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A NON-EQUILIBRIUM MULTI-COMPONENT MODEL WITH MISCIBLE CONDITIONS*

JEAN BUSSAC†

Abstract. This paper concerns the study of a full non-equilibrium model for a compressible mixture of any number of phases. Miscible conditions are considered in one phase, which lead to non-symmetric constraints on the statistical fractions. These models are subject to the choice of interfacial and source terms. We show that under a standard assumption on the interfacial velocity, the interfacial pressures are uniquely defined. The model is hyperbolic and symmetrizable under non-resonance conditions. Classes of entropy-consistent source terms are then proposed.

Keywords. Multiphase flows; Baer-Nunziato; Nonconservative; Closure laws.

AMS subject classifications. 35L60; 35L65; 76T30.

1. Introduction

The present paper concerns the modelling of multiphase compressible flows which arise in many industrial applications, especially in the nuclear safety framework. Among the several scenarios that are studied (see IRSN website [19]), the Loss Of Coolant Accident (LOCA) involves these types of flows. It is a brutal rupture in the coolant circuit that creates phase transition waves in the loop and leads to the appearance of vapor inside the liquid. These phases can interact with the ambient air at the breach. Other types of accidents require to consider more phases, for instance in vapor explosion type scenarios [3]. Consequently, a model must account for the dynamical and the thermodynamical disequilibrium.

A wide range of multiphase flow models has been proposed since decades, especially in the two-phase flow situation, see for instance [1,4,7,11,20] for the Baer-Nunziato two-fluid approach and [2,17,22] for the homogeneous (in terms of velocity) models. More recently, three-phase flows have been investigated with full non-equilibrium models in the immiscible [10,13,14] and miscible [12] conditions, and also with homogeneous models [2,17,22]. Among all these models, two classes can be distinguished: the first one is based on the multifluid approach [1,4,10–13,20], where each component has its own velocity field. The second one corresponds to the homogeneous models [2,18,22] where all the component share the same velocity.

In the multifluid approach, the dynamics of each component is described by an Euler-type model and the different systems are coupled through interfacial non-conservative terms and source terms which model the return to the equilibrium. Closure conditions have been investigated in the immiscible case in [4,13] respectively for two-phase and three-phase flows. Analysis of this model for a two-phase flow has been done for instance in [6,8], and a generalization with all these features has been proposed in [23]. More recently, these type of results have been investigated with hybrid miscible-immiscible conditions in [11,12]. In the continuity of these works, this paper is a generalization of [23] by adding miscible components, and of [11,12] by considering an arbitrary number of phases.

For the sake of clarity, let us specify that the term "phase" refers to a state of the

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†Laboratoire de Mathématiques Jean Leray, Nantes Université, F-44000 Nantes, France (jean.bussac@univ-nantes.fr).

matter, "component" to a chemical substance and "field" to a component in a given phase. For instance, a hybrid mixture of liquid water, vapor water and an inert gas is composed of three fields, two phases and two components, see [11]. This configuration will be given as the main example throughout the article.

The present work focuses on the modelling of a N-phase mixture of M fields, with $N \leq M$, where the whole miscible part of the fluid is contained in the N-th phase. That is a physically relevant situation where this latter phase would be the gaseous phase, and no miscibility can be observed for another state of the matter. These miscible components are then supposed to be perfectly intricate like ideal gases, and so all their statistical fractions are equal

$$\forall k \geq N,\ \alpha_k = \alpha_N. \tag{1.1}$$

Since we consider that no void can occur, the global volume constraint is

$$\sum_{k=1}^{N} \alpha_k = 1, \tag{1.2}$$

with $\alpha_k \in [0;1]$ for all $k = 1, ..., N$. The equations are written in the one-dimensional case for the sake of clarity. This work can be extended to the multi-dimensional situation.

The outline of the article is the following.

The first part presents the model by introducing the thermodynamical framework, the system of equations and the evolution equation of the mixture entropy. Then we demonstrate the uniqueness of the interfacial pressure terms under classical hypotheses concerning the interfacial velocity. In the third part, the analysis of the convective system is investigated, by studying its hyperbolicity and the symmetrization. Finally, admissible forms of source terms are detailed, in order to satisfy the second principle of thermodynamics.

2. Model

2.1. Thermodynamical framework.

The fluid is composed of $N \geq 2$ phases that are immiscible, where only the N-th phase can contain several components of number $K \geq 1$. The fluid is then composed of $M = N + K - 1$ fields. In other words, we consider a mixture of N phases and M fields, where the phase N contains all the miscible components of the fluid. We define $\mathcal{K}$ the set of fields, and thus we have $\#\mathcal{K} = M$.

1	2	3	...	N, N+1, ..., M

In the hybrid two-phase configuration, M is equal to 3, N is equal to 2 and the schematic representation is | l | v, g |. Later on, we will refer to this configuration as the $\{N = 2, M = 3\}$ case, where $l = 1$, $v = 2$ and $g = 3$.

2.1.1. Equations of state.

Each field $k \in \mathcal{K}$ is depicted by its phasic specific volume $\tau_k > 0$ and its specific energy $e_k > 0$. The thermodynamical behaviour of each phase k is fully described by its intensive entropy function $(\tau_k, e_k) \mapsto s_k(\tau_k, e_k)$ defined on $\Omega_k \subset (\mathbb{R}_*^+)^2$ that is supposed to be a convex set. Such an entropy function is a complete equation of state and it will always be supposed to be concave. By adopting the Gibbs formalism, each entropy function s_k complies with the following differential form

$$T_k ds_k = de_k + p_k d\tau_k, \tag{2.1}$$

where the phasic temperature $T_k = T_k(\tau_k, e_k)$ and pressure $p_k = p_k(\tau_k, e_k)$ are defined by

$$\frac{1}{T_k} = \left.\frac{\partial s_k}{\partial e_k}\right|_{\tau_k}, \qquad p_k = T_k \left.\frac{\partial s_k}{\partial \tau_k}\right|_{e_k}, \tag{2.2}$$

and the phasic chemical potential by the relation

$$\mu_k = -T_k s_k + p_k \tau_k + e_k. \tag{2.3}$$

The phasic temperatures T_k will be supposed to be stricly positive.

2.1.2. Volume constraints and mixture entropy. We now turn to the description of the thermodynamical behaviour of the mixture. The volume conservation and the miscibility hypotheses give the following relations on the statistical fractions

$$\begin{cases} \sum_{k=1}^{N} \alpha_k = 1 \\ \alpha_k = \alpha_N \text{ for } k \geq N, \end{cases} \tag{2.4}$$

thus the derivatives satisfy

$$\sum_{k=1}^{N-1} \partial_x \alpha_k = -\partial_x \alpha_N. \tag{2.5}$$

In the sequel, several mathematical proofs rely on these constraints. We define the state vector of the field k by

$$\mathbf{Y}_k = (\alpha_k, \rho_k, v_k, e_k),$$

where $\rho_k = 1/\tau_k$ and v_k is the phasic velocity. For a given mixture state (τ, e), the mixture entropy is defined as a combination of the phasic entropies with weights $m_k = \alpha_k \rho_k$. Denoting $\mathbf{Y} = \bigcup_{k \in \mathcal{K}} \mathbf{Y}_k$, it reads

$$\sigma(\mathbf{Y}) = \sum_{k \in \mathcal{K}} m_k s_k(\mathbf{Y}_k). \tag{2.6}$$

Finally, the specific total energy of the phase k is noted $E_k = e_k + v_k^2/2$.

