GLOBAL STRONG SOLUTIONS TO
THE COMPRESSIBLE NAVIER-STOKES SYSTEM WITH
POTENTIAL TEMPERATURE TRANSPORT*

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Abstract. We study the global strong solutions to the compressible Navier-Stokes system with potential temperature transport in $\mathbb{R}^n$. Different from the Navier-Stokes-Fourier system, the pressure being a nonlinear function of the density and the potential temperature, we can not exploit the special quasi-diagonalization structure of this system to capture any dissipation of the density. Some new ideas and delicate analysis involving high or low frequency decomposition in the Besov spaces have to be made to close the energy estimates.

1. Introduction and the main results

In this paper, we are concerned with the Cauchy problem of the compressible Navier-Stokes system with potential temperature transport. The system has the following form:

$$\begin{cases} \partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u} + \nabla P(\rho, \theta) = 0, \\ \partial_t (\rho \theta) + \text{div}(\rho \theta \mathbf{u}) = 0, \\ (\rho, \mathbf{u}, \theta)|_{t=0} = (\rho_0, \mathbf{u}_0, \theta_0). \end{cases} \tag{1.1}$$

Here, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $t \geq 0$ are the space and time variables, respectively. The unknown functions ρ is the fluid density, $\mathbf{u}$ is the velocity field, P is the scalar pressure, θ is the fluid potential temperature. The viscosity coefficients μ and λ are subject to the standard physical restrictions:

$$\mu > 0 \quad \text{and} \quad n\lambda + 2\mu > 0.$$

$\gamma > 1$ is the adiabatic index, the pressure state equation reads

$$P(\rho, \theta) = A(\rho\theta)^\gamma, \quad A > 0. \tag{1.2}$$

System (1.1) with (1.2) governs the motion of viscous compressible fluids with potential temperature, where diabatic processes and the influence of molecular transport on potential temperature are excluded. It's often used in meteorological applications, see, e.g., [17,19] and the references therein. Due to the importance of the system, it has drawn much attention recently. For any $\gamma > 3/2, n = 3$, Michálek [24] studied the stability of weak solutions of (1.1)–(1.2), see also [19, Chapter 5 and Chapter 8], in which Lions investigated the stability of weak solutions for the compressible Navier-Stokes

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equations with a scalar transport for $\gamma \geq 9/5$. For any $\gamma \geq 9/5, n = 3$ or $\gamma > 1, n = 2$, Maltese *et al.* [22] obtained the existence of global-in-time weak solutions to (1.1)–(1.2) with $\theta^\gamma = s^\gamma$ (s is the entropy). Feireisl *et al.* [10] analyzed the singular limit in the low Mach/Froude number regime of the above Navier-Stokes system with $\gamma > 3/2$. By analyzing the convergence of a suitable numerical scheme, the mixed finite element-finite volume method, Lukáčová-Medvid'ová and Schömer [20] proved the global-in-time existence of DMV (dissipative measure-valued) solutions for all adiabatic indices $\gamma > 1$ for $n = 2, 3$. Later, they [21] further established a DMV-strong uniqueness result for (1.1)–(1.2). Moreover they showed that strong solutions are stable in the class of DMV solutions. However, to the authors' knowledge, there are few results about the strong solutions of (1.1) with (1.2).

If the effect of the temperature is neglected and thus the pressure is a function of ρ, Eq. (1.1) reduces to the following isentropic compressible Navier-Stokes equations,

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \mathrm{div}\,\mathbf{u} + \nabla \rho^\gamma = 0. \end{cases} \tag{1.3}$$

The above system (1.3) has been widely studied (see [2, 3, 5, 7, 12, 27] and the references therein). For arbitrary initial data and $\bar\rho = 0$, the breakthrough was made by Lions [19], where the author proved the global existence of weak solutions for $P = A\rho^\gamma$ for $\gamma \geq 9/5$. Later, Feireisl *et al.* [11] extended Lions' result to the case of $\gamma > 3/2$. Jiang and Zhang [15, 16] improved the global existence of weak solution for any $\gamma > 1$ for the spherically symmetric or axisymmetric initial data. However, the question of the regularity and uniqueness of weak solutions is completely open even in the case of two dimensional space.

Compared to the weak solutions, there are relative fruitful results on the strong solutions. Nash [25] proved the local existence and uniqueness of smooth solution of the isentropic compressible Navier-Stokes equations for smooth initial data without vacuum. The global classical solutions were first obtained by Matsumura and Nishida [23] for initial data $(\rho_0, \mathbf{u}_0)$ close to an equilibrium $(\bar\rho, 0)$ in $H^3 \times H^3(\mathbb{R}^3)$, $\bar\rho > 0$. This result was further generalized by Huang *et al.* [14] with constant state as far field which could be either vacuum or non-vacuum in $\mathbb{R}^3$ with smooth initial data. Moreover, the initial data are of small total energy but possibly large oscillations, see also Li and Xin [18] for further developments.

