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DETERMINATION FOR THE 2D INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN LIPSCHITZ DOMAIN*

XIN-GUANG YANG†, MENG HU‡, TO FU MA§, AND JINYUN YUAN¶

Abstract. The number of determining modes is estimated for the 2D Navier-Stokes equations subject to an inhomogeneous boundary condition in Lipschitz domains by using an appropriate set of points in the configuration space to represent the flow by virtue of the Grashof number and the measure of Lipschitz boundary based on a stream function and some delicate estimates. The asymptotic determination via finite functionals for 2D autonomous Navier-Stokes equations in Lipschitz domains has been derived for the trajectories inside global attractor with finite Hausdorff dimension, which leads to this fluid flow reducing to a functional ordinary differential equation.

Keywords. Navier-Stokes equations; Lipschitz domain; determining modes; Grashof number.

AMS subject classifications. 35Q30; 35B40; 35B41; 76D03; 76D05.

1. Introduction

The two-dimensional incompressible Navier-Stokes equations constitute the conservation law of fluid flows such as water, see [14] for the physical background in hydrodynamics. The global weak and strong well-posedness in mathematical theory can be seen in [7, 12, 15, 16, 21, 23, 25]. For the purpose of a better understanding of turbulence and chaos in view of mathematical theory, the infinite-dimensional dynamics of the 2D Navier-Stokes equations in bounded smooth domains or unbounded domains, with the Poincaré inequality holding, has attracted considerable attention from 1980s, see [2, 7, 13, 17, 18, 20, 21, 24]. To the best of our knowledge, the existence of invariant manifold for 2D Navier-Stokes equations is still unsolved, which needs to deal with the spectrum of nonlinear operator or search a new way to overcome the difficulty of convective term. Moreover, the global well-posedness and stability of the 2D Navier-Stokes equations in non-smooth manifold is another interesting topic to describe the real fluid flow. Our main interest here is to determining modes of 2D incompressible Navier-Stokes equations subject to inhomogeneous boundary condition for a bounded Lipschitz domain (non-smooth case), which is based on the known results (such as the determining modes and nodes, even finite volume elements) in bounded domain with at least C^2-boundary, see [3, 6–10].

Let $\Omega\subset\mathbb{R}^d$ be a bounded set. The domain Ω is Lipschitz if the boundary $\partial\Omega$ can be covered by finite balls $B_i=B(Q_i,r_0)$ centered at the points $Q_i\in\partial\Omega$ such that for each ball B_i, there exist a rectangular coordinate system and a Lipschitz continuous function $\Psi:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying

$$B(Q_i,r_0)\cap\Omega=\{(x_1,x_2,\cdots,x_d)|\Psi(x_1,x_2,\cdots,x_{d-1})-\beta<x_d<\Psi(x_1,x_2,\cdots,x_{d-1})+\beta\}$$

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†Department of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, Henan, P.R. China (yangxinguang@hotmail.com).

‡Department of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, Henan, P.R. China (huhumeng75@163.com).

§Department of Mathematics, University of Brasília, Brasília, DF 70910-900, Brazil (matofu@mat.unb.br).

¶School of Computer Science and Technology, Dongguan University of Technology, Dongguan, P.R. China; Department of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, Henan, P.R. China (yuanjy@gmail.com).

and

$$B(Q_i,r_0)\cap\partial\Omega=\{(x_1,\cdots,x_{d-1},x_d)|x_d=\Psi(x_1,\cdots,x_{d-1})\}$$

with $|(x_1,\cdots,x_{d-1})|<r_0$ for positive constants $r_0>0$, $\beta>0$, the details can be found in [19, 23]. Suppose that $\Omega\subset\mathbb{R}^2$ is a Lipschitz domain occupied by the fluid. Consider the 2D non-autonomous incompressible Navier-Stokes equations in Ω subject to inhomogeneous boundary condition and initial data as

$$\begin{cases} \frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f(t,x),\ (x,t)\in\Omega_0,\\ \mathrm{div}u=0,\ (x,t)\in\Omega_0,\\ u(t,x)|_{\partial\Omega}=\varphi,\quad \varphi\cdot\overrightarrow{n}=0,\ (x,t)\in\partial\Omega_0,\\ u(t=0,x)=u_0(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega_0=\Omega\times(0,+\infty)$, $\partial\Omega_0=\partial\Omega\times(0,+\infty)$. The functions $u=(u_1(t,x),u_2(t,x))$ and p are the unknown velocity field and pressure of the fluid, respectively. The positive constant ν is the kinematic viscosity of the fluid, $f=f(t,x)$ is a non-autonomous external force, $\varphi\in L^\infty(\partial\Omega)$ and $\varphi\cdot\overrightarrow{n}=0$ a.e. on $\partial\Omega$, where $\overrightarrow{n}$ is the unit outward normal to $\partial\Omega$.

The determination of 2D incompressible Navier-Stokes equations in non-smooth manifolds has never been investigated, although this topic is important for understanding the inertial manifold and turbulence of fluid flow, and is the main motivation of this work. For the 2D incompressible Navier-Stokes fluid flow driven by inhomogeneous boundary conditions in regular domains, the finite dimensional global attractor was obtained under the assumptions $\partial\Omega\in C^3$ and $|\nabla\varphi|\in L^\infty(\partial\Omega)$ in [16–18] based on stream function to deal with the inhomogeneous boundary condition. For the Lipschitz domain, the finite fractal dimensional universal attractor was obtained based on estimates of the Stokes problem by means of the stream function from [1, 4, 22]. Inspired by [1, 17, 18, 27, 28, 30], by means of the stream function ψ in Lipschitz domain Ω satisfying

$$\begin{cases} \mathrm{div}\psi=0 & \text{for}\quad x\in\Omega,\\ \psi=\varphi & \text{on}\quad \partial\Omega, \end{cases} \tag{1.2}$$

with $v=u-\psi$, (1.1) can be transformed into the following equivalent homogeneous boundary problem

$$\begin{cases} \frac{\partial v}{\partial t}-\nu\Delta v+(v\cdot\nabla)v+(v\cdot\nabla)\psi+(\psi\cdot\nabla)v+\nabla(p-\nu q\eta_\varepsilon) & \\ =f(t,x)+\nu F-(\psi\cdot\nabla)\psi, & (x,t)\in\Omega_0,\\ \mathrm{div}v=0, & (x,t)\in\Omega_0,\\ v=0, & (x,t)\in\partial\Omega_0,\\ v(t=0,x)=v_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where F and $q\eta_\varepsilon$ are generated by the stream function and defined in Section 2.2. Based on the estimate of stream function ψ constructed by [1], using the weak and improved version of generalized Gronwall's inequality (Lemma 2.2) in [6, 9], respectively, we shall here prove the determining modes for weak and regular solutions of (1.3) with the following features:

(a) *For non-autonomous system* (1.1) *defined on Lipschitz domain*, the regularity of global solutions for its equivalent form (1.3) is not good enough to obtain strong solution as in generic smooth domain, i.e., the solution belongs to V (see definition in Section 2), which leads to some trouble for estimates of trilinear operators. Let A be the Stokes operator. It follows from the uniform boundedness of weak solution in $D(A^{1/4})$ from [1] and estimates of Galerkin's projection in Hilbert space that the decay for tail term of velocity field holds in H and $D(A^{1/4})$, which implies the determination for the problem.

Compared with results of 2D Navier-Stokes equations in generic smooth domains, the estimate of determining modes for Lipschitz domain depends on not only Grashof number ($m \geq cGr^2$) but also the value of stream function ψ and the measure for boundary $\partial\Omega$ although it is compact for bounded Lipschitz manifold. The difficulty here lies in the estimates of trilinear operators containing ψ as $b(\psi,\cdot,\cdot)$ and $b(\cdot,\psi,\cdot)$.

The determination of problem (1.3) provides a theoretical foundation for the computational hydrodynamics, the existence of inertial and approximated inertial manifolds. Moreover, the obtained results also state the complexity of fluid flow in Lipschitz domain by using the Grashof number.

(b) The global well-posedness and dynamics for *2D autonomous Navier-Stokes system and corresponding Stokes equations on Lipschitz domain* were investigated by [1, 4, 5, 22] and references therein, especially the existence of finite dimensional global (universal) attractor. One way to understand complexity of 2D hydrodynamic flow is the Hausdorff/fractal dimension via box-counting theorem, whose finite dimensional property can be interpreted by Mané projection theorem. The other one is the existence of inertial manifold for 2D Navier-Stokes equations defined on smooth domain, even the Lipschitz case, which needs to verify invariant smooth mapping at least Lipschitz, exponential tracking property and spectral gap condition. However, the inertial manifold for (1.1) remains open. Originating from [11] and [26], the asymptotic determination and Lyapunov-Schmitt reduction of trajectories inside global attractor for problem (1.1) have been derived under some constraint on Grashof number and m-modes, which is also true for bounded smooth domains. Our results also give a partly positive answer to Problem 6.1 in [11], and present the existence of Lipschitz mapping for reduced system, i.e., the equivalent equation for autonomous problem (1.1) can be uniquely determined by a functional ordinary differential equation by Mané projection theorem. However, whether the Takens delayed embedding theorem can be used to describe asymptotic determination of dynamic systems for (1.1) is still unknown as in [11].