2.2. Set of partial differential equations. Let $\boldsymbol{W}$ be the main unknowns vector, defined by

$$\boldsymbol{W} = (\alpha_1, \ldots, \alpha_N, \boldsymbol{W}_1^\top, \ldots \boldsymbol{W}_M^\top)^\top, \quad \boldsymbol{W}_k = \begin{pmatrix} m_k \\ m_k v_k \\ m_k E_k \end{pmatrix}. \tag{2.7}$$

Following the models in [11–13, 23], the fluid equations contain N transport equations on α_k, for $k = 1, \ldots, N$

$$\partial_t \alpha_k + V_I(\mathbf{Y}) \partial_x \alpha_k = \Phi_k(\mathbf{Y}), \tag{2.8}$$

plus, for $k \in \mathcal{K}$, the following Euler-type systems

$$\partial_t \mathbf{W}_k + \partial_x \mathbf{f}(\mathbf{W_k}) + \boldsymbol{C}_k \boldsymbol{\partial}_x \boldsymbol{\alpha} = \mathbf{S}_k, \tag{2.9}$$

where

$$\mathbf{f}(\mathbf{W})_k=\begin{pmatrix} m_k v_k \\ m_k v_k^2+\alpha_k p_k \\ m_k v_k(E_k+\frac{p_k}{\rho_k}) \end{pmatrix},\ \boldsymbol{\alpha}=\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix},\ \mathbf{S}_k=\begin{pmatrix} \Gamma_k(\mathbf{Y}) \\ S_{q,k}(\mathbf{Y}) \\ S_{E,k}(\mathbf{Y}) \end{pmatrix} \tag{2.10}$$

and

$$\boldsymbol{C}_k=\begin{pmatrix} 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ P_{k,1} & \cdots & P_{k,k-1} & 0 & P_{k,k+1} & \cdots & P_{k,M} \\ V_I P_{k,1} & \cdots & V_I P_{k,k-1} & 0 & V_I P_{k,k+1} & \cdots & V_I P_{k,M} \end{pmatrix}, \tag{2.11}$$

where the k-th column only contains null terms. In the sequel, the vector $\mathbf{Y}=\bigcup_{k\in\mathcal{K}}\mathbf{Y}_k$ will be refered as the secondary unkowns vector.

In order to identify this class of models to practical cases, we refer the reader to [11] for the case $\{N=2,M=3\}$ and [12] for the case $\{N=3,M=4\}$.

Source terms Φ_k, Γ_k, $S_{q,k}$, $S_{E,k}$ must be chosen in agreement with the second law of thermodynamics. Let us emphasize that the $(\Phi_k)_{k=1,\dots,N}$ are only defined for the N first statistical fractions.

Exchanges between the different fields are modeled by non-conservative terms in Equations (2.8) and (2.9), involving interfacial velocity V_I and interfacial pressures $(P_{k,l})$, defined for $k\neq l$. These terms must be specified in order to close the system, which will be done in Section 2.3.

The set of equations for the water-gas case $\{N=2,M=3\}$ is

$$\partial_t\alpha_l+V_I(\mathbf{Y})\partial_x\alpha_l=\Phi_l(\mathbf{Y}),$$

plus

$$\begin{cases} \partial_t(m_l)+\partial_x(m_l v_l)=\Gamma_l(\mathbf{Y}), \\ \partial_t(m_l v_l)+\partial_x(m_l v_l^2+\alpha_l p_l)+\big(P_{l,v}(\mathbf{Y})+P_{l,g}(\mathbf{Y})\big)\partial_x\alpha_l=S_{q,l}(\mathbf{Y}), \\ \partial_t(m_l E_l)+\partial_x(m_l v_l(E_l+\frac{p_l}{\rho_l}))+\big(P_{l,v}(\mathbf{Y})+P_{l,g}(\mathbf{Y})\big)V_I(\mathbf{Y})\partial_x\alpha_l=S_{E,l}(\mathbf{Y}), \end{cases}$$

$$\begin{cases} \partial_t(m_v)+\partial_x(m_v v_v)=\Gamma_v(\mathbf{Y}), \\ \partial_t(m_v v_v)+\partial_x(m_v v_v^2+\alpha_v p_v)-\big(P_{v,l}(\mathbf{Y})+P_{v,g}(\mathbf{Y})\big)\partial_x\alpha_l=S_{q,v}(\mathbf{Y}), \\ \partial_t(m_v E_v)+\partial_x(m_v v_v(E_v+\frac{p_v}{\rho_v}))-\big(P_{v,l}(\mathbf{Y})+P_{v,g}(\mathbf{Y})\big)V_I(\mathbf{Y})\partial_x\alpha_l=S_{E,v}(\mathbf{Y}), \end{cases}$$

$$\begin{cases} \partial_t(m_g)+\partial_x(m_g v_g)=\Gamma_g(\mathbf{Y}), \\ \partial_t(m_g v_g)+\partial_x(m_g v_g^2+\alpha_g p_g)-\big(P_{g,l}(\mathbf{Y})+P_{g,v}(\mathbf{Y})\big)\partial_x\alpha_l=S_{q,g}(\mathbf{Y}), \\ \partial_t(m_g E_g)+\partial_x(m_g v_g(E_g+\frac{p_g}{\rho_g}))-\big(P_{g,l}(\mathbf{Y})+P_{g,v}(\mathbf{Y})\big)V_I(\mathbf{Y})\partial_x\alpha_l=S_{E,g}(\mathbf{Y}), \end{cases}$$

where $\Gamma_g(\mathbf{Y})=0$ since the gas is inert.

REMARK 2.1. Following recent works [11, 12], we consider that source terms and interfacial closures only depend on $\mathbf{Y}$. These could depend on the derivatives of $\mathbf{Y}$, as suggested for instance in [16, 25].

Since we consider a closed system, the total mass, momentum and energy exchanges must balance, thus the source terms must satisfy

$$\sum_{k\in\mathcal{K}}\Gamma_k(\mathbf{Y})=0,\quad \sum_{k\in\mathcal{K}}S_{q,k}(\mathbf{Y})=0,\quad \sum_{k\in\mathcal{K}}S_{E,k}(\mathbf{Y})=0. \tag{2.12}$$

Moreover, as we consider no vacuum occurence, the source terms on the void fractions satisfy

$$\sum_{k=1}^{N}\Phi_k(\mathbf{Y})=0. \tag{2.13}$$

Besides, the interfacial pressure terms $(P_{k,l})$ should cancel each other

$$\sum_{k\in\mathcal{K}}\sum_{\substack{l\in\mathcal{K}\\ l\neq k}}P_{k,l}(\mathbf{Y})\partial_x\alpha_l=0, \tag{2.14}$$

in order to preserve the mixture conservative equations on momentum and energy.

