{\gamma}{2}\int_{\mathbb{R}^3}|\nabla^j\varrho^h|^2\mathrm{d}x\lesssim\|\nabla^j u\|_{L^2}^2+\|\nabla^{j+1}u\|_{L^2}^2+\|\nabla^j\tau\|_{L^2}^2. \qquad (3.39)
$$

Step 6. Closing the estimates. Now, we are in a position to close the estimates. To do this, for $1\le k\le\ell$, we define the temporal energy functional

$$
\mathcal{E}_k^\ell(t)=\frac{D_1}{2}\sum_{j=k}^{\ell}\int_{\mathbb{R}^3}\left(\mu_2\gamma|\nabla^j\varrho|^2+\mu_2|\nabla^j u|^2+\mu_1|\nabla^j\tau|^2\right)\mathrm{d}x+\sum_{j=k}^{\ell}\int_{\mathbb{R}^3}\nabla^{j-1}u^h\cdot\nabla\nabla^{j-1}\varrho^h\mathrm{d}x,
$$

Note that $\mathcal{E}_k^\ell(t)$ is equivalent to $\|\nabla^k(\varrho,u,\tau)\|_{H^{\ell-k}}^2$ if we choose D_2 large enough. Substituting (3.39) into

$$
D_1\times[\mu_2\times(\gamma\times(3.21)+(3.27))+\mu_1\times(3.32)],
$$

and summing up the resulting inequality from $k\le j\le\ell$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_k^\ell(t)+\frac{1}{D_1}\mathcal{E}_k^\ell(t)\lesssim\|\nabla^k(\varrho^l,u^l,\tau^l)\|_{H^{\ell-k}}^2.
$$

Applying Gronwall's inequality and (3.17) to the above inequality, we can infer that

$$\mathcal{E}_k^\ell(t) \le C(N_0)(1+t)^{-\frac{3}{2}-k}. \tag{3.40}$$

Consequently, (3.14) follows from (3.40) and the definition of $\mathcal{E}_k^\ell(t)$ given in (3.5) immediately. Therefore, we have completed the proof of Lemma 3.2. ∎

Lemma 3.3. *Under the assumption of Theorem 1.1, for any $t\ge 0$, and $0\le k\le \ell-1$, it holds that*

$$\left\|\nabla^k\tau(t)\right\|_{H^{\ell-k}} \le CK_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}. \tag{3.41}$$

Proof. Applying ∇^k to $(2.1)_3$ and multiplying the resultant identity by $\nabla^k\tau$, then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^k\tau|^2dx + a\int_{\mathbb{R}^3}|\nabla^k\tau|^2dx\\
=&-\int_{\mathbb{R}^3}\nabla^k(u\cdot\nabla\tau+Q(\tau,\nabla u)+b\mathrm{tr}\tau\tau)\cdot\nabla^k\tau dx+\mu_2\int_{\mathbb{R}^3}\nabla^k D(u)\cdot\nabla^k\tau dx\\
:=&\sum_{i=1}^4 P_i.
\end{aligned}\tag{3.42}$$

We now estimate the term P_1, by Lemma A.2 and Hölder's inequality, we get

$$\begin{aligned}
P_1=&-\int_{\mathbb{R}^3}\nabla^k(u\cdot\nabla\tau)\cdot\nabla^k\tau dx\\
=&-\int_{\mathbb{R}^3}\nabla^k(u\cdot\nabla\nabla^k\tau+[\nabla^k,u]\nabla\tau)\cdot\nabla^k\tau dx\\
\lesssim&\|\nabla u\|_{L^\infty}\|\nabla^k\tau\|_{L^2}^2+\|\nabla^k u\|_{L^6}\|\nabla\tau\|_{L^3}\|\nabla^k\tau\|_{L^2}\\
\lesssim&\delta(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2)\\
\lesssim&\delta\|\nabla^k\tau\|_{L^2}^2+(1+t)^{-\frac{5}{2}-k}.
\end{aligned}\tag{3.43}$$