In the framework of critical spaces, a breakthrough was made by Danchin [5] for the isentropic compressible Navier-Stokes equations, where the author proved the local wellposedness with large initial data and global solutions with small initial data. This result was further extended by Charve and Danchin [2], Chen *et al.* [3], and Haspot [12]. Recently, more and more researchers are devoted to the global solutions of the compressible Navier-Stokes equations with different class of large initial data in the critical spaces. Based on the spectral analysis for the linearized system and Hoff's energy method, Wang *et al.* [26] proved a global existence result of three dimensional compressible Navier–Stokes equations for a class of initial data, which may have large oscillation for the density and large energy for the velocity. Making use of the dispersive estimates for the system of acoustics, Fang *et al.* [9] constructed the global strong solutions to (1.3) in $\mathbb{R}^n$ which allows the low frequency part of the initial velocity field to be large. He *et al.* [13] studied the global-in-time stability of (1.3) and proved that any perturbed solution remains close to the reference solution if they are initially close to each other. As a byproduct, they constructed also the global large solutions to (1.3) which allow the vertical component of the initial velocity to be arbitrarily large. Zhai

et al. [27] also constructed global solutions in $\mathbb{R}^3$ with another class of large initial data satisfying nonlinear smallness which allows each component of the incompressible part of initial velocity to be arbitrarily large. Let $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{I} - \mathcal{Q} = \mathcal{I} - \nabla\Delta^{-1}\text{div}$ be the Stokes projection operator. Recently, Danchin and Mucha [8] obtained the global existence of regular solutions to (1.3) with arbitrary large initial incompressible velocity $\mathcal{P}\mathbf{u}_0$, almost constant density ρ_0, and large viscosity λ. This result was further extended by Chen and Zhai [4] in a critical L^p framework, which implies that the highly oscillating initial data are allowed.

Our main goal is to solve the Cauchy problem of the compressible Navier-Stokes system with potential temperature transport. We will concentrate on the local well-posedness issue for large data with no vacuum, on the global well-posedness issue for small perturbations of a constant steady equilibrium, in the critical regularity framework. By criticality, we mean that the scaling transformation which keeps the norms of the function space of the solution also leaves (1.1) invariant. In the case of compressible fluids, it is easy to see that the transformation

$$(\rho(t,x),\mathbf{u}(t,x),\theta(t,x)) \mapsto (\rho(\ell^2 t,\ell x),\ell\mathbf{u}(\ell^2 t,\ell x),\theta(\ell^2 t,\ell x)), \quad \ell > 0,$$

possesses such property provided the pressure term has been changed accordingly. One can check that the product space $\dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n) \times \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$ is a critical space for the system (1.1).

To overcome the difficulties arising from the strong nonlinearity of the pressure, combining with the renormalized equations of ρ and θ, we will transform (1.1) into another form in terms of variables $\rho, \mathbf{u}, P$,

$$\begin{cases} \partial_t\rho + \text{div}(\rho\mathbf{u}) = 0, \\ \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\text{div}\,\mathbf{u} + \nabla P = 0, \\ \partial_t P + \mathbf{u}\cdot\nabla P + \gamma P\text{div}\,\mathbf{u} = 0. \end{cases} \tag{1.4}$$

Moreover, to deal with the lack of dissipation on ρ and θ, we shall restrict the initial data of (1.4) to satisfy

$$(\rho,\mathbf{u},P)(0,x) = (\rho_0,\mathbf{u}_0,P_0)(x) \to (\bar{\rho},\mathbf{0},\bar{P}) \quad \text{as} \quad |x| \to \infty,$$

where $\bar{\rho}$ and $\bar{P}$ are two positive constants. For convenience, denote $\mu = 1, \lambda = 0, \bar{\rho} = \bar{P} = 1$ and define

$$\mathcal{A}\mathbf{u} \stackrel{\text{def}}{=} \Delta\mathbf{u} + \nabla\text{div}\,\mathbf{u}, \quad \rho \stackrel{\text{def}}{=} 1 + a, \quad P \stackrel{\text{def}}{=} 1 + b, \quad \text{and} \quad I(a) \stackrel{\text{def}}{=} \frac{a}{1+a},$$

we can rewrite (1.4) into the following new form

$$\begin{cases} \partial_t a + \text{div}\,\mathbf{u} + \mathbf{u}\cdot\nabla a + a\,\text{div}\,\mathbf{u} = 0, \\ \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathcal{A}\mathbf{u} + \nabla b = I(a)\nabla b - I(a)\mathcal{A}\mathbf{u}, \\ \partial_t b + \gamma\text{div}\,\mathbf{u} + \mathbf{u}\cdot\nabla b + \gamma b\,\text{div}\,\mathbf{u} = 0, \\ (a,\mathbf{u},b)|_{t=0} = (a_0,\mathbf{u}_0,b_0). \end{cases} \tag{1.5}$$

The above system (1.5) has a similar structure to the isentropic compressible Navier-Stokes equations if we regard a and b as a whole, hence, we can follow the method used in [3,5] and [6] to prove the local wellposedness. For readers' convenience, we only state the theorem as follows without detailed proof.

THEOREM 1.1 (Local wellposedness). *Let $n \geq 2$. Then for any $\mathbf{u}_0 \in \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$, $(a_0, b_0) \in \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$ with $1 + a_0$ bounded away from zero, there exists a positive time T such that the system* (1.5) *has a unique solution with*

$$(a, b) \in C([0,T]; \dot{B}_{2,1}^{\frac{n}{2}}), \quad \mathbf{u} \in C([0,T]; \dot{B}_{2,1}^{\frac{n}{2}-1}) \cap L^1([0,T]; \dot{B}_{2,1}^{\frac{n}{2}+1}).$$

REMARK 1.1. We can generalize the above theorem to an L^p-critical framework as [3] and [6].