2.2.1. Entropy production. The mixture entropy has been defined in (2.6), and by manipulating evolution equations on the phasic densities, kinetic energies and internal energies [12, Appendix A], we can derive the following equation

$$\partial_t\sigma(\mathbf{Y})+\partial_x f_\sigma(\mathbf{Y})=\mathcal{A}_\sigma(\mathbf{Y},\partial_x\mathbf{Y})+RHS_\sigma(\mathbf{Y}), \tag{2.15}$$

where $f_\sigma(\mathbf{Y})=\sum_{k\in\mathcal{K}}m_k s_k v_k$ is the entropy flux and the production terms are defined by

$$\mathcal{A}_\sigma(\mathbf{Y},\partial_x\mathbf{Y})=\sum_{k\in\mathcal{K}}\frac{1}{T_k}(v_k-V_I)\Big(\sum_{\substack{l\in\mathcal{K}\\ l\neq k}}P_{k,l}(\mathbf{Y})\partial_x\alpha_l+p_k\partial_x\alpha_k\Big), \tag{2.16}$$

that correspond to the interfacial contribution, and

$$\begin{aligned}RHS_\sigma(\mathbf{Y})=\sum_{k\in\mathcal{K}}\frac{1}{T_k}\Big(&S_{E,k}(\mathbf{Y})-\Gamma_k(\mathbf{Y})e_k-v_k(S_{q,k}(\mathbf{Y})\\ &-\frac{1}{2}\Gamma_k(\mathbf{Y})v_k)+\rho_k\frac{\partial e_k}{\partial\rho_k}(\rho_k\Phi_k(\mathbf{Y})-\Gamma_k(\mathbf{Y}))\Big)\\ &+\sum_{k\in\mathcal{K}}\Big(s_k\Gamma_k(\mathbf{Y})+\rho_k\frac{\partial s_k}{\partial\rho_k}(\Gamma_k(\mathbf{Y})-\rho_k\Phi_k(\mathbf{Y}))\Big),\end{aligned}$$

which correspond to the source terms contribution. According to the second law of thermodynamics, these production terms have to be non-negative. By following the work in [4,13] in the two- and three-phase flow context, our concern here is to determine constraints such that

$$\mathcal{A}_\sigma(\mathbf{Y},\partial_x\mathbf{Y})=0, \tag{2.17}$$

which defines the so-called minimal entropy dissipation model [11], which is the subject of the following section. The second term RHS_σ relies on the source terms, which we will study later on, see Section 4.

2.3. Definition of the interfacial pressures. The system of partial differential Equations (2.8)-(2.9) requires a unique definition of the interfacial pressure terms. Derivation of these terms has been done for immiscible two, three and N-phase flow in [4,13,23], or more recently for hybrid mixtures in [11,12].

For this purpose, we postulate that the interfacial velocity is a convex combination (using Galilean invariance) of the phasic velocities

$$V_I(\mathbf{Y}) = \sum_{k\in\mathcal{K}} \beta_k(\mathbf{Y}) v_k, \tag{2.18}$$

with

$$\begin{cases} \sum_{k\in\mathcal{K}} \beta_k(\mathbf{Y}) = 1, \\ \beta_k(\mathbf{Y}) \geq 0 \ \forall k \in \mathcal{K}. \end{cases} \tag{2.19}$$

Determining the interfacial pressures $(P_{k,l})$ as it is done in [23] is out of reach due to the miscibility constaints, that correspond mathematically to adding unkowns without adding equations. However, these constraints allow to consider another set of pressure terms that we define here.

Let us introduce the interfacial terms $(K_{k,l})$, defined for $k=1,...,M$ and $l=1,...,N-1$. For the sake of clarity, we allow ourselves to write the initial pressure terms $(P_{k,l})$ for $k=l$.

Definition 2.1. *For $k=1,...,M$ and $l=1,...,N-1$, we define*

$$K_{k,l} = P_{k,l}(1-\delta_{k,l}) - \sum_{j=N}^{M} P_{k,j}(1-\delta_{k,j}). \tag{2.20}$$

The coefficients $(\chi_{k,l})$ are defined for the same indices by

$$\chi_{k,l} = \delta_{k,l} - \sum_{j=N}^{M} \delta_{k,j} = \begin{cases} 0 \text{ if } k \leq N-1 \text{ and } k \neq l, \\ 1 \text{ if } k = l, \\ -1 \text{ if } k \geq N \text{ for all } l \leq N-1, \end{cases} \tag{2.21}$$

where $\delta_{k,l}$ is the Kronecker symbol.

For the hybrid two-phase case $\{N=2, M=3\}$, there are only 3 coefficients $K_{k,l}$ that are :

$$K_{l,l} = -\left(P_{l,v} + P_{l,g}\right), \quad K_{v,l} = P_{v,l} - P_{v,g}, \quad K_{g,l} = P_{g,l} - P_{g,v}. \tag{2.22}$$

We now introduce the following lemma that contains fundamental rewriting techniques used in this paper.

Lemma 2.1. *For all $k=1,...,M$, we have the two relations*

$$\sum_{\substack{l\in\mathcal{K}\\ l\neq k}} P_{k,l}\partial_x\alpha_l = \sum_{l=1}^{N-1} K_{k,l}\partial_x\alpha_l, \tag{2.23}$$

$$\sum_{\substack{l\in\mathcal{K}\\ l\neq k}} P_{k,l}\partial_x\alpha_l + p_k\partial_x\alpha_k = \sum_{l=1}^{N-1} (K_{k,l} + \chi_{k,l}p_k)\partial_x\alpha_l. \tag{2.24}$$

Proof. By using miscibility constraints (2.4) and relation (2.5), we can rewrite the sum as

$$\sum_{\substack{l\in\mathcal{K}\\ l\neq k}} P_{k,l}\partial_x\alpha_l = \sum_{l=1}^{N-1} P_{k,l}(1-\delta_{k,l})\partial_x\alpha_l + \sum_{j=N}^{M} P_{k,j}(1-\delta_{k,j})\partial_x\alpha_j$$

$$
\begin{aligned}
&= \sum_{l=1}^{N-1} P_{k,l}(1-\delta_{k,l})\partial_x \alpha_l + \partial_x \alpha_N \sum_{j=N}^{M} P_{k,j}(1-\delta_{k,j}) \\
&= \sum_{l=1}^{N-1} P_{k,l}(1-\delta_{k,l})\partial_x \alpha_l - \sum_{l=1}^{N-1} \partial_x \alpha_l \sum_{j=N}^{M} P_{k,j}(1-\delta_{k,j}) \\
&= \sum_{l=1}^{N-1} \Big(P_{k,l}(1-\delta_{k,l}) - \sum_{j=N}^{M} P_{k,j}(1-\delta_{k,j}) \Big) \partial_x \alpha_l \\
&= \sum_{l=1}^{N-1} K_{k,l} \partial_x \alpha_l, \qquad (2.25)
\end{aligned}
$$

where $K_{k,l}$ has been defined in (2.20).