The rest of this paper is organized as follows. In Section 2, some preliminaries are given which will be used in what follows. The existence and uniqueness of solutions for the discussed problem are derived in Section 3. The determination of global solutions is studied in Section 4. Finally some future work and conclusion are given in the last section.

2. Functional setting and preliminaries

2.1. Preliminaries.

• **Functional spaces:** Let $E := \{u | u \in (C_0^\infty(\Omega))^2, \mathrm{div} u = 0\}$ and H be the closure of E in the $(L^2(\Omega))^2$ topology. Denote $\|\cdot\|_2$ and $(\cdot,\cdot)$ as the norm and inner product in H respectively, i.e.,

$$\|u\|_2^2 = (u,u), \quad (u,v) = \sum_{j=1}^{2} \int_\Omega u_j(x)v_j(x)dx, \ \forall\ u,v \in H.$$

Assume that V is the closure of E in the $(H^1(\Omega))^2$ topology, and $\|\cdot\|$ and $((\cdot,\cdot))$ the norm and inner product in V respectively, i.e.,

$$\|u\|^2=((u,u)),\quad ((u,v))=\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx,\ \forall\ u,v\in V.$$

Clearly, $V\hookrightarrow H\equiv H'\hookrightarrow V'$, H' and V' being the dual spaces of H and V respectively, where the injections are dense and continuous. The norm $\|\cdot\|_*$ and $\langle\cdot,\cdot\rangle$ denote the norm in V' and the dual product between V and V' (or H to itself), respectively.

Let P be the Helmholz-Leray orthogonal projection from $(L^2(\Omega))^2$ to H, and $A:=-P\Delta$ as the Stokes operator. By the theory of Sturm-Liouville problem for Stokes's operator with non-slip boundary, there exists an eigenvalue sequence $\{\lambda_j\}_{j=1}^\infty$ $(0<\lambda_1\le\lambda_2\le\cdots)$ for corresponding orthonormal eigenfunctions $\{\omega_j\}_{j=1}^\infty\subset H$ of A. Then for every $u=\sum_{j=1}^{\infty}(u,\omega_j)\omega_j$ in H, define

$$A^s u=\sum_{j=1}^{\infty}\lambda_j^s(u,\omega_j)\omega_j \tag{2.1}$$

for $s\in(0,1)$ with the domain

$$V^{2s}=D(A^s)=\Big\{A^s u\in H,\ \sum_{j=1}^{\infty}\lambda_j^{2s}|(u,\omega_j)|^2<+\infty\Big\}. \tag{2.2}$$

Here the norm and inner product for $D(A^s)$ are defined by $\|A^\sigma u\|_2=\Big(\sum_{j=1}^{\infty}\lambda_j^{2\sigma}|(u,\omega_j)|^2\Big)^{1/2}$ and $(u,v)_{D(A^s)}=\sum_{j=1}^{\infty}\lambda_j^{2s}(u,\omega_j)(v,\omega_j)$, respectively for $u,v\in D(A^s)$. Especially, $H=V^0$ and $V=D(A^{1/2})$.

- **The bilinear and trilinear operators:**

We define the bilinear and trilinear operators as follows (see [24]).

$$B(u,v):=P((u\cdot\nabla)v),\quad \forall\ u,v\in V,$$

$$b(u,v,w)=(B(u,v),w)=\sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j dx,$$

and denote $B(u)=B(u,u)$ for short, where $B(u,v)$ is a bilinear continuous operator from V to V'. In addition, $b(u,v,w)$ satisfies

$$\begin{cases} b(u,v,v) &= 0, & \forall\ u,v,w\in V,\\ b(u,v,w) &= -b(u,w,v), & \forall\ u,v,w\in V,\\ |b(u,v,w)| &\le C_1\|u\|_2^{\frac12}\|u\|^{\frac12}\|v\|\|w\|_2^{\frac12}\|w\|^{\frac12}, & \forall\ u\in V,\ v\in V,\ w\in V. \end{cases} \tag{2.3}$$

By using the Agmon-Ladyzhenskaya inequality in [12], we can choose C_1 as $2^{1/4}$ in two dimensional bounded domain.

• **Some lemmas:**

Lemma 2.1 (See [1, 4, 22]).

(1) *Hardy's inequality:*

$$\int_\Omega \frac{|u(x)|^2}{[dist(x,\partial\Omega)]^2}dx \leq C\int_\Omega |\nabla u(x)|^2 dx,\ \forall u\in V. \tag{2.4}$$

(2) *For* $u\in D(A^{1/4})$*, there exists a constant* C *such that*

$$\|u\|_{L^4(\Omega)} \leq C\|A^{1/4}u\|_2 \tag{2.5}$$

and

$$\int_\Omega \frac{|u(x)|^2}{dist(x,\partial\Omega)}dx \leq C\|A^{1/4}u\|_2^2 \tag{2.6}$$

hold, where C *is independent of the domain* Ω*.*

Lemma 2.2 (See [6, 9]). *Let* $\alpha=\alpha(t)$ *and* $\beta=\beta(t)$ *be locally integrable real-valued functions defined on* $[0,+\infty)$ *satisfying the following conditions*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)d\tau = \gamma > 0, \tag{2.7}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)d\tau = \Gamma < \infty, \tag{2.8}$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau = 0, \tag{2.9}$$

for some $T>0$*, where* $\alpha^-(t)=\max\{-\alpha(t),0\}$*,* $\beta^+(t)=\max\{\beta(t),0\}$*. If* $\xi=\xi(t)$ *is an absolutely continuous non-negative function on* $[0,+\infty)$ *satisfying the inequality*

$$\frac{d\xi}{dt}+\alpha\xi \leq \beta \tag{2.10}$$

on $[0,\infty)$*, then* $\lim_{t\to 0}\xi(t)=0$*.*

2.2. The stream function for the Stokes problem in Lipschitz domain. Originated by the stream function in bounded domain with C^2-boundary in [17] and [18], for the Lipschitz domain given in [1, 4], we can construct a stream function ψ, solvers of the Stokes system in Lipschitz domain

$$\begin{cases} -\nu\triangle u+\nabla q=0, \text{ in } \Omega, \\ \mathrm{div} u=0, \text{ in } \Omega, \\ u=\varphi \text{ a.e. on } \partial\Omega \text{ in the sense of nontangential convergence.} \end{cases} \tag{2.11}$$

Assume that $u=(u_1,u_2)$ is the solution of problem (2.11) with $\varphi\in L^\infty(\partial\Omega)$ and $\varphi\cdot\overrightarrow{n}=0$. It follows from the incompressible condition and Green's theorem that $u=(\frac{\partial g}{\partial x_2},-\frac{\partial g}{\partial x_1})$, where $g(x)=\int_P^x(-u_2,u_2)\cdot T ds$ for fixed $P\in\partial\Omega$, T is the unit tangent vector to the path from P to $x=(x_1,x_2)$.

Assume that $\varepsilon\in(0,C\text{diam}(\Omega))$ is an arbitrary small parameter. Let $\eta_\varepsilon\in C_0^\infty(\mathbb{R}^2)$ satisfy

$$\begin{cases}\eta_\varepsilon=1, \text{ in } \{x\in\mathbb{R}^2|\text{dist}(x,\partial\Omega)\le C_1\varepsilon\},\\ \eta_\varepsilon=0, \text{ in } \{x\in\mathbb{R}^2|\text{dist}(x,\partial\Omega)\ge C_2\varepsilon\},\end{cases}$$

and $0\le\eta_\varepsilon\le 1$ for the rest region. Then ψ can be constructed as

$$\psi=\psi_\varepsilon=(\frac{\partial}{\partial x_2}(g\eta_\varepsilon),-\frac{\partial}{\partial x_1}(g\eta_\varepsilon)).$$

It enjoys the same nonhomogeneous boundary conditions as in (1.1), namely,

$$\begin{cases}\text{div}\psi=0,\ x\in\Omega;\ \psi=u,\ x\in\{x\in\Omega;\ \text{dist}(x,\partial\Omega)<C_1'\varepsilon\},\\ \psi=\varphi \text{ on } \partial\Omega \text{ in the sense of nontangential convergence.}\end{cases}$$

Moreover, the stream function ψ has the compact support property

$$\text{Supp}\psi\subset\{x\in\bar{\Omega};\ \text{dist}(x,\partial\Omega)<C_2'\varepsilon\}$$

and satisfies the estimates

$$\begin{aligned}&\sup_{x\in\Omega}|\psi(x)|+\sup_{x\in\Omega}|\nabla\psi(x)|\text{dist}(x,\partial\Omega)\le C\|\varphi\|_{L^\infty(\partial\Omega)},\\ &\||\nabla\psi|\text{dist}(\cdot,\partial\Omega)^{1-\frac{1}{p}}\|_{L^p(\Omega)}\le C\|\varphi\|_{L^p(\partial\Omega)},\ 2\le p\le\infty,\end{aligned}$$

which results in

$$\|\psi\|_{L^\infty(\Omega)}\le C\|\varphi\|_{L^\infty(\partial\Omega)}.$$

In addition, the stream function ψ can be represented by

$$\Delta\psi=\nabla(q\eta_\varepsilon)+F, \tag{2.12}$$

where

$$\begin{aligned}&\text{Supp}\psi\subset\{x\in\Omega;\ C_1'\varepsilon\le\text{dist}(x,\partial\Omega)<C_2'\varepsilon\}.\\ &\|F\|_2\le C/\varepsilon^{\frac{3}{2}}\|\varphi\|_2,\ \nabla q=\triangle u.\end{aligned}$$

3. Main results

3.1. Existence, uniqueness of global weak solution. With $v_0\in H$, by applying the Helmholz-Leray projector P_L to (1.3), the equivalent abstract form of (1.3) can be given by

$$\begin{cases}\frac{\partial v}{\partial t}+\nu Av+B(v,v)+B(v,\psi)+B(\psi,v)=P_L(f(t,x)+\nu F)-B(\psi),\\ v|_{\partial\Omega}=0,\\ v(0)=v_0,\end{cases} \tag{3.1}$$

which is the problem for our study.