Before stating the second theorem, we introduce some notations. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions over $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be its dual space. For any $z \in \mathcal{S}'(\mathbb{R}^n)$, the lower and higher frequency parts are expressed as

$$z^\ell \overset{\text{def}}{=} \sum_{j \leq j_0} \dot{\Delta}_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} \sum_{j > j_0} \dot{\Delta}_j z$$

for some fixed integer $j_0 \geq 1$ (the value of j_0 is dependent in the proof of the main theorems). The corresponding truncated semi-norms are defined as follows:

$$\|z\|_{\dot{B}_{2,1}^s}^\ell \overset{\text{def}}{=} \|z^\ell\|_{\dot{B}_{2,1}^s} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^s}^h \overset{\text{def}}{=} \|z^h\|_{\dot{B}_{2,1}^s}.$$

The second main result of the paper is stated as follows.

THEOREM 1.2 (Global well-posedness). *Let $n \geq 2$. Then for any $(a_0^\ell, \mathbf{u}_0, b_0^\ell) \in \dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$, and $(a_0^h, b_0^h) \in \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$, there exists a positive constant c_0 such that if,*

$$\|(a_0^\ell, \mathbf{u}_0, b_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a_0^h, b_0^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \leq c_0, \tag{1.6}$$

the system (1.5) *has a unique global solution $(a, \mathbf{u}, b)$ so that*

$$a^\ell \in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}-1}), \quad a^h \in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}}),$$
$$b^h \in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}}),$$
$$(b^\ell, \mathbf{u}) \in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{n}{2}+1}).$$

Moreover, there exists some constant C such that

$$\|(a^\ell, b^\ell, \mathbf{u})\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(a^h, b^h)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(b^\ell, \mathbf{u})\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \leq C c_0.$$

REMARK 1.2. Due to lack of dissipation of the density, it may be a challenge to generalize the above theorem to more general Besov spaces related to the L^p with $p \neq 2$.

Strategy of the proof of Theorem 1.2. Now let us outline the main points of the study and explain some of the major difficulties and techniques presented in this article. By the continuity argument, the existence of the global solutions can be proven by combining the local existence and the *a priori* estimates. The local well-posedness can be proven by the standard compact argument as [5]. The key point is to obtain the *a priori* estimates of the strong solutions. More specifically, since the dissipative variables b and $\mathbf{u}$ satisfy (1.5)$_3$ and (1.5)$_2$ whose linear parts possess the same structure as that of the compressible isentropic Navier-Stokes Equations (1.3). By spectral analysis just like [3,4,7], the variable b has smoothing effect in the low frequency and damping effect

in the high frequency. Moreover, the uniform bound of $b, \mathbf{u}$ can be established by a direct energy method as in [3]-[5]. Due to the appearance of the non-dissipative variable a, the main difficulty lies in how to enclose the energy estimates of the variable a. Different from the isentropic compressible Navier-Stokes Equations (1.3), there is a missing term ∇a in the momentum equation, so, the density function a possesses neither smoothing effect in the low frequency nor damping effect in the high frequency. Moreover, we even cannot directly control the low frequency part of a in the spaces $\dot{B}_{2,1}^{\frac{n}{2}-1}(\mathbb{R}^n)$ as the linear term $\operatorname{div} \mathbf{u}$ appears in the first equation of (1.5). Indeed, for the first equation of (1.5), if we make the energy estimate for a^ℓ in the space $\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})$, we have to control the linear term $\|\mathbf{u}^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}$. However, we can only obtain $\|\mathbf{u}^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}$ from the previous energy argument of $(b^\ell, \mathbf{u}^\ell)$. This leads to the loss of control for the nonlinear terms $I(a)\nabla b$ and $I(a)\mathcal{A}\mathbf{u}$. To overcome this difficulty, we need to introduce a combination function $\phi \overset{\text{def}}{=} \gamma a - b$ to annihilate the linear term $\operatorname{div} \mathbf{u}$. Exploiting some delicate energy estimates, we can first get the bound of $\|\phi^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}$ and then obtain the control of $\|b^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}$, which further implies the bound of $\|a^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}$. This enables us to obtain the energy estimates of the non-dissipative variable a and thus to prove the global wellposedness.

The rest of this paper is arranged as follows. In the second section, we recall some basic facts about Littlewood-Paley theory. In the third section, we use four subsections to prove Theorem 1.2. In the first subsection, we obtain the estimates of $\mathcal{P}\mathbf{u}$. In the second and third subsections, we obtain the estimates of compressible part of $(a, b, \mathcal{Q}\mathbf{u})$ in the low frequency and high frequency, respectively. In the last subsection, we use the continuity argument to close the energy estimates and thus complete the proof of Theorem 1.2.

Let us introduce some notations. For two operators A and B, we denote $[A, B] = AB - BA$, the commutator between A and B. The letter C stands for a generic constant whose meaning is clear from the context. We denote $\langle a, b \rangle$ the $L^2(\mathbb{R}^n)$ inner product of a and b and write $a \lesssim b$ instead of $a \leq Cb$. Given a Banach space X, we shall denote $\|(a, b)\|_X = \|a\|_X + \|b\|_X$.