The second relation can be deduced from the first one, we just need to express the term $p_k \partial_x \alpha_k$ as a function of the $(\partial_x \alpha_l)_{1 \le l \le N-1}$, that is

$$
p_k \partial_x \alpha_k = \sum_{l=1}^{N-1} \chi_{k,l} p_k \partial_x \alpha_l, \tag{2.26}
$$

where $\chi_{k,l}$ is defined by (2.21). Then we deduce the relation (2.24). ∎

These relations allow us to rewrite any expression involving $(\partial_x \alpha_l)_{1 \le l \le M}$ in terms of the first $N-1$ quantities $(\partial_x \alpha_l)_{l \le N-1}$.

Using Lemma 2.1, the momentum and energy equations of (2.9) can be rewritten

$$
\partial_t(\alpha_k \rho_k v_k) + \partial_x(\alpha_k \rho_k v_k^2 + \alpha_k p_k) + \sum_{k=1}^{N-1} K_{k,l} \partial_x \alpha_l = 0, \tag{2.27}
$$

$$
\partial_t(\alpha_k \rho_k E_k) + \partial_x(\alpha_k \rho_k v_k (E_k + p_k/\rho_k)) + \sum_{k=1}^{N-1} K_{k,l} V_I \partial_x \alpha_l = 0. \tag{2.28}
$$

Thereby, we have reduced the number of interfacial unknowns. Indeed, there were $M(M-1)$ coefficients $(P_{k,l})$ and now there are $M(N-1)$ effective coefficients $(K_{k,l})$. We can now rewrite the minimal entropy dissipation condition (2.17) as a function of the $(K_{k,l})$.

Lemma 2.2. *The term $\mathcal{A}_\sigma(\mathbf{Y}, \partial_x \mathbf{Y})$ reads*

$$
\mathcal{A}_\sigma(\mathbf{Y}, \partial_x \mathbf{Y}) = \sum_{k \in \mathcal{K}} \frac{1}{T_k}(v_k - V_I) \sum_{l=1}^{N-1} (K_{k,l} + \chi_{k,l} p_k) \partial_x \alpha_l, \tag{2.29}
$$

with $(\chi)_{k,l}$ defined by (2.21).

Proof. It consists in rewriting (2.17) as a function of the $N-1$ firsts $(\partial_x \alpha_k)$, using Lemma 2.1. ∎

Let us now give the main result of this paper.

Proposition 2.1 (Minimal entropy production due to the interfacial states). *Let us assume that (2.14) hold true. If all the phasic temperatures are positive, then for any convex combination (2.18), the interfacial pressure terms $(K_{k,l})$ are uniquely defined.*

Proof. The structure of the proof is similar to [23]. We determine equations thanks to

(1) The independence of the derivatives $(\partial_x \alpha_k)$

(2) The independence of the relative velocities $(v_{k+1} - v_k)$

Then we obtain $N-1$ linear systems, one for each $l = 1,...,N-1$, in the variables $(K_{k,l})_{k=1,...,M}$. All of them are defined by the same matrix, thus we have to demonstrate that it is regular.

According to (2.4), the $(\partial_x \alpha_l)_{1\le l\le N-1}$ must be independent. Using this and Lemma 2.2, imposing $\mathcal{A}_\sigma(\mathbf{Y}, \partial_x \mathbf{Y}) = 0$ is equivalent to writing, for $l = 1,...,N-1$ the following relations

$$\sum_{k\in\mathcal{K}} \frac{1}{T_k}(v_k - V_I)(K_{k,l} + \chi_{k,l} p_k) = 0. \tag{2.30}$$

Using the convex combination (2.18), we rewrite the difference $v_k - V_I$ as

$$v_k - V_I = \sum_{i=1}^{k-1}\sum_{j=1}^{i}(-\beta_j)(v_i - v_{i+1}) + \sum_{i=k}^{M-1}\sum_{j=i+1}^{M}(\beta_j)(v_i - v_{i+1}), \tag{2.31}$$

which allows us to rearrange (2.30) in terms of the independent differences $(v_i - v_{i+1})$. Let us introduce the coefficients c_i and c^i defined for $i = 1,...,M-1$

$$c_i = \sum_{j=1}^{i} \beta_j, \; c^i = \sum_{j=i+1}^{M} \beta_j. \tag{2.32}$$

We obtain the following relations

$$\sum_{k=1}^{i} \frac{1}{T_k} c^i K_{k,l} - \sum_{k=i+1}^{M} \frac{1}{T_k} c_i K_{k,l} = \sum_{k=1}^{i} \frac{1}{T_k} c^i \chi_{k,l} p_k - \sum_{k=i+1}^{M} \frac{1}{T_k} c_i \chi_{k,l} p_k, \tag{2.33}$$

for $l = 1,...,N-1$ and $i = 1,...,M-1$. We define for the same indices

$$d_l^i = \sum_{k=1}^{i} a_k c^i \chi_{k,l} p_k - \sum_{k=i+1}^{M} a_k c_i \chi_{k,l} p_k \text{ and } a_i = \frac{1}{T_i}.$$

We obtain the final system defined by the following equations for $l = 1,...,N-1$ and $i = 1,...,M-1$

$$\sum_{k=1}^{i} c^i a_k K_{k,l} - \sum_{k=i+1}^{M} c_i a_k K_{k,l} = d_l^i. \tag{2.34}$$

In other words, for any $l = 1,...,N-1$, there are $M-1$ equations for M unknowns that are the $(K_{k,l})_{1\le k\le M}$. This can be balanced by adding the constraint (2.14) that gives, using (2.20) and the independence of $(\partial_x \alpha_l)_{l\le N-1}$, for $l = 1,...,N-1$

$$\sum_{k\in\mathcal{K}} K_{k,l} = 0. \tag{2.35}$$

Then we have $N-1$ linear systems of size $M \times M$, for each $l \in \{1,...,N-1\}$, that are

$$\boldsymbol{A}\boldsymbol{K}_l = \boldsymbol{d}_l, \tag{2.36}$$

with

- $\boldsymbol{A} = \begin{pmatrix} c^1 a_1 & -c_1 a_2 & -c_1 a_3 & \cdots & -c_1 a_M \\ c^2 a_1 & c^2 a_2 & -c_2 a_3 & \cdots & -c_2 a_M \\ \vdots & & & & \vdots \\ c^{M-1} a_1 & c^{M-1} a_2 & \cdots & c^{M-1} a_{M-1} & -c_{M-1} a_M \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}$,
- $\boldsymbol{K}_l = (K_{1,l}, K_{2,l}, ..., K_{M,l})^\top$,
- $\boldsymbol{d}_l = (d_l^1, d_l^2, ..., d_l^M)^\top$.

Finally, we need to show that $\boldsymbol{A}$ is regular so the $(K_{k,l})$ are uniquely determined. Let us note

$$D = \det \boldsymbol{A}. \tag{2.37}$$

The idea of the proof is to expand this determinant along the last row, then express the minors of size $M-1$ with a second order determinant thanks to $M-3$ developments.