The existence of a global weak solution can be stated as follows.

THEOREM 3.1 (See [1]). *Suppose that* $v_0\in H$, *and* $f\in L^2(0,T;H)$. *Then, problem* (3.1) *possesses a global weak solution satisfying*

$$v\in C(0,T;H)\cap L^2(0,T;V),\quad \frac{dv}{dt}\in L^2(0,T;V'),\quad v(0)=v_0 \tag{3.2}$$

and

$$\begin{aligned}&\frac{d}{dt}(v(t),\mathfrak{z})+\nu((v(t),\mathfrak{z}))+b(v(t),v(t),\mathfrak{z})+b(v(t),\psi,\mathfrak{z})+b(\psi,v(t),\mathfrak{z})\\ =&(f(t)+\nu F,\mathfrak{z})-b(\psi,\psi,\mathfrak{z})\end{aligned} \tag{3.3}$$

for a.e. $t\in[0,T]$ *and* $\mathfrak{z}\in V$. *In addition, the regularity* $v\in L^2(0,T;D(A^{\frac{1}{4}}))$ *comes from* $\varphi\in L^\infty(\partial\Omega)$.

DEFINITION 3.1 (Weak Hadamard solution of (1.1)). *Assume that* $u_0\in H$ *and* $f\in L^2(0,T;H)$. *Let* $\varphi\in L^\infty(\partial\Omega)$ *and* $\varphi\cdot\overrightarrow{n}=0$ *on* $\partial\Omega$. *Then, the function* u *is called as a weak solution of* (1.1) *provided that*

(a) $u\in C([0,T];H)\cap L^2(0,T;V)$ *and* $\frac{du}{dt}\in L^2(0,T;V')$ *with* $u(0)=u_0$.

(b) *The equation*

$$\frac{d}{dt}\langle u,\mathfrak{V}\rangle+\nu a(u,\mathfrak{V})+b(u,u,\mathfrak{V})=\langle f,\mathfrak{V}\rangle$$

holds for every $\mathfrak{V}\in E$ *as distribution on* $(0,T)$.

(c) *There exist functions* $\psi\in C^2(\Omega)\cap L^\infty(\overline{\Omega})$, $q\in C^1(\Omega)$ *and* $g\in L^2(\Omega)$ *such that*

$$\begin{cases}\nu\triangle\psi=\nabla q+g, \ \text{in } \Omega,\\ \text{div}u=0, \ \text{in } \Omega,\\ \psi=\varphi \ \text{a.e. on } \partial\Omega \text{ in the sense of nontangential convergence}\end{cases}$$

holds.

THEOREM 3.2 (See [1, 30]). *Suppose that* $u_0\in H$, $f\in L^2(0,T;H)$ *and* $\psi\in C^2(\Omega)\cap L^\infty(\overline{\Omega})$. *Then problem* (1.1) *possesses at least a global weak solution satisfying*

$$u\in C(0,T;H),\quad u-\psi\in L^2(0,T;V),\quad u(0)=u_0 \tag{3.4}$$

and Definition 3.1 for a.e. $t\in[0,T]$. *Moreover, the weak solution is unique when* $\varphi\in L^\infty(\partial\Omega)$ *with* $\varphi\cdot\overrightarrow{n}=0$ *on* $\partial\Omega$.

THEOREM 3.3. *Assume that* $f\in L^2_{loc}(\mathbb{R};H)$ *and* $v_0\in H$. *Then, the solution* v *to problem* (3.1) *satisfies that*

$$\|v(t)\|_2^2\le\|v_0\|_2^2e^{-\nu t}+\int_0^t K_0^2ds \tag{3.5}$$

and

$$\int_0^t\|v(s)\|^2ds\le\frac{2}{\nu}\Big(\|v_0\|_2^2e^{-\nu t}+\int_0^t K_0^2ds\Big), \tag{3.6}$$

where

$$K_0^2=\frac{4}{\nu}\Big[\frac{\|f\|_2^2}{\lambda_1}+\frac{C\nu^2}{\varepsilon}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}+C\varepsilon\ |\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]. \tag{3.7}$$

Proof. The estimates (3.5) and (3.6) follow from Hardy's inequality and some delicate estimates. The details of a similar technique can be found in [28, 29], which is omitted here. ∎

3.2. Determination of solutions for non-autonomous system.

• **The determining modes of weak solution.** Considering the Sturm-Liouville problem $Av=\lambda v$ with Dirichlet boundary condition and the eigenfunctions of H consisting of $\{w_1,\, w_2,\cdots,\, w_m,\cdots\}$, we can see that $\{w_m\}_{m\geq 1}$ are also the eigenfunctions of the Stokes operator A with the corresponding eigenvalues $0<\lambda_1\leq\lambda_2\leq\cdots$, i.e., $Aw_m=\lambda_m w_m$. Let $H_m=span\{w_1,w_2,\cdots,w_m\}$, $P_m:H\to H_m$ be the Galerkin orthogonal projection, and denote $Q_m=I-P_m$ as the orthogonal complement operator of P_m.

DEFINITION 3.2. *Suppose that $\tilde{v}$ and v are two solvers of (3.1), that is,*

$$\begin{cases} \frac{dv}{dt}+\nu Av+B(v,v)+B(v,\psi)+B(\psi,v)=P_L(f(t,x)+\nu F)-B(\psi,\psi), \\ v(t=0,x)=v_0 \end{cases} \tag{3.8}$$

and

$$\begin{cases} \frac{d\tilde{v}}{dt}+\nu A\tilde{v}+B(\tilde{v},\tilde{v})+B(\tilde{v},\psi)+B(\psi,\tilde{v})=P_L(g(t,x)+\nu F)-B(\psi,\psi), \\ \tilde{v}(t=0,x)=\tilde{v}_0, \end{cases} \tag{3.9}$$

respectively, here $f(t,x)$ and $g(t,x)$ are given external forces in $L^\infty_{loc}(\mathbb{R}^+;H)$, the stream function ψ and F are defined in Section 2.2.

For

$$\lim_{t\to\infty}\|f(t,x)-g(t,x)\|_2=0 \tag{3.10}$$

and

$$\lim_{t\to\infty}\|P_m v(t)-P_m\tilde{v}(t)\|_2=0, \tag{3.11}$$

the system has determination property and $\{\omega_j\}_{m=1}^\infty$ is called determination modes provided that

$$\lim_{t\to\infty}\|v(t)-\tilde{v}(t)\|_2=0. \tag{3.12}$$

THEOREM 3.4. *Suppose that $m\in\mathbb{N}$ satisfies*

$$m\geq\frac{24}{c'}G^2+\frac{6}{\nu^3c'\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big], \tag{3.13}$$

where $G=\frac{\bar{F}}{\nu^2\lambda_1}$ is the Grashof number with $\bar{F}=\limsup_{t\to\infty}\Big(\int_\Omega|f(x,t)|^2dx\Big)^{1/2}$ for the fluid flow and $c'>0$ is a constant depending only on the shape of Ω, φ is defined in Section 2.2. Then the first m modes are determining in the sense of Definition 3.2 for (3.1), that is, for $f(t),g(t)\in L^\infty_{loc}(\mathbb{R}^+;H)$, $v_0\in H$, the difference of two solutions as $w=v-\tilde{v}$ decays to 0 as t goes to infinity.

Proof. See, e.g., Section 4.1 for more details. ∎

By the equivalent form of (1.1) as (3.1), we can obtain the similar result as follows.

THEOREM 3.5. *Suppose that $m\in\mathbb{N}$ satisfies (3.13). Then the first m modes are determining in the sense of Definition 3.2 with u instead of v for the two dimensional incompressible Navier-Stokes Equation (1.1) on Lipschitz domain with inhomogeneous* *non-slip boundary condition, i.e., for* $f(t), g(t) \in L^\infty_{loc}(\mathbb{R}; H)$, $u_0 \in H$, *the difference of two solutions for* (1.1) *as* $u - \tilde{u}$ *decays to* 0 *as* t *goes to infinity.*

Proof. Noting that $u = v - \psi$, combining the Stokes problem and the estimates in Section 2.2 with Theorem 3.4, the result can be proved. ∎

REMARK 3.1. The uniform boundedness of the external force here does not require more regularity, which only means the average. It is worth to point out that the formula (4.15) does not imply the convergence of $\beta(t)$ as $t \to \infty$, but its average.