2. Preliminaries

For readers' convenience, in this section, we list some basic knowledge on Littlewood-Paley theory. The Littlewood-Paley decomposition plays a central role in our analysis. To define it, fix some smooth radial nonincreasing function χ supported in the ball $B(0, \frac{4}{3})$ of $\mathbb{R}^n$, and with value 1 on, say, $B(0, \frac{3}{4})$, then set $\psi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$. We have

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j} \cdot) = 1 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \operatorname{Supp} \psi \subset \left\{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined on tempered distributions by

$$\dot{\Delta}_j u \overset{\text{def}}{=} \psi(2^{-j} D) u \overset{\text{def}}{=} \mathcal{F}^{-1}(\psi(2^{-j} \cdot) \mathcal{F} u).$$

Let us remark that, for any homogeneous function A of order 0 smooth outside 0, we have

$$\forall p \in [1, \infty], \qquad \|\dot{\Delta}_j(A(D) u)\|_{L^p} \leq C \|\dot{\Delta}_j u\|_{L^p}.$$

DEFINITION 2.1. *Let p,r be in $[1,+\infty]$ and s in $\mathbb{R}$, $u \in \mathcal{S}'(\mathbb{R}^n)$. We define the Besov norm by*

$$\|u\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_j\|_{\ell^r(\mathbb{Z})}.$$

We then define the spaces $\dot{B}^s_{p,r} \overset{\text{def}}{=} \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n), \big| \|u\|_{\dot{B}^s_{p,r}} < \infty \right\}$, where $u \in \mathcal{S}'_h(\mathbb{R}^n)$ means that $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ (see Definition 1.26 of [1]).

When employing parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm before performing the summation for computing the Besov norm. So we next introduce the following Besov-Chemin-Lerner space $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ (see [1]):

$$\widetilde{L}^q_T(\dot{B}^s_{p,r}) = \left\{ u \in (0,+\infty) \times \mathcal{S}'_h(\mathbb{R}^n) : \|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \overset{\text{def}}{=} \big\|2^{ks}\|\dot{\Delta}_k u(t)\|_{L^q(0,T;L^p)}\big\|_{\ell^r}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\widetilde{\mathcal{C}}_b([0,T];\dot{B}^s_{p,r})$ the subset of functions of $\widetilde{L}^\infty_T(\dot{B}^s_{p,r})$ which are also continuous from $[0,T]$ to $\dot{B}^s_{p,r}$.

The following Bernstein's lemma will be repeatedly used throughout this paper.

LEMMA 2.1. *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^n$. A constant C exists so that for any positive real number λ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (p,q) with $1 \leq p \leq q \leq \infty$, there hold*

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+n(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p},$$

$$\text{Supp}\,\hat{u} \subset \lambda\mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m}\lambda^{m+n(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}.$$

Next, we give some important inequalities on Besov spaces.

LEMMA 2.2 ([5]). *Let $s_1 \leq \frac{n}{2}$, $s_2 \leq \frac{n}{2}$ and $s_1 + s_2 > 0$. For any $u \in \dot{B}^{s_1}_{2,1}(\mathbb{R}^n)$, $v \in \dot{B}^{s_2}_{2,1}(\mathbb{R}^n)$, we have*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{2}}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,1}}.$$

LEMMA 2.3 (Lemma 2.100 in [1]). *Let $-1-\frac{n}{2} < s \leq 1+\frac{n}{2}$, $v \in \dot{B}^s_{2,1}(\mathbb{R}^n)$ and $u \in \dot{B}^{\frac{n}{2}+1}_{2,1}(\mathbb{R}^n)$ with $\text{div}\,u = 0$. Then there holds*

$$\|[\dot{\Delta}_j, u \cdot \nabla]v\|_{L^2} \lesssim d_j 2^{-js}\|\nabla u\|_{\dot{B}^{\frac{n}{2}}_{2,1}}\|v\|_{\dot{B}^s_{2,1}}.$$

LEMMA 2.4 ([1]). *Let G with $G(0) = 0$ be a smooth function defined on an open interval I of $\mathbb{R}$ containing 0. Then the following estimates*

$$\|G(f)\|_{\dot{B}^s_{2,1}} \lesssim \|f\|_{\dot{B}^s_{2,1}} \quad \text{and} \quad \|G(f)\|_{\widetilde{L}^q_T(\dot{B}^s_{2,1})} \lesssim \|f\|_{\widetilde{L}^q_T(\dot{B}^s_{2,1})}$$

hold true for $s > 0$, $1 \leq q \leq \infty$ and f valued in a bounded interval $J \subset I$.

LEMMA 2.5 ([1]).   *Let $\sigma \in \mathbb{R}$, $T > 0$, and $1 \le q_2 \le q_1 \le \infty$. Let u satisfy the heat equation*

$$\partial_t u - \Delta u = f, \quad u_{|t=0} = u_0.$$

Then there holds the following a priori estimate

$$\|u\|_{\widetilde{L}_T^{q_1}(\dot{B}_{2,1}^{\sigma+\frac{2}{q_1}})} \lesssim \|u_0\|_{\dot{B}_{2,1}^{\sigma}} + \|f\|_{\widetilde{L}_T^{q_2}(\dot{B}_{2,1}^{\sigma-2+\frac{2}{q_2}})}.$$

3. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 by the following four subsections.

3.1. Estimates for incompressible part of the velocity $\mathcal{P}\mathbf{u}$.

First, we apply the operator $\mathcal{P}$ to the second equation of (1.5) to get

$$\partial_t \mathcal{P}\mathbf{u} - \Delta \mathcal{P}\mathbf{u} = -\mathcal{P}(\mathbf{u} \cdot \nabla \mathbf{u}) + \mathcal{P}(I(a)\nabla b - I(a)\mathcal{A}\mathbf{u}).$$

By Lemma 2.5, there holds

$$\|\mathcal{P}\mathbf{u}\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\mathcal{P}\mathbf{u}\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}$$
$$\lesssim \|\mathcal{P}\mathbf{u}_0\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|I(a)\nabla b\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|I(a)\mathcal{A}\mathbf{u}\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}. \quad (3.1)$$

3.2. Estimates for the low frequency part of $(a, b, \mathcal{Q}u)$.

We cannot use directly the equation of a to obtain $\|a^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})}$ as there is no control $\|\operatorname{div} \mathbf{u}^\ell\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}$. Here, we introduce an unknown good function

$$\phi \stackrel{\text{def}}{=} \gamma a - b$$

to overcome the difficulty.