Two relations must be reminded here. For $i = 1, ..., M-1$, we have

$$c_i + c^i = 1, \; c^{i+1} - c^i = -\beta_i. \tag{2.38}$$

By developing D on the last row and factorizing by $\bar{a}_i$, we get

$$D = \sum_{i=1}^{M} (-1)^{M+i} \bar{a}_i D_i, \tag{2.39}$$

where D_i is the minor formed by deleting the last row and the i-th column, and $\bar{a}_i = \prod_{j \neq i}^{M} a_j$. For $i = 1, ..., M-1$, we have

$$D_i = \det \begin{pmatrix} c^1 & -c_1 & \cdots & -c_1 & -c_1 & \cdots & -c_1 \\ c^2 & c^2 & & \cdots & \cdots & & -c_2 \\ & & \ddots & -c_{i-1} & -c_{i-2} & & \\ \vdots & & \ddots & c^{i-1} & -c_{i-1} & & \vdots \\ & & & c^i & -c_i & & \\ & & & c^{i+1} & c^{i+1} & & \\ & & & \vdots & \vdots & \ddots & \\ c^{M-1} & c^{M-1} & \cdots & c^{M-1} & c^{M-1} & \cdots & -c_{M-1} \end{pmatrix}, \tag{2.40}$$

that we can rewrite with the columns $\mathcal{C}_j = (-c_1, ..., -c_{j-1}, c^j, ..., c^{M-1})^\top \in \mathbb{R}^{M-1}$ as

$$D_i = \det \big(\mathcal{C}_1 \; \mathcal{C}_2 \; ... \; \mathcal{C}_{i-1} \; \mathcal{C}_{i+1} ... \; \mathcal{C}_M \big). \tag{2.41}$$

Then, D_i can be reduced to a second order determinant with $M-3$ operations:

- $\mathcal{C}_1 \leftarrow \mathcal{C}_1 - \mathcal{C}_2$ and a development along $\mathcal{C}_1$ is done $i-2$ times,
- $\mathcal{C}_n \leftarrow \mathcal{C}_n - \mathcal{C}_{n-1}$ and a development along $\mathcal{C}_n$ is done $M-(i+1)$ times, where n is the size of the considered minor.

A factor (-1) appears for each second type operation, thus we have

$$D_i = (-1)^{M-(i+1)} \det \begin{pmatrix} c^{i-1} & -c_{i-1} \\ c^i & -c_i \end{pmatrix}, \tag{2.42}$$

so, thanks to relations (2.32), we can express the latter determinant as

$$\begin{aligned} D_i &= (-1)^{M-(i+1)}(-\beta_i) \\ &= (-1)^{M-i}\beta_i, \end{aligned} \tag{2.43}$$

for $i=2,\ldots,M-1$. On the other hand, the cases $i=1$ and $i=M$ give $D_1=c_1$ and $D_M=c^{M-1}$.

Coming back to (2.39), we get

$$D = \bar{a_1}c_1 + \bar{a_M}c^{M-1} + \sum_{i=2}^{M-1}(-1)^{M+i}\bar{a_i}(-1)^{M-i}\beta_i, \tag{2.44}$$

that simplifies into

$$D = \bar{a_1}c_1 + \bar{a_M}c^{M-1} + \sum_{i=2}^{M-1}\bar{a_i}\beta_i > 0. \tag{2.45}$$

Since $\sum_i \beta_i = 1$ and $a_i > 0$ for all $i=1,\ldots,M$, then $\det \boldsymbol{A} > 0$ and the interfacial terms $(K_{k,l})$ are fully determined. □

Let us give an example for $\{N=2, M=3\}$, studied in [11]. As seen earlier in (2.22), we only have to determine the three terms $(K_{k,l})_{k=l,v,g}$ since $N=2$. We obtain the simple expressions

$$K_{l,l} = -(p_v+p_g), \quad K_{v,l} = p_v, \quad K_{g,l} = p_g. \tag{2.46}$$

We refer to [11, Prop. 1] for the complete proof. The case $\{N=3, M=4\}$ is more intricate and is detailed in [12, Prop. 1].

Remark 2.2 (Preservation of the pressure equilibria). An important feature is the preservation of an initial steady state where the fluid is supposed to be at the thermodynamical equilibrium: the phasic pressures satisfy Dalton's law, the phasic temperatures are equal to each other and the velocities are null. A detailed explanation is given in [15, Appendix A] in the three-phase immiscible case.

3. Analysis

In this section, we investigate the hyperbolicity and the symmetrization of the system. The hyperbolicity sets a known framework in order to solve the system of equations. The symmetrization is important to show the existence of a local-in-time smooth solution for a Cauchy problem, by applying Kato's theorem [21].

- In the conservative case, the existence of an entropy function gives the symmetrization of the system, thanks to the Godunov-Mock theorem [9, Theorem 3.1].
- In the nonconservative case, this theorem does not apply. Consequently, the symmetrization must be proved by hand.

3.1. Hyperbolicity. First, we consider the convective system associated to (2.8)-(2.9) and rewrite it in the primitive variables $\boldsymbol{w} = (\alpha_1,\ldots,\alpha_{N-1},w_1,\ldots,w_M)$, where $\boldsymbol{w}_k = (\rho_k, v_k, p_k)$. For $k=1,\ldots,N$ we have

$$\partial_t \alpha_k + V_I \partial_x \alpha_k = 0, \tag{3.1}$$

and for $k\in\mathcal{K}$

$$\partial_t\rho_k+\frac{\rho_k}{\alpha_k}(v_k-V_I)\partial_x\alpha_k+v_k\partial_x\rho_k+\rho_k\partial_x v_k=0, \tag{3.2}$$

$$\partial_t v_k+\frac{1}{\alpha_k\rho_k}\left(\sum_{l\in\mathcal{K}}P_{k,l}\partial_x\alpha_l+p_k\partial_x\alpha_k\right)+v_k\partial_x v_k+\frac{1}{\rho_k}\partial_x p_k=0, \tag{3.3}$$

$$\partial_t p_k+\frac{\rho_k}{\alpha_k}(v_k-V_I)\sum_{\substack{l\in\mathcal{K}\\ l\neq k}}C_{k,l}^2\partial_x\alpha_l+\rho_k c_k^2\partial_x v_k+v_k\partial_x p_k=0, \tag{3.4}$$

where $C_{k,l}$ and c_k are respectively the interfacial sound speed and the phasic sound speed, defined by

$$\begin{aligned} C_{k,l}^2&=-\big((\partial p_k/\partial e_k)P_{k,l}/\rho_k^2+(\partial p_k/\partial\rho_k)\big),\\ c_k^2&=\partial p_k/\partial\rho_k+p_k/\rho_k+p_k/\rho_k^2(\partial p_k/\partial e_k).\end{aligned} \tag{3.5}$$

