COMMUN. MATH. SCI.
Vol. 21, No. 8, pp. 2261–2299

STABILITY OF PLANAR RAREFACTION WAVE FOR VISCOUS VASCULOGENESIS MODEL*

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Abstract. In this paper, we are concerned with a two-dimensional quasi-linear hyperbolic-parabolic-elliptic system modelling vasculogenesis. We first derive a two-dimensional inviscid system as the asymptotic equations in large time by ignoring all the viscous terms. Then we show that this inviscid system admits a planar rarefaction wave when the pressure function satisfies some suitable structure conditions. By using elaborate energy estimates, we further prove that the solution of the concerned system will asymptotically converge to this planar rarefaction wave under the same assumptions on pressure function.

Keywords. Vasculogenesis model; planar rarefaction wave; asymptotic stability.

AMS subject classifications. 35Q92; 35L65; 35B40; 35B35.

1. Introduction and main result

The term vasculogenesis refers to the formation of the first blood vessels by endothelial cells or their precursors, angioblasts. In order to describe the formation of the vascular network, Gamba et al. in [1] proposed a viscous vasculogenesis model,

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = \lambda \Delta \boldsymbol{u} + \mu \rho \nabla c, \\ c_t = D\Delta c + a\rho - bc. \end{cases} \tag{1.1}$$

Here the model describes the cell population as a continuous density field ρ and velocity $\boldsymbol{u}$; it also assumes the presence of a concentration field c of soluble factor. $p(\rho)$ is density dependent pressure function. a and b are positive constants, representing the rate of release and the reciprocal of characteristic degradation time of the chemotactic factor. $D>0$ and $\lambda>0$ denote the diffusivity of the chemoattractant and the velocity, respectively. The parameter $\mu>0$ measures the strength of the cell response.

When the small velocity diffusion λ (or the fast signal diffusion D) is involved, we may denote

$$\epsilon = \frac{\lambda}{D}$$

by the ratio between the diffusivity of the velocity and of the chemoattractant, and make the following transformation:

$$\hat{t} = \lambda t, \quad \rho^*(\hat{t}, \boldsymbol{x}) = \rho(t, \boldsymbol{x}), \ \boldsymbol{u}^*(\hat{t}, \boldsymbol{x}) = \frac{1}{\lambda} \boldsymbol{u}(t, \boldsymbol{x}), \ c^*(\hat{t}, \boldsymbol{x}) = c(t, \boldsymbol{x}), \ p^*(\rho^*) = \frac{1}{\lambda^2} p(\rho), \tag{1.2}$$

and denote the new parameters by

$$\hat{\mu} = \frac{\mu}{\lambda^2}, \ \hat{a} = \frac{a}{D}, \ \hat{b} = \frac{b}{D}, \tag{1.3}$$

*Received: October 18, 2022; Accepted (in revised form): April 02, 2023. Communicated by Yaguang Wang.

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then the original Equations (1.1) become (dropping the asterisks and circumflexes for algebraic simplicity),

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = \Delta \boldsymbol{u} + \mu \rho \nabla c, \\ \epsilon c_t = \Delta c + a\rho - bc. \end{cases} \tag{1.4}$$

Corresponding to a fast relaxation of the chemical substance c, i.e., $\epsilon \to 0$, system (1.4) is formally reduced to the following hyperbolic-parabolic-elliptic system,

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = \Delta \boldsymbol{u} + \mu \rho \nabla c, \\ 0 = \Delta c + a\rho - bc. \end{cases} \tag{1.5}$$

In this paper, we are concerned with the above viscous vasculogenesis model (1.5) in two-dimensional space, that is $\boldsymbol{x} = (x,y)$, with $x \in \mathbb{R}$ being the real line and $y \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ being a one-dimensional unit flat torus. Here the pressure function $p(\rho)$ is assumed to satisfy

$$p(\rho) \in C^3(\mathbb{R} \times \mathbb{T}), \quad p'(\rho) - \frac{a\mu}{b}\rho > 0 \quad \text{and} \quad p''(\rho) - \frac{a\mu}{b} > 0 \tag{1.6}$$

for any $\rho > 0$. The initial data to (1.5) is given by

$$(\rho, \boldsymbol{u})(0,x,y) = (\rho, u, v)(0,x,y) = (\rho_0, u_0, v_0)(x,y), \tag{1.7}$$

with the far field conditions in the x-direction,

$$(\rho_0, u_0, v_0)(x,y) \to (\rho_\pm, u_\pm, 0), \qquad \text{as} \quad x \to \pm\infty. \tag{1.8}$$

The far field data of $c(t,x,y)$ in the x-direction are given by

$$c(t,x,y) \to c_\pm, \qquad \text{as} \quad x \to \pm\infty. \tag{1.9}$$

Here $\rho_\pm > 0, u_\pm$ and $c_\pm$ are prescribed constants, and the periodic boundary conditions are imposed on $y \in \mathbb{T}$ for the solution $(\rho, \boldsymbol{u}, c)(t,x,y)$. From the third equation of (1.5), we have the following compatible condition

$$c_\pm = \frac{a}{b}\rho_\pm. \tag{1.10}$$

Understanding how blood vessels form in the organism has been a central issue in biomedical research during the last decades. Besides the viscous vasculogenesis model (1.1) where cell adhesion is considered, another typical model, which is referred as the PEC model (Persistence and Endogenous Chemotaxis model), has been proposed in [2,3]. The only difference between the PEC model and (1.1) is that the viscous term $\lambda \Delta \boldsymbol{u}$ in (1.1) is replaced by the damping term $-\alpha \rho \boldsymbol{u}$ to explain the drag between cells and the substrate material. We also mention a model proposed in [13], where the viscous term and damping term are both considered. Although the diffusive structure of these models for velocity are different, the coupling between concentration c and the fluid quantity ρ, $\boldsymbol{u}$ is the same. Hence, the analysis results for PEC model can give us some hints. In [32,33], the authors proved the global-in-time existence and the large-time behavior of solutions to the PEC model if the initial data is a small perturbation of a small enough constant equilibrium, where the smallness of the equilibrium plays a crucial role to deal with the coupling. Later, instead of small assumption on the constant equilibrium for density as in [32, 33], under the restriction on the coefficients that $bP'(\bar{\rho}) - a\mu\bar{\rho} > 0$, the authors obtained the asymptotic stability of linear and nonlinear diffusion waves for the PEC model in [25, 26] respectively, where the nonlinear diffusion wave can connect distinct far field states. And the stability of transition layer solutions of the PEC model on $\mathbb{R}_+ = [0, +\infty)$ was established in [10] under the same restriction on the coefficients.

The large-time behaviors of the PEC model motivate us to consider similar things for the viscous vasculogenesis model (1.1) or the simplified model (1.5), especially when the far field states are different. Inviscid Euler equation should be a good choice based on the similarity of (1.1) or (1.5) and Navier-Stokes equations or Navier-Stokes-Poisson equations. But the parabolic structure with damping terms for c in (1.1) makes it difficult to construct an approximate conservation law as the large-time behavior. Then it is necessary to seek a proper profile to connect distinct far field states for (1.1), we will leave it for study in the future. Fortunately, this has been realized for the simplified model (1.5) in one-dimensional space in [27]. They showed that the rarefaction wave of the corresponding inviscid Euler equation, where the dissipation effects are neglected, can describe the large-time behavior of (1.5). In this paper, we hope this asymptotic stability of rarefaction wave can be generalized to high-dimensional system (1.5).

The stability of three basic wave patterns, shock wave, rarefaction wave, and contact discontinuity for the high-dimensional viscous scalar conservation laws, Navier-Stokes equations or other Navier-Stokes type equations has been extensively studied. For the viscous scalar conservation laws, one can refer to [11, 30, 39] for the stability of planar rarefaction wave and its extended results, and refer to [7] for the stability of planar shock front solutions. For the Navier-Stokes equations, Li and Wang [18] proved the asymptotic stability of the planar rarefaction wave in the two-dimensional region $\mathbb{R} \times \mathbb{T}$. Later, they generalized this result to three-dimensional full equations in $\mathbb{R} \times \mathbb{T}^2$ (cf. [19]) and three-dimensional isentropic equations in half space with Navier boundary conditions (cf. [37]). For the superposition of a planar boundary layer and a planar rarefaction wave for compressible Navier-Stokes system with outflow boundary condition, one can refer to [34]. For the stability of other wave patterns, such as planar contact discontinuity, one can refer to [12]. For the other models, such as three-dimensional Boltzmann equation, 3-D bipolar Vlasov-Poisson-Boltzmann (VPB) system, Navier-Stokes-Korteweg equations, viscous compressible two-phase flow etc., one can refer to [8, 14, 16, 17, 21, 23, 35, 36, 38]. Multi-dimensional vanishing dissipation limit to the planar rarefaction wave is another interesting problem, please refer to [9, 15, 20, 22]. All the stability results of wave patterns for high-dimensional Navier-Stokes equations are based on the existence results for the multi-dimensional Riemann problem of inviscid Euler system and stability results of three basic wave patterns for Navier-Stokes in one dimensional space. We only mention the works that are most related to our topic in this paper. The 1-D rarefaction wave solutions were shown to be unique in the class of bounded entropy solutions to the multidimensional compressible isentropic Euler system [5], complete Euler system without vacuum [6], and complete Euler system with vacuum [4]. The stability results of rarefaction waves for Navier-Stokes equations in one dimensional space can be found in [24, 28, 29, 31].

Inspired by the work in [18], we anticipate that the solution to (1.5)-(1.9) in large time may behave as the solution to the Riemann problem of the following two-dimensional equations

$$
\begin{cases}
\rho_t + (\rho u)_x + (\rho v)_y = 0, & (x,y)\in\mathbb{R}\times\mathbb{T},\ t>0,\\
(\rho u)_t + (\rho u^2 + p(\rho))_x + (\rho uv)_y - \mu\rho c_x = 0,\\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p(\rho))_y - \mu\rho c_y = 0,\\
a\rho - bc = 0.
\end{cases}
\tag{1.11}
$$

Riemann initial data is given by

$$
(\rho,u,v)(0,x,y) = (\rho_0^r, u_0^r, v_0^r)(x,y) = \begin{cases} (\rho_-, u_-, 0), & x<0,\\ (\rho_+, u_+, 0), & x>0. \end{cases}
\tag{1.12}
$$

Plugging $c = \frac{a}{b}\rho$ into the second and the third equation of (1.11) and introducing a new pressure function

$$
p_1(\rho) = p(\rho) - \frac{\mu a}{2b}\rho^2,
\tag{1.13}
$$

then according to the condition on p in (1.6), we have $p_1'(\rho)>0$, $p_1''(\rho)>0$, and inviscid system (1.11) is reduced to two-dimensional Euler equations,

$$
\begin{cases}
\rho_t + (\rho u)_x + (\rho v)_y = 0, & (x,y)\in\mathbb{R}\times\mathbb{T},\ t>0,\\
(\rho u)_t + (\rho u^2 + p_1(\rho))_x + (\rho uv)_y = 0,\\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p_1(\rho))_y = 0,
\end{cases}
\tag{1.14}
$$

with Riemann initial data

$$
(\rho,u,v)(0,x,y) = (\rho_0^r, u_0^r, v_0^r)(x,y) = \begin{cases} (\rho_-, u_-, 0), & x<0,\\ (\rho_+, u_+, 0), & x>0, \end{cases}
\tag{1.15}
$$

which admits a unique planar rarefaction wave [5], and the planar rarefaction wave solution to (1.14)-(1.15) is defined in the following way,

$$
[\rho,u,v](t,x,y) = \left[\rho^r\left(\frac{x}{t}\right), u^r\left(\frac{x}{t}\right), 0\right]
$$

with $(\rho^r,u^r)(\frac{x}{t})$ being the rarefaction wave for the following one-dimensional Euler system,

$$
\begin{cases}
\rho_t + (\rho u)_x = 0, & x\in\mathbb{R},\ t>0,\\
(\rho u)_t + (\rho u^2 + p_1(\rho))_x = 0,
\end{cases}
\tag{1.16}
$$

with the Riemann initial data

$$
(\rho_0^r, u_0^r)(x) = \begin{cases} (\rho_-, u_-), & x<0,\\ (\rho_+, u_+), & x>0. \end{cases}
\tag{1.17}
$$

Now we discuss the rarefaction wave solution $(\rho^r,u^r)(\frac{x}{t})$ to (1.16)-(1.17) and state our main theorem in this paper. It's widely known that for $\rho>0$, the inviscid Euler system (1.16) is strictly hyperbolic and has two different eigenvalues

$$
\lambda_1(\rho,u) = u - \sqrt{p_1'(\rho)} = u - \sqrt{p'(\rho) - \frac{a\mu\rho}{b}},\quad \lambda_2(\rho,u) = u + \sqrt{p_1'(\rho)} = u + \sqrt{p'(\rho) - \frac{a\mu\rho}{b}}
$$

under the constriction $p'(\rho)-\frac{a\mu}{b}\rho>0$ in (1.6). For two distinct eigenvalues, respectively, we have two right eigenvectors $r_1(\rho,u)$ and $r_2(\rho,u)$, and both characteristic fields are genuinely nonlinear. Moreover, the i-Riemann invariant $z_i(\rho,u)(i=1,2)$ is determined by

$$z_i(\rho,u)=u+(-1)^{i+1}\int^{\rho}\frac{\sqrt{p'(s)-\frac{a\mu s}{b}}}{s}ds,$$

which satisfies $\nabla_{(\rho,u)}z_i(\rho,u)\cdot r_i(\rho,u)\equiv 0$ $(i=1,2)$ for all ρ, u. Without loss of generality, in this paper we only consider the case of 2-rarefaction wave for simplicity. Similarly, one can prove the cases of 1-rarefaction wave and the superposition of two rarefaction waves. If the end states $(\rho_\pm,u_\pm)$ satisfy

$$u_+-\int_{\rho_-}^{\rho_+}\frac{\sqrt{p'(s)-\frac{a\mu s}{b}}}{s}ds=u_-,\qquad\qquad \lambda_2(\rho_+,u_+)>\lambda_2(\rho_-,u_-), \tag{1.18}$$

that is to say, 2-Riemann invariant $z_2(\rho,u)$ is constant and the second eigenvalue $\lambda_2(\rho,u)$ is expanding along the 2-rarefaction wave curve, then a self-similar wave fan $(\rho^r,u^r)(\frac{x}{t})$ would be admitted to the Riemann problem (1.16)-(1.17). After that, the planar rarefaction wave solution to (1.11)-(1.12) is defined by

$$[\rho,u,v,c](t,x,y)=\left[\rho^r\left(\frac{x}{t}\right),u^r\left(\frac{x}{t}\right),0,c^r\left(\frac{x}{t}\right)\right]=\left[\rho^r\left(\frac{x}{t}\right),u^r\left(\frac{x}{t}\right),0,\frac{a}{b}\rho^r\left(\frac{x}{t}\right)\right]$$

where $(\rho^r,u^r,c^r)(\frac{x}{t})$ satisfies the following equations,

$$\begin{cases}\rho_t+(\rho u)_x=0, & x\in\mathbb{R},\ t>0,\\ (\rho u)_t+(\rho u^2+p(\rho))_x-\mu\rho c_x=0, \\ a\rho-bc=0,\end{cases} \tag{1.19}$$

with the Riemann initial data

$$(\rho_0^r,u_0^r)(x)=\begin{cases}(\rho_-,u_-), & x<0,\\ (\rho_+,u_+), & x>0.\end{cases} \tag{1.20}$$

To formulate our main result, we assume that

$$\begin{aligned}(\rho_0-{\rho_0}^r,u_0-{u_0}^r,v_0)\in L^2(\mathbb{R}\times\mathbb{T}),\\ (\nabla\rho_0,\nabla u_0,\nabla v_0)\in H^1(\mathbb{R}\times\mathbb{T}),\end{aligned} \tag{1.21}$$

and we set

$$\Phi_0^2=\|(\rho_0-\rho_0^r,u_0-u_0^r,v_0)\|^2+\|(\nabla\rho_0,\nabla u_0,\nabla v_0)\|_1^2+|(\rho_+-\rho_-,u_+-u_-)|^2. \tag{1.22}$$

Here the notations will be introduced at the end of this section. Our main theorem is stated as follows.