It's straightforward to deduce from (1.5) that ϕ satisfies

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi + \gamma(a - b)\operatorname{div} \mathbf{u} = 0. \quad (3.2)$$

Applying $\dot{\Delta}_j$ to both sides of (3.2) and using a commutator's argument give rise to

$$\partial_t \dot{\Delta}_j \phi + \mathbf{u} \cdot \nabla \dot{\Delta}_j \phi + [\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\phi + \gamma \dot{\Delta}_j((a - b)\operatorname{div} \mathbf{u}) = 0.$$

Taking the L^2 inner product with $\dot{\Delta}_j \phi$ and multiplying by $1/\|\dot{\Delta}_j \phi\|_{L^2} 2^{j(\frac{n}{2}-1)}$ formally on both sides, then integrating the resultant inequality from 0 to t, we can get by summing up about $j \le j_0$ that

$$\|\phi^\ell\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|\phi_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t \|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} ds$$
$$+ \int_0^t \sum_{j \le j_0} 2^{(\frac{n}{2}-1)j} \|[\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\phi\|_{L^2} ds + \gamma \int_0^t \|((a-b)\operatorname{div} \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} ds. \quad (3.3)$$

Thanks to Lemmas 2.2, 2.3 and the embedding relation $\dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, there holds

$$\|\operatorname{div} \mathbf{u}\|_{L^\infty} \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \sum_{j \le j_0} 2^{(\frac{n}{2}-1)j} \|[\dot{\Delta}_j, \mathbf{u} \cdot \nabla]\phi\|_{L^2} + \gamma \|((a-b)\operatorname{div} \mathbf{u})^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$\lesssim (\|\phi\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \gamma \|a-b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}) \|\nabla \mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}}, \tag{3.4}$$

from which we can get

$$\|\phi^\ell\|_{\widetilde{L}_t^\infty (\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|\phi_0^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t (\|(a^\ell, b^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a^h, b^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} \, ds. \tag{3.5}$$

For studying the coupling between b and $\mathcal{Q}\mathbf{u}$, it is convenient to set $\varphi = \Lambda^{-1} \mathrm{div}\, \mathbf{u}$ (with $\Lambda^s z \overset{\mathrm{def}}{=} \mathcal{F}^{-1}(|\xi|^s \mathcal{F}z)(s \in \mathbb{R})$), keeping in mind that, bounding φ or $\mathcal{Q}\mathbf{u}$ is equivalent, as one can go from φ to $\mathcal{Q}\mathbf{u}$ or from $\mathcal{Q}\mathbf{u}$ to φ by means of a 0 order homogeneous Fourier multiplier. Now, one can infer from (1.5) that

$$\begin{cases} \partial_t b + \gamma \mathrm{div}\, \mathbf{u} + \mathbf{u} \cdot \nabla b + \gamma b \mathrm{div}\, \mathbf{u} = 0, \\ \partial_t \varphi + \Lambda^{-1} \mathrm{div}(\mathbf{u} \cdot \nabla \mathbf{u}) - 2\Delta \varphi - \Lambda b = \Lambda^{-1} \mathrm{div}\, \mathbf{F}(a, \mathbf{u}, b), \end{cases} \tag{3.6}$$

with $\mathbf{F}(a, \mathbf{u}, b) \overset{\mathrm{def}}{=} I(a)\nabla b - I(a)\mathcal{A}\mathbf{u}$.

The estimates on the dissipation of $b, \mathcal{Q}\mathbf{u}$ in the low frequency part are presented in the following lemma.

LEMMA 3.1. *Under the assumption of Theorem 1.2, we have*

$$\|(b^\ell, \mathcal{Q}\mathbf{u}^\ell)\|_{\widetilde{L}_t^\infty (\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(b^\ell, \mathcal{Q}\mathbf{u}^\ell)\|_{L_t^1 (\dot{B}_{2,1}^{\frac{n}{2}+1})}$$
$$\lesssim \|(b_0^\ell, \mathbf{u}_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{u} \cdot \nabla b\|_{L_t^1 (\dot{B}_{2,1}^{\frac{n}{2}-1})}$$
$$+ \gamma \|b \mathrm{div}\, \mathbf{u}\|_{L_t^1 (\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L_t^1 (\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|\mathbf{F}(a, \mathbf{u}, b)\|_{L_t^1 (\dot{B}_{2,1}^{\frac{n}{2}-1})}. \tag{3.7}$$

Proof. The linear equation of b, φ in (3.6) coincides with the compressible Navier-Stokes equations, hence, we can follow the method used in [5] to get the desired estimates. Here, we sketch its proof for readers' convenience. We first rewrite (3.6) into

$$\begin{cases} \partial_t b + \gamma \Lambda \varphi = f_1, \\ \partial_t \varphi - 2\Delta \varphi - \Lambda b = f_2, \end{cases} \tag{3.8}$$

with

$$f_1 \overset{\mathrm{def}}{=} -\mathbf{u} \cdot \nabla b - \gamma b \mathrm{div}\, \mathbf{u}, \quad f_2 \overset{\mathrm{def}}{=} \Lambda^{-1} \mathrm{div}(-\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{F}(a, \mathbf{u}, b)). \tag{3.9}$$