Similarly, the terms P_2-P_4 can be estimated as

$$P_2\lesssim\delta(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2)\lesssim\delta\|\nabla^k\tau\|_{L^2}^2+(1+t)^{-\frac{5}{2}-k}. \tag{3.44}$$

$$P_3\lesssim\delta\|\nabla^k\tau\|_{L^2}^2. \tag{3.45}$$

$$P_4\lesssim\frac{a}{2}\|\nabla^k\tau\|_{L^2}^2+\frac{\mu_2^2}{2a}\|\nabla^{k+1}u\|_{L^2}^2\le\frac{a}{2}\|\nabla^k\tau\|_{L^2}^2+C(1+t)^{-\frac{5}{2}-k}. \tag{3.46}$$

Combining (3.43) – (3.46), we deduce from (3.42) that

$$\frac{d}{dt}\|\nabla^k\tau\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2\le C(1+t)^{-\frac{5}{2}-k}.$$

This, together with the Gronwall's inequality implies (3.41). ∎

In what follows, we devote ourselves to deducing the lower bound on the convergence rate of the global solution to complete the proof of Theorem 1.1.

THEOREM 3.3. *Assume that the hypotheses of Theorem 1.1 and (1.9) are in force. Then there is a positive constant c_1 independent of t such that for any large enough t and $0 \le k \le \ell$, it holds that*

$$\min\{\|\nabla^k(\varrho,u)(t)\|_{L^2}\} \ge c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{3.47}$$

$$\|\nabla^k\tau(t)\|_{L^2} \ge c_1(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{3.48}$$

$$\min\{\|\nabla^k(\varrho,u)(t)\|_{L^p}\} \ge c_1(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{3.49}$$

$$\|\nabla^k\tau(t)\|_{L^p} \ge c_1(1+t)^{-\frac{3}{2}(\frac{4}{3}-\frac{1}{p})}. \tag{3.50}$$

Proof. If t is large enough, it follows from (3.9), Proposition 2.6, Lemma 3.4 and Lemma A.4 that

$$\begin{aligned}
\left\|\Lambda^{-1}(\varrho,u)(t)\right\|_{L^2} \le& \left\|\Lambda^{-1}(\varrho^l,u^l)(t)\right\|_{L^2} + \left\|\Lambda^{-1}(\varrho^h,u^h)(t)\right\|_{L^2} \\
\le& C(N_0)(1+t)^{-\frac{1}{4}} + \int_0^t (1+t-s)^{-\frac{1}{4}} \|\mathcal{S}(s)\|_{L^1}\,\mathrm{d}s + \left\|(\varrho^h,u^h)(t)\right\|_{L^2} \\
\le& C(N_0)\left((1+t)^{-\frac{1}{4}} + \int_0^t (1+t-s)^{-\frac{1}{4}}(1+s)^{-\frac{3}{2}}\,\mathrm{d}s\right) \\
\le& C(N_0)(1+t)^{-\frac{1}{4}},
\end{aligned}$$

$$\begin{aligned}
\left\|\Lambda^{-1}\tau(t)\right\|_{L^2} \le& \left\|\Lambda^{-1}\tau^l(t)\right\|_{L^2} + \left\|\Lambda^{-1}\tau^h(t)\right\|_{L^2} \\
\le& C(N_0)(1+t)^{-\frac{3}{4}} + \int_0^t (1+t-s)^{-\frac{3}{4}} \|\mathcal{S}(s)\|_{L^1}\,\mathrm{d}s + \left\|\tau^h(t)\right\|_{L^2} \\
\le& C(N_0)\left((1+t)^{-\frac{3}{4}} + \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{2}}\,\mathrm{d}s\right) \\
\le& C(N_0)(1+t)^{-\frac{3}{4}},
\end{aligned}$$

and

$$\begin{aligned}
\min\|(\varrho,u)(t)\|_{L^2} \ge& \min\left\|(\varrho^l,u^l)(t)\right\|_{L^2} \\
\ge& C_1\sqrt{\delta}N_0(1+t)^{-\frac{3}{4}} - \int_0^t (1+t-s)^{-\frac{3}{4}} \|\mathcal{S}(s)\|_{L^1}\,\mathrm{d}s \\
\ge& C_1\sqrt{\delta}N_0(1+t)^{-\frac{3}{4}} - C\delta N_0(1+t)^{-\frac{3}{4}} \\
\ge& c_2(1+t)^{-\frac{3}{4}}.
\end{aligned}$$