THEOREM 1.1. *Let the conditions on pressure p in (1.6) hold, and $(\rho^r,u^r,0,c^r)(\frac{x}{t})$ be the planar 2-rarefaction wave to (1.11)-(1.12) which connects the constant states $(\rho_\pm,u_\pm,0,c_\pm)$ satisfying (1.18) with $\rho_\pm>0$ and $c_\pm=\frac{a}{b}\rho_\pm$. There exists a positive constant ε_0 such that if the initial perturbation around the planar rarefaction wave and the* *wave strength satisfy* $\Phi_0 < \varepsilon_0$, *then the initial value problem* (1.5)-(1.9) *admits a unique global smooth solution* $(\rho, \boldsymbol{u}, c) = (\rho, u, v, c)$ *satisfying*

$$\begin{cases} (\rho - \rho^r, u - u^r, v, c - c^r, \nabla c) \in C^0\left(0, +\infty; L^2(\mathbb{R}\times\mathbb{T})\right), \\ (\nabla \rho, \nabla u, \nabla v) \in C^0\left(0, +\infty; H^1(\mathbb{R}\times\mathbb{T})\right), \\ (\nabla^2 \rho, \nabla^2 c) \in L^2\left(0, +\infty; L^2(\mathbb{R}\times\mathbb{T})\right), \\ \nabla^2 \boldsymbol{u} \in L^2\left(0, +\infty; H^1(\mathbb{R}\times\mathbb{T})\right), \end{cases} \tag{1.23}$$

and the time-asymptotic stability of the planar 2-rarefaction wave holds true in the sense that

$$\lim_{t\to+\infty} \sup_{(x,y)\in\mathbb{R}\times\mathbb{T}} \left|(\rho, u, v, c)(t, x, y) - (\rho^r, u^r, 0, c^r)\left(\frac{x}{t}\right)\right| = 0. \tag{1.24}$$

We now give some remarks on the above theorem. The first remark is about the existence of rarefaction wave. At the first glance, inviscid system (1.11) is not a conservation law. But the linear relationship between $c^r(\frac{x}{t})$ and $\rho^r(\frac{x}{t})$ in $(1.11)_3$ helps us to rewrite system (1.11) into a conservative law. Thanks to the two constraints $p'(\rho) - \frac{a\mu}{b}\rho > 0$ and $p''(\rho) - \frac{a\mu}{b} > 0$, the system (1.11) is strictly hyperbolic and its both characteristic fields are genuinely nonlinear for $\rho > 0$. Then we can construct the rarefaction wave solution with the help of Euler system.

The second remark is about energy estimates. Since the simplified viscous vasculogenesis model (1.5) can be viewed as the Navier-Stokes equation with a chemotactic body force $\mu\rho\nabla c$, then our proof of the theorem enters in the framework proposed by Li and Wang in [18]. The difficulties are still the propagation of rarefaction waves in the y-direction and the interactions between x and y directions. That's the reason why we consider the same case $y \in \mathbb{T}$ as the classical Navier-Stokes equation without any external force. However, on the one hand, we need to treat the estimates on those terms related to the chemoattractant c. As mentioned above, the constraint $p'(\rho) - \frac{a\mu}{b}\rho > 0$ plays an important role in dealing with the linear terms produced by the source term $\mu\rho\nabla c$ in $(1.5)_2$ and $a\rho$ in $(1.5)_3$. Then the linear terms will be absorbed or eliminated in the energy estimates. On the other hand, the presence of non-trivial approximated smooth rarefaction wave $\bar{u}$ still poses some new challenges. It is hard to control $\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \bar{u}_x \tilde{c}^2 dxdydt$ on the right-hand side since the perturbed chemoattractant $\tilde{c}$ itself is not time-space integrable. If it is placed on the left-hand side, it enjoys the negative sign, so it is still hard to control it. We explore the structure of the system, especially the coupling between the perturbed density ϕ and perturbed chemoattractant $\tilde{c}$, to rewrite the trouble term into a "good" term under another constraint $p''(\rho) - \frac{a\mu}{b} > 0$ and other terms which can be dealt with (see details in Lemma 4.1). Here, we emphasize that the conditions on the pressure p are only sufficient conditions for the existence of rarefaction wave and the stability of the perturbed system.

The rest of this paper is organized as follows. In Section 2, we construct a smooth approximate rarefaction wave which tends to the rarefaction wave fan uniformly as the time t goes to infinity. In Section 3, we reformulate a perturbed system around the approximate rarefaction wave and state the global-in-time existence of the solutions to the perturbed equations. In the last section, the *a priori* estimates for the perturbed system are established by using an elementary L^2 energy method.

At the end of this section, we introduce some notations that appear frequently throughout the paper.

Notations: The notation $(\cdot\ ,\ \cdot)^t$ means the transpose of the vector $(\cdot\ ,\ \cdot)$. The differential operator $\mathrm{div}\boldsymbol{f} := \partial_x f_1 + \partial_y f_2$ with $\boldsymbol{f} = (f_1, f_2)^t$, $\nabla := (\partial_x, \partial_y)^t$ and $\Delta := \partial_x^2 + \partial_y^2$. $H^k(\mathbb{R}\times\mathbb{T})(k\geq 0, k\in\mathbb{Z})$ denotes the usual Sobolev space with the norm $\|\cdot\|_k$ .We denote $L^2(\mathbb{R}\times\mathbb{T}) = H^0(\mathbb{R}\times\mathbb{T})$ and set $\|\cdot\| = \|\cdot\|_0$ for convenience. C denotes a generic positive constant where C may vary in the context. For scalar f and g, we denote

$$\nabla f\cdot\nabla g := \begin{pmatrix} f_x & f_y \end{pmatrix}\cdot\begin{pmatrix} g_x \\ g_y \end{pmatrix} = f_x g_x + f_y g_y,$$

and

$$\nabla^2 f\cdot\nabla^2 g := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\cdot\begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} = f_{xx}g_{xx} + 2f_{xy}g_{xy} + f_{yy}g_{yy}.$$

For two vectors $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$, we denote

$$\boldsymbol{a}\otimes\boldsymbol{b} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}.$$

2. Smooth approximate rarefaction wave

Since the rarefaction wave solution is only Lipschitz continuous, we need to construct a smooth approximate rarefaction wave to study the asymptotic stability. In the same way as in [18], we start with the following Riemann problem of the inviscid Burgers equation (refer to [28] and [39]):

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x<0, \\ w_+, & x>0, \end{cases} \end{cases} \tag{2.1}$$

with $w_- < w_+$. Then it is well known that (2.1) has the rarefaction wave fan $w^r(x/t)$ which is given by

$$w^r(t,x) = w^r(\frac{x}{t}) = \begin{cases} w_-, & x < w_- t, \\ \frac{x}{t}, & w_- t\leq x\leq w_+ t, \\ w_+, & x > w_+ t. \end{cases} \tag{2.2}$$

Moreover, $w^r(x/t)$ can be approximated by the smooth solution of the following Burgers' equation

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2}\tanh x, \end{cases} \tag{2.3}$$

then the solution $w(t,x)$ to the Cauchy problem (2.3) enjoys the following properties and their proofs can be found in [28].

LEMMA 2.1. *Let $\tilde{w} = w_+ - w_- > 0$, then the problem (2.3) has a unique smooth solution w(t,x) which satisfies the following properties:*

(1) *$w_- < w(t,x) < w_+$, $w_x > 0$ for $x\in\mathbb{R}$ and $t\geq 0$;*

(2) *For any $1\leq p\leq +\infty$, there exists a constant C such that for all $t>0$,*

$$\|w_x(t,\cdot)\|_{L^p} \leq C\min(|\tilde{w}|, |\tilde{w}|^{\frac{1}{p}} t^{-1+\frac{1}{p}}),$$

$$\|w_{xx}(t,\cdot)\|_{L^p} \le C\min(|\tilde{w}|, t^{-1}),$$

$$\|w_{xxx}(t,\cdot)\|_{L^p} \le C\min(|\tilde{w}|, t^{-1});$$

(3) $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbb{R}}\left|w(t,x)-w^r(\frac{x}{t})\right|=0.$

We now turn to the approximate rarefaction wave for the inviscid system (1.19)-(1.20). Throughout this paper, we consider the case that the constant states $(\rho_\pm, u_\pm, c_\pm)$ are connected by the 2-rarefaction wave. Here, we let $w_- = \lambda_2(\rho_-, u_-)$, $w_+ = \lambda_2(\rho_+, u_+)$, and set the rarefaction wave strength $\alpha := |\rho_+ - \rho_-| + |u_+ - u_-|$. As to the Riemann problem (1.19)-(1.20), the 2-rarefaction wave $(\rho^r, u^r)(t,x) = (\rho^r, u^r)(x/t)$ is given implicitly by using Implicit Function Theorem through the following relationship

$$\begin{aligned}
&\lambda_2(\rho^r, u^r)(t,x) = w^r(t,x),\\
&z_2(\rho^r, u^r)(t,x) = z_2(\rho_\pm, u_\pm),
\end{aligned} \tag{2.4}$$

and $c^r(x/t)$ is determined linearly by $\rho^r(x/t)$ through

$$c^r(x/t) = \frac{a}{b}\rho^r(x/t).$$

Correspondingly, the 2-rarefaction wave fan $(\rho^r, u^r, c^r)(t,x)$ can be approximated by a smooth rarefaction wave $(\bar{\rho}, \bar{u}, \bar{c})(t,x)$ which is constructed by

$$\begin{aligned}
&\lambda_2(\bar{\rho}, \bar{u})(t,x) = w(1+t, x),\\
&z_2(\bar{\rho}, \bar{u})(t,x) = z_2(\rho_\pm, u_\pm),\\
&\bar{c} = \frac{a}{b}\bar{\rho},
\end{aligned} \tag{2.5}$$

where $w(t,x)$ is the smooth solution in (2.3). Moreover, one can easily check that the smooth approximate rarefaction wave $(\bar{\rho}, \bar{u}, \bar{c})(t,x)$ satisfies the following system

$$\begin{cases}
\bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \qquad x\in\mathbb{R},\ t>0,\\
(\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + p(\bar{\rho}))_x - \mu\bar{\rho}\bar{c}_x = 0,\\
a\bar{\rho} - b\bar{c} = 0,\\
(\bar{\rho}, \bar{u}, \bar{c})(0,x) = (\bar{\rho}_0, \bar{u}_0, \frac{a}{b}\bar{\rho}_0)(x).
\end{cases} \tag{2.6}$$

Then the smooth approximate profile $(\bar{\rho}, \bar{u}, \bar{c})(t,x)$ enjoys the following properties according to Lemma 2.1 and Implicit Function Theorem.

Lemma 2.2. *The smooth approximate rarefaction wave $(\bar{\rho}, \bar{u}, \bar{c})(t,x)$ given by (2.6) satisfies the following properties:*

(1) $\rho_- \le \bar{\rho} \le \rho_+$.

(2) $\bar{\rho}_x = \frac{2\bar{\rho}\sqrt{p_1'(\bar{\rho})}}{\bar{\rho}p_1''(\bar{\rho}) + 2p_1'(\bar{\rho})} w_x > 0$, $\bar{u}_x = \frac{\sqrt{p_1'(\bar{\rho})}}{\bar{\rho}}\bar{\rho}_x > 0$, *and* $\bar{c}_x = \frac{a}{b}\bar{\rho}_x > 0$ *for* $x\in\mathbb{R}$, $t\ge 0$, *where* p_1 *is defined in* (1.13) *and satisfies that* $p_1'(\rho) > 0$ *and* $p_1''(\rho) > 0$.

(3) *For any* $1\le p\le\infty$, *there exists a constant* C *such that for all* $t\ge 0$,

$$\|(\bar{\rho}_x, \bar{u}_x, \bar{c}_x)\|_{L^p} \le C\min\{\alpha, \alpha^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},$$

$$\|(\bar{\rho}_{xx}, \bar{u}_{xx}, \bar{c}_{xx})\|_{L^p} \le C\min\{\alpha, (1+t)^{-1}\},$$

$$\|(\bar{\rho}_{xxx}, \bar{u}_{xxx}, \bar{c}_{xxx})\|_{L^p} \le C\min\{\alpha, (1+t)^{-1}\}.$$

(4) $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbb{R}}\left|(\bar{\rho}, \bar{u}, \bar{c})(t,x) - (\rho^r, u^r, c^r)(\frac{x}{t})\right| = 0.$

3. Reformulation of the problem

Now we are in a position to reformulate the original problem around the approximate rarefaction wave $(\bar{\rho},\bar{u},0,\bar{c})$. Define the perturbation by

$$\begin{aligned}&\phi(t,x,y)=\rho(t,x,y)-\bar{\rho}(t,x),\\&\Psi(t,x,y)=(\varphi,\psi)^t(t,x,y)=(u,v)^t(t,x,y)-(\bar{u},0)^t(t,x),\\&\tilde{c}(t,x,y)=c(t,x,y)-\bar{c}(t,x),\end{aligned}$$

with $(\rho,u,v,c)(t,x,y)$ being the solution to the problem (1.5)-(1.9). Direct calculations show that $(\phi,\Psi,\tilde{c})$ satisfies the perturbation system:

$$\begin{cases}\phi_t+\rho\mathrm{div}\Psi+\rho_y\psi+u\phi_x+\bar{\rho}_x\varphi+\bar{u}_x\phi=0,\\ \rho\Psi_t+\rho u\Psi_x+\rho\psi\Psi_y+(\rho\bar{u}_x\varphi,0)^t+p'(\rho)\nabla\phi+((p'(\rho)-\frac{\rho}{\bar{\rho}}p'(\bar{\rho}))\bar{\rho}_x,0)^t\\ =\Delta\Psi+\mu\rho\nabla\tilde{c}+(\bar{u}_{xx},0)^t,\\ \Delta\tilde{c}+\bar{c}_{xx}+a\phi-b\tilde{c}=0,\end{cases} \tag{3.1}$$

with the initial data

$$\begin{aligned}(\phi,\Psi)(0,x,y)&=(\phi_0,\Psi_0)(x,y)=(\phi_0,\varphi_0,\psi_0)(x,y)\\&:=(\rho_0-\bar{\rho}_0,u_0-\bar{u}_0,v_0)(x,y)\in H^2(\mathbb{R}\times\mathbb{T}),\end{aligned} \tag{3.2}$$

and the far field data for $\tilde{c}(t,x,y)$ in the x-direction is given by

$$\tilde{c}(t,x,y)\to 0,\quad \text{as}\quad x\to\pm\infty. \tag{3.3}$$

It can be noted that the condition (1.21) assures (3.2) holds true.

We first choose a positive constant E_0 such that

$$\sup_{(x,y)\in\mathbb{R}\times\mathbb{T}}|f(x,y)|\le\frac{1}{2}\rho_-\qquad\text{for any } f\in H^2(\mathbb{R}\times\mathbb{T}),\ \|f\|_2\le E_0. \tag{3.4}$$

By virtue of the two-dimensional Sobolev's inequality,

$$\sup_{(x,y)\in\mathbb{R}\times\mathbb{T}}|f(x,y)|\le C\Big(\|f\|^{\frac12}\|f_x\|^{\frac12}+\|f_y\|^{\frac12}\|f_{xy}\|^{\frac12}\Big),\ \text{for any } f\in H^2(\mathbb{R}\times\mathbb{T}), \tag{3.5}$$

(3.4) is obviously true if E_0 is suitably small. Then the solution to (3.1)-(3.3) can be sought in a functional space $X(0,+\infty)$, and for given $0<T\le+\infty$, we define

$$\begin{aligned}X(0,T)=\Big\{(\phi,\Psi,\tilde{c})\Big|&(\phi,\Psi)\in C^0\left(0,T;H^2\right),\ \tilde{c}\in C^0\left(0,T;H^1\right),\ (\nabla\phi,\nabla\tilde{c})\in L^2\left(0,T;H^1\right),\\&\nabla\Psi\in L^2\left(0,T;H^2\right),\ \text{and}\ \sup_{0\le t\le T}\|(\phi,\Psi)\|_2+\sup_{0\le t\le T}\|\tilde{c}\|_1\le E_0\Big\}.\end{aligned}$$

Combining the condition $\sup_{0\le t\le T}\|(\phi,\Psi)(t)\|_2\le E_0$ with (3.4), we know that $|\phi|$, $|\Psi|\le\frac12\rho_-$ and $|\boldsymbol{u}|=|(u,v)|\le C$, where C is a positive constant which only depends on ρ_- and $u_\pm$. What's more, the density function $\rho(t,x,y):=\bar{\rho}(t,x)+\phi(t,x,y)$ satisfies that

$$0<\frac{1}{2}\rho_-\le\rho\le\frac{1}{2}\rho_-+\rho_+, \tag{3.6}$$

since $0<\rho_-\le\bar{\rho}\le\rho_+$. Thanks to the uniform positive upper and lower boundedness of the density function $\rho(t,x,y)$ in (3.6), the momentum equation $(1.5)_2$ is strictly parabolic, which is crucial for the local and global-in-time existence of the classical solution to the system (1.5).

PROPOSITION 3.1. *Suppose that all the conditions in Theorem 1.1 hold. There exist positive constants ε_0 and C such that if*

$$\|(\phi_0,\Psi_0)\|_2+\alpha\le\varepsilon_0,$$

then the reformulated problem (3.1)-(3.3) admits a unique global solution $(\phi,\Psi,\tilde{c})\in X(0,+\infty)$ satisfying

$$\begin{aligned}&\|(\phi,\Psi)\|_2^2+\|\tilde{c}\|_1^2+\int_0^{+\infty}\|\bar{u}_x^{\frac12}(\varphi,\phi,\tilde{c}_x,\phi_x,\Psi_x,\nabla\phi_x)\|^2dt\\&\quad+\int_0^{+\infty}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|_1^2dt+\int_0^{+\infty}\|\nabla^3\Psi\|^2dt\le C(\|(\phi_0,\Psi_0)\|_2^2+\alpha^{\frac14}).\end{aligned}\tag{3.7}$$

The local-in-time existence and uniqueness of the classical solutions of (3.1)-(3.3) can be established by the standard iteration argument. The existence of global solution follows from the standard continuity argument based on the local existence and the *a priori* estimate in Proposition 3.2. In this article, for the proof of Proposition 3.1, we only devote ourselves to showing the following *a priori* estimates.