Let k_0 be some integer, and $z^\ell \overset{\mathrm{def}}{=} \dot{S}_{k_0} z$. Denote $f_k = \dot{\Delta}_k f$, applying the operator $\dot{\Delta}_k S_{k_0}$ to the equations in (3.8), then multiplying $(3.8)_1$ by b_k^ℓ, $(3.8)_2$ by $\gamma \varphi_k^\ell$, respectively, we can get

$$\frac{1}{2} \frac{d}{dt} (\|b_k^\ell\|_{L^2}^2 + \gamma \|\varphi_k^\ell\|_{L^2}^2) + 2\gamma \|\Lambda \varphi_k^\ell\|_{L^2}^2 = \langle (f_1)_k^\ell, b_k^\ell \rangle + \langle (f_2)_k^\ell, \gamma \varphi_k^\ell \rangle. \tag{3.10}$$

To capture the dissipation of b, we have to consider the time derivative of the mixed terms involved in $\langle \varphi_k^\ell, \Lambda b_k^\ell \rangle$:

$$-\frac{1}{2} \frac{d}{dt} \langle \varphi_k^\ell, \Lambda b_k^\ell \rangle + \frac{1}{2} \|\Lambda b_k^\ell\|_{L^2}^2 - \frac{\gamma}{2} \|\Lambda \varphi_k^\ell\|_{L^2}^2 = -\langle \Delta \varphi_k^\ell, \Lambda b_k^\ell \rangle - \frac{1}{2} \langle (\Lambda f_1)_k^\ell, \varphi_k^\ell \rangle - \frac{1}{2} \langle (f_2)_k^\ell, \Lambda b_k^\ell \rangle. \tag{3.11}$$

To eliminate the highest order term appearing on the right-hand side of (3.11), we next want an estimate for $\|\Lambda b_k^\ell\|_{L^2}^2$. From (3.8), we have

$$\partial_t \Lambda b_k^\ell + \gamma \Lambda^2 \varphi_k^\ell = \Lambda(f_1)_k^\ell.$$

Testing that equation by Λb_k^ℓ yields

$$\frac{1}{2\gamma}\frac{d}{dt}\|\Lambda b_k^\ell\|_{L^2}^2 = \langle \Delta \varphi_k^\ell, \Lambda b_k^\ell \rangle + \frac{1}{\gamma}\langle (f_1)_k^\ell, \Lambda^2 b_k^\ell \rangle. \tag{3.12}$$

Denote

$$\mathcal{L}_k^2 \overset{\text{def}}{=} \|b_k^\ell\|_{L^2}^2 + \gamma\|\varphi_k^\ell\|_{L^2}^2 + \frac{1}{\gamma}\|\Lambda b_k^\ell\|_{L^2}^2 - \langle \varphi_k^\ell, \Lambda b_k^\ell \rangle.$$

Summing up (3.10), (3.11), and (3.12), we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_k^2 + \frac{3\gamma}{2}\|\Lambda \varphi_k^\ell\|_{L^2}^2 + \frac{1}{2}\|\Lambda b_k^\ell\|_{L^2}^2 = \langle (f_1)_k^\ell, b_k^\ell \rangle + \langle (f_2)_k^\ell, \gamma \varphi_k^\ell \rangle$$
$$- \frac{1}{2}\langle (\Lambda f_1)_k^\ell, \varphi_k^\ell \rangle - \frac{1}{2}\langle (f_2)_k^\ell, \Lambda b_k^\ell \rangle + \frac{1}{\gamma}\langle (f_1)_k^\ell, \Lambda^2 b_k^\ell \rangle.$$

It's straightforward to deduce from the low-frequency cut-off and Young inequality that

$$\mathcal{L}_k^2 \approx \left\|\left(b_k^\ell, \gamma\varphi_k^\ell, \frac{1}{\gamma}\Lambda b_k^\ell\right)\right\|_{L^2}^2 \approx \|(b_k^\ell, \gamma\varphi_k^\ell)\|_{L^2}^2,$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_k^2 + 2^{2k}\mathcal{L}_k^2 \lesssim \|((f_1)_k^\ell, (f_2)_k^\ell)\|_{L^2}\mathcal{L}_k. \tag{3.13}$$

Multiplying the above inequality by $2^{(\frac{n}{2}-1)j}/\mathcal{L}_k$ formally on both sides, and then integrating from 0 to t, summing up about $j \leq j_0$, we finally get that

$$\|(b^\ell, \gamma\varphi^\ell)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(b^\ell, \gamma\varphi^\ell)\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}$$
$$\lesssim \|(b_0^\ell, \gamma\varphi_0^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t \left\|((f_1)^\ell, (f_2)^\ell)\right\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} ds. \tag{3.14}$$

The combination of (3.14) and (3.9) implies (3.7). This proves the lemma. □

3.3. Estimates for the high frequency part of $(a, b, \mathcal{Q}u)$. We first present the estimate of the high frequency part of a. Taking similar processes as the derivation of (3.5), one can infer from the first equation of (1.5) that

$$\|a^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} \lesssim \|a_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} ds + \int_0^t \left(\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}\right)\|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} ds. \tag{3.15}$$