Then, we express the $(\partial_x\alpha_l)_{1\leq l\leq M}$ as a function of the $(\partial_x\alpha_l)_{l\leq N-1}$ by using Lemma 2.1, it reads

$$\partial_t\alpha_k+V_I\partial_x\alpha_k=0, \tag{3.6}$$

$$\partial_t\rho_k+\frac{\rho_k}{\alpha_k}(v_k-V_I)\left((1-\sum_{j=N}^{M}\delta_{k,j})\partial_x\alpha_k-\left(\sum_{j=N}^{M}\delta_{k,j}\right)\sum_{l=1}^{N-1}\partial_x\alpha_l\right)+v_k\partial_x\rho_k+\rho_k\partial_x v_k=0, \tag{3.7}$$

$$\partial_t v_k+\frac{1}{\alpha_k\rho_k}\sum_{l=1}^{N-1}(K_{k,l}+\chi_{k,l}p_k)\partial_x\alpha_l+v_k\partial_x v_k+\frac{1}{\rho_k}\partial_x p_k=0, \tag{3.8}$$

$$\partial_t p_k+\rho_k c_k^2\partial_x v_k+\frac{\rho_k}{\alpha_k}(v_k-V_I)\sum_{l=1}^{N-1}\gamma_{k,l}\partial_x\alpha_l+v_k\partial_x p_k=0, \tag{3.9}$$

with $\gamma_{k,l}=C_{k,l}^2(1-\delta_{k,l})-\sum_{j=N}^{M}C_{k,j}^2(1-\delta_{k,j})$ and $K_{k,l}$ defined by (2.20). Note that we used the argument from Lemma 2.2 for the velocity equation.

Thus, we have the following quasi-linear system

$$\partial_t\boldsymbol{w}+\boldsymbol{B}(\boldsymbol{w})\partial_x\boldsymbol{w}=0, \tag{3.10}$$

where $\boldsymbol{B}(\boldsymbol{w})\in\mathcal{M}_{N-1+3M}(\mathbb{R})$ is the block matrix

$$\boldsymbol{B}(\boldsymbol{w})=\left(\begin{array}{c|ccc} V_I\boldsymbol{I}_{N-1} & 0 & \dots & 0\\ \hline \boldsymbol{A}_1 & \boldsymbol{B}_1 & & \\ \dots & & \dots & \\ \boldsymbol{A}_M & & & \boldsymbol{B}_M\end{array}\right). \tag{3.11}$$

The matrices $\boldsymbol{A}_k\in\mathcal{M}_{3,N-1}(\mathbb{R})$ and $\boldsymbol{B}_k\in\mathcal{M}_3(\mathbb{R})$ are defined by

$$\boldsymbol{A}_k=\begin{pmatrix}\frac{\rho_k}{\alpha_k}(v_k-V_I)(\boldsymbol{e}_k^\top(1-\sum_{j=N}^{M}\delta_{k,j})-\boldsymbol{1}_{N-1}^\top\sum_{j=N}^{M}\delta_{k,j})\\ \frac{1}{\alpha_k\rho_k}\boldsymbol{\beta}_k^\top\\ \frac{\rho_k}{\alpha_k}(v_k-V_I)\boldsymbol{\gamma}_k^\top\end{pmatrix}, \tag{3.12}$$

$$\boldsymbol{B}_k = \begin{pmatrix} v_k & \rho_k & 0 \\ 0 & v_k & 1/\rho_k \\ 0 & \rho_k c_k^2 & v_k \end{pmatrix}, \tag{3.13}$$

where

- $\boldsymbol{e}_k$ is the k-th unit vector in $\mathbb{R}^{N-1}$ when $k<N$, and with the convention $\boldsymbol{e}_k=\boldsymbol{0}$ if $k\geq N$,
- $\boldsymbol{1}_{N-1}$ is the vector in $\mathbb{R}^{N-1}$ whose components are all equal to 1,
- the vectors $\boldsymbol{\beta}_k$, $\boldsymbol{\gamma}_k\in\mathbb{R}^{N-1}$ are defined, for $l=1,...,N-1$, by

$$\boldsymbol{\beta}_k = \left(K_{k,1}+\chi_{k,1}p_k,\ \dots\ ,K_{k,N-1}+\chi_{k,N-1}p_k\right)^\top, \tag{3.14}$$

$$\boldsymbol{\gamma}_k = \left(\gamma_{k,1},\ \dots\ ,\gamma_{k,N-1}\right)^\top. \tag{3.15}$$

We notice that the phasic matrices $\boldsymbol{B}_k$ do not differ from the immiscible case [23], but the $\boldsymbol{A}_k$ matrices do change. The eigenstructure of $\boldsymbol{B}$ is exactly the same as in this latter case and its eigenvalues are:

- $\lambda_{I,k}=V_I,\ k=1,...,N-1$
- $\lambda_k=v_k$ and $\lambda_{k,\pm}=v_k\pm c_k,\ k=1,...,M.$

REMARK 3.1. The nature and properties of these characteristic fields are as expected, see [8] for their analysis in the two-phase framework. We emphasize that the nature of V_I remains unknown, and so its Riemann invariants, which obviously depend on its definition. One can refer to [4] where different closures for V_I are investigated for a two-phase mixture, allowing to define unique jump relations.

We now turn to the determination of the eigenvectors. The matrix $\boldsymbol{R}$ composed of the right eigenvectors has a block structure and reads

$$\boldsymbol{R} = \left(\begin{array}{c|ccc} \boldsymbol{R}_I^0 & 0 & \dots & 0 \\ \hline \boldsymbol{R}_I^1 & \boldsymbol{R}_1 & & \\ \dots & & \dots & \\ \boldsymbol{R}_I^M & & & \boldsymbol{R}_M \end{array}\right). \tag{3.16}$$

The phasic problems are well known, and thus matrices $\boldsymbol{R}_k$ containing the associated right eigenvectors are classical. For $\boldsymbol{R}_I^k$, we introduce the following notations

$$\kappa_0 = \prod_{l=1}^{M}\alpha_l\sigma_l,\ \kappa_k = \prod_{l=1,\neq k}^{M}\alpha_l\sigma_l,\ k=1,...,M \tag{3.17}$$

$$\sigma_k=\delta_k^2-c_k^2 \text{ and } \delta_k=v_k-V_I. \tag{3.18}$$

Thus we have

$$\boldsymbol{R}_I^k = \kappa_k\rho_k \begin{pmatrix} \alpha_k c_k^2\boldsymbol{\beta}_k^\top-\delta_k^2\boldsymbol{\gamma}_k^\top \\ -(\alpha_k\boldsymbol{\beta}_k^\top-\boldsymbol{\gamma}_k^\top)\delta_k/\rho_k \\ -\sigma_k(\boldsymbol{e}_k^\top(1-\sum_{j=N}^M\delta_{k,j})-\boldsymbol{1}_{N-1}^\top\sum_{j=N}^M\delta_{k,j})+\alpha_k\boldsymbol{\beta}_k^\top-\boldsymbol{\gamma}_k^\top \end{pmatrix}, \tag{3.19}$$

$$\boldsymbol{R}_I^0=\kappa_0\boldsymbol{I}_{N-1}, \qquad \boldsymbol{R}_k = \begin{pmatrix} 1 & 1 & 1 \\ -c_k/\rho_k & 0 & c_k/\rho_k \\ c_k^2 & 0 & c_k^2 \end{pmatrix}. \tag{3.20}$$

The matrix $\boldsymbol{R}$ is regular under the following well known condition

$$\sigma_k \neq 0 \iff V_I \neq v_k \pm c_k,\ \forall k = 1,\ldots,M \tag{3.21}$$

called the *non-resonance condition.*

PROPOSITION 3.1. *The system (2.8)-(2.9) is hyperbolic under the non-resonance condition (3.21).*

The non-resonance condition appears when a genuinely non-linear field associated to $v_k \pm c_k$ overlaps the coupling wave V_I. In such a situation, the vector space is not any more spanned by the eigenvectors.