PROPOSITION 3.2 (A priori estimate). *Assume that the reformulated problem (3.1)-(3.3) has a solution $(\phi,\Psi,\tilde{c})\in X(0,T)$ for some $T(>0)$. Then there exist positive constants ε_1 and C which are independent of T such that if*

$$\sup_{0\le t\le T}\|(\phi,\Psi)\|_2+\sup_{0\le t\le T}\|\tilde{c}\|_1+\alpha\le\varepsilon_1,$$

then it holds

$$\begin{aligned}&\sup_{0\le t\le T}\|(\phi,\Psi)\|_2^2+\sup_{0\le t\le T}\|\tilde{c}\|_1^2+\int_0^{T}\|\bar{u}_x^{\frac12}(\varphi,\phi,\tilde{c}_x,\phi_x,\Psi_x,\nabla\phi_x)\|^2dt\\&\quad+\int_0^{T}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|_1^2dt+\int_0^{T}\|\nabla^3\Psi\|^2dt\le C\left(\|(\phi_0,\Psi_0)\|_2^2+\alpha^{\frac14}\right).\end{aligned}\tag{3.8}$$

4. Stability of the planar rarefaction wave

In this section, we will show the stability of the planar rarefaction wave. We divide this section into two subsections. First, we prove the *a priori* estimate in Proposition 3.2, and then we prove the large-time behavior of the solution in Theorem 1.1. Throughout this section, we assume that $(\rho_\pm,u_\pm)$ satisfies (1.18) with $\rho_\pm>0$, $u_\pm\in\mathbb{R}$ being fixed and for some $T(>0)$, the problem (3.1)-(3.3) has a solution $(\phi,\Psi,\tilde{c})\in X(0,T)$. We set

$$E=\sup_{0\le t\le T}\|(\phi,\Psi)(t)\|_2+\sup_{0\le t\le T}\|\tilde{c}(t)\|_1.$$

4.1. *A priori* estimates.

The first lemma is about zero-order energy estimate.

LEMMA 4.1. *There exists a positive constant C such that for $0\le t\le T$,*

$$\begin{aligned}&\sup_{0\le t\le T}\|(\phi,\Psi)\|^2(t)+\sup_{0\le t\le T}\|\tilde{c}\|_1^2(t)+\int_0^T\|\nabla\Psi\|^2dt+\int_0^T\|\bar{u}_x^{\frac12}(\varphi,\phi,\tilde{c}_x)\|^2dt\\&\le C\left(\|(\phi_0,\Psi_0)\|^2+\|\tilde{c}_0\|_1^2\right)+C(E+\alpha^{\frac13})\int_0^T\|(\nabla\phi,\nabla\tilde{c})\|^2dt+C\alpha^{\frac14}.\end{aligned}\tag{4.1}$$

Proof. Define the potential energy by

$$\Phi(\rho,\bar\rho)=\int_{\bar\rho}^{\rho}\frac{p(s)-p(\bar\rho)}{s^2}ds.$$

Through direct calculation, we get

$$\begin{aligned}(\rho\Phi)_t+\operatorname{div}[\rho\boldsymbol{u}\Phi+(p(\rho)-p(\bar\rho))\Psi]+\bar u_x[p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi]&\\ -p'(\rho)\nabla\phi\cdot\Psi-\left(p'(\rho)-\frac{\rho}{\bar\rho}p'(\bar\rho)\right)\bar\rho_x\varphi=0.&\end{aligned}\tag{4.2}$$

Multiplying the second equation of (3.1) by Ψ, we have

$$\begin{aligned}\left(\frac12\rho|\Psi|^2\right)_t+\frac12\operatorname{div}\left(\rho\boldsymbol{u}|\Psi|^2\right)-\operatorname{div}(\varphi\nabla\varphi+\psi\nabla\psi)+\rho\bar u_x\varphi^2+p'(\rho)\nabla\phi\cdot\Psi&\\ +\left(p'(\rho)-\frac{\rho}{\bar\rho}p'(\bar\rho)\right)\bar\rho_x\varphi+|\nabla\Psi|^2-\mu\rho\nabla\tilde c\cdot\Psi=\bar u_{xx}\varphi,&\end{aligned}\tag{4.3}$$

where the last term $-\mu\rho\nabla\tilde c\cdot\Psi$ on the left-hand side can be rewritten by using equation $(3.1)_1$ as follows,

$$\begin{aligned}-\mu\rho\nabla\tilde c\cdot\Psi&=-\operatorname{div}(\mu\rho\tilde c\Psi)+\mu\tilde c\operatorname{div}(\rho\Psi)\\&=-\operatorname{div}(\mu\rho\tilde c\Psi)-\mu\tilde c\phi_t-\mu\tilde c(\bar u\phi)_x\\&=-\operatorname{div}(\mu\rho\tilde c\Psi)-\mu(\tilde c\phi)_t+\mu\tilde c_t\phi-\mu(\tilde c\bar u\phi)_x+\mu\tilde c_x\bar u\phi.\end{aligned}\tag{4.4}$$

Multiplying the third equation of (3.1) by $-\frac{\mu}{a}\tilde c_t$, one has

$$\left[\frac{\mu b}{2a}\tilde c^2+\frac{\mu}{2a}|\nabla\tilde c|^2+\frac{\mu}{a}\tilde c_x\bar c_x\right]_t-\frac{\mu}{a}\operatorname{div}(\tilde c_t\nabla\tilde c)-\frac{\mu}{a}(\tilde c_t\bar c_x)_x-\frac{\mu}{a}\tilde c_x\bar c_{xt}-\mu\tilde c_t\phi=0,\tag{4.5}$$

and multiplying the third equation of (3.1) again by $-\frac{\mu}{a}\tilde c_x\bar u$, one has

$$\begin{aligned}\left[\frac{\mu}{2a}\bar u|\nabla\tilde c|^2+\frac{\mu b}{2a}\bar u\tilde c^2\right]_x-\frac{\mu}{a}\operatorname{div}\left(\bar u\tilde c_x\nabla\tilde c\right)-\frac{\mu}{2a}\bar u_x|\nabla\tilde c|^2+\frac{\mu}{a}\bar u_x\tilde c_x^2&\\ -\frac{\mu}{a}\bar u\tilde c_x\bar c_{xx}-\mu\tilde c_x\bar u\phi-\frac{\mu b}{2a}\bar u_x\tilde c^2=0,&\end{aligned}\tag{4.6}$$

i.e.,

$$\begin{aligned}\left[\frac{\mu}{2a}\bar u|\nabla\tilde c|^2+\frac{\mu b}{2a}\bar u\tilde c^2\right]_x-\frac{\mu}{a}\operatorname{div}\left(\bar u\tilde c_x\nabla\tilde c\right)-\frac{\mu}{2a}\bar u_x\tilde c_y^2+\frac{\mu}{2a}\bar u_x\tilde c_x^2&\\ -\frac{\mu}{a}\bar u\tilde c_x\bar c_{xx}-\mu\tilde c_x\bar u\phi-\frac{\mu b}{2a}\bar u_x\tilde c^2=0.&\end{aligned}\tag{4.7}$$

Substituting (4.4) into (4.3), taking summation of (4.2), (4.3), (4.5), and (4.7), and noticing that

$$\begin{aligned}&\bar u_x\Big(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi\Big)-\frac{\mu b}{2a}\bar u_x\tilde c^2\\ =&\bar u_x\left(\frac{p''(\bar\rho)}{2}-\frac{\mu a}{2b}\right)\phi^2+\bar u_x\left(\frac{\mu a}{2b}\phi^2-\frac{\mu b}{2a}\tilde c^2\right)+\bar u_x\left(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi-\frac{p''(\bar\rho)}{2}\phi^2\right),\end{aligned}\tag{4.8}$$

we arrive at the following equality after integrating the resulting equation over $[0,t]\times\mathbb{T}\times\mathbb{R}$,

$$
\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\frac{1}{2}\rho|\Psi|^2+\rho\Phi-\mu\tilde{c}\phi+\frac{\mu b}{2a}\tilde{c}^2+\frac{\mu}{2a}|\nabla\tilde{c}|^2+\frac{\mu}{a}\tilde{c}_x\bar{c}_x\Big](t)dxdy\\
&\qquad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\nabla\Psi|^2dxdydt+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big(\frac{\mu}{2a}\bar{u}_x\tilde{c}_x^2+\rho\bar{u}_x\varphi^2\Big)dxdydt\\
&\qquad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{u}_x\Big(\frac{p''(\bar{\rho})}{2}-\frac{\mu a}{2b}\Big)\phi^2dxdydt\\
=&\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\frac{1}{2}\rho|\Psi|^2+\rho\Phi-\mu\tilde{c}\phi+\frac{\mu b}{2a}\tilde{c}^2+\frac{\mu}{2a}|\nabla\tilde{c}|^2+\frac{\mu}{a}\tilde{c}_x\bar{c}_x\Big]\Big|_{t=0}dxdy\\
&\qquad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{2a}\bar{u}_x\tilde{c}_y^2dxdydt+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}\tilde{c}_x(\bar{c}_{xt}+\bar{u}\bar{c}_{xx})dxdydt\\
&\qquad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{u}_{xx}\varphi dxdydt-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{u}_x\Big(\frac{\mu a}{2b}\phi^2-\frac{\mu b}{2a}\tilde{c}^2\Big)dxdydt\\
&\qquad-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{u}_x\Big(p(\rho)-p(\bar{\rho})-p'(\bar{\rho})\phi-\frac{p''(\bar{\rho})}{2}\phi^2\Big)dxdydt. \qquad (4.9)
\end{aligned}
$$

Firstly, we estimate each term on the left-hand side of (4.9). It follows from the condition on p in (1.6) that $p'(\bar{\rho})>\frac{a\mu}{b}\bar{\rho}$, which implies that the following matrix is positive definite

$$
\begin{pmatrix} \frac{p'(\bar{\rho})}{2\bar{\rho}} & -\frac{\mu}{2} \\ -\frac{\mu}{2} & \frac{\mu b}{2a} \end{pmatrix}. \qquad (4.10)
$$

Then

$$
\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\rho\Phi-\mu\tilde{c}\phi+\frac{\mu b}{2a}\tilde{c}^2\Big]dxdy\\
=&\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\frac{p'(\bar{\rho})}{2\bar{\rho}}\phi^2-\mu\tilde{c}\phi+\frac{\mu b}{2a}\tilde{c}^2\Big]dxdy+\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\rho\Phi-\frac{p'(\bar{\rho})}{2\bar{\rho}}\phi^2\Big]dxdy\\
\geq& C\|(\phi,\tilde{c})\|^2(t)-CE\|\phi\|^2(t), \qquad (4.11)
\end{aligned}
$$

and the condition on p in (1.6) that $\frac{p''(\bar{\rho})}{2}>\frac{\mu a}{2b}$ implies

$$
\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{u}_x\Big(\frac{p''(\bar{\rho})}{2}-\frac{\mu a}{2b}\Big)\phi^2dxdydt\geq C\int_0^t\|\bar{u}_x^{\frac{1}{2}}\phi\|^2dt. \qquad (4.12)
$$

Due to the uniform lower and upper boundedness of the density function $\rho(t,x,y)$, one has

$$
\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\frac{1}{2}\rho|\Psi|^2+\frac{\mu}{2a}|\nabla\tilde{c}|^2\Big](t)dxdy\geq C\|(\Psi,\nabla\tilde{c})\|^2(t). \qquad (4.13)
$$

By Cauchy's inequality and Lemma 2.2, we have

$$\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\mu}{a}\tilde{c}_x\bar{c}_x dxdy \geq -\varepsilon\|\tilde{c}_x\|^2 - C_\varepsilon\alpha^2. \tag{4.14}$$

Similarly to (4.13), we have

$$\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \left(|\nabla\Psi|^2 + \rho\bar{u}_x\varphi^2 + \frac{\mu}{2a}\bar{u}_x\tilde{c}_x^2\right)dxdydt \geq C\int_0^t \left(\|\nabla\Psi\|^2 + \|\bar{u}_x^{\frac{1}{2}}(\varphi,\tilde{c}_x)\|^2\right)dt. \tag{4.15}$$

Furthermore, we estimate each term on the right-hand side of (4.9). It follows from Cauchy's inequality and Lemma 2.2 that

$$\begin{aligned}&\left|\int_{\mathbb{T}}\int_{\mathbb{R}} \left[\frac{1}{2}\rho|\Psi|^2 + \rho\Phi - \mu\tilde{c}\phi + \frac{\mu b}{2a}\tilde{c}^2 + \frac{\mu}{2a}|\nabla\tilde{c}|^2 + \frac{\mu}{a}\tilde{c}_x\bar{c}_x\right]\Big|_{t=0} dxdy\right| \\ \leq & C\|(\Psi_0,\phi_0,\tilde{c}_0,\nabla\tilde{c}_0)\|^2 + C\alpha^2. \end{aligned}\tag{4.16}$$

By Cauchy's inequality and Lemma 2.2, we have

$$\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\mu}{2a}\bar{u}_x\tilde{c}_y^2 dxdydt \leq C\alpha\int_0^t \|\nabla\tilde{c}\|^2 dt, \tag{4.17}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\mu}{a}\tilde{c}_x(\bar{c}_{xt} + \bar{u}\bar{c}_{xx})dxdydt \\ \leq & C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \left(|\bar{c}_{xt}|^{\frac{1}{3}}|\bar{c}_{xt}|^{\frac{2}{3}} + |\bar{c}_{xx}|^{\frac{1}{3}}|\bar{c}_{xx}|^{\frac{2}{3}}\right)|\tilde{c}_x|dxdydt \\ \leq & C\alpha^{\frac{1}{3}}\int_0^t\int_{\mathbb{T}} \left(\|\bar{c}_{xt}\|_{L_x^{\frac{4}{3}}}^{\frac{4}{3}} + \|\bar{c}_{xx}\|_{L_x^{\frac{4}{3}}}^{\frac{4}{3}}\right)dydt + C\alpha^{\frac{1}{3}}\int_0^t \|\nabla\tilde{c}\|^2 dt \\ \leq & C\alpha^{\frac{1}{3}} + C\alpha^{\frac{1}{3}}\int_0^t \|\nabla\tilde{c}\|^2 dt. \end{aligned}\tag{4.18}$$

As in [18], by the one-dimensional Sobolev's inequality, Hölder's inequality, Young's inequality and Lemma 2.2, we have,

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_{xx}\varphi dxdydt\right| \leq & C\int_0^t\int_{\mathbb{T}} \|\bar{u}_{xx}\|_{L_x^1}\|\varphi\|_{L_x^\infty} dydt \\ \leq & C\int_0^t\int_{\mathbb{T}} \|\bar{u}_{xx}\|_{L_x^1}\|\varphi\|_{L_x^2}^{\frac{1}{2}}\|\varphi_x\|_{L_x^2}^{\frac{1}{2}} dydt \\ \leq & \frac{1}{4}\int_0^t \|\varphi_x\|^2 dt + C\int_0^t\int_{\mathbb{T}} \|\bar{u}_{xx}\|_{L_x^1}^{\frac{4}{3}}\|\varphi\|_{L_x^2}^{\frac{2}{3}} dydt \\ \leq & \frac{1}{4}\int_0^t \|\varphi_x\|^2 dt + C\int_0^t \left(\int_{\mathbb{T}} \|\bar{u}_{xx}\|_{L_x^1}^2 dy\right)^{\frac{2}{3}}\left(\int_{\mathbb{T}} \|\varphi\|_{L_x^2}^2 dy\right)^{\frac{1}{3}} dt \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{4}\int_0^t \|\varphi_x\|^2 dt + C\int_0^t \|\bar{u}_{xx}\|^{\frac{4}{3}}\|\varphi\|^{\frac{2}{3}} dt\\
&\le \frac{1}{4}\int_0^t \|\varphi_x\|^2 dt + C\sup_{0\le t\le T}\|\varphi\|^{\frac{2}{3}}\int_0^t \left(\alpha^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\right)^{\frac{4}{3}} dt\\
&\le \frac{1}{4}\int_0^t \|\varphi_x\|^2 dt + \frac{1}{3}\sup_{0\le t\le T}\|\varphi\|^2 + C\alpha^{\frac{1}{4}}.
\end{aligned} \tag{4.19}
$$

With $(3.1)_3$ and integration by parts, the fifth term on the right-hand side of (4.9) can be rewritten as three terms,

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x\left(\frac{\mu a}{2b}\phi^2 - \frac{\mu b}{2a}\tilde{c}^2\right) dxdydt\\
=&-\frac{\mu}{2ab}\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x(a\phi-b\tilde{c})(a\phi+b\tilde{c}) dxdydt\\
=&\frac{\mu}{2ab}\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x(a\phi+b\tilde{c})(\Delta\tilde{c}+\bar{c}_{xx}) dxdydt\\
=&\underbrace{-\frac{\mu}{2ab}\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x(a\nabla\phi+b\nabla\tilde{c})\nabla\tilde{c}\, dxdydt}_{I_1}\underbrace{-\frac{\mu}{2ab}\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_{xx}(a\phi+b\tilde{c})\tilde{c}_x dxdydt}_{I_2}\\
&\underbrace{+\frac{\mu}{2ab}\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x(a\phi+b\tilde{c})\bar{c}_{xx} dxdydt}_{I_3}.
\end{aligned}
$$