We next deal with the high frequency estimates of $b, \mathcal{Q}u$. By using the operator $\mathcal{Q}$, we infer from (1.5) that

$$\begin{cases} \partial_t b + \gamma\,\mathrm{div}\,\mathbf{u} = -\mathbf{u}\cdot\nabla b - \gamma b\,\mathrm{div}\,\mathbf{u}, \\ \partial_t \mathcal{Q}\mathbf{u} - 2\Delta\mathcal{Q}\mathbf{u} + \nabla b = -\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathcal{Q}\mathbf{F}(a, \mathbf{u}, b). \end{cases} \tag{3.16}$$

Subsequently, we perform the energy argument in terms of the effective velocity by following the approach used in [4,7], and [12] that

$$\mathbf{G} \overset{\text{def}}{=} \mathcal{Q}\mathbf{u} - \frac{1}{2}\Delta^{-1}\nabla b, \tag{3.17}$$

then $\mathbf{G}$ satisfies

$$\partial_t \mathbf{G} - 2\Delta\mathbf{G} = \frac{\gamma}{2}\mathbf{G} + \frac{\gamma}{4}\Delta^{-1}\nabla b + \frac{1}{2}\mathcal{Q}(b\mathbf{u}) + \frac{\gamma-1}{2}\Delta^{-1}\nabla(b\,\mathrm{div}\,\mathbf{u}) - \mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathcal{Q}\mathbf{F}(a,\mathbf{u},b).$$

By taking $\sigma = \frac{n}{2} - 1$, $q_1 = \infty$, or $q_1 = 1$, and $q_2 = 1$ in Lemma 2.5 respectively, we can get the estimate of $\mathbf{G}$ in the high frequencies as follows

$$\|\mathbf{G}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + 2\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}$$

$$\lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t \|\mathbf{G}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds + \int_0^t \|\Delta^{-1}\nabla b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds + \int_0^t \|\mathcal{Q}(b\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds$$

$$+ \int_0^t \|(\Delta^{-1}\nabla(b\,\mathrm{div}\,\mathbf{u}))^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds + \int_0^t \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds + \int_0^t \|\mathbf{F}(a,\mathbf{u},b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}\,ds. \tag{3.18}$$

The important point is that, owing to the high frequency cut-off at $|\xi| \sim 2^{j_0}$,

$$\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim 2^{-2j_0}\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} \quad \text{and} \quad \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-2})} \lesssim 2^{-2j_0}\|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}.$$

Hence, if j_0 is large enough then the term $\|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}-1})}$ may be absorbed in the right-hand side.

In view of (3.17), one can get the equation of b

$$\partial_t b + \frac{\gamma}{2}b + \mathbf{u}\cdot\nabla b = -\gamma\,\mathrm{div}\,\mathbf{G} - \gamma b\,\mathrm{div}\,\mathbf{u}. \tag{3.19}$$

Applying $\dot{\Delta}_j$ to both sides of (3.19) and using a commutator's argument give rise to

$$\partial_t \dot{\Delta}_j b + \frac{\gamma}{2}\dot{\Delta}_j b + \mathbf{u}\cdot\nabla\dot{\Delta}_j b = -\gamma\dot{\Delta}_j\,\mathrm{div}\,\mathbf{G} - [\dot{\Delta}_j, \mathbf{u}\cdot\nabla]b - \dot{\Delta}_j(\gamma b\,\mathrm{div}\,\mathbf{u}).$$

A standard energy argument leads to

$$\|\dot{\Delta}_j b(t)\|_{L^2} + \frac{\gamma}{2}\int_0^t \|\dot{\Delta}_j b\|_{L^2}\,ds$$

$$\lesssim \|\dot{\Delta}_j b_0\|_{L^2} + \int_0^t \|\dot{\Delta}_j\,\mathrm{div}\,\mathbf{G}\|_{L^2}\,ds$$

$$+ \int_0^t \|\mathrm{div}\,\mathbf{u}\|_{L^\infty}\|\dot{\Delta}_j b\|_{L^2}\,ds + \int_0^t \big\|[\dot{\Delta}_j, \mathbf{u}\cdot\nabla]b\big\|_{L^2}\,ds + \int_0^t \|\dot{\Delta}_j(b\,\mathrm{div}\,\mathbf{u})\|_{L^2}\,ds$$

from which and Lemma 2.3 we can further get

$$\|b^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \frac{\gamma}{2}\|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} \lesssim \|b_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t \|\mathbf{G}^h\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}\,ds + \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}}\|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}}\,ds. \tag{3.20}$$

Multiplying by a suitable large constant on both sides of (3.18) and then summing up the resultant and (3.20), we can infer that

$$\|\mathbf{G}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|b^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}$$

$$\lesssim \|\mathbf{G}_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|b_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \, ds + \int_0^t \|(b\mathrm{div}\,\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-2}} \, ds$$

$$+ \int_0^t (\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(b\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}) \, ds + \int_0^t \|\mathbf{F}(a,\mathbf{u},b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \, ds. \qquad (3.21)$$

In view of $\mathbf{G} \overset{\mathrm{def}}{=} \mathcal{Q}\mathbf{u} - \frac{1}{2}\Delta^{-1}\nabla b$ and the embedding relation in the high frequency, there hold

$$\|\mathcal{Q}\mathbf{u}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} \lesssim \|\mathbf{G}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|b^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})},$$

$$\|\mathcal{Q}\mathbf{u}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} \lesssim \|\mathbf{G}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}.$$