Similarly, we can prove

$$\|\tau(t)\|_{L^2} \ge \left\|\tau^l(t)\right\|_{L^2} \ge c_2(1+t)^{-\frac{5}{4}}.$$

Since N_0 is sufficiently small. These together with the interpolation inequality

$$\|f\|_{L^2} \le C\|\Lambda^{-1}f\|_{L^2}^{\frac{k}{k+1}} \|\nabla^k f\|_{L^2}^{\frac{1}{k+1}}$$

and

$$\|f\|_{L^2} \le C\|\Lambda^{-1}f\|_{L^2}^{\frac{3p-6}{5p-6}} \|f\|_{L^p}^{\frac{2p}{5p-6}}$$

imply (3.47)-(3.50) immediately, and thus the proof of Theorem 3.3 is completed. ∎

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Appendix. Analytic tools. We recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

LEMMA A.1. *Let $0\le i,j\le k$, then we have*

$$\|\nabla^i f\|_{L^p}\lesssim \|\nabla^j f\|_{L^q}^{1-a}\|\nabla^k f\|_{L^r}^{a}$$

where a satisfies

$$\frac{i}{3}-\frac{1}{p}=\left(\frac{j}{3}-\frac{1}{q}\right)(1-a)+\left(\frac{k}{3}-\frac{1}{r}\right)a.$$

Especially, while $p=q=r=2$, we have

$$\|\nabla^i f\|_{L^2}\lesssim \|\nabla^j f\|_{L^2}^{\frac{k-i}{k-j}}\|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}.$$

Proof. *This is a special case of [22, pp. 125, Theotem].* ∎

Next, to estimate the L^p–norm of the spatial derivatives of the product of two functions, we shall recall the following estimate:

LEMMA A.2 ([12]). *Let $m\ge 1$ be an integer and define the commutator*

$$[\nabla^m, f]g=\nabla^m(fg)-f\nabla^m g,$$

then we have

$$\|[\nabla^m,f]g\|_{L^p}\lesssim \|\nabla f\|_{L^{p_1}}\|\nabla^{m-1}g\|_{L^{p_2}}+\|\nabla^m f\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{A.1}$$

and for $m\ge 0$

$$\|\nabla^m(fg)\|_{L^p}\lesssim \|f\|_{L^{p_1}}\|\nabla^m g\|_{L^{p_2}}+\|\nabla^m f\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{A.2}$$

where $p,p_2,p_3\in[1,\infty]$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$.

LEMMA A.3 ([26]). *Assume that $\|\varrho\|_{L^\infty}\le 1$ and $p>1$. Let $g(\varrho)$ be a smooth function of ϱ with bounded derivatives of any order, then for any integer $m\ge 1$, we have*

$$\|\nabla^m g(\varrho)\|_{L^p}\lesssim \|\nabla^m \varrho\|_{L^p}.$$

Finally, the following two lemmas concern the estimate for the low-frequency part and the high-frequency part of f.

LEMMA A.4. *If $f\in L^p(\mathbb{R}^3)$ for any $2\le p\le\infty$, then we have*

$$\|f^l\|_{L^p}+\|f^h\|_{L^p}\lesssim \|f\|_{L^p}.$$

Proof. For $2 \le p \le \infty$, by virtue of Young's inequality for convolutions, for the low frequency, it holds that

$$\|f^l\|_{L^p} \lesssim \|\mathfrak{F}^{-1}\phi\|_{L^1}\|f\|_{L^p} \lesssim \|f\|_{L^p},$$

and hence

$$\|f^h\|_{L^p} \lesssim \|f\|_{L^p} + \|f^l\|_{L^p} \lesssim \|f\|_{L^p}.$$

∎

Lemma A.5. *Let $f \in H^k(\mathbb{R}^3)$ for any integer $k \ge 2$. Then there exists a positive constant C_0 such that*

$$\|\nabla^j f^h\|_{L^2} \le C_0\|\nabla^{j+1} f\|_{L^2},$$

and

$$\|\nabla^{j+1} f^l\|_{L^2} \le C_0\|\nabla^j f\|_{L^2},$$

for any $0 \le j \le k-1$.

Proof. This lemma can be shown directly by the definitions of the low–frequency and high–frequency of f and the Plancherel theorem, and thus we omit the details. ∎

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