Then we estimate I_j for $1\le j\le 3$. For I_1, by Hölder's inequality, Cauchy's inequality and Lemma 2.2, it holds that

$$
|I_1|\le C\int_0^t \|\bar{u}_x\|_{L^\infty}(\|\nabla\phi\|^2+\|\nabla\tilde{c}\|^2)dt \le C\alpha\int_0^t (\|\nabla\phi\|^2+\|\nabla\tilde{c}\|^2)dt. \tag{4.20}
$$

By Hölder's inequality, interpolation inequality, Cauchy's inequality, and Lemma 2.2, one has

$$
\begin{aligned}
|I_2|&\le C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}} (|\bar{u}_{xx}\phi\tilde{c}_x|+|\bar{u}_{xx}\tilde{c}\tilde{c}_x|) dxdydt\\
&\le C\int_0^t \|\bar{u}_{xx}\|_{L^4}\|\phi\|_{L^4}\|\tilde{c}_x\| dt + C\int_0^t \|\bar{u}_{xx}\|_{L^4}\|\tilde{c}\|_{L^4}\|\tilde{c}_x\| dt\\
&\le C\int_0^t \|\bar{u}_{xx}\|_{L^4}^{\frac{1}{2}}\|\bar{u}_{xx}\|_{L^4}^{\frac{1}{2}}\|\phi\|_{L^2}^{\frac{1}{2}}\|\nabla\phi\|_{L^2}^{\frac{1}{2}}\|\tilde{c}_x\| dt\\
&\quad + C\int_0^t \|\bar{u}_{xx}\|_{L^4}^{\frac{1}{2}}\|\bar{u}_{xx}\|_{L^4}^{\frac{1}{2}}\|\tilde{c}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{c}\|_{L^2}^{\frac{1}{2}}\|\tilde{c}_x\| dt\\
&\le C\int_0^t \|\bar{u}_{xx}\|_{L^4}\|\tilde{c}_x\|^2 dt + C\int_0^t \|\bar{u}_{xx}\|_{L^4}^2 dt + \int_0^t \left(\|\phi\|_{L^2}^2\|\nabla\phi\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2\|\nabla\tilde{c}\|_{L^2}^2\right) dt\\
&\le C(\alpha+E)\int_0^t \|(\nabla\tilde{c},\nabla\phi)\|^2 dt + C\alpha^{\frac{1}{2}}.
\end{aligned} \tag{4.21}
$$

For I_3, we obtain

$$
\begin{aligned}
|I_3| \leq & C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} (|\bar{u}_x\phi\bar{c}_{xx}|+|\bar{u}_x\tilde{c}\bar{c}_{xx}|)dxdydt \\
\leq & C\int_0^t (\|\bar{c}_{xx}\|^2+\|\bar{u}_x\phi\|^2+\|\bar{u}_x\tilde{c}\|^2)dt \\
\leq & C\int_0^t (\|\bar{c}_{xx}\|^2+\|\phi\|_{L^4}^4+\|\tilde{c}\|_{L^4}^4+\|\bar{u}_x\|_{L^4}^4)dt \\
\leq & C\alpha^{\frac{2}{3}}+CE\int_0^t \|(\nabla\phi,\nabla\tilde{c})\|^2dt+C\alpha.
\end{aligned} \tag{4.22}
$$

The last term can be estimated by using Taylor expansion,

$$
\begin{aligned}
\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \bar{u}_x\left[p(\rho)-p(\bar{\rho})-p'(\bar{\rho})\phi-\frac{p''(\bar{\rho})}{2}\phi^2\right]dxdydt \leq & C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\bar{u}_x\phi^3|dxdydt \\
\leq & C\int_0^t \|\phi\|_{L^\infty}\|\bar{u}_x^{\frac{1}{2}}\phi\|^2dt \\
\leq & CE\int_0^t \|\bar{u}_x^{\frac{1}{2}}\phi\|^2dt.
\end{aligned} \tag{4.23}
$$

Substituting the estimates of (4.11)-(4.23) into (4.9), and taking E and α suitably small, we can prove (4.1) in Lemma 4.1. □

LEMMA 4.2. *There exists a positive constant C such that for $0\leq t\leq T$,*

$$
\begin{aligned}
\|(\phi,\Psi,\tilde{c})\|^2(t)+\|(\nabla\phi,\nabla\tilde{c})\|^2(t)+\int_0^t \|\bar{u}_x^{\frac{1}{2}}(\varphi,\phi,\tilde{c}_x,\phi_x)\|^2dt+\int_0^t \|(\nabla\Psi,\nabla\phi,\nabla\tilde{c})\|^2dt \\
+\int_0^t \|\nabla^2\tilde{c}\|^2dt \leq C\left(\|(\phi_0,\Psi_0,\tilde{c}_0)\|^2+\|(\nabla\phi_0,\nabla\tilde{c}_0)\|^2+\alpha^{\frac{1}{4}}\right).
\end{aligned} \tag{4.24}
$$

Proof. We apply the operator ∇ to the first equation of (3.1) and then multiply the resulting equation by $\frac{\nabla\phi}{\rho^2}$ to give

$$
\begin{aligned}
&\left(\frac{|\nabla\phi|^2}{2\rho^2}\right)_t+\mathrm{div}\left(\frac{\boldsymbol{u}|\nabla\phi|^2}{2\rho^2}\right)+\frac{\bar{u}_x{\phi_x}^2}{\rho^2}+\frac{\nabla\mathrm{div}\Psi\cdot\nabla\phi}{\rho} \\
=&-\frac{\phi_x\nabla\phi\cdot\nabla\varphi}{\rho^2}-\frac{\phi_y\nabla\phi\cdot\nabla\psi}{\rho^2}-\frac{\bar{\rho}_x\nabla\phi\cdot\nabla\varphi}{\rho^2}-\frac{\bar{\rho}_x\phi_x\mathrm{div}\Psi}{\rho^2} \\
&+\frac{\mathrm{div}\Psi|\nabla\phi|^2}{2\rho^2}+\frac{\bar{u}_x|\nabla\phi|^2}{2\rho^2}-\frac{(\bar{\rho}_{xx}\varphi+\bar{u}_{xx}\phi)\phi_x}{\rho^2}.
\end{aligned} \tag{4.25}
$$

In order to deal with the last term on the left-hand side of (4.25), we multiply the second equation of (3.1) by $\frac{\nabla\phi}{\rho}$ to get

$$
\begin{aligned}
&(\Psi\cdot\nabla\phi)_t-\mathrm{div}(\Psi\phi_t)-(u\phi\psi_y)_x+(u\psi_x\phi)_y+\frac{p'(\rho)}{\rho}|\nabla\phi|^2-\frac{1}{\rho}\Delta\Psi\cdot\nabla\phi \\
=&\rho(\mathrm{div}\Psi)^2+\bar{\rho}_x\varphi\mathrm{div}\Psi+\mu\nabla\tilde{c}\cdot\nabla\phi+\frac{1}{\rho}\bar{u}_{xx}\phi_x-\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_x\phi_x \\
&+\phi(\varphi_y\psi_x-\varphi_x\psi_y)+\psi(\phi_y\varphi_x-\phi_x\varphi_y)+\bar{u}_x(\phi\varphi_x-\varphi\phi_x).
\end{aligned} \tag{4.26}
$$

In order to deal with the cross term $\mu\nabla\tilde{c}\cdot\nabla\phi$ in (4.26), we multiply $(3.1)_3$ by $\frac{\mu}{a}\Delta\tilde{c}$ to get

$$\frac{\mu}{a}(\Delta\tilde{c})^2+\frac{\mu b}{a}|\nabla\tilde{c}|^2+\mathrm{div}\left(\mu\phi\nabla\tilde{c}-\frac{\mu b}{a}\tilde{c}\nabla\tilde{c}\right)=\mu\nabla\tilde{c}\cdot\nabla\phi-\frac{\mu}{a}\bar{c}_{xx}\Delta\tilde{c}. \tag{4.27}$$

Similar to [18], the last term $\frac{\nabla\mathrm{div}\Psi\cdot\nabla\phi}{\rho}$ on the left-hand side of (4.25) has an equivalent form,

$$\begin{aligned}&\frac{\nabla\mathrm{div}\Psi\cdot\nabla\phi}{\rho}\\ =&\frac{\nabla\phi\cdot\Delta\Psi}{\rho}+\left(\frac{\phi_y\varphi_y}{\rho}\right)_x+\left(\frac{\phi_x\psi_x}{\rho}\right)_y-\left(\frac{\phi_x\varphi_y}{\rho}\right)_y-\left(\frac{\phi_y\psi_x}{\rho}\right)_x+\frac{\bar{\rho}_x\phi_y(\varphi_y-\psi_x)}{\rho^2}.\end{aligned} \tag{4.28}$$

Now substituting the equivalent form (4.28) into (4.25), adding (4.25), (4.26) and (4.27) together, and then integrating the final equality over $[0,t]\times\mathbb{T}\times\mathbb{R}$ lead to

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{R}}\left(\frac{1}{2\rho^2}|\nabla\phi|^2+\Psi\cdot\nabla\phi\right)(t)dxdy+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{1}{\rho^2}\bar{u}_x\phi_x^2dxdydt+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}(\Delta\tilde{c})^2dxdydt\\ &\quad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left[\frac{p'(\rho)}{\rho}|\nabla\phi|^2-2\mu\nabla\phi\cdot\nabla\tilde{c}+\frac{\mu b}{a}|\nabla\tilde{c}|^2\right]dxdydt\\ &=\int_{\mathbb{T}}\int_{\mathbb{R}}\left(\frac{1}{2\rho^2}|\nabla\phi|^2+\Psi\cdot\nabla\phi\right)\Big|_{t=0}dxdy-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}\bar{c}_{xx}\Delta\tilde{c}dxdydt\\ &\quad+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\rho(\mathrm{div}\Psi)^2dxdydt+\sum_{i=1}^{8}G_i+\sum_{i=1}^{6}H_i,\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}\sum_{i=1}^{8}G_i=&\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\psi(\phi_y\varphi_x-\phi_x\varphi_y)+\bar{u}_x(\phi\varphi_x-\varphi\phi_x)+\bar{\rho}_x\varphi\mathrm{div}\Psi\\ &-\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_x\phi_x+\frac{1}{\rho}\bar{u}_{xx}\phi_x-\frac{1}{\rho^2}\phi_x(\bar{\rho}_{xx}\varphi+\bar{u}_{xx}\phi)\\ &-\frac{1}{\rho^2}\bar{\rho}_x\nabla\phi\cdot\nabla\varphi-\frac{1}{\rho^2}\phi_x\nabla\phi\cdot\nabla\varphi\Big]dxdydt,\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\sum_{i=1}^{6}H_i=&\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\phi(\varphi_y\psi_x-\varphi_x\psi_y)-\frac{1}{\rho^2}\bar{\rho}_x\phi_y(\varphi_y-\psi_x)-\frac{1}{\rho^2}\bar{\rho}_x\phi_x\mathrm{div}\Psi\\ &+\frac{1}{2\rho^2}\bar{u}_x|\nabla\phi|^2-\frac{1}{\rho^2}\phi_y\nabla\phi\cdot\nabla\psi+\frac{1}{2\rho^2}\mathrm{div}\Psi|\nabla\phi|^2\Big]dxdydt.\end{aligned} \tag{4.31}$$

Furthermore, by Lemma 4.1, we get

$$\|\nabla\phi\|^2+\int_0^t\|(\nabla\phi,\nabla\tilde{c})\|^2dt+\int_0^t\|\Delta\tilde{c}\|^2dt+\int_0^t\|\bar{u}_x^{\frac{1}{2}}\phi_x\|^2dt$$

$$\leq C\left(\|(\phi_0,\tilde{c}_0)\|_1^2+\|\Psi_0\|^2+\alpha^{\frac{1}{4}}\right)-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}\bar{c}_{xx}\Delta\tilde{c}dxdydt$$
$$+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\rho(\mathrm{div}\Psi)^2dxdydt+\sum_{i=1}^{8}G_i+\sum_{i=1}^{6}H_i. \tag{4.32}$$

Here we have used the fact that the matrix (4.10) is positive definite. The combination of (4.1) and (4.32) leads to

$$\|(\phi,\Psi,\tilde{c})\|^2+\|(\nabla\phi,\nabla\tilde{c})\|^2+\int_0^t\|(\nabla\phi,\nabla\tilde{c},\nabla\Psi)\|^2dt+\int_0^t\|\bar{u}_x^{\frac{1}{2}}(\varphi,\phi,\tilde{c}_x,\phi_x)\|^2dt$$
$$+\int_0^t\|\Delta\tilde{c}\|^2dt\leq C\left(\|(\phi_0,\tilde{c}_0)\|_1^2+\|\Psi_0\|^2+\alpha^{\frac{1}{4}}\right)-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}\bar{c}_{xx}\Delta\tilde{c}dxdydt$$
$$+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\rho(\mathrm{div}\Psi)^2dxdydt+\sum_{i=1}^{8}G_i+\sum_{i=1}^{6}H_i. \tag{4.33}$$

In the following part, we estimate each term on the right-hand side of (4.33). By Young's inequality and Lemma 2.2, one has

$$\left|-\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\mu}{a}\bar{c}_{xx}\Delta\tilde{c}dxdydt\right|\leq\frac{1}{80}\int_0^t\|\Delta\tilde{c}\|^2dt+C\int_0^t\|\bar{c}_{xx}\|^2dt$$
$$\leq\frac{1}{80}\int_0^t\|\Delta\tilde{c}\|^2dt+C\alpha^{\frac{2}{3}}. \tag{4.34}$$

It follows from Lemma 4.1 that

$$\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\rho(\mathrm{div}\Psi)^2dxdydt\right|\leq C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\nabla\Psi|^2dxdydt$$
$$\leq C\left(\|(\phi_0,\Psi_0)\|^2+\|\tilde{c}_0\|_1^2+\alpha^{\frac{1}{4}}\right)+C(E+\alpha^{\frac{1}{3}})\int_0^t\|(\nabla\phi,\nabla\tilde{c})\|^2dt. \tag{4.35}$$

Next we estimate $G_i(i=1,\cdots,8)$. And $H_i(i=1,\cdots,6)$ can be estimated similarly and the details will be omitted for brevity. By Young's inequality and the interpolation inequality, one has

$$|G_1|=\Big|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\psi(\phi_y\varphi_x-\phi_x\varphi_y)dxdydt\Big|$$
$$\leq\frac{1}{80}\int_0^t\|(\phi_y,\varphi_y)\|^2dt+C\int_0^t\left(\|\psi\|_{L^4}^4+\|\varphi_x\|_{L^4}^4+\|\phi_x\|_{L^4}^4\right)dt$$
$$\leq\frac{1}{80}\int_0^t\|(\phi_y,\varphi_y)\|^2dt+C\int_0^t\left(\|\psi\|^2\|\nabla\psi\|^2+\|\varphi_x\|^2\|\nabla\varphi_x\|^2+\|\phi_x\|^2\|\nabla\phi_x\|^2\right)dt$$
$$\leq\frac{1}{80}\int_0^t\|(\nabla\phi,\nabla\Psi)\|^2dt+CE\int_0^t\left(\|\nabla\psi\|^2+\|\varphi_x\|^2+\|\phi_x\|^2\right)dt. \tag{4.36}$$

H_1 can be estimated in the same way. By Young's inequality and Lemma 2.2, one has

$$\begin{aligned}|G_2| &= \Big|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \bar{u}_x(\phi\varphi_x - \varphi\phi_x)dxdydt\Big| \\ &\leq \frac{1}{80}\int_0^t \|(\varphi_x,\phi_x)\|^2 dt + C\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \Big[(\bar{u}_x\phi)^2 + (\bar{u}_x\varphi)^2\Big]dxdydt \\ &\leq \frac{1}{80}\int_0^t \|(\nabla\phi,\nabla\Psi)\|^2 dt + C\alpha\int_0^t \|\bar{u}_x^{\frac{1}{2}}(\phi,\varphi)\|^2 dt. \end{aligned} \tag{4.37}$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}|G_3| &= \Big|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \bar{\rho}_x\varphi \mathrm{div}\Psi dxdydt\Big| \leq \int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \Big(|\bar{\rho}_x\varphi\varphi_x| + |\bar{\rho}_x\varphi\psi_y|\Big)dxdydt \\ &\leq \frac{1}{80}\int_0^t \Big(\|\varphi_x\|^2 + \|\psi_y\|^2\Big)dt + C\int_0^t \Big(\|\bar{\rho}_x\|_{L^4}^4 + \|\varphi\|_{L^4}^4\Big)dt \\ &\leq \frac{1}{80}\int_0^t \Big(\|\varphi_x\|^2 + \|\psi_y\|^2\Big)dt + C\int_0^t \Big(\alpha^{\frac{1}{4}}(1+t)^{-\frac{3}{4}}\Big)^4 dt + C\int_0^t \|\varphi\|^2\|\nabla\varphi\|^2 dt \\ &\leq \frac{1}{80}\int_0^t \Big(\|\varphi_x\|^2 + \|\psi_y\|^2\Big)dt + C\alpha + CE\int_0^t \|\nabla\varphi\|^2 dt, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned}|G_4| &= \Big|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \Big(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}}\Big)\bar{\rho}_x\phi_x dxdydt\Big| \\ &\leq C\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} |\phi\bar{\rho}_x\phi_x| dxdydt \\ &\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \Big(\|\bar{\rho}_x\|_{L^4}^4 + \|\phi\|_{L^4}^4\Big)dt \\ &\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\alpha + CE\int_0^t \|\nabla\phi\|^2 dt. \end{aligned} \tag{4.39}$$