- **The determining modes of regular solution.**

DEFINITION 3.3. *For* $f \in L^2_{loc}(\mathbb{R}; H)$, $v_0 \in D(A^{\frac{1}{4}})$, *the function* $v(t,x)$ *is called as regular solution if it is the global weak solution and bounded in* $C([0,T]; D(A^{1/4})) \cap L^2(0,T; D(A^{3/4}))$ *for arbitrary* $T > 0$.

Let H be Hilbert space. Assume that $D(A^\alpha)$ is the subspace of H with the special basis $\{w_1, \ldots\}$ given in [16]. Denote $H_m = span\{w_1, w_2, \cdots, w_m\}$, subspace of $D(A^\alpha)$. Suppose that $P_m : H \to H_m$ is the Galerkin orthogonal projection, and then $Q_m = I - P_m$ the orthogonal complement operator of P_m.

DEFINITION 3.4. *Consider two solvers* $\tilde{v}$ *and* v *of* (3.1) *defined by* (3.8) *and* (3.9), *respectively, here* $f(t,x)$ *and* $g(t,x)$ *are given external forces in* $L^\infty_{loc}(\mathbb{R}^+; H)$. *Let the stream function* ψ *and* F *be defined in Section 2.2. For*

$$\lim_{t\to\infty} \|f(t,x) - g(t,x)\|_2 = 0 \tag{3.14}$$

and

$$\lim_{t\to\infty} \|P_m v(t) - P_m \tilde{v}(t)\|_{D(A^{1/4})} = 0, \tag{3.15}$$

the set of modes $\{\omega_j\}_{m=1}^\infty$ *is called determination modes provided that*

$$\lim_{t\to\infty} \|v(t) - \tilde{v}(t)\|_{D(A^{1/4})} = 0. \tag{3.16}$$

THEOREM 3.6. *Suppose that* $m \in \mathbb{N}$ *satisfies*

$$\begin{aligned}
\frac{\lambda_{m+1}}{\lambda_1} \geq m \geq \frac{81}{\tilde{c}} \Big[12 + 4\big(\frac{C}{\nu\lambda_1^{\frac{2}{3}}} + 2C\|\varphi\|^2_{L^\infty(\partial\Omega)}\big)\Big] G^4 \\
+ \Big(\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2\tilde{c}} + \frac{9\varepsilon^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2\tilde{c}}\Big)^2 \\
+ \frac{81}{2\nu^6\tilde{c}} \Big(\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}} + 4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big) \Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big] \\
+ \Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)} + \frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\Big)^2,
\end{aligned} \tag{3.17}$$

where $G = \frac{F}{\nu^2\lambda_1}$ *is the generalized Grashof number for the fluid flow and* $c' > 0$ *is a constant depending only on the shape of* Ω, φ *is defined in Section 2.2. Then, the first* m *modes are determining in the sense of Definition 3.4 for* (3.1), *that is, for* $f(t), g(t) \in L^\infty_{loc}(\mathbb{R}^+; H)$, $v_0 \in H$, *the difference of two solutions as* $w = v - \tilde{v}$ *decays to* 0 *as* t *goes to infinity.*

Proof. See the detailed proof in Section 4.2. □

The similar result comes from the equivalence of (1.1) and (3.1), which leads to Theorem 3.6 as follows.

THEOREM 3.7. *Suppose that $m\in\mathbb{N}$ satisfies (3.17). Then the first m modes are determining for the two dimensional incompressible Navier-Stokes Equation (1.1) on Lipschitz domain with inhomogeneous non-slip boundary condition.*

Proof. Using the similar technique of the proof of Theorem 3.6, and combining some estimate of stream function, the results can be derived directly, here we omit the details. □

3.3. The asymptotic determination for (1.1) with autonomous external force $f(x)$. Considering the 2D autonomous Navir-Stokes equations defined on Lipschitz domain as its equivalent form as

$$\begin{cases} \frac{\partial v}{\partial t}+\nu Av+F(v)=P_L(f(x)+\nu F)-B(\psi),\\ v|_{\partial\Omega}=0,\\ v(0)=v_0 \end{cases} \tag{3.18}$$

with $\tilde{F}(v)=B(v,v)+B(v,\psi)+B(\psi,v)$, we will prove $\tilde{F}(\cdot)$ is globally Lipschitz in H via the restriction on Grashof number, and then show the asymptotic determination of trajectories inside global attractor $\mathcal{A}$ and present the Lyapunov-Schmidt reduction of equivalent autonomous system (3.18) of (1.1), which give a positive answer partly for the open problem in [11].

• **Existence of finite dimensional global attractor for autonomous problem (1.1).** Based on the well-posedness of (1.1) with autonomous external force $f(x)\in H$, Brown, Perry and Shen [1] presented the existence of global attractor with finite dimension as following.

THEOREM 3.8. *Suppose that the hypothesis in Theorem 3.2 is satisfied, and $f(x)\in H$, then the dynamic system $(S(t),H)$ generalized by global weak solution possesses a universal attractor $\mathcal{A}$ with finite fractal dimension as $dim_F\mathcal{A}\leq C_1Gr+C_2Re^{3/2}+1$ with the Grashof and Reynolds numbers $Gr=\frac{\|f\|_2}{\nu^2\lambda_1}$, $Re=\frac{\|\phi\|_{L^\infty(\partial\Omega)}}{\nu\lambda^{1/2}}$ respectively for $C_1,C_2>0$.*

Proof. See, e.g., Brown, Perry and Shen [1]. □

REMARK 3.2. Theorem 3.8 implies there exists a bounded absorbing ball $\mathcal{B}$ with a radius ρ such that there exists a time $\tilde{T}>0$, all trajectories inside $\mathcal{A}$ satisfying $\|v(t)\|_2\leq\rho$ for $t\geq\tilde{T}$.

• **Asymptotic determination of trajectories for autonomous system.** Let $\mathcal{F}=\{F_1,\cdots,F_m\}$ be linear functional system generated by corresponding Fourier modes $F_i=(v,w_i)$ for $v=\sum_{i=1}^{\infty}(v,w_i)w_i$, which is possibly nonlinear, we shall show the asymptotic determination for our problem if the Grashof number satisfies suitable assumption as following.

DEFINITION 3.5. *Let $\tilde{v}$ and v be two trajectories inside global attractor $\mathcal{A}$ generated by the autonomous system (3.1) with $f=f(x)$. The system $\mathcal{F}$ with $F_k:H\to\mathbb{R}$ for all $k=1,\cdots,m$ is called asymptotically determining for the dynamic system $(S(t),H)$ for* *autonomous system* (3.1) *if*

$$F_k(v)-F_k(\tilde{v})\to 0, \quad as \quad t\to\infty$$

implies

$$\lim_{t\to\infty}\|v-\tilde{v}\|_2=0.$$

Next we will prove the asymptotic determination of system $\mathcal{F}$ if m is large enough.

THEOREM 3.9. *Let the assumptions of Theorem 3.8 hold, the Grashof number and m satisfy*

$$m\geq\frac{4\rho^2}{3C'}Gr^2+\frac{\rho^2}{3C'\nu^3\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]. \tag{3.19}$$

Then the system $\mathcal{F}=\{F_1,\cdots,F_m\}$ defined above is asymptotically determining for the dynamic system $(S(t),H)$ generated by autonomous system (3.1).

Proof. See, Section 4.3 for more details. □

• **Lyapunov-Schmidt reduction of autonomous system.** By using the Galerkin projection, denoting $g(x)=P_L(f(x)+\nu\tilde{F})-B(\psi)$, $g_+=P_mg$ and $g_-=Q_mg$, we can rewrite our autonomous system as following lower and higher frequency Fourier modes

$$\begin{cases}(E1)\ \dfrac{d}{dt}v_+(t)+\nu A(v_+(t))+P_m\tilde{F}(v_+(t)+v_-(t))=g_+,\\[2mm] (E2)\ \dfrac{d}{dt}v_-(t)+\nu A(v_-(t))+Q_m\tilde{F}(v_+(t)+v_-(t))=g_-,\end{cases} \tag{3.20}$$

where $v_+(t)=P_mv(t)$ and $v_-(t)=(I-P_m)v(t)=Q_mv(t)$, we will show the higher frequency modes $v_-(t)$ are uniquely determined if the corresponding lower frequency modes $v_+(t)$ are determined, i.e., the following Lyapunov-Schmidt reduction theorem.

THEOREM 3.10. *Assume that*

$$\begin{aligned}m\geq\max\Big\{&\frac{32}{C'}Gr+\frac{8}{C'\nu^3}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big],\\ &\frac{4\rho^2}{3C'}Gr^2+\frac{\rho^2}{3C'\nu^3\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\Big\}\end{aligned} \tag{3.21}$$

and hypothesis in Theorem 3.8 are true.