3.2. Symmetrization. The proof of the symmetrization relies on the same arguments as in [6, 11] and is exactly the same as in the generalized immiscible case of N phases [23]. We build a symmetric positive definite matrix $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{w})$ such that $\boldsymbol{PB}$ is symmetric, by using the block structure of $\boldsymbol{B}$ and its left and right eigenvectors matrices. We remind here the general idea.

First we define $\boldsymbol{P}_k$ the symmetrizer of the phasic problem by

$$\boldsymbol{P}_k = \begin{pmatrix} 1 & 0 & -1/c_k^2 \\ 0 & 0.5(\rho_k/c_k)^2 & 0 \\ -1/c_k^2 & 0 & 1.5/c_k^4 \end{pmatrix}, \tag{3.22}$$

that is a symmetric positive definite matrix. Moreover, it is such that $\boldsymbol{P}_k\boldsymbol{B}_k$ is symmetric. We then define $\boldsymbol{P}_{k,\alpha}$ under the non-resonance condition (3.21) by

$$\boldsymbol{P}_{k,\alpha} = \boldsymbol{L}_k^\top (\boldsymbol{\Lambda}_k - V_I \boldsymbol{I}_3)^{-1} \boldsymbol{R}_k^\top \boldsymbol{P}_k \boldsymbol{A}_k, \tag{3.23}$$

where $\boldsymbol{L}_k$ is the left eigenvectors matrix of $\boldsymbol{B}_k$ and $\boldsymbol{\Lambda}_k$ is the eigenvalues matrix of $\boldsymbol{B}_k$. We remind that these matrices satisfy $\boldsymbol{L}_k\boldsymbol{B}_k\boldsymbol{R}_k = \boldsymbol{\Lambda}_k$ and $\boldsymbol{L}_k\boldsymbol{R}_k = \boldsymbol{I}_3$. Thus, we can define the symmetrizer for $\boldsymbol{B}$, that is

$$\boldsymbol{P} = \left(\begin{array}{c|ccc} NP_\alpha \boldsymbol{I}_{N-1} & \boldsymbol{P}_{1,\alpha}^\top & \ldots & \boldsymbol{P}_{M,\alpha}^\top \\ \hline \boldsymbol{P}_{1,\alpha} & \boldsymbol{P}_1 & & \\ \ldots & & \ldots & \\ \boldsymbol{P}_{M,\alpha} & & & \boldsymbol{P}_M \end{array}\right), \tag{3.24}$$

where $P_\alpha \boldsymbol{I}_{N-1}$ must be specified.

The cornerstone is that $(V_1\boldsymbol{P}_{k,\alpha} + \boldsymbol{P}_k\boldsymbol{A}_k)^\top = \boldsymbol{P}_{k,\alpha}^\top \boldsymbol{B}_k$, which corresponds to the symmetry of the first row and column of block of $\boldsymbol{PB}$.

It relies on the fact that $\boldsymbol{P}_{k,\alpha}^\top \boldsymbol{A}_k$ is a symmetric matrix, since $\boldsymbol{P}_k^\top \boldsymbol{R}_k = \boldsymbol{L}_k^\top$.

Now we prove that $\boldsymbol{P}$ is positive definite. Let $\boldsymbol{a} \in \mathbb{R}^{N-1+3M}$ such as $\boldsymbol{a} = (\boldsymbol{a}_\alpha, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_M)$, with $\boldsymbol{a}_\alpha \in \mathbb{R}^{N-1}$. We have

$$\boldsymbol{a}^\top \boldsymbol{P}\boldsymbol{a} = P_\alpha \sum_{k=1}^{N-1}\sum_{i=1}^{N-1} \left(a_{\alpha,i} + (\boldsymbol{P}_{k,\alpha}^\top \boldsymbol{a}_k)_i / P_\alpha\right)^2 + \sum_{k=1}^{M} \frac{1}{P_\alpha} \boldsymbol{a}_k^\top \boldsymbol{Q}\boldsymbol{a}_k, \tag{3.25}$$

where $\boldsymbol{Q} = P_\alpha \boldsymbol{P}_k - \boldsymbol{P}_{k,\alpha}\boldsymbol{P}_{k,\alpha}^\top$. Let us determine a condition on P_α such that the terms $\boldsymbol{a}_k^\top \boldsymbol{Q}\boldsymbol{a}_k$ are non-negative.

We consider the Cholesky decomposition of $\boldsymbol{P}_k = \boldsymbol{C}_k \boldsymbol{C}_k^\top$, and we define $\boldsymbol{E}_k = \boldsymbol{C}_k^{-1} \boldsymbol{P}_{k,\alpha}^\top \boldsymbol{P}_{k,\alpha} \boldsymbol{C}_k^\top$, that is a symmetric matrix. Thus, there exists $\boldsymbol{T}_k$ so that $\boldsymbol{T}_k \boldsymbol{E}_k \boldsymbol{T}_k^\top = \boldsymbol{D}_k = \mathrm{diag}(\mu_1^k, \mu_2^k, \mu_3^k)$, where $(\mu_i^k)_{1 \le i \le 3}$ are the eigenvalues of $\boldsymbol{E}_k$. Thus we have

$$\boldsymbol{a}_k^\top \boldsymbol{Q} \boldsymbol{a}_k = \boldsymbol{b}_k^\top (P_\alpha \boldsymbol{I}_3 - \boldsymbol{D}_k) \boldsymbol{b}_k = \sum_{i=1}^{3} b_{k,i} (P_\alpha - \mu_i^k), \tag{3.26}$$

with $\boldsymbol{b}_k = \boldsymbol{T}_k^\top \boldsymbol{C}_k^\top \boldsymbol{a}_k$. Choosing $P_\alpha > \max_{i,k} |\mu_i^k| > 0$ the term $\boldsymbol{a}_k^\top \boldsymbol{Q} \boldsymbol{a}_k$ is non-negative and is positive for $\boldsymbol{a}_k \neq 0$, so $\boldsymbol{a}^\top \boldsymbol{Q} \boldsymbol{a}$ is.

Finally, the system (3.6)-(3.9) is symmetrizable and according to Kato's theorem [21], there exists a local-in-time smooth solution to the associated Cauchy problem.

4. Admissible source terms

We investigate in this section some conditions on the source terms in order to satisfy the second principle of thermodynamics.

We omit the dependence on $\boldsymbol{Y}$ of the source terms for the sake of readability. We first give the form of each source term, and then we determine conditions on each contribution.

Firstly, let us remark that the mechanical transfer source terms Φ_k satisfy

$$\forall k \ge N, \Phi_k = \Phi_N, \quad \sum_{k=1}^{N} \Phi_k = 0, \tag{4.1}$$

which is a consequence of the miscibility and saturation constraints. This allows us to use a similar rewriting to (2.23) and (2.24), which will be useful later.