By Young's inequality and Lemma 2.2, one has

$$\begin{aligned}|G_5| &= \Big|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \bar{u}_{xx}\phi_x dxdydt\Big| \\ &\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \|\bar{u}_{xx}\|^2 dt \\ &\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \Big(\alpha^{\frac{1}{3}}(1+t)^{-\frac{2}{3}}\Big)^2 dt \\ &\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\alpha^{\frac{2}{3}}. \end{aligned} \tag{4.40}$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$|G_6| = \Big|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{1}{\rho^2}\phi_x(\bar{\rho}_{xx}\varphi + \bar{u}_{xx}\phi)dxdydt\Big| \leq C\int_0^t \Big(|\phi_x\bar{\rho}_{xx}\varphi| + |\phi_x\bar{u}_{xx}\phi|\Big)dxdydt$$

$$
\begin{aligned}
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \left(\|\bar\rho_{xx}\|_{L^4}^4 + \|\bar u_{xx}\|_{L^4}^4 + \|\varphi\|_{L^4}^4 + \|\phi\|_{L^4}^4\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \left(\alpha^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right)^4 dt + C\int_0^t \left(\|\varphi\|^2\|\nabla\varphi\|^2 + \|\phi\|^2\|\nabla\phi\|^2\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\alpha^2 + CE\int_0^t \|(\nabla\varphi,\nabla\phi)\|^2 dt. \qquad (4.41)
\end{aligned}
$$

By Young's inequality and Lemma 2.2, one has

$$
\begin{aligned}
|G_7| &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{1}{\rho^2}\bar\rho_x \nabla\phi\cdot\nabla\varphi\, dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\nabla\phi\|^2 dt + C\int_0^t \|\bar\rho_x\nabla\varphi\|^2 dt \\
&\leq \frac{1}{80}\int_0^t \|\nabla\phi\|^2 dt + C\alpha\int_0^t \|\nabla\varphi\|^2 dt. \qquad (4.42)
\end{aligned}
$$

H_2, H_3 and H_4 can be estimated in the same way. By Young's inequality and the interpolation inequality, one has

$$
\begin{aligned}
|G_8| &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{1}{\rho^2}\phi_x \nabla\phi\cdot\nabla\varphi\, dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \left(\|\nabla\phi\|_{L^4}^4 + \|\nabla\varphi\|_{L^4}^4\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + C\int_0^t \left(\|\nabla\phi\|^2\|\nabla^2\phi\|^2 + \|\nabla\varphi\|^2\|\nabla^2\varphi\|^2\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_x\|^2 dt + CE\int_0^t \left(\|\nabla\phi\|^2 + \|\nabla\varphi\|^2\right) dt. \qquad (4.43)
\end{aligned}
$$

H_5 and H_6 can be estimated in the same way. Substituting the estimates of (4.34)-(4.43) and the estimates of $H_i(i=1,\cdots,6)$ into (4.33), and taking E and α suitably small, we can prove (4.24) in Lemma 4.2. □

LEMMA 4.3. *There exists a positive constant C such that for $0\leq t\leq T$,*

$$
\begin{aligned}
\|(\phi,\Psi,\tilde c)\|_1^2(t) &+ \int_0^t \|\bar u_x^{\frac{1}{2}}(\varphi,\phi,\Psi_x,\phi_x,\tilde c_x)\|^2 dt + \int_0^t \|(\nabla\phi,\nabla\Psi,\nabla\tilde c)\|^2 dt \\
&+ \int_0^t \|(\nabla^2\tilde c,\nabla^2\Psi)\|^2 dt \leq C\left(\|(\phi_0,\Psi_0,\tilde c_0)\|_1^2 + \alpha^{\frac{1}{4}}\right). \qquad (4.44)
\end{aligned}
$$

Proof. We multiply the second equation of (3.1) by $-\Delta\Psi/\rho$ to give

$$
\begin{aligned}
&\left(\frac{|\nabla\Psi|^2}{2}\right)_t - \mathrm{div}(\varphi_t\nabla\varphi + \psi_t\nabla\psi) + \left(\frac{u}{2}\Psi_y^2 - \frac{u}{2}\Psi_x^2\right)_x - (u\Psi_x\Psi_y)_y + \frac{\bar u_x}{2}\Psi_x^2 + \frac{1}{\rho}(\Delta\Psi)^2 \\
&= -\frac{1}{2}\varphi_x\Psi_x^2 + \frac{1}{2}\varphi_x\left(\psi_y^2 - \varphi_y^2\right) - \varphi_y\psi_x\psi_y + \frac{\bar u_x}{2}\Psi_y^2 + \psi\Psi_y\cdot\Delta\Psi + \bar u_x\varphi\Delta\varphi \\
&\quad + \frac{p'(\rho)}{\rho}\nabla\phi\cdot\Delta\Psi + \left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar\rho)}{\bar\rho}\right)\bar\rho_x\Delta\varphi - \mu\nabla\tilde c\cdot\Delta\Psi - \frac{1}{\rho}\bar u_{xx}\Delta\varphi. \qquad (4.45)
\end{aligned}
$$

Integrating the above equation over $[0,t]\times\mathbb{T}\times\mathbb{R}$ yields

$$\|\nabla\Psi\|^2(t)+\int_0^t\|\bar{u}_x^{\frac{1}{2}}\Psi_x\|^2dt+\int_0^t\|\Delta\Psi\|^2dt\le C\|\nabla\Psi_0\|^2+\sum_{i=1}^{10}I_i, \tag{4.46}$$

where

$$\begin{aligned}\sum_{i=1}^{10}I_i:=&\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Bigg(\left|\frac{1}{2}\varphi_x\Psi_x^2\right|+\left|\frac{1}{2}\varphi_x(\psi_y^2-\varphi_y^2)\right|+|\varphi_y\psi_x\psi_y|\\&+\left|\frac{\bar{u}_x}{2}\Psi_y^2\right|+|\psi\Psi_y\cdot\Delta\Psi|+|\bar{u}_x\varphi\Delta\varphi|+\left|\frac{p'(\rho)}{\rho}\nabla\phi\cdot\Delta\Psi\right|\\&+\left|\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_x\Delta\varphi\right|+|\mu\nabla\tilde{c}\cdot\Delta\Psi|+\left|\frac{1}{\rho}\bar{u}_{xx}\Delta\varphi\right|\Bigg)dxdydt.\end{aligned}\tag{4.47}$$

We estimate I_i $(i=1,\ldots,10)$ term by term as follows. By Cauchy's inequality, the interpolation inequality, and Lemma 2.2, one has

$$\begin{aligned}|I_1|=&C\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{1}{2}\varphi_x\Psi_x^2dxdydt\right|\\\le&C\int_0^t\|\varphi_x\|^2dt+C\int_0^t\|\Psi_x\|_{L^4}^4dt\\\le&C\int_0^t\|\varphi_x\|^2dt+C\int_0^t\|\Psi_x\|^2\|\nabla\Psi_x\|^2dt\\\le&C\int_0^t\|\varphi_x\|^2dt+CE\int_0^t\|\Psi_x\|^2dt,\end{aligned}\tag{4.48}$$

$$\begin{aligned}|I_2|=&C\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{1}{2}\varphi_x\left(\psi_y^2-\varphi_y^2\right)dxdydt\right|\\\le&C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left(|\varphi_x\psi_y^2|+|\varphi_x\varphi_y^2|\right)dxdydt\\\le&C\int_0^t\|\varphi_x\|^2dt+CE\int_0^t\left(\|\psi_y\|^2+\|\varphi_y\|^2\right)dt,\end{aligned}\tag{4.49}$$

$$|I_3|=C\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\varphi_y\psi_x\psi_ydxdydt\right|\le C\int_0^t\|\varphi_y\|^2dt+CE\int_0^t\|\nabla\psi\|^2dt, \tag{4.50}$$

$$|I_4|=C\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{\bar{u}_x}{2}\Psi_y^2dxdydt\right|\le C\alpha\int_0^t\|\Psi_y\|^2dt, \tag{4.51}$$

$$|I_5|=C\left|\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\psi\Psi_y\cdot\Delta\Psi dxdydt\right|\le\frac{1}{80}\int_0^t\|\Delta\Psi\|^2dt+CE\int_0^t\left(\|\nabla\psi\|^2+\|\Psi_y\|^2\right)dt, \tag{4.52}$$

$$
\begin{aligned}
|I_6| &= C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \bar{u}_x \varphi \Delta\varphi dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\int_0^t \|\bar{u}_x\varphi\|^2 dt \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\alpha\int_0^t \|\bar{u}_x^{\frac{1}{2}}\varphi\|^2 dt,
\end{aligned} \tag{4.53}
$$

$$
|I_7| = C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{p'(\rho)}{\rho}\nabla\phi\cdot\Delta\Psi dxdydt\right| \leq \frac{1}{80}\int_0^t \|\Delta\Psi\|^2 dt + C\int_0^t \|\nabla\phi\|^2 dt, \tag{4.54}
$$

$$
\begin{aligned}
|I_8| &= C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_x \Delta\varphi dxdydt\right| \\
&\leq C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \phi\bar{\rho}_x\Delta\varphi dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\int_0^t \left(\|\bar{\rho}_x\|_{L^4}^4 + \|\phi\|_{L^4}^4\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\alpha + CE\int_0^t \|\nabla\phi\|^2 dt,
\end{aligned} \tag{4.55}
$$

$$
|I_9| = C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \mu\nabla\tilde{c}\cdot\Delta\Psi dxdydt\right| \leq \frac{1}{80}\int_0^t \|\Delta\Psi\|^2 dt + C\int_0^t \|\nabla\tilde{c}\|^2 dt, \tag{4.56}
$$

and

$$
\begin{aligned}
|I_{10}| &= C\left|\int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{1}{\rho}\bar{u}_{xx}\Delta\varphi dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\int_0^t \|\bar{u}_{xx}\|^2 dt \\
&\leq \frac{1}{80}\int_0^t \|\Delta\varphi\|^2 dt + C\alpha^{\frac{2}{3}}.
\end{aligned} \tag{4.57}
$$

Substituting the above estimates for I_i $(i=1,...,10)$ into (4.46), using the elliptic estimate $\|\Delta\Psi\| \sim \|\nabla^2\Psi\|$, and taking E and α suitably small, we have

$$
\begin{aligned}
\|\nabla\Psi\|^2 + \int_0^t \|\bar{u}_x^{\frac{1}{2}}\Psi_x\|^2 dt + \int_0^t \|\nabla^2\Psi\|^2 dt &\leq C\left(\|\nabla\Psi_0\|^2 + \alpha^{\frac{2}{3}}\right) \\
&\quad + C\int_0^t \left(\|\bar{u}_x^{\frac{1}{2}}\varphi\|^2 + \|(\nabla\phi,\nabla\tilde{c},\nabla\Psi)\|^2\right) dt.
\end{aligned} \tag{4.58}
$$

Combining (4.24) and (4.58), we complete the proof of Lemma 4.3. □

LEMMA 4.4. *There exists a positive constant C such that for $0\le t\le T$,*

$$\|(\phi,\Psi,\tilde{c})\|_1^2(t)+\|\nabla^2\phi\|^2(t)+\int_0^t\|\bar{u}_x^{\frac{1}{2}}(\varphi,\phi,\phi_x,\tilde{c}_x,\Psi_x,\nabla\phi_x)\|^2dt+\int_0^t\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|_1^2dt$$
$$\le C\left(\|(\phi_0,\Psi_0,\tilde{c}_0)\|_1^2+\|\nabla^2\phi_0\|^2+\alpha^{\frac{1}{4}}\right)+CE\int_0^t\|\nabla^3\Psi\|^2dt. \tag{4.59}$$

Proof. Before giving the proof of Lemma 4.4, we will introduce some basic calculations for the quick understanding of the proof. For the scalar functions f, g, and h,

(1) $\nabla^2(fg)=g\nabla^2 f+\nabla f\otimes\nabla g+\nabla g\otimes\nabla f+f\nabla^2 g$,

(2) $(\nabla f\otimes\nabla g)\cdot\nabla^2 h+(\nabla g\otimes\nabla f)\cdot\nabla^2 h=2(\nabla f\otimes\nabla g)\cdot\nabla^2 h$,

(3) $(\nabla f\otimes\nabla g)\cdot\nabla^2 h=f_x\nabla g\cdot\nabla h_x+f_y\nabla g\cdot\nabla h_y=g_x\nabla f\cdot\nabla h_x+g_y\nabla f\cdot\nabla h_y$.

We apply the operator ∇^2 on the first equation of (3.1) and then multiply the resulting equation by $\nabla^2\phi/\rho^2$ to give

$$\left(\frac{|\nabla^2\phi|^2}{2\rho^2}\right)_t+\mathrm{div}\left(\frac{\boldsymbol{u}|\nabla^2\phi|^2}{2\rho^2}\right)+\left(\frac{\nabla\phi_y\cdot\nabla\varphi_y}{\rho}\right)_x+\left(\frac{\nabla\phi_x\cdot\nabla\psi_x}{\rho}\right)_y-\left(\frac{\nabla\phi_y\cdot\nabla\psi_x}{\rho}\right)_x$$
$$-\left(\frac{\nabla\phi_x\cdot\nabla\varphi_y}{\rho}\right)_y+\frac{2\bar{u}_x|\nabla\phi_x|^2}{\rho^2}+\frac{\nabla^2\phi\cdot\nabla\Delta\Psi}{\rho}=K(t,x,y)+L(t,x,y), \tag{4.60}$$

where

$$K(t,x,y)=\frac{\mathrm{div}\Psi|\nabla^2\phi|^2}{2\rho^2}-\frac{2\psi_x\nabla\phi_y\cdot\nabla\phi_x}{\rho^2}-\frac{2\mathrm{div}\Psi_x\nabla\phi\cdot\nabla\phi_x}{\rho^2}-\frac{\phi_x\nabla^2\varphi\cdot\nabla^2\phi}{\rho^2}$$
$$+\frac{\bar{u}_x|\nabla^2\phi|^2}{2\rho^2}-\frac{2\mathrm{div}\Psi_x\bar{\rho}_x\phi_{xx}}{\rho^2}-\frac{\mathrm{div}\Psi\bar{\rho}_{xx}\phi_{xx}}{\rho^2}-\frac{\varphi\bar{\rho}_{xxx}\phi_{xx}}{\rho^2}, \tag{4.61}$$

and

$$L(t,x,y)=-\frac{2\psi_y|\nabla\phi_y|^2}{\rho^2}-\frac{2\varphi_x|\nabla\phi_x|^2}{\rho^2}-\frac{2\varphi_y\nabla\phi_x\cdot\nabla\phi_y}{\rho^2}-\frac{2\mathrm{div}\Psi_y\nabla\phi\cdot\nabla\phi_y}{\rho^2}$$
$$-\frac{\phi_y\nabla^2\psi\cdot\nabla^2\phi}{\rho^2}+\frac{\phi_x\nabla\psi_x\cdot\nabla\phi_y}{\rho^2}+\frac{\phi_y\nabla\varphi_y\cdot\nabla\phi_x}{\rho^2}-\frac{\phi_x\nabla\varphi_y\cdot\nabla\phi_y}{\rho^2}$$
$$-\frac{\phi_y\nabla\psi_x\cdot\nabla\phi_x}{\rho^2}-\frac{2\mathrm{div}\Psi_y\bar{\rho}_x\phi_{xy}}{\rho^2}-\frac{\bar{\rho}_x\nabla^2\varphi\cdot\nabla^2\phi}{\rho^2}+\frac{\bar{\rho}_x\nabla\psi_x\cdot\nabla\phi_y}{\rho^2}$$
$$-\frac{\bar{\rho}_x\nabla\varphi_y\cdot\nabla\phi_y}{\rho^2}-\frac{\phi_x\bar{u}_{xx}\phi_{xx}}{\rho^2}-\frac{2\bar{\rho}_{xx}\nabla\varphi\cdot\nabla\phi_x}{\rho^2}-\frac{2\bar{u}_{xx}\nabla\phi\cdot\nabla\phi_x}{\rho^2}$$
$$-\frac{\phi\bar{u}_{xxx}\phi_{xx}}{\rho^2}. \tag{4.62}$$