Then, for every $v_+(t)\in C_b(R_+,H)$, there exists a unique solution $v_-(t)\in C_b(R_+,H)$ for $(E2)$ in (3.20).

Moreover, there exists a Lipschitz continuous mapping $\mathcal{L}:C_b(R_+,H)\to C_b(R_+,H)$ satisfying Lyapunov-Schmidt reduction $\mathcal{L}(v_+(t))=v_-(t)$ in following sense

$$||\mathcal{L}(v_+^1(t))-\mathcal{L}(v_+^2(t))||_{C_b(R_+,H)}\leq\frac{C}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}e^{-\bar{\eta}(t-s)/2}\sup_{s\in\mathbb{R}_-}\|v_+^1(s)-v_+^2(s)\|_2^2, \tag{3.22}$$

where $\bar{\eta}>0$ is a constant independent of $v_+^1(t)$ and $v_+^2(t)$ in $C_b(R_+,H)$.

Proof. See, Section 4.3 for more details. □

REMARK 3.3. Using the idea from [11], based on the results in Theorem 3.10, we can see that $v_-(t)$ is uniquely determined by $\mathcal{L}$ when the trajectory v_+ inside global attractor $\mathcal{A}$ is known, which can be represented as

$$v_-(t)=\mathcal{L}((v_+)_t(0)) \tag{3.23}$$

with $(v_+)_t(0)=\mathcal{L}_0((v_+)_t)$ for $(v_+)_t\in C_b(\mathbb{R}_+,H)$, where $(v_+)_t(s)=v_+(t+s)$ is a delay. Based on (3.23), the trajectories of dynamic system $(S(t),H)$ for problem (3.18) inside $\mathcal{A}$ can be reduced to a m-order functional ordinary differential equation as

$$\frac{d}{dt}(v_+)+\nu Av_+ + P_m\tilde{F}(v_+(t)+\mathcal{L}_0(v_+)_t)=g. \tag{3.24}$$

Since the delay term is infinite, the reduced Equation (3.24) is still an infinite dimensional ODE. Furthermore, the inertial manifold for system (3.18) is still a challenging problem since the spectral gap condition is not easy to verify.

REMARK 3.4. The asymptotic determination and reduction in Theorems 3.9 and 3.10 also hold for problem (1.1), which is determined by (3.19) or (3.21), and the equilibrium as stream function for Stokes problem (2.11).

• **Further results of asymptotic determination for non-autonomous system.**

REMARK 3.5. Assume that $f(t)\in L^2_b(\mathbb{R};H)$ is pullback translation bounded for non-autonomous system (3.1) and $v_\tau\in H$, consider the existence of unique $\mathcal{D}_\nu$-family of pullback attractors $\mathcal{A}'_{\mathcal{D}_\nu}=\{\mathcal{A}'_{\mathcal{D}_\nu}(t)\}$ in H constructed in [28], then the results in Theorems 3.9 and 3.10 hold except some revision on parameters and $f(t)$ for matching with generalized Grashof number for the trajectories inside pullback attractors $\mathcal{A}'_{\mathcal{D}_\nu}$.

4. Proof of Determination

4.1. Proof of Theorem 3.4.

Proof.

Step 1: The estimate of Q_mw. Throughout this paper, we will suppose that v and $\tilde{v}$ solve problems (3.8) and (3.9), respectively, and also $\|f(t)-g(t)\|_2\to 0$ as $t\to+\infty$. Denoting $w=v-\tilde{v}$, by the assumption, we know that

$$\|P_mw(t)\|_2\to 0 \tag{4.1}$$

as $t\to 0$, our objective next is to show that $Q_mw(t)\to 0$ as well, where $w(t)$ satisfies the following equivalent functional form

$$\begin{cases} \frac{dw}{dt}+\nu Aw+B(v,w)+B(w,\tilde{v})+B(w,\psi)+B(\psi,w)=P_L(f(t,x)-g(t,x)),\\ w(t=0)=v_0-\tilde{v}_0, \end{cases} \tag{4.2}$$

It follows from inner product of (4.2) with Q_mw in H that

$$\frac{1}{2}\frac{d}{dt}\|Q_mw\|_2^2+\nu\|Q_mw\|^2+b(v,w,Q_mw)+b(w,\tilde{v},Q_mw)+b(w,\psi,Q_mw)+b(\psi,w,Q_mw)$$
$$=(f(t)-g(t),Q_mw). \tag{4.3}$$

In terms of Lemma 2.2, it is enough that we can prove (2.7)-(2.10) for $\xi(t)=\|Q_mw(t)\|_2^2$. For this purpose, it follows from (2.3) that the trilinear terms can be written as

$$\begin{aligned} b(v,w,Q_mw)&=b(v,P_mw,Q_mw),\\ b(w,\tilde{v},Q_mw)&=b(P_mw,\tilde{v},Q_mw)+b(Q_mw,\tilde{v},Q_mw),\\ b(w,\psi,Q_mw)&=b(P_mw,\psi,Q_mw)+b(Q_mw,\psi,Q_mw),\\ b(\psi,w,Q_mw)&=b(\psi,P_mw,Q_mw). \end{aligned}$$

By the Hardy and Hölder inequalities, using the estimates on ψ in Section 2.2 and choosing ε such that $C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \le \frac{\nu}{\tilde{C}}$ with $\tilde{C}$ determined by different estimate, from (2.3), we find that

$$|b(v,P_m w,Q_m w)| \le 2^{1/4}\|v\|_2^{1/2}\|v\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\|, \tag{4.4}$$

$$|b(P_m w,\tilde{v},Q_m w)| \le 2^{1/4}\|\tilde{v}\|_2^{1/2}\|\tilde{v}\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\|, \tag{4.5}$$

$$\begin{aligned}|b(Q_m w,\tilde{v},Q_m w)| &\le 2^{1/4}\|\tilde{v}\|\|Q_m w\|_2\|Q_m w\| \\ &\le \frac{\nu}{6}\|Q_m w\|^2 + \frac{3}{2^{1/2}\nu}\|Q_m w\|_2^2\|\tilde{v}\|^2,\end{aligned} \tag{4.6}$$

$$\begin{aligned}|b(P_m w,\psi,Q_m w)| &\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\int_{dist(x,\partial\Omega)\le C\varepsilon}\frac{|P_m w||Q_m w|}{dist^2(x,\partial\Omega)}dx \\ &\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|P_m w\|\|Q_m w\| \\ &\le \frac{\nu}{6}\|Q_m w\|^2 + \frac{C}{\nu}\|P_m w\|^2,\end{aligned} \tag{4.7}$$

$$\begin{aligned}|b(Q_m w,\psi,Q_m w)| &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\int_{dist(x,\partial\Omega)\le C\varepsilon}\frac{|Q_m w|^2}{dist(x,\partial\Omega)}dx \\ &\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|Q_m w\|^2 \\ &\le \frac{\nu}{6}\|Q_m w\|^2\end{aligned} \tag{4.8}$$

and

$$b(\psi,P_m w,Q_m w) \le C\|\varphi\|_{L^\infty(\partial\Omega)}\|Q_m w\|\|P_m w\|_2, \tag{4.9}$$

since $\|\psi\|_{L^\infty(\partial\Omega)} \le C\|\varphi\|_{L^\infty(\partial\Omega)}$.

Moreover, by the Cauchy-Schwarz inequality, we obtain

$$|(f(t)-g(t),Q_m w)| \le \|f(t)-g(t)\|_2\|Q_m w\|_2. \tag{4.10}$$

Combining (4.3)-(4.10), we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Q_m w\|_2^2 + \frac{\nu}{2}\|Q_m w\|^2 - \frac{3}{2^{1/2}\nu}\|Q_m w\|^2\|\tilde{v}\|^2 \\ &\le 2^{1/4}\|v\|_2^{1/2}\|v\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\| \\ &\quad + 2^{1/4}\|\tilde{v}\|_2^{1/2}\|\tilde{v}\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\| \\ &\quad + C\|\varphi\|_{L^\infty(\partial\Omega)}\|Q_m w\|\|P_m w\|_2 + \|f(t)-g(t)\|_2\|Q_m w\|_2.\end{aligned} \tag{4.11}$$

Using the Poincaré inequality $\lambda_{m+1}\|Q_m w\|_2^2 \le \|Q_m w\|^2$ and denoting that $\xi(t) = \|Q_m w(t)\|_2^2$ as in Lemma 2.2, it follows that

$$\frac{d}{dt}\xi(t) + \alpha(t)\xi(t) \le \beta(t) \tag{4.12}$$

with

$$\alpha(t) = \nu\lambda_{m+1} - \frac{6}{2^{1/2}\nu}\|\tilde{v}\|^2 \tag{4.13}$$

and

$$\begin{aligned}\beta(t) = {} & 2^{5/4}\|v\|_2^{1/2}\|v\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\| \\ & + 2^{5/4}\|\tilde{v}\|_2^{1/2}\|\tilde{v}\|^{1/2}\|P_m w\|_2^{1/2}\|P_m w\|^{1/2}\|Q_m w\| \\ & + C\|\varphi\|_{L^\infty(\partial\Omega)}\|Q_m w\|\|P_m w\|_2 + 2\|f(t) - g(t)\|_2\|Q_m w\|_2.\end{aligned} \tag{4.14}$$

Since the solutions v and $\tilde{v}$ are uniformly bounded in H and V for t bounded away from 0, and also $\|P_m w(t)\|_2 \to 0$ as $t \to \infty$, there is

$$\frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau \to 0 \quad \text{as} \quad t \to \infty. \tag{4.15}$$

We shall verify the other conditions in next steps.