As a result, we can rewrite (3.21) into

$$\|b^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\mathcal{Q}\mathbf{u}^h\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})} + \|\mathcal{Q}\mathbf{u}^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})}$$

$$\lesssim \|b_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathcal{Q}\mathbf{u}_0^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \, ds + \int_0^t \|(b\mathrm{div}\,\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-2}} \, ds$$

$$+ \int_0^t (\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(b\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{F}(a,\mathbf{u},b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}) \, ds. \qquad (3.22)$$

Combining (3.5), (3.7), (3.15), with (3.22) gives

$$\|(\phi^\ell, b^\ell, \mathbf{u})\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(a^h, b^h)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(b^\ell, \mathbf{u})\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}$$

$$\lesssim \|(a_0^\ell, b_0^\ell, \mathbf{u}_0)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a_0^h, b_0^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$+ \int_0^t \|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \, ds + \int_0^t \|(b\mathrm{div}\,\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-2}} \, ds + \int_0^t \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \, ds$$

$$+ \int_0^t (\|(a^\ell, b^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a,b)^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} \, ds + \int_0^t \|(b\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \, ds$$

$$+ \int_0^t (\|\mathbf{u}\cdot\nabla b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \gamma\|b\mathrm{div}\,\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|\mathbf{F}(a,\mathbf{u},b)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}) \, ds. \qquad (3.23)$$

We now bound each term on the right-hand side of (3.23). First, according to product law in Lemma 2.2, we have

$$\|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|(b\mathrm{div}\,\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|b\mathrm{div}\,\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$\lesssim (\|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}},$$

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}. \qquad (3.24)$$

By Lemma 2.2, the embedding relation in high frequency, the Young inequality, and the interpolation inequality, there holds

$$\|\mathbf{u}\cdot\nabla b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \gamma\|b\mathrm{div}\,\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(b\mathbf{u})^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$\lesssim \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \gamma \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|\operatorname{div}\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|b\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$\lesssim \|b\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2$$

$$\lesssim \|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}^2 + \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}. \tag{3.25}$$

At last, we deal with each term in $\mathbf{F}(a,\mathbf{u},b)$. We first use the fact that $I(a)=a-aI(a)$ and Lemmas 2.2, 2.4 to get

$$\|I(a)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|a\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|I(a)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) + (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|a\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + 1)(\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}).$$

Hence, in view of $I(a)\nabla b = I(a)\nabla b^\ell + I(a)\nabla b^h$, we further obtain

$$\|I(a)\nabla b\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|I(a)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\nabla b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|I(a)\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|\nabla b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$\lesssim \|I(a)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \|\nabla b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|a\|_{\dot{B}_{2,1}^{\frac{n}{2}}} \|\nabla b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}$$

$$\lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$+ (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + 1)(\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}. \tag{3.26}$$

Similarly,

$$\|(I(a)(\Delta\mathbf{u} + \nabla\operatorname{div}\mathbf{u}))^h\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim (\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}. \tag{3.27}$$

From the definition of ϕ, we have

$$\|a^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} \lesssim \|\phi^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}}.$$

Hence, collecting the above estimates (3.24)–(3.27), we can finally get from (3.23) that

$$\|(a^\ell,b^\ell,\mathbf{u})\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(a^h,b^h)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(b^\ell,\mathbf{u})\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}$$

$$\lesssim \|(a_0^\ell,b_0^\ell,\mathbf{u}_0)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a_0^h,b_0^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}$$

$$+ \int_0^t (\|(a^\ell,b^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}})(\|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + \|\mathbf{u}\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}}) \, ds$$

$$+ \int_0^t (\|(b^\ell,\mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}})(\|(b^\ell,\mathbf{u})\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} + \|b^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \, ds$$

$$+ \int_0^t (\|(a^\ell,b^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}} + 1)(\|(a^\ell,b^\ell)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|a^h\|_{\dot{B}_{2,1}^{\frac{n}{2}}}) \|b^\ell\|_{\dot{B}_{2,1}^{\frac{n}{2}+1}} \, ds. \tag{3.28}$$

3.4. Continuity argument.

Proof. **(Proof of Theorem 1.2.)** In this subsection, we complete the proof of Theorem 1.2 by the continuity arguments. Denote

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \|(a^\ell,b^\ell,\mathbf{u})\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}-1})} + \|(a^h,b^h)\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{n}{2}})} + \|(b^\ell,\mathbf{u})\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}+1})} + \|b^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{n}{2}})}$$

$$\mathcal{E}_0 \overset{\text{def}}{=} \|(a_0^\ell, b_0^\ell, \mathbf{u}_0)\|_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(a_0^h, b_0^h)\|_{\dot{B}_{2,1}^{\frac{n}{2}}}.$$

Subsequently, we deduce from (3.28) that

$$\mathcal{E}(t) \le \mathcal{E}_0 + C(\mathcal{E}(t))^2(1 + C\mathcal{E}(t)). \tag{3.29}$$

Under the setting of initial data in Theorem 1.2, there exists a positive constant C_0 such that $\mathcal{E}_0 \le C_0\varepsilon$. Due to the local existence result which has been achieved by Theorem 1.1, there exists a positive time T such that

$$\mathcal{E}(t) \le 2C_0\,\varepsilon, \quad \forall\, t \in [0,T]. \tag{3.30}$$

Let T^* be the largest possible time of T for which (3.30) holds. Now, we only need to show $T^* = \infty$. By (3.29), we can use a standard continuation argument to prove that $T^* = \infty$ provided that ε is small enough. We omit the details here. Hence, we finish the proof of Theorem 1.2. □

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