The fact that

$$\frac{\nabla^2\phi\cdot\nabla^2\mathrm{div}\Psi}{\rho}=\frac{\nabla^2\phi\cdot\nabla\Delta\Psi}{\rho}+\left(\frac{\nabla\phi_y\cdot\nabla\varphi_y}{\rho}\right)_x+\left(\frac{\nabla\phi_x\cdot\nabla\psi_x}{\rho}\right)_y$$
$$-\left(\frac{\nabla\phi_y\cdot\nabla\psi_x}{\rho}\right)_x-\left(\frac{\nabla\phi_x\cdot\nabla\varphi_y}{\rho}\right)_y-\frac{\phi_y\nabla\varphi_y\cdot\nabla\phi_x}{\rho^2}-\frac{\phi_x\nabla\psi_x\cdot\nabla\phi_y}{\rho^2}$$

$$-\frac{\bar{\rho}_x\nabla\psi_x\cdot\nabla\phi_y}{\rho^2}+\frac{\phi_y\nabla\psi_x\cdot\nabla\phi_x}{\rho^2}+\frac{\phi_x\nabla\varphi_y\cdot\nabla\phi_y}{\rho^2}+\frac{\bar{\rho}_x\nabla\varphi_y\cdot\nabla\phi_y}{\rho^2} \tag{4.63}$$

was used in the derivation of (4.60). In order to deal with the last term on the left-hand side of (4.60), we divide the second equation of (3.1) by ρ, apply the operator ∇ to the resulting equation, and then multiply the final equation by $\nabla^2\phi$ to get

$$\begin{aligned}&\big(\nabla\Psi\cdot\nabla^2\phi\big)_t-\mathrm{div}\,(\phi_{xt}\nabla\varphi+\phi_{yt}\nabla\psi)-(u\nabla\phi\cdot\Psi_{yy})_x+(u\nabla\phi\cdot\Psi_{xy})_y+\frac{p'(\rho)}{\rho}|\nabla^2\phi|^2\\&\quad-\frac{1}{\rho}\nabla^2\phi\cdot\nabla\Delta\Psi=\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi+\mu\nabla^2\tilde{c}\cdot\nabla^2\phi+M(t,x,y)+N(t,x,y),\end{aligned} \tag{4.64}$$

where $\nabla\Psi\cdot\nabla^2\phi=\nabla\varphi\cdot\nabla\phi_x+\nabla\psi\cdot\nabla\phi_y$, since ϕ_x corresponds to φ, and ϕ_y corresponds to ψ from the equation $(3.1)_2$. And, $M(t,x,y)$ and $N(t,x,y)$ are defined as follows,

$$\begin{aligned}M(t,x,y)\sim&\,\mathrm{div}\Psi\nabla\phi\cdot\Delta\Psi+\psi\nabla\phi_y\cdot\Delta\Psi-\psi\nabla\Psi_y\cdot\nabla^2\phi+\bar{\rho}_x\mathrm{div}\Psi\Delta\varphi\\&-\varphi_x\bar{u}_x\phi_{xx}+\bar{\rho}_{xx}\varphi\Delta\varphi-\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\phi_{xx}-\phi(\bar{\rho}_x)^2\phi_{xx}\\&-\frac{1}{\rho^2}\Delta\varphi\bar{\rho}_x\phi_{xx}-\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\phi_x-\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x\phi_{xx}+\frac{1}{\rho}\bar{u}_{xxx}\phi_{xx}\\&-\frac{1}{\rho^2}\bar{u}_{xx}\nabla\phi\cdot\nabla\phi_x,\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}N(t,x,y)\sim&\,\phi_x\nabla\varphi\cdot\Delta\Psi+\phi_y\nabla\psi\cdot\Delta\Psi-\varphi_x\nabla\varphi\cdot\nabla\phi_x-\psi_x\nabla\varphi\cdot\nabla\phi_y\\&-\varphi_x\nabla\phi\cdot\Psi_{yy}+\varphi_y\nabla\phi\cdot\Psi_{xy}-\varphi_y\nabla\psi\cdot\nabla\phi_x-\psi_y\nabla\psi\cdot\nabla\phi_y\\&-\phi_x\nabla\phi\cdot\nabla\phi_x-\phi_y\nabla\phi\cdot\nabla\phi_y+\bar{u}_x\phi_x\Delta\varphi+\bar{\rho}_x\nabla\varphi\cdot\Delta\Psi\\&+\bar{u}_x\nabla\phi\cdot\Delta\Psi-\psi_x\bar{u}_x\phi_{xy}-\bar{u}_x\nabla\phi\cdot\Psi_{yy}-\bar{u}_x\nabla\varphi\cdot\nabla\phi_x\\&-\bar{\rho}_x\nabla\phi\cdot\nabla\phi_x-\phi_x\bar{\rho}_x\phi_{xx}-\phi_y\bar{\rho}_x\phi_{xy}+\bar{u}_{xx}\phi\Delta\varphi-\varphi\bar{u}_{xx}\phi_{xx}\\&-\frac{1}{\rho^2}\Delta\psi\bar{\rho}_x\phi_{xy}-\frac{1}{\rho^2}\Delta\psi\nabla\phi\cdot\nabla\phi_y.\end{aligned} \tag{4.66}$$

Now combining (4.60) and (4.64) together, and then integrating the final equation over $[0,t]\times\mathbb{T}\times\mathbb{R}$ lead to

$$\begin{aligned}&\left[\int_{\mathbb{T}}\int_{\mathbb{R}}\Big(\frac{1}{2\rho^2}|\nabla^2\phi|^2+\nabla\Psi\cdot\nabla^2\phi\Big)dxdy\right]\Bigg|_0^t+\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left[\frac{2}{\rho^2}\bar{u}_x|\nabla\phi_x|^2+\frac{p'(\rho)}{\rho}|\nabla^2\phi|^2\right]dxdydt\\&=\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big[\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi+\mu\nabla^2\tilde{c}\cdot\nabla^2\phi+K(t,x,y)+L(t,x,y)\\&\qquad\qquad+M(t,x,y)+N(t,x,y)\Big]dxdydt.\end{aligned} \tag{4.67}$$

Furthermore, the combination of (4.67) and (4.44) in Lemma 4.3 leads to

$$\begin{aligned}&\|(\phi,\Psi,\tilde{c})\|_1^2(t)+\|\nabla^2\phi\|^2+\int_0^t\|\bar{u}_x^{\frac{1}{2}}(\varphi,\phi,\Psi_x,\phi_x,\tilde{c}_x,\nabla\phi_x)\|^2dt\\&\quad+\int_0^t\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|_1^2dt\le C\left(\|(\phi_0,\Psi_0,\tilde{c}_0)\|_1^2+\|\nabla^2\phi_0\|^2+\alpha^{\frac{1}{4}}\right)\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Big[|\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi|+|\mu\nabla^2\tilde{c}\cdot\nabla^2\phi|+|K(t,x,y)|\\
&+|L(t,x,y)|+|M(t,x,y)|+|N(t,x,y)|\Big]dxdydt.
\end{aligned}\tag{4.68}
$$

We now estimate the terms on the right-hand side of (4.68). By Cauchy's inequality, one has

$$
C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}|\rho\nabla\mathrm{div}\Psi\cdot\Delta\Psi|dxdydt\le C\int_0^t\Big(\|\nabla\mathrm{div}\Psi\|^2+\|\Delta\Psi\|^2\Big)dt\le C\int_0^t\|\nabla^2\Psi\|^2dt.\tag{4.69}
$$

Young's inequality guarantees

$$
C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}|\mu\nabla^2\tilde{c}\cdot\nabla^2\phi|dxdydt\le\varepsilon\int_0^t\|\nabla^2\phi\|^2dt+C\int_0^t\|\nabla^2\tilde{c}\|^2dt.\tag{4.70}
$$

We denote the items in $K(t,x,y)$ as

$$
\begin{aligned}
&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}|K(t,x,y)|dxdydt\\
\le&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Bigg[\left|\frac{\mathrm{div}\Psi|\nabla^2\phi|^2}{2\rho^2}\right|+\left|\frac{2\psi_x\nabla\phi_y\cdot\nabla\phi_x}{\rho^2}\right|\\
&+\left|\frac{2\mathrm{div}\Psi_x\nabla\phi\cdot\nabla\phi_x}{\rho^2}\right|+\left|\frac{\phi_x\nabla^2\varphi\cdot\nabla^2\phi}{\rho^2}\right|+\left|\frac{\bar{u}_x|\nabla^2\phi|^2}{2\rho^2}\right|+\left|\frac{2\mathrm{div}\Psi_x\bar{\rho}_x\phi_{xx}}{\rho^2}\right|\\
&+\left|\frac{\mathrm{div}\Psi\bar{\rho}_{xx}\phi_{xx}}{\rho^2}\right|+\left|\frac{\varphi\bar{\rho}_{xxx}\phi_{xx}}{\rho^2}\right|\Bigg]dxdydt:=\sum_{i=1}^{8}K_i.
\end{aligned}\tag{4.71}
$$

Similarly, we get $\sum\limits_{i=1}^{17}L_i$. We estimate each term in (4.71) as follows. The terms in $L(t,x,y)$ can be handled similarly and the details will be omitted. By Hölder's inequality, Sobolev's inequality and Cauchy's inequality, it holds that

$$
\begin{aligned}
K_1=&\left|\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\frac{\mathrm{div}\Psi|\nabla^2\phi|^2}{2\rho^2}dxdydt\right|\\
\le&C\int_0^t\|\mathrm{div}\Psi\|_{L^\infty}\|\nabla^2\phi\|\|\nabla^2\phi\|dt\\
\le&CE\int_0^t\|\nabla\Psi\|_{L^\infty}\|\nabla^2\phi\|dt\le CE\int_0^t\|\nabla\Psi\|_{H^2}\|\nabla^2\phi\|dt\\
\le&CE\int_0^t\Big(\|\nabla\Psi\|+\|\nabla^2\Psi\|+\|\nabla^3\Psi\|\Big)\|\nabla^2\phi\|dt\\
\le&CE\int_0^t\Big(\|\nabla\Psi\|^2+\|\nabla^2\Psi\|^2+\|\nabla^3\Psi\|^2+\|\nabla^2\phi\|^2\Big)dt,
\end{aligned}\tag{4.72}
$$

and

$$
K_2=\left|\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\frac{2\psi_x\nabla\phi_y\cdot\nabla\phi_x}{\rho^2}dxdydt\right|
$$

$$
\begin{aligned}
&\le C\int_0^t \|\psi_x\|_{L^\infty}\|\nabla\phi_y\|\|\nabla\phi_x\|dt\\
&\le CE\int_0^t \|\nabla\psi\|_{H^2}\|\nabla^2\phi\|dt\\
&\le CE\int_0^t \Big(\|\nabla\psi\|+\|\nabla^2\psi\|+\|\nabla^3\psi\|\Big)\|\nabla^2\phi\|dt\\
&\le CE\int_0^t \Big(\|\nabla\psi\|^2+\|\nabla^2\psi\|^2+\|\nabla^3\psi\|^2+\|\nabla^2\phi\|^2\Big)dt. \qquad (4.73)
\end{aligned}
$$

L_1, L_2 and L_3 can be estimated in the same way. By Young's inequality and the interpolation inequality, one has

$$
\begin{aligned}
K_3 &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{2\mathrm{div}\Psi_x \nabla\phi\cdot\nabla\phi_x}{\rho^2}dxdydt\right|\\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_x\|^2dt + C\int_0^t \Big(\|\mathrm{div}\Psi_x\|_{L^4}^4+\|\nabla\phi\|_{L^4}^4\Big)dt\\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_x\|^2dt + C\int_0^t \Big(\|\mathrm{div}\Psi_x\|^2\|\nabla\mathrm{div}\Psi_x\|^2+\|\nabla\phi\|^2\|\nabla^2\phi\|^2\Big)dt\\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_x\|^2dt + CE\int_0^t \Big(\|\nabla\mathrm{div}\Psi_x\|^2+\|\nabla^2\phi\|^2\Big)dt, \qquad (4.74)
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\phi_x \nabla^2\varphi\cdot\nabla^2\phi}{\rho^2}dxdydt\right|\\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + C\int_0^t \Big(\|\phi_x\|_{L^4}^4+\|\nabla^2\varphi\|_{L^4}^4\Big)dt\\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + C\int_0^t \Big(\|\phi_x\|^2\|\nabla\phi_x\|^2+\|\nabla^2\varphi\|^2\|\nabla^3\varphi\|^2\Big)dt\\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + CE\int_0^t \Big(\|\nabla\phi_x\|^2+\|\nabla^3\varphi\|^2\Big)dt. \qquad (4.75)
\end{aligned}
$$

$L_i (i=4,\ldots,9)$ can be estimated in the same way. It follows from Lemma 2.2 and Young's inequality that

$$
\begin{aligned}
K_5 &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\bar{u}_x|\nabla^2\phi|^2}{2\rho^2}dxdydt\right|\\
&\le C\int_0^t \|\bar{u}_x\|_{L^\infty}\|\nabla^2\phi\|^2dt\\
&\le C\alpha\int_0^t \|\nabla^2\phi\|^2dt, \qquad (4.76)
\end{aligned}
$$

and

$$
K_6 = \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{2\mathrm{div}\Psi_x \bar{\rho}_x\phi_{xx}}{\rho^2}dxdydt\right|
$$

$$
\begin{aligned}
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\int_0^t \|\bar{\rho}_x \mathrm{div}\Psi_x\|^2 dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\alpha\int_0^t \|\mathrm{div}\Psi_x\|^2 dt. \qquad (4.77)
\end{aligned}
$$

$L_i(i=10,...,13)$ can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$
\begin{aligned}
K_7 &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\mathrm{div}\Psi \bar{\rho}_{xx}\phi_{xx}}{\rho^2} dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\int_0^t \left(\|\mathrm{div}\Psi\|_{L^4}^4 + \|\bar{\rho}_{xx}\|_{L^4}^4\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\int_0^t \|\mathrm{div}\Psi\|^2 \|\nabla \mathrm{div}\Psi\|^2 dt + C\int_0^t \left[\alpha^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right]^4 dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + CE\int_0^t \|\nabla \mathrm{div}\Psi\|^2 dt + C\alpha^2. \qquad (4.78)
\end{aligned}
$$

$L_i(i=14,...,16)$ can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$
\begin{aligned}
K_8 &= \left|\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{\varphi \bar{\rho}_{xxx}\phi_{xx}}{\rho^2} dxdydt\right| \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\int_0^t \left(\|\varphi\|_{L^4}^4 + \|\bar{\rho}_{xxx}\|_{L^4}^4\right) dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + C\int_0^t \|\varphi\|^2 \|\nabla\varphi\|^2 dt + C\int_0^t \left[\alpha^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right]^4 dt \\
&\leq \frac{1}{80}\int_0^t \|\phi_{xx}\|^2 dt + CE\int_0^t \|\nabla\varphi\|^2 dt + C\alpha^2. \qquad (4.79)
\end{aligned}
$$

L_{17} can be estimated in the same way. Next, we will estimate $M(t,x,y)$ and $N(t,x,y)$ and we only show the estimate of $M(t,x,y)$ for brevity. The terms in $M(t,x,y)$ can be denoted as

$$
\begin{aligned}
&C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |M(t,x,y)| dxdydt \leq C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \Big[|\mathrm{div}\Psi \nabla\phi\cdot\Delta\Psi| + |\psi\nabla\phi_y\cdot\Delta\Psi| \\
&+|\psi\nabla\Psi_y\cdot\nabla^2\phi| + |\bar{\rho}_x \mathrm{div}\Psi\Delta\varphi| + |\varphi_x\bar{u}_x\phi_{xx}| + |\bar{\rho}_{xx}\varphi\Delta\varphi| + \left|\left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\phi_{xx}\right| \\
&+|\phi(\bar{\rho}_x)^2\phi_{xx}| + \left|\frac{1}{\rho^2}\Delta\varphi\bar{\rho}_x\phi_{xx}\right| + \left|\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\phi_x\right| + \left|\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x\phi_{xx}\right| + \left|\frac{1}{\rho}\bar{u}_{xxx}\phi_{xx}\right| \\
&+\left|\frac{1}{\rho^2}\bar{u}_{xx}\nabla\phi\cdot\nabla\phi_x\right|\Big] dxdydt := \sum_{i=1}^{13} M_i. \qquad (4.80)
\end{aligned}
$$

Similarly, we get $\sum_{i=1}^{23} N_i$. We estimate the right-hand side of (4.80) term by term. By

Young's inequality and the interpolation inequality, one has

$$
\begin{aligned}
M_1 &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\mathrm{div}\Psi\nabla\phi\cdot\Delta\Psi|dxdydt \\
&\le \frac{1}{80}\int_0^t \|\Delta\Psi\|^2dt + C\int_0^t \left(\|\nabla\phi\|_{L^4}^4+\|\mathrm{div}\Psi\|_{L^4}^4\right)dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\Psi\|^2dt + C\int_0^t \left(\|\nabla\phi\|^2\|\nabla^2\phi\|^2+\|\mathrm{div}\Psi\|^2\|\nabla\mathrm{div}\Psi\|^2\right)dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\Psi\|^2dt + CE\int_0^t \left(\|\nabla^2\phi\|^2+\|\nabla\mathrm{div}\Psi\|^2\right)dt. \qquad (4.81)
\end{aligned}
$$