Step 2: The estimate of $\frac{1}{T}\int_t^{t+T}\|v(s)\|^2 ds$. Based on the well-posedness, it follows from inner product of (3.1) with v and $b(\cdot,v,v) = 0$, that

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \nu\|v\|^2 \le |b(v,\psi,v)| + |b(\psi,\psi,v)| + |(f(t) + \nu F, v)|. \tag{4.16}$$

By virtue of the Hardy and Hölder inequalities, choosing ε such that $C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \le \frac{\nu}{C_0}$ with C_0 determined by different estimate, we derive

$$|b(v,\psi,v)| \le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|v\|^2 \le \frac{\nu}{8}\|v\|^2, \tag{4.17}$$

$$|b(\psi,\psi,v)| \le \frac{\nu}{8}\|v\|^2 + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu} \tag{4.18}$$

and

$$\nu| < F, v > | \le \frac{\nu}{8}\|v\|^2 + \frac{\nu}{\varepsilon}\|\varphi\|^2_{L^2(\partial\Omega)} \le \frac{\nu}{8}\|v\|^2 + \frac{\nu C}{\varepsilon}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}, \tag{4.19}$$

since $\|\varphi\|_{L^2(\partial\Omega)} \le C|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}$. Moreover,

$$|(f(t),v)| \le \frac{\nu}{8}\|v\|^2 + \frac{2}{\nu\lambda_1}\|f(t)\|_2^2. \tag{4.20}$$

Combining (4.16)-(4.20), we obtain

$$\frac{d}{dt}\|v\|_2^2 + \nu\|v\|^2 \le \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu} + \frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{2}{\nu\lambda_1}\|f(t)\|_2^2. \tag{4.21}$$

From the Gronwall inequality, for $0 \le t_0 \le t \le \tilde{T}$, there are estimates as follows.

$$\begin{aligned}\|v(t)\|_2^2 \le {} & \|v(t_0)\|_2^2 e^{-\nu\lambda_1(t-t_0)} + \frac{2}{\nu^2\lambda_1^2}\|f(t)\|^2_{L^\infty(0,T;H)}(1 - e^{-\nu\lambda_1(t-t_0)}) \\ & + \frac{1}{\nu\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big](1 - e^{-\nu\lambda_1(t-t_0)})\end{aligned} \tag{4.22}$$

and

$$\int_{t_0}^{t}\|v(s)\|^2ds\leq\frac{1}{\nu}\|v(t_0)\|_2^2+\frac{2}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t_0,t;H)}(t-t_0)$$
$$+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big](t-t_0) \tag{4.23}$$

hold, which implies that

$$\frac{1}{T}\int_t^{t+T}\|v(s)\|^2ds$$
$$\leq\frac{4}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t,t+T;H)}+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big] \tag{4.24}$$

holds for sufficiently large $T>0$, this is also true for $\tilde{v}$.

Step 3: The determining modes. It follows from the Grashof number

$$G=\frac{\bar{F}}{\nu^2\lambda_1}, \tag{4.25}$$

and the estimates (4.22)-(4.24) to verify (2.7) and (2.8) in Lemma 2.2 as

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau$$
$$=\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}(\nu\lambda_{m+1}-\frac{6}{2^{1/2}\nu}\|\tilde{v}(\tau)\|^2)d\tau$$
$$\geq\nu\lambda_{m+1}-\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{6}{2^{1/2}\nu}\|\tilde{v}(\tau)\|^2d\tau$$
$$\geq\nu\lambda_{m+1}-\Big(\frac{24F^2}{2^{1/2}\nu^3\lambda_1}+\frac{6}{2^{1/2}\nu^2}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\Big). \tag{4.26}$$

Then, (2.7) and (2.8) hold for m sufficiently large provided that

$$\lambda_{m+1}>\frac{24F^2}{2^{1/2}\nu^4\lambda_1}+\frac{6}{2^{1/2}\nu^3}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]. \tag{4.27}$$

Hence,

$$\xi(t)=\|Q_mw\|_2^2\to 0 \tag{4.28}$$

as $t\to\infty$.

In terms of $\lambda_{m+1}\sim c'\lambda_1 m$ with some non-dimensional constant c' for $m\to\infty$, the model is determining if

$$m\geq\frac{24}{c'}G^2+\frac{6}{\nu^3c'\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big], \tag{4.29}$$

where c' is dependent on the shape of the domain Ω only.

Therefore, the proof is completed. ∎

4.2. Proof of Theorem 3.6.

• Some estimates of regular solution.

LEMMA 4.1. *For $0\le t_0\le t\le \tilde{T}$, $f\in L^\infty_{loc}(\mathbb{R}^+;H)$ and $v_0\in D(A^{1/4})$, the following estimates*

$$\begin{aligned}\int_{t_0}^t \|v(s)\|^2 ds \le& \frac{1}{\nu}\|A^{1/4}v(t_0)\|_2^2+\frac{6}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t_0,t;H)}(t-t_0)\\ &+\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big(\frac{1}{\nu}\|v(t_0)\|_2^2+\frac{2}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t_0,t;H)}(t-t_0)\\ &+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big](t-t_0)\Big)\\ &+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big](t-t_0),\end{aligned}\tag{4.30}$$

and

$$\begin{aligned}\frac{1}{T}\int_t^{t+T}\|v(s)\|^2ds\le& \frac{1}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t,t+T;H)}\Big(12+4\Big(\frac{C}{\nu\lambda_1^{\frac{2}{3}}}+2C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big)\\ &+\frac{1}{\nu}\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\\ &+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\end{aligned}\tag{4.31}$$

hold for sufficiently large $T>0$.

Proof. Taking inner product of (3.1) with $A^{1/2}v$ in H, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{\frac14}v\|_2^2+\nu\|A^{\frac34}v\|_2^2\le& |(B(v,v),A^{1/2}v)|+|(B(v,\psi),A^{1/2}v)|+|(B(\psi,v),A^{1/2}v)|\\ &+|(B(\psi,\psi),A^{1/2}v)|+|\langle Pf,A^{1/2}v\rangle|+|\langle \nu PF,A^{1/2}v\rangle|.\end{aligned}\tag{4.32}$$

Using the same techniques as in the proof in [28], we have

$$|(B(v,v),A^{1/2}v)|\le \frac{\nu}{12}\|A^{\frac34}v\|_2^2+\frac{C}{\nu\lambda_1^{\frac23}}\|v\|^2,\tag{4.33}$$

$$|(B(v,\psi),A^{1/2}v)|\le \frac{\nu}{12}\|A^{\frac34}v\|_2^2+C\|\varphi\|^2_{L^\infty(\partial\Omega)}\|v\|^2,\tag{4.34}$$

$$|(B(\psi,v),A^{1/2}v)|\le \frac{\nu}{12}\|A^{\frac34}v\|_2^2+C\|\varphi\|^2_{L^\infty(\partial\Omega)}\|v\|^2,\tag{4.35}$$

$$|(B(\psi,\psi),A^{1/2}v)|\le \frac{\nu}{12}\|A^{\frac34}v\|_2^2+\frac{1}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|,\tag{4.36}$$

$$|\langle Pf,A^{1/2}v\rangle|\le \frac{3}{\nu\lambda_1}\|f\|_2^2+\frac{\nu}{12}\|A^{\frac34}v\|_2^2\tag{4.37}$$

and

$$|\langle \nu PF,A^{1/2}v\rangle|\le \frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{\nu}{12}\|A^{\frac34}v\|_2^2.\tag{4.38}$$

Thus, combining (4.32)–(4.38), we can conclude

$$\begin{aligned}\frac{d}{dt}\|A^{\frac14}v\|_2^2+\nu\|A^{\frac34}v\|_2^2\le&\Big(\frac{2C}{\nu\lambda_1^{\frac23}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\|v\|^2+\frac{2}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|\\ &+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{6}{\nu\lambda_1}\|f\|_2^2,\end{aligned}\tag{4.39}$$