The mass transfer term Γ_k is defined as a sum of dyadic contributions

$$\Gamma_k = \sum_{\substack{l \in \mathcal{K} \\ l \neq k}} \Gamma_{kl}, \tag{4.2}$$

where Γ_{kl} represents the mass transfer from phase k towards phase l. In practical modelling, some of these terms may equal zero, for example if the phase k is a non-condensable gas.

The momentum contribution is decomposed into a drag term and a mass transfer term

$$S_{q,k} = \sum_{l \neq k} D_{kl} + \sum_{l \neq k} \Gamma_{kl} v_{kl}. \tag{4.3}$$

Last, the total energy source term contains thermal transfer, drag effects and mass transfer

$$S_{E,k} = \sum_{l \neq k} \Psi_{kl} + \sum_{l \neq k} v_{kl} D_{kl} + \sum_{l \neq k} \Gamma_{kl} H_{kl} + \sum_{l \neq k} P_{k,l} \Phi_l, \tag{4.4}$$

plus the last term $\sum_{l \neq k} P_{k,l} \Phi_l$, coming from the choice of modelling we made at the beginning of the paper. Indeed, by choosing the derivatives in space for α_l in the energy equation from (2.9), the associated source term $S_{E,k}$ contains a mechanical contribution, which is not the case for example in [11, 12].

Let us remark some relations on these contributions. First, the dyadic contributions of (4.2), (4.3) and (4.4) must satisfy

$$\Gamma_{kl} = -\Gamma_{lk}, \quad D_{kl} = -D_{lk}, \quad \Psi_{kl} = -\Psi_{lk}. \tag{4.5}$$

Those from (4.4) give

$$v_{kl}D_{kl} = -v_{lk}D_{lk},$$

which impose $v_{kl} = v_{lk}$, thanks to the independence of the D_{kl}. Similarly, we have $H_{kl} = H_{lk}$.

Now we can regroup the contributions according to their nature

$$RHS_\sigma = RHS_\sigma^\Phi + RHS_\sigma^\Gamma + RHS_\sigma^D + RHS_\sigma^\Psi. \tag{4.6}$$

In order to ensure the entropy growth, each of these four terms must be positive.

- **Admissible mechanical contribution Φ**

$$\begin{aligned} RHS_\sigma^\Phi &= \sum_{k\in\mathcal{K}} a_k\Big(\sum_{l\neq k} P_{k,l}\Phi_l + \rho_k^2\frac{\partial e_k}{\partial \rho_k}\Phi_k\Big) - \rho_k^2\frac{\partial s_k}{\partial \rho_k}\Phi_k \\ &= \sum_{k\in\mathcal{K}} a_k\Big(\sum_{l\neq k} P_{k,l}\Phi_l + p_k\Phi_k\Big) \\ &= \sum_{k\in\mathcal{K}} a_k\Big(\sum_{l\leq N-1} (K_{k,l}+\chi_{k,l}p_k)\Phi_l\Big) \\ &= \sum_{l=1}^{N-1}\Big(\sum_{k=1}^{M} a_k(K_{k,l}+\chi_{k,l}p_k)\Big)\Phi_l. \end{aligned}$$

We obtain the condition, for all $l = 1,...,N-1$

$$\Big(\sum_{k=1}^{M} a_k(K_{k,l}+\chi_{k,l}p_k)\Big)\Phi_l \geq 0. \tag{4.7}$$

- **Admissible mass transfer contribution Γ**

By using the relation $\mu_k = e_k - T_k s_k + p_k/\tau_k$, we have

$$RHS_\sigma^\Gamma = \sum_k a_k \sum_{l\neq k}(H_{kl} + \frac{v_k^2}{2} - v_k v_{kl})\Gamma_{kl} - \sum_k\sum_{l\neq k} a_k\mu_k\Gamma_{kl}.$$

By setting $H_{kl} = \frac{v_k v_l}{2}$ and $v_{kl} = \frac{v_k+v_l}{2}$, the first sum is null. Hence, it imposes

$$\forall k\neq l,\ (a_k\mu_k - a_l\mu_l)\Gamma_{kl} \geq 0, \tag{4.8}$$

that is equivalent to

$$\forall k\neq l,\ (\frac{T_l}{\mu_l} - \frac{T_k}{\mu_k})\Gamma_{kl} \geq 0. \tag{4.9}$$

- **Admissible drag effects contribution**

$$RHS_\sigma^D = \sum_{k\in\mathcal{K}} a_k\Big(\sum_{l\neq k} v_{kl}D_{kl} - v_k\sum_{l\neq k} D_{kl}\Big).$$

We assume the following form for v_{kl} to comply with the Galilean invariance principle

$$v_{kl} = \beta_{kl}v_k + (1-\beta_{kl})v_l, \text{ with } \beta_{kl}+\beta_{lk} = 1 \text{ and } \beta_{kl}\in[0;1].$$

Then we impose for $k \neq l$

$$\begin{aligned} & a_k(1-\beta_{kl}+a_l(1-\beta_{lk}))(v_l-v_k)D_{kl} \geq 0 \\ \iff & a_k(\beta_{lk}+a_l(1-\beta_{lk}))(v_l-v_k)D_{kl} \geq 0. \end{aligned}$$

Knowing that $a_k(\beta_{lk}+a_l(1-\beta_{lk}))>0$, the final condition is

$$(v_l-v_k)D_{kl} \geq 0. \tag{4.10}$$

- **Admissible thermal transfer contribution**

$$RHS_\sigma^\Psi = \sum_{k\in\mathcal{K}}\sum_{l>k}\Psi_{kl}(a_k-a_l),$$

and so the constraint is that for $k \neq l$,

$$\Psi_{kl}(T_l-T_k) \geq 0. \tag{4.11}$$

Finally, these classes of source terms comply with the second principle of thermodynamics. In practical cases, see [11, 12] for instance, the condition on the mechanical contribution Φ_l can be given more precisely, and we refer to these papers for more details.

5. Conclusion

We have addressed in this paper the study of a Baer-Nunziato-like model for a compressible N-phase flow with miscibility conditions. The main result concerns the closure laws of the interfacial pressure terms. We demonstrate that under classic conditions on the interfacial velocity, the interfacial pressure terms are uniquely defined. Explicit expressions of these terms can be given in practical situations for given values of M and N, see for instance [11, 12].

Then, the hyperbolicity and symmetrization of the convective system have been investigated. The system is hyperbolic under the classical non-resonance condition. The symmetrization gives the local-in-time existence of a smooth solution.

Finally, we have determined constraints on the source terms in order to satisfy the second principle of thermodynamics. Results are the same as in practical models where M and N are given. The missing explicit expressions of the interfacial terms Φ_k may be overcome in practical situations, see [11, 12].

If we turn to the approximation of solutions of system (2.8)-(2.9), recent works investigated relaxation schemes dedicated to the two-phase Baer-Nunziato model with strong properties, see [5], as well as the three-phase immiscible barotropic case [24]. It would be interesting to extend these works to the two-phase three-field case and to begin a numerical study.

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