$N_i(i=1,...,10)$ can be estimated in the same way. By Young's inequality and the interpolation inequality, one has

$$
\begin{aligned}
M_2 &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\psi\nabla\phi_y\cdot\Delta\Psi|dxdydt \\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_y\|^2dt + C\int_0^t \left(\|\psi\|_{L^4}^4+\|\Delta\Psi\|_{L^4}^4\right)dt \\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_y\|^2dt + C\int_0^t \left(\|\psi\|^2\|\nabla\psi\|^2+\|\Delta\Psi\|^2\|\nabla\Delta\Psi\|^2\right)dt \\
&\le \frac{1}{80}\int_0^t \|\nabla\phi_y\|^2dt + CE\int_0^t \left(\|\nabla\psi\|^2+\|\nabla\Delta\Psi\|^2\right)dt, \qquad (4.82)
\end{aligned}
$$

and

$$
\begin{aligned}
M_3 &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\psi\nabla\Psi_y\cdot\nabla^2\phi|dxdydt \\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + C\int_0^t \left(\|\psi\|_{L^4}^4+\|\nabla\Psi_y\|_{L^4}^4\right)dt \\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + C\int_0^t \left(\|\psi\|^2\|\nabla\psi\|^2+\|\nabla\Psi_y\|^2\|\nabla^2\Psi_y\|^2\right)dt \\
&\le \frac{1}{80}\int_0^t \|\nabla^2\phi\|^2dt + CE\int_0^t \left(\|\nabla\psi\|^2+\|\nabla^2\Psi_y\|^2\right)dt. \qquad (4.83)
\end{aligned}
$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$
\begin{aligned}
M_4 &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\bar\rho_x\mathrm{div}\Psi\Delta\varphi|dxdydt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi\|^2dt + C\int_0^t \left(\|\bar\rho_x\|_{L^4}^4+\|\mathrm{div}\Psi\|_{L^4}^4\right)dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi\|^2dt + C\int_0^t \left[\alpha^{\frac14}(1+t)^{-\frac34}\right]^4dt + C\int_0^t \|\mathrm{div}\Psi\|^2\|\nabla\mathrm{div}\Psi\|^2dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi\|^2dt + C\alpha + CE\int_0^t \|\nabla\mathrm{div}\Psi\|^2dt, \qquad (4.84)
\end{aligned}
$$

and

$$\begin{aligned}M_5&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\varphi_x\bar{u}_x\phi_{xx}|dxdydt\\&\le\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\alpha+CE\int_0^t\|\nabla\varphi_x\|^2dt.\end{aligned}\tag{4.85}$$

$N_i(i=11,...,19)$ can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}M_6&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\bar{\rho}_{xx}\varphi\Delta\varphi|dxdydt\\&\le\frac{1}{80}\int_0^t\|\Delta\varphi\|^2dt+C\int_0^t\left(\|\bar{\rho}_{xx}\|_{L^4}^4+\|\varphi\|_{L^4}^4\right)dt\\&\le\frac{1}{80}\int_0^t\|\Delta\varphi\|^2dt+C\int_0^t\left[\alpha^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\right]^4dt+C\int_0^t\|\varphi\|^2\|\nabla\varphi\|^2dt\\&\le\frac{1}{80}\int_0^t\|\Delta\varphi\|^2dt+C\alpha^2+CE\int_0^t\|\nabla\varphi\|^2dt,\end{aligned}\tag{4.86}$$

and

$$\begin{aligned}M_7&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left|\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\phi_{xx}\right|dxdydt\\&\le\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\int_0^t\left(\|\bar{\rho}_{xx}\|_{L^4}^4+\|\phi\|_{L^4}^4\right)dt\\&\le\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\alpha^2+CE\int_0^t\|\nabla\phi\|^2dt.\end{aligned}\tag{4.87}$$

N_{20} and N_{21} can be estimated in the same way. It follows from Young's inequality and Lemma 2.2 that

$$\begin{aligned}M_8&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\phi(\bar{\rho}_x)^2\phi_{xx}|dxdydt\\&\le CE\int_0^t\|\phi_{xx}\|^2dt+CE\int_0^t\|\bar{\rho}_x\|_{L^4}^4dt\\&\le CE\int_0^t\|\phi_{xx}\|^2dt+C\alpha.\end{aligned}\tag{4.88}$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}M_9&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left|\frac{1}{\rho^2}\Delta\varphi\bar{\rho}_x\phi_{xx}\right|dxdydt\\&\le\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\int_0^t\left(\|\bar{\rho}_x\|_{L^4}^4+\|\Delta\varphi\|_{L^4}^4\right)dt\\&\le\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\alpha+CE\int_0^t\|\nabla\Delta\varphi\|^2dt.\end{aligned}\tag{4.89}$$

N_{22} can be estimated in the same way. By Young's inequality and the interpolation inequality, one has

$$\begin{aligned}M_{10}=&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Big|\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\phi_x\Big|dxdydt\\ \leq&\frac{1}{80}\int_0^t\|\nabla\phi_x\|^2dt+CE\int_0^t\Big(\|\nabla^2\phi\|^2+\|\nabla\Delta\varphi\|^2\Big)dt.\end{aligned}\tag{4.90}$$

N_{23} can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}M_{11}=&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Big|\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x\phi_{xx}\Big|dxdydt\\ \leq&\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\int_0^t\|\bar{u}_{xx}\|_{L^4}^4dt+C\int_0^t\|\bar{\rho}_x\|_{L^4}^4dt\\ \leq&\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\alpha^2+C\alpha.\end{aligned}\tag{4.91}$$

It follows from Young's inequality and Lemma 2.2 that

$$\begin{aligned}M_{12}=&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Big|\frac{1}{\rho}\bar{u}_{xxx}\phi_{xx}\Big|dxdydt\\ \leq&\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\int_0^t\|\bar{u}_{xxx}\|^2dt\\ \leq&\frac{1}{80}\int_0^t\|\phi_{xx}\|^2dt+C\alpha^{\frac{2}{3}}.\end{aligned}\tag{4.92}$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}M_{13}=&C\int_0^t\int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}}\Big|\frac{1}{\rho^2}\bar{u}_{xx}\nabla\phi\cdot\nabla\phi_x\Big|dxdydt\\ \leq&\frac{1}{80}\int_0^t\|\nabla\phi_x\|^2dt+C\int_0^t\|\bar{u}_{xx}\|_{L^4}^4dt+C\int_0^t\|\nabla\phi\|_{L^4}^4dt\\ \leq&\frac{1}{80}\int_0^t\|\nabla\phi_x\|^2dt+C\alpha^2+CE\int_0^t\|\nabla^2\phi\|^2dt.\end{aligned}\tag{4.93}$$

Substituting the estimates of (4.69)-(4.93), $L(t,x,y)$ and $N(t,x,y)$ into (4.68), using the elliptic estimates $\|\Delta\Psi\|\sim\|\nabla^2\Psi\|$ and $\|\nabla\Delta\Psi\|\sim\|\nabla^3\Psi\|$, taking E and α suitably small, and using (4.44) in Lemma 4.3, we end the proof of Lemma 4.4. ∎

LEMMA 4.5. *There exists a positive constant C such that for $0\leq t\leq T$,*

$$\begin{aligned}\|(\phi,\Psi)\|_2^2(t)+\|\tilde{c}\|_1^2(t)+\int_0^t\|\bar{u}_x^{\frac{1}{2}}(\varphi,\phi,\phi_x,\tilde{c}_x,\Psi_x,\nabla\phi_x)\|^2dt+\int_0^t\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|_1^2dt\\ +\int_0^t\|\nabla^3\Psi\|^2dt\leq C\Big(\|(\phi_0,\Psi_0)\|_2^2+\|\tilde{c}_0\|_1^2+\alpha^{\frac{1}{4}}\Big).\end{aligned}\tag{4.94}$$

Proof. We divide the second equation of (3.1) by ρ, apply the operator ∇ to the resulting equation and then multiply the final equation by $-\nabla\Delta\Psi$ to give

$$\begin{aligned}&\left(\frac{|\nabla^2\Psi|^2}{2}\right)_t-\operatorname{div}\Big(\nabla\varphi_t\nabla^2\varphi+\nabla\psi_t\nabla^2\psi\\&\qquad+u\nabla\varphi_x\Delta\varphi+u\nabla\psi_x\Delta\psi\Big)+\left(\frac{u}{2}|\Delta\Psi|^2\right)_x+\frac{1}{\rho}|\nabla\Delta\Psi|^2\\&=-\mu\nabla^2\tilde{c}\cdot\nabla\Delta\Psi+\frac{p'(\rho)}{\rho}\nabla^2\phi\cdot\nabla\Delta\Psi+Q(t,x,y)+R(t,x,y),\end{aligned}\tag{4.95}$$

where

$$\begin{aligned}Q(t,x,y)\sim&\,\varphi_x\nabla\varphi\cdot\nabla\Delta\varphi+\phi_x\nabla\phi\cdot\nabla\Delta\varphi+\varphi_x\bar{u}_x\Delta\varphi_x+\bar{u}_x\nabla\varphi\cdot\nabla\Delta\varphi\\&-\Delta\varphi\nabla\varphi\cdot\nabla\varphi_x+\frac{\varphi_x}{2}(\Delta\varphi)^2-\bar{u}_x\varphi_{xx}\Delta\varphi+\frac{\bar{u}_x}{2}(\Delta\varphi)^2+\psi\nabla\varphi_y\cdot\nabla\Delta\varphi\\&+\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\Delta\varphi_x+\phi(\bar{\rho}_x)^2\Delta\varphi_x+\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\Delta\varphi\\&+\frac{1}{\rho^2}\bar{\rho}_x\Delta\varphi\Delta\varphi_x-\frac{1}{\rho}\bar{u}_{xxx}\Delta\varphi_x+\frac{1}{\rho^2}\bar{u}_{xx}\nabla\phi\cdot\nabla\Delta\varphi+\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x\Delta\varphi_x,\end{aligned}\tag{4.96}$$

and

$$\begin{aligned}R(t,x,y)\sim&\,\psi_x\nabla\varphi\cdot\nabla\Delta\psi+\varphi_y\nabla\psi\cdot\nabla\Delta\varphi+\psi_y\nabla\psi\cdot\nabla\Delta\psi+\phi_y\nabla\phi\cdot\nabla\Delta\psi\\&+\psi_x\bar{u}_x\Delta\psi_x+\bar{\rho}_x\phi_x\Delta\varphi_x+\bar{\rho}_x\phi_y\Delta\psi_x+\bar{\rho}_x\nabla\phi\cdot\nabla\Delta\varphi\\&-\Delta\psi\nabla\varphi\cdot\nabla\psi_x+\frac{\varphi_x}{2}(\Delta\psi)^2-\bar{u}_x\psi_{xx}\Delta\psi+\frac{\bar{u}_x}{2}(\Delta\psi)^2\\&+\psi\nabla\psi_y\cdot\nabla\Delta\psi+\varphi\bar{u}_{xx}\Delta\varphi_x+\frac{1}{\rho^2}\Delta\psi\nabla\phi\cdot\nabla\Delta\psi+\frac{1}{\rho^2}\bar{\rho}_x\Delta\psi\Delta\psi_x.\end{aligned}\tag{4.97}$$

Integrating (4.95) over $[0,t]\times\mathbb{T}\times\mathbb{R}$ leads to

$$\begin{aligned}&\|\nabla^2\Psi\|^2(t)+\int_0^t\|\nabla\Delta\Psi\|^2dt\le C\|\nabla^2\Psi_0\|^2+C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\mu\nabla^2\tilde{c}\cdot\nabla\Delta\Psi|dxdydt\\&+C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left|\frac{p'(\rho)}{\rho}\nabla^2\phi\cdot\nabla\Delta\Psi\right|dxdydt+C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big(|Q(t,x,y)|+|R(t,x,y)|\Big)dxdydt.\end{aligned}\tag{4.98}$$

We are now in a position to estimate the terms on the right-hand side of (4.98). First, by Young's inequality, one has

$$C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\mu\nabla^2\tilde{c}\cdot\nabla\Delta\Psi|dxdydt\le\frac{1}{80}\int_0^t\|\nabla\Delta\Psi\|^2dt+C\int_0^t\|\nabla^2\tilde{c}\|^2dt,\tag{4.99}$$

and

$$C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\left|\frac{p'(\rho)}{\rho}\nabla^2\phi\cdot\nabla\Delta\Psi\right|dxdydt\le\frac{1}{80}\int_0^t\|\nabla\Delta\Psi\|^2dt+C\int_0^t\|\nabla^2\phi\|^2dt.\tag{4.100}$$

In the following, since the estimate for $R(t,x,y)$ is similar to $Q(t,x,y)$, we only show the estimate of $Q(t,x,y)$ for brevity. The terms in $Q(t,x,y)$ can be denoted simplicitly as

$$\begin{aligned}
&C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|Q(t,x,y)|dxdydt\le C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}\Big(|\varphi_x\nabla\varphi\cdot\nabla\Delta\varphi|+|\phi_x\nabla\phi\cdot\nabla\Delta\varphi|\\
&\quad+|\varphi_x\bar{u}_x\Delta\varphi_x|+|\bar{u}_x\nabla\varphi\cdot\nabla\Delta\varphi|+|\Delta\varphi\nabla\varphi\cdot\nabla\varphi_x|+\left|\frac{\varphi_x}{2}(\Delta\varphi)^2\right|+|\bar{u}_x\varphi_{xx}\Delta\varphi|\\
&\quad+\left|\frac{\bar{u}_x}{2}(\Delta\varphi)^2\right|+|\psi\nabla\varphi_y\cdot\nabla\Delta\varphi|+\left|\left(\frac{p'(\rho)}{\rho}-\frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\Delta\varphi_x\right|+|\phi(\bar{\rho}_x)^2\Delta\varphi_x|\\
&\quad+|\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\Delta\varphi|+\left|\frac{1}{\rho^2}\bar{\rho}_x\Delta\varphi\Delta\varphi_x\right|+\left|\frac{1}{\rho}\bar{u}_{xxx}\Delta\varphi_x\right|+\left|\frac{1}{\rho^2}\bar{u}_{xx}\nabla\phi\cdot\nabla\Delta\varphi\right|\\
&\quad+\left|\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x\Delta\varphi_x\right|\Big)dxdydt:=\sum_{i=1}^{16}Q_i.
\end{aligned}\tag{4.101}$$

Similarly, we get $\sum\limits_{i=1}^{16}R_i$. Each term in (4.101) will be estimated as follows. By Young's inequality and the interpolation inequality, we can obtain

$$\begin{aligned}
Q_1&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\varphi_x\nabla\varphi\cdot\nabla\Delta\varphi|dxdydt\\
&\le\frac{1}{80}\int_0^t\|\nabla\Delta\varphi\|^2dt+C\int_0^t\left(\|\varphi_x\|_{L^4}^4+\|\nabla\varphi\|_{L^4}^4\right)dt\\
&\le\frac{1}{80}\int_0^t\|\nabla\Delta\varphi\|^2dt+C\int_0^t\|\nabla\varphi\|^2\|\nabla^2\varphi\|^2dt\\
&\le\frac{1}{80}\int_0^t\|\nabla\Delta\varphi\|^2dt+CE\int_0^t\|\nabla^2\varphi\|^2dt,
\end{aligned}\tag{4.102}$$

and

$$\begin{aligned}
Q_2&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\phi_x\nabla\phi\cdot\nabla\Delta\varphi|dxdydt\\
&\le\frac{1}{80}\int_0^t\|\nabla\Delta\varphi\|^2dt+C\int_0^t\|\nabla\phi\|_{L^4}^4dt\\
&\le\frac{1}{80}\int_0^t\|\nabla\Delta\varphi\|^2dt+CE\int_0^t\|\nabla^2\phi\|^2dt.
\end{aligned}\tag{4.103}$$

$R_i(i=1,...,4)$ can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}
Q_3&=C\int_0^t\int_{\mathbb{T}}\int_{\mathbb{R}}|\varphi_x\bar{u}_x\Delta\varphi_x|dxdydt\\
&\le\frac{1}{80}\int_0^t\|\Delta\varphi_x\|^2dt+C\int_0^t\left(\|\varphi_x\|_{L^4}^4+\|\bar{u}_x\|_{L^4}^4\right)dt\\
&\le\frac{1}{80}\int_0^t\|\Delta\varphi_x\|^2dt+C\int_0^t\|\varphi_x\|^2\|\nabla\varphi_x\|^2dt+C\int_0^t\left[\alpha^{\frac14}(1+t)^{-\frac34}\right]^4dt
\end{aligned}$$

$$\leq \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + CE\int_0^t \|\nabla\varphi_x\|^2 dt + C\alpha, \tag{4.104}$$

and

$$\begin{aligned} Q_4 &= C\int_0^t \int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}} |\bar{u}_x \nabla\varphi\cdot\nabla\Delta\varphi| dxdydt \\ &\leq \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\int_0^t \left(\|\nabla\varphi\|_{L^4}^4 + \|\bar{u}_x\|_{L^4}^4\right) dt \\ &\leq \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + CE\int_0^t \|\nabla^2\varphi\|^2 dt + C\alpha. \end{aligned} \tag{4.105}$$