Using the Gronwall inequality and (4.22), we obtain that for $0\le t_0\le t\le \tilde{T}$

$$\begin{aligned}\|A^{\frac14}v\|_2^2\le&\|A^{\frac14}v_0\|_2^2e^{-\nu\lambda_1(t-t_0)}+\frac{6}{\nu\lambda_1}\|f(t)\|^2_{L^\infty(0,\tilde{T};H)}(1-e^{-\nu\lambda_1(t-t_0)})\\&+\Big(\frac{2C}{\nu\lambda_1^{\frac23}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big(\frac{1}{\nu}\|v(t_0)\|_2^2+\frac{2}{\nu^2\lambda_1}\|f\|^2_{L^\infty(0,\tilde{T};H)}(t-t_0)\\&+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big](t-t_0)\Big)\\&+\frac{1}{\nu\lambda_1}\Big(\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)(1-e^{-\nu\lambda_1(t-t_0)})\end{aligned}\tag{4.40}$$

and

$$\begin{aligned}\int_{t_0}^t\|A^{3/4}v(s)\|_2^2ds\le&\frac{1}{\nu}\|A^{1/4}v(t_0)\|_2^2+\frac{6}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t_0,t;H)}(t-t_0)\\&+\Big(\frac{2C}{\nu\lambda_1^{\frac23}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big(\frac{1}{\nu}\|v(t_0)\|_2^2+\frac{2}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t_0,t;H)}(t-t_0)\\&+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big](t-t_0)\Big)\\&+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big](t-t_0)\end{aligned}\tag{4.41}$$

hold, which implies that

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\|A^{1/4}v(s)\|_2^2ds\\&\le\frac{1}{\nu^2\lambda_1}\|f\|^2_{L^\infty(t,t+T;H)}\Big(12+4\Big(\frac{C}{\nu\lambda_1^{\frac23}}+2C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big)\\&\quad+\frac{1}{\nu}\Big(\frac{2C}{\nu\lambda_1^{\frac23}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\\&\quad+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\end{aligned}\tag{4.42}$$

for sufficiently large $T>0$. This means we get the desired results. □

- **Proof of Theorem 3.6.**

Proof.

Step 1: The estimate of $Q_m w$. It is well-known that for $h\in D(A^\alpha)$ defined on bounded Lipschitz-like domain, $h(x,t)$ can be represented as $h(x,t)=\sum_{i=1}^\infty (h,\omega_i)\omega_i$ and $A^\alpha h(x,t)=\sum_{i=1}^\infty \lambda_i^\alpha (h,\omega_i)\omega_i$ with $\alpha\in\mathbb{R}$ where $\{w_m\}_{m\ge1}$ are the eigenfunctions of the Stokes operator A with the corresponding eigenvalues $0<\lambda_1\le\lambda_2\le\cdots$. Throughout rest of the paper, we shall suppose that v and $\tilde{v}$ solve problems (3.9) and (3.8) as weak solutions which are bounded in $D(A^{1/4})$, respectively, and also $\|f(t)-g(t)\|_2\to0$ as $t\to+\infty$. Setting $w=v-\tilde{v}=\sum_{i=1}^\infty (w,\omega_i)\omega_i$, by the assumption, we know that

$$\|P_m w(t)\|_{D(A^{1/4})}\to 0\tag{4.43}$$

as $t\to 0$. Next is to show that $Q_m w(t)\to 0$ as in Theorem 3.4, where $w(t)$ satisfies the following equivalent functional form

$$\begin{cases} \frac{dw}{dt}+\nu Aw+B(v,w)+B(w,\tilde{v})+B(w,\psi)+B(\psi,w)=P_L(f(t,x)-g(t,x)), \\ w(t=0)=v_0-\tilde{v}_0, \end{cases} \tag{4.44}$$

It follows from inner product of (4.2) with $A^{1/2}Q_m w=\sum_{j=m+1}^{\infty}\lambda_j^{1/2}(w,\omega_j)\omega_j$ in H that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A^{1/4}Q_m w\|_2^2+\nu\|A^{3/4}Q_m w\|_2^2+b(v,w,A^{1/2}Q_m w) \\ &\quad +b(w,\tilde{v},A^{1/2}Q_m w)+b(w,\psi,A^{1/2}Q_m w)+b(\psi,w,A^{1/2}Q_m w) \\ &=(f(t)-g(t),A^{1/2}Q_m w). \end{aligned} \tag{4.45}$$

To use Lemma 2.2 with $\xi(t)=\|A^{1/4}Q_m w(t)\|_2^2$, we rewrite the trilinear terms as

$$\begin{aligned} b(v,w,A^{1/2}Q_m w)&=b(v,Q_m w,A^{1/2}Q_m w)+b(v,P_m w,A^{1/2}Q_m w), \\ b(w,\tilde{v},A^{1/2}Q_m w)&=b(P_m w,\tilde{v},A^{1/2}Q_m w)+b(Q_m w,\tilde{v},A^{1/2}Q_m w), \\ b(w,\psi,A^{1/2}Q_m w)&=b(P_m w,\psi,A^{1/2}Q_m w)+b(Q_m w,\psi,A^{1/2}Q_m w), \\ b(\psi,w,A^{1/2}Q_m w)&=b(\psi,P_m w,A^{1/2}Q_m w)+b(\psi,Q_m w,A^{1/2}Q_m w). \end{aligned}$$

By the Hardy and Hölder inequalities, from (2.5)-(2.6), using the estimates on ψ in Section 2.2, from (2.3), choosing appropriate m such that $\lambda_{m+1}\geq 1$ by the property of λ_m, we obtain

$$\begin{aligned} &|b(v,Q_m w,A^{1/2}Q_m w)| \\ &\leq \|v\|_4\|A^{1/2}Q_m w\|_2\|A^{1/2}Q_m w\|_4 \\ &\leq 2^{1/4}\|A^{1/4}v\|_2\|A^{1/2}Q_m w\|_2\|A^{3/4}Q_m w\|_2 \\ &\leq \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9}{2^{1/2}\nu\lambda_{m+1}^{1/2}}\|A^{1/4}v\|_2^2\|A^{3/4}Q_m w\|_2^2, \end{aligned} \tag{4.46}$$

$$\begin{aligned} |b(v,P_m w,A^{1/2}Q_m w)|&\leq 2^{1/4}\|A^{1/4}v\|_2\|A^{1/2}P_m w\|_2\|A^{3/4}Q_m w\|_2 \\ &\leq 2^{1/4}\|A^{1/4}v\|_2\|A^{1/4}P_m w\|_2^{1/2}\|A^{3/4}P_m w\|_2^{1/2}\|A^{3/4}Q_m w\|_2 \\ &\leq \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9}{2^{1/2}\nu}\|A^{1/4}v\|_2^2\|A^{1/4}P_m w\|_2\|A^{3/4}P_m w\|_2 \\ &\leq \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}v\|_2^2\|A^{1/4}P_m w\|_2^2, \end{aligned} \tag{4.47}$$

$$\begin{aligned} |b(P_m w,\tilde{v},A^{1/2}Q_m w)|&\leq 2^{1/4}\|A^{1/2}\tilde{v}\|_2\|A^{3/4}Q_m w\|_2\|A^{1/4}P_m w\|_2 \\ &\leq \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9}{2^{1/2}\nu}\|\tilde{v}\|^2\|A^{1/4}P_m w\|_2^2, \end{aligned} \tag{4.48}$$

$$\begin{aligned} |b(Q_m w,\tilde{v},A^{1/2}Q_m w)|&\leq 2^{1/4}\|A^{1/2}\tilde{v}\|_2\|A^{3/4}Q_m w\|_2\|A^{1/4}P_m w\|_2 \\ &\leq \frac{\nu}{18}\|A^{3/4}Q_m w\|^2+\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}P_m w\|_2^2\|\tilde{v}\|^2, \end{aligned} \tag{4.49}$$

$$\begin{aligned} b(P_m w,\psi,A^{1/2}Q_m w) &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\int_{dist(x,\partial\Omega)\le C\varepsilon}\frac{|P_m w||A^{1/2}Q_m w|}{dist(x,\partial\Omega)}dx \\ &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\|P_m w\|\|A^{1/2}Q_m w\|_2 \\ &\le \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}\lambda_1^{1/2}}\|A^{1/4}P_m w\|_2^2, \end{aligned} \tag{4.50}$$

$$\begin{aligned} b(Q_m w,\psi,A^{1/2}Q_m w) &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\Big(\int_\Omega\frac{|Q_m w|^2}{dist(x,\partial\Omega)}dx\Big)^{1/2}\Big(\int_\Omega\frac{|A^{1/2}Q_m w|^2}{dist(x,\partial\Omega)}dx\Big)^{1/2} \\ &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\|A^{1/4}Q_m w\|_2\|A^{3/4}Q_m w\|_2 \\ &\le \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}}\|A^{3/4}Q_m w\|_2^2, \end{aligned} \tag{4.51}$$

$$\begin{aligned} b(\psi,P_m w,A^{1/2}Q_m w) &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\|P_m w\|\|A^{1/2}Q_m w\|_2 \\ &\le \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\|A^{1/4}P_m w\|_2^2 \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} b(\psi,Q_m w,A^{1/2}Q_m w) &\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\Big(\int_\Omega\frac{|Q_m w|^2}{dist(x,\partial\Omega)}dx\Big)^{1/2}\Big(\int_\Omega\frac{|A^{1/2}Q_m w|^2}{dist(x,\partial\Omega)}dx\Big)^{1/2} \\ &\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|A^{1/4}Q_m w\|_2\|A^{3/4}P_m w\|_2 \\ &\le \frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9\varepsilon^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}}\|A^{3/4}Q_m w\|_2^2. \end{aligned} \tag{4.53}$$