$R_i (i=5,...,8)$ can be estimated in the same way. By Young's inequality, the interpolation inequality, Hölder's inequality and Sobolev's inequality, one has

$$\begin{aligned} Q_5 &= C\int_0^t \int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}} |\Delta\varphi\nabla\varphi\cdot\nabla\varphi_x| dxdydt \\ &\leq C\int_0^t \|\nabla\varphi_x\|^2 dt + C\int_0^t \left(\|\Delta\varphi\|_{L^4}^4 + \|\nabla\varphi\|_{L^4}^4\right) dt \\ &\leq C\int_0^t \|\nabla\varphi_x\|^2 dt + C\int_0^t \left(\|\Delta\varphi\|^2\|\nabla\Delta\varphi\|^2 + \|\nabla\varphi\|^2\|\nabla^2\varphi\|^2\right) dt \\ &\leq C\int_0^t \|\nabla\varphi_x\|^2 dt + CE\int_0^t \left(\|\nabla\Delta\varphi\|^2 + \|\nabla^2\varphi\|^2\right) dt, \end{aligned} \tag{4.106}$$

and

$$\begin{aligned} Q_6 &= C\int_0^t \int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}} \left|\frac{\varphi_x}{2}(\Delta\varphi)^2\right| dxdydt \\ &\leq C\int_0^t \|\varphi_x\|_{L^\infty}\|\Delta\varphi\|\|\Delta\varphi\| dt \\ &\leq CE\int_0^t \|\varphi_x\|_{H^2}\|\Delta\varphi\| dt \\ &\leq CE\int_0^t \left(\|\varphi_x\| + \|\nabla\varphi_x\| + \|\nabla^2\varphi_x\|\right)\|\Delta\varphi\| dt \\ &\leq CE\int_0^t \left(\|\varphi_x\|^2 + \|\nabla\varphi_x\|^2 + \|\nabla^2\varphi_x\|^2 + \|\Delta\varphi\|^2\right) dt. \end{aligned} \tag{4.107}$$

R_9 and R_{10} can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned} Q_7 = C\int_0^t \int\limits_{\mathbb{T}}\int\limits_{\mathbb{R}} |\bar{u}_x\varphi_{xx}\Delta\varphi| dxdydt &\leq C\int_0^t \|\varphi_{xx}\|^2 dt + C\int_0^t \left(\|\Delta\varphi\|_{L^4}^4 + \|\bar{u}_x\|_{L^4}^4\right) dt \\ &\leq C\int_0^t \|\varphi_{xx}\|^2 dt + C\int_0^t \|\Delta\varphi\|^2\|\nabla\Delta\varphi\|^2 dt + C\alpha \\ &\leq C\int_0^t \|\varphi_{xx}\|^2 dt + CE\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\alpha, \end{aligned} \tag{4.108}$$

and

$$\begin{aligned} Q_8 =& C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{\bar{u}_x}{2}(\Delta\varphi)^2\right| dxdydt \\ \le& C\int_0^t \|\bar{u}_x\|_{L^\infty}\|\Delta\varphi\|^2 dt \le C\alpha\int_0^t \|\Delta\varphi\|^2 dt. \end{aligned} \tag{4.109}$$

R_{11} and R_{12} can be estimated in the same way. It follows from Young's inequality and the interpolation inequality that

$$\begin{aligned} Q_9 =& C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\psi\nabla\varphi_y\cdot\nabla\Delta\varphi| dxdydt \\ \le& \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\int_0^t \left(\|\psi\|_{L^4}^4 + \|\nabla\varphi_y\|_{L^4}^4\right) dt \\ \le& \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\int_0^t \left(\|\psi\|^2\|\nabla\psi\|^2 + \|\nabla\varphi_y\|^2\|\nabla^2\varphi_y\|^2\right) dt \\ \le& \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + CE\int_0^t \left(\|\nabla\psi\|^2 + \|\nabla^2\varphi_y\|^2\right) dt. \end{aligned} \tag{4.110}$$

R_{13} can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned} Q_{10} =& C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_{xx}\Delta\varphi_x\right| dxdydt \\ \le& \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\int_0^t \left(\|\phi\|_{L^4}^4 + \|\bar{\rho}_{xx}\|_{L^4}^4\right) dt \\ \le& \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\alpha^2 + CE\int_0^t \|\nabla\phi\|^2 dt. \end{aligned} \tag{4.111}$$

R_{14} can be estimated in the same way. By Young's inequality, Sobolev's inequality and Lemma 2.2, one has

$$\begin{aligned} Q_{11} = C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} |\phi(\bar{\rho}_x)^2\Delta\varphi_x| dxdydt \le& CE\int_0^t \|\Delta\varphi_x\|^2 dt + CE\int_0^t \|\bar{\rho}_x\|_{L^4}^4 dt \\ \le& CE\int_0^t \|\Delta\varphi_x\|^2 dt + C\alpha. \end{aligned} \tag{4.112}$$

It follows from Young's inequality and the interpolation inequality that

$$\begin{aligned} Q_{12} =& C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{1}{\rho^2}\Delta\varphi\nabla\phi\cdot\nabla\Delta\varphi\right| dxdydt \\ \le& \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\int_0^t \left(\|\Delta\varphi\|_{L^4}^4 + \|\nabla\phi\|_{L^4}^4\right) dt \\ \le& \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + CE\int_0^t \left(\|\nabla\Delta\varphi\|^2 + \|\nabla^2\phi\|^2\right) dt. \end{aligned} \tag{4.113}$$

R_{15} can be estimated in the same way. By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}
Q_{13} &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{1}{\rho^2}\bar{\rho}_x \Delta\varphi \Delta\varphi_x\right| dxdydt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\int_0^t \left(\|\Delta\varphi\|_{L^4}^4 + \|\bar{\rho}_x\|_{L^4}^4\right) dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + CE\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\alpha. \qquad (4.114)
\end{aligned}$$

R_{16} can be estimated in the same way. By Young's inequality and Lemma 2.2, one has

$$\begin{aligned}
Q_{14} &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{1}{\rho}\bar{u}_{xxx} \Delta\varphi_x\right| dxdydt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\int_0^t \|\bar{u}_{xxx}\|^2 dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\alpha^{\frac{2}{3}}. \qquad (4.115)
\end{aligned}$$

By Young's inequality, the interpolation inequality and Lemma 2.2, one has

$$\begin{aligned}
Q_{15} &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{1}{\rho^2}\bar{u}_{xx} \nabla\phi\cdot\nabla\Delta\varphi\right| dxdydt \\
&\le \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\int_0^t \left(\|\bar{u}_{xx}\|_{L^4}^4 + \|\nabla\phi\|_{L^4}^4\right) dt \\
&\le \frac{1}{80}\int_0^t \|\nabla\Delta\varphi\|^2 dt + C\alpha^2 + CE\int_0^t \|\nabla^2\phi\|^2 dt, \qquad (4.116)
\end{aligned}$$

and

$$\begin{aligned}
Q_{16} &= C\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \left|\frac{1}{\rho^2}\bar{u}_{xx}\bar{\rho}_x \Delta\varphi_x\right| dxdydt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\int_0^t \left(\|\bar{u}_{xx}\|_{L^4}^4 + \|\bar{\rho}_x\|_{L^4}^4\right) dt \\
&\le \frac{1}{80}\int_0^t \|\Delta\varphi_x\|^2 dt + C\alpha^2 + C\alpha. \qquad (4.117)
\end{aligned}$$

Substituting the estimates of (4.99), (4.100), $Q(t,x,y)$ and $R(t,x,y)$ into (4.98) and using the elliptic estimates $\|\Delta\Psi\| \sim \|\nabla^2\Psi\|$ and $\|\nabla\Delta\Psi\| \sim \|\nabla^3\Psi\|$ give

$$\begin{aligned}
\|\nabla^2\Psi\|^2(t) + \int_0^t \|\nabla^3\Psi\|^2 dt &\le C\left(\|\nabla^2\Psi_0\|^2 + \alpha^{\frac{2}{3}}\right) + C\int_0^t \|(\nabla^2\phi, \nabla^2\tilde{c}, \nabla^2\Psi)\|^2 dt \\
&+ C(E+\alpha)\int_0^t \left(\|(\nabla\phi,\nabla\Psi)\|_1^2 + \|\nabla^3\Psi\|^2\right) dt. \qquad (4.118)
\end{aligned}$$

Taking E and α suitably small, combining (4.59) and (4.118), we end the proof of Lemma 4.5. □

Multiplying $(3.1)_3$ by $-\tilde{c}$, we have

$$-\mathrm{div}(\tilde{c}\nabla\tilde{c})+|\nabla\tilde{c}|^2+b|\tilde{c}|^2=\bar{c}_{xx}\tilde{c}+a\phi\tilde{c}.$$

It follows from integration over $\mathbb{T}\times\mathbb{R}$ and Cauchy's inequality that

$$\|\tilde{c}(t)\|_{H^1}^2\leq C(\|\phi(t)\|^2+\|\bar{c}_{xx}(t)\|^2)\leq C(\|\phi(t)\|^2+\alpha^2),$$

and hence in particular,

$$\|\tilde{c}(0)\|_{H^1}^2\leq C(\|\phi(0)\|^2+\alpha^2).$$

Thus the proof of Proposition 3.2 is completed, which implies that (3.8) holds true.

4.2. Proof of Theorem 1.1. The global existence follows from Proposition 3.1. Therefore, it suffices to show the large-time behavior of the solution as $t\to+\infty$, that is to prove (1.24). For this, by the two-dimensional Sobolev's inequality, it holds that

$$\sup_{(x,y)\in\mathbb{R}\times\mathbb{T}}|\phi|\leq C\Big(\|\phi\|^{\frac{1}{2}}\|\phi_x\|^{\frac{1}{2}}+\|\phi_y\|^{\frac{1}{2}}\|\phi_{xy}\|^{\frac{1}{2}}\Big). \tag{4.119}$$

According to $(\phi,\Psi)\in H^2(\mathbb{R}\times\mathbb{T})$ and the third equation of (3.1), we can deduce that $\tilde{c}\in H^2(\mathbb{R}\times\mathbb{T})$. Then we need to prove

$$\lim_{t\to+\infty}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|=0, \tag{4.120}$$

which can be guaranteed by the following claim,

$$\int_0^{+\infty}\left[\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|^2+\left|\frac{d}{dt}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|^2\right|\right]dt<+\infty. \tag{4.121}$$

In fact, owing to Proposition 3.1, we get

$$\int_0^{+\infty}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|^2dt<+\infty. \tag{4.122}$$

Next we only need to show

$$\int_0^{+\infty}\left|\frac{d}{dt}\|(\nabla\phi,\nabla\Psi,\nabla\tilde{c})\|^2\right|dt<+\infty. \tag{4.123}$$

By (3.1), Cauchy's inequality, Lemma 2.2 and (3.7), one has

$$\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\|\nabla\phi\|^2\right|dt=\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}2\nabla\phi\cdot\nabla\phi_t dxdy\right|dt\\
&=2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}\mathrm{div}(\phi_t\nabla\phi)-\phi_t\Delta\phi dxdy\right|dt\\
&=2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}\phi_t\Delta\phi dxdy\right|dt
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}(\rho\mathrm{div}\Psi+\psi\phi_y+u\phi_x+\bar{\rho}_x\varphi+\bar{u}_x\phi)\Delta\phi dxdy\right|dt\\
&\le C\int_0^{+\infty}\int_{\mathbb{T}}\int_{\mathbb{R}}\left(\varphi_x^2+\psi_y^2+{\phi_y}^2+{\phi_x}^2+\bar{u}_x\varphi^2+\bar{u}_x\phi^2+(\Delta\phi)^2\right)dxdydt\\
&\le C\left(\|(\phi_0,\Psi_0)\|_2^2+\alpha^{\frac14}\right)<+\infty, \qquad (4.124)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\|\nabla\Psi\|^2\right|dt\\
&=\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}2\nabla\Psi\cdot\nabla\Psi_t dxdy\right|dt\\
&=2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}\mathrm{div}(\Psi_t\nabla\Psi)-\Psi_t\cdot\Delta\Psi dxdy\right|dt\\
&=2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}\Psi_t\cdot\Delta\Psi dxdy\right|dt\\
&\le C\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}(|\nabla\Psi|^2+\bar{u}_x\varphi^2+|\nabla\phi|^2+\bar{u}_x\phi^2+|\nabla^2\Psi|^2+|\nabla\tilde{c}|^2+\bar{u}_{xx}^2)dxdy\right|dt\\
&\le C\left(\|(\phi_0,\Psi_0)\|_2^2+\alpha^{\frac14}+\alpha^{\frac23}\right)<+\infty, \qquad (4.125)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\|\nabla\tilde{c}\|^2\right|dt\\
&=2\int_0^{+\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}\nabla\tilde{c}\cdot\nabla\tilde{c}_t dxdy\right|dt\le\int_0^{+\infty}\left(\|\nabla\tilde{c}\|^2+\|\nabla\tilde{c}_t\|^2\right)dt. \qquad (4.126)
\end{aligned}
$$

Referring to (3.7), we know that $\|\nabla\tilde{c}\|^2$ is time-integrable. It suffices to prove $\|\nabla\tilde{c}_t\|^2$ is time-integrable. Taking derivative on the third equation of (3.1) with respect to t, one has

$$
\Delta\tilde{c}_t+\bar{c}_{xxt}+a\phi_t-b\tilde{c}_t=0. \qquad (4.127)
$$

Multiplying (4.127) by $-\tilde{c}_t$ and then integrating the resulting equation over $\mathbb{T}\times\mathbb{R}$, we have

$$
-\int_{\mathbb{T}}\int_{\mathbb{R}}\tilde{c}_t\Delta\tilde{c}_t dxdy-\int_{\mathbb{T}}\int_{\mathbb{R}}\bar{c}_{xxt}\tilde{c}_t dxdy-a\int_{\mathbb{T}}\int_{\mathbb{R}}\phi_t\tilde{c}_t dxdy+b\int_{\mathbb{T}}\int_{\mathbb{R}}\tilde{c}_t^2 dxdy=0. \qquad (4.128)
$$

It follows from integration by parts and Cauchy's inequality that

$$
\|\nabla\tilde{c}_t\|^2+\|\tilde{c}_t\|^2\le\eta\|\tilde{c}_t\|^2+C_\eta\|\bar{c}_{xxt}\|^2+C_\eta\|\phi_t\|^2, \qquad (4.129)
$$

which implies that

$$
\|\nabla\tilde{c}_t\|^2\le C\left(\|\bar{c}_{xxt}\|^2+\|\phi_t\|^2\right). \qquad (4.130)
$$

By Lemma 2.2, we have

$$\int_0^{+\infty} \|\bar{c}_{xxt}\|^2 dt \le C \int_0^{+\infty} (1+t)^{-2} dt < +\infty. \tag{4.131}$$

As to $\int_0^{+\infty} \|\phi_t\|^2 dt$, by the first equation of (3.1), we have

$$\begin{aligned}
\int_0^{+\infty} \|\phi_t\|^2 dt &= \int_0^{+\infty} \|(\rho \operatorname{div}\Psi + \psi\phi_y + u\phi_x + \bar{\rho}_x\varphi + \bar{u}_x\phi)\|^2 dt \\
&\le C \int_0^{+\infty} \int\limits_{\mathbb{T}} \int\limits_{\mathbb{R}} \left({\varphi_x}^2 + {\psi_y}^2 + {\phi_y}^2 + {\phi_x}^2 + \bar{u}_x\varphi^2 + \bar{u}_x\phi^2\right) dxdydt \\
&\le C\left(\|(\phi_0,\Psi_0)\|_2^2 + \alpha^{\frac{1}{4}}\right) < +\infty.
\end{aligned} \tag{4.132}$$

Combining (4.130), (4.131) and (4.132), we obtain that

$$\int_0^{+\infty} \left|\frac{d}{dt}\|\nabla\tilde{c}\|^2\right| dt \le \int_0^{+\infty} \left(\|\nabla\tilde{c}\|^2 + \|\nabla\tilde{c}_t\|^2\right) dt < +\infty. \tag{4.133}$$

Now we get (4.123). Furthermore, we end the proof of (4.120). So we have

$$\lim_{t\to+\infty} \sup_{(x,y)\in\mathbb{R}\times\mathbb{T}} \|(\phi,\Psi,\tilde{c})\| = 0. \tag{4.134}$$

Thus, combining (4.134) and Lemma 2.2(3), we obtain the desired time-asymptotic behavior of the solution that

$$\lim_{t\to+\infty} \sup_{(x,y)\in\mathbb{R}\times\mathbb{T}} \left|(\rho,u,v,c)(t,x,y) - (\rho^r,u^r,0,c^r)\left(\frac{x}{t}\right)\right| = 0. \tag{4.135}$$

Acknowledgments. Q.Q. Liu was supported by the National Natural Science Foundation of China (No. 12071153), Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515012360, and No. 2020B1515310015) and Guangzhou Municipal Science and Technology Project (No. 202102021137).

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