Moreover, by the Cauchy-Schwarz inequality, we arrive at

$$|(f(t)-g(t),A^{1/2}Q_m w)|\le\frac{\nu}{18}\|A^{3/4}Q_m w\|_2^2+\frac{9}{2\lambda_{m+1}^{1/2}\nu}\|f(t)-g(t)\|_2^2. \tag{4.54}$$

Combining (4.3)-(4.10), we conclude that

$$\begin{aligned} &\tfrac{d}{dt}\|A^{1/4}Q_m w\|_2^2+\Big(\nu-\tfrac{9}{2^{1/2}\nu\lambda_{m+1}^{1/2}}\|A^{1/4}v\|_2^2-\tfrac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}}-\tfrac{9\varepsilon^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}}\Big)\|A^{3/4}Q_m w\|_2^2 \\ &\le\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}v\|_2^2\|A^{1/4}P_m w\|_2^2+\frac{9}{2^{1/2}\nu}\|\tilde v\|^2\|A^{1/4}P_m w\|_2^2+\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}P_m w\|_2^2\|\tilde v\|^2 \\ &\quad+\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_{m+1}^{1/2}\lambda_1^{1/2}}\|A^{1/4}P_m w\|_2^2+\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\|A^{1/4}P_m w\|_2^2+\frac{9}{2\lambda_{m+1}^{1/2}\nu}\|f(t)-g(t)\|_2^2. \end{aligned} \tag{4.55}$$

It follows from the Poincaré inequality $\lambda_{m+1}\|A^{1/4}Q_m w\|_2^2\le\|A^{3/4}Q_m w\|_2^2$ and Lemma 2.2 for $\xi(t)=\|A^{1/4}Q_m w(t)\|_2^2$ that

$$\frac{d}{dt}\xi(t)+\alpha(t)\xi(t)\le\beta(t) \tag{4.56}$$

with

$$\alpha(t)=\lambda_{m+1}\Big(\nu-\frac{9}{2^{1/2}\nu\lambda_{m+1}^{1/2}}\|A^{1/4}v\|_2^2-\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}-\frac{9\varepsilon^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}\Big) \tag{4.57}$$

and

$$\begin{aligned}\beta(t)=&\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}v\|_2^2\|A^{1/4}P_mw\|_2^2+\frac{9}{2^{1/2}\nu}\|\tilde{v}\|^2\|A^{1/4}P_mw\|_2^2+\frac{9\lambda_m^{1/2}}{2^{1/2}\nu}\|A^{1/4}P_mw\|_2^2\|\tilde{v}\|^2\\&+\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}\lambda_1^{1/2}}\|A^{1/4}P_mw\|_2^2+\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\|A^{1/4}P_mw\|_2^2+\frac{9}{2\lambda_{m+1}^{1/2}\nu}\|f(t)-g(t)\|_2^2.\end{aligned} \tag{4.58}$$

$$\frac{1}{T}\int_t^{t+T}\beta^+(\tau)d\tau\to 0 \quad\text{as}\quad t\to\infty \tag{4.59}$$

because of $\|A^{1/4}P_mw(t)\|_2\to 0$ as $t\to\infty$.

Step 2: The estimate of $\frac{1}{T}\int_t^{t+T}\|A^{1/4}v(s)\|_2^2ds$. Based on the well-posedness, for sufficiently large $T>0$, we have

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\|A^{1/4}v(s)\|_2^2ds\\&\le\frac{1}{\nu^2\lambda_1}\|f\|_{L^\infty(t,t+T;H)}^2\Big(12+4\Big(\frac{C}{\nu\lambda_1^{\frac{2}{3}}}+2C\|\varphi\|_{L^\infty(\partial\Omega)}^2\Big)\Big)\\&\quad+\frac{1}{\nu}\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|_{L^\infty(\partial\Omega)}^2\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big]\\&\quad+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^4+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2\Big].\end{aligned} \tag{4.60}$$

Step 3: The determining modes. Suppose that $F=\limsup_{t\to\infty}\Big(\int_\Omega|f(x,t)|^2dx\Big)^{1/2}$ and the Grashof number $G=\frac{F}{\nu^2\lambda_1}$. For the estimates (4.22)-(4.24), we need to verify (2.7) and (2.8) in Lemma 2.2 as

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau\\&=\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\lambda_{m+1}\Big(\nu-\frac{9}{2^{1/2}\nu\lambda_{m+1}^{1/2}}\|A^{1/4}v\|_2^2-\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}-\frac{9\varepsilon^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}\Big)d\tau\\&\ge\lambda_{m+1}\Big[\Big(\nu-\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}-\frac{9\varepsilon^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu\lambda_{m+1}^{1/2}}\Big)-\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{9}{2^{1/2}\nu\lambda_{m+1}^{1/2}}\|A^{1/4}\tilde{v}(\tau)\|_2^2d\tau\Big]\\&\ge\lambda_{m+1}\Big\{\nu-\frac{1}{\lambda_{m+1}^{1/2}}\Big[\Big(\frac{9\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu}+\frac{9\varepsilon^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{2\nu}\Big)\\&\quad-\frac{9}{2^{1/2}\nu}\Big(\frac{1}{\nu^2\lambda_1}\|f\|_{L^\infty(t,t+T;H)}^2\Big(12+4\Big(\frac{C}{\nu\lambda_1^{\frac{2}{3}}}+2C\|\varphi\|_{L^\infty(\partial\Omega)}^2\Big)\Big)\\&\quad+\frac{1}{\nu}\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|_{L^\infty(\partial\Omega)}^2\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big]\end{aligned}$$

$$+\frac{1}{\nu}\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\Big)\Big]\Big\}, \tag{4.61}$$

which implies that (2.7) and (2.8) hold for m sufficiently large provided that

$$\begin{aligned}\lambda^{1/2}_{m+1}>&\frac{9\Big[12+4(\frac{C}{\nu\lambda_1^{\frac{2}{3}}}+2C\|\varphi\|^2_{L^\infty(\partial\Omega)})\Big]F^2}{2^{1/2}\nu^4\lambda_1}+\Big(\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2}+\frac{9\varepsilon^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2}\Big)\\&+\frac{9}{2^{1/2}\nu^3}\Big(\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\\&+\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\Big).\end{aligned} \tag{4.62}$$

Then,

$$\xi(t)=\|A^{1/4}Q_mw\|^2_2\to 0 \tag{4.63}$$

as $t\to\infty$.

It follows from $\lambda_{m+1}\sim c'\lambda_1 m$ with some non-dimensional constant c' for $m\to\infty$ that the model is determining if

$$\begin{aligned}m\geq&\frac{81}{\tilde c}\Big[12+4(\frac{C}{\nu\lambda_1^{\frac{2}{3}}}+2C\|\varphi\|^2_{L^\infty(\partial\Omega)})\Big]G^4\\&+\Big(\frac{9\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2\tilde c}+\frac{9\varepsilon^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu^2\tilde c}\Big)^2\\&+\frac{81}{2\nu^6\tilde c}\Big(\Big(\frac{2C}{\nu\lambda_1^{\frac{2}{3}}}+4C\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big]\\&+\Big[\frac{2|\partial\Omega|}{\nu\lambda_1}\|\varphi\|^4_{L^\infty(\partial\Omega)}+\frac{2C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big]\Big)^2,\end{aligned} \tag{4.64}$$

where $\tilde c$ is dependent on the shape of the domain Ω only. Thus the proof is completed. □

4.3. Proof of asymptotic determination for autonomous system. In this section, originating from the idea in Kalantarov, Kostianko and Zelik [11], we will illustrate the asymptotic determination by using the estimates as in Section 4.1 via the Grashof number. Based on asymptotic determination of trajectories, the Lyapunov-Schmidt reduction can be used to achieve Theorem 3.10.

• **Proof of Theorem 3.9.**

Proof. Let v and $\tilde v$ be two trajectories inside the finite fractal dimensional global attractor $\mathcal{A}$ in Theorem 3.8 with initial data v_0 and $\tilde v_0$ such that $P_mv(t)=P_m\tilde v(t)$ for $t\in\mathbb{R}$. Denote $w(t)=v(t)-\tilde v(t)$, then it is easy to check that w satisfies

$$\begin{cases}\partial_t w(t)+\nu Aw(t)=\tilde F(\tilde v(t))-\tilde F(v(t)),\\ w(0)=v_0-\tilde v_0.\end{cases} \tag{4.65}$$

Suppose $F(\cdot)$ satisfies the assumption

$$\begin{cases}1)\ \|\tilde F(v)\|_{V'}\leq C,\\ 2)\ \|\tilde F(v)-\tilde F(\tilde v)\|_{V'}\leq L\|v-\tilde v\|_H,\end{cases} \tag{4.66}$$

where $v,\ \tilde{v}\in H$, L is an undetermined variable parameter.

Taking inner product of (4.65) with $w(t)$, noting that $P_m w(t)=0$ and from the (4.66), we drive

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2+\nu\|w(t)\|^2\le|b(w,v,w)|+|b(w,\psi,w)|, \tag{4.67}$$

where

$$\begin{aligned}|b(w,v,w)|&\le 2^{1/4}\|v\|_2^{1/2}\|v\|^{1/2}\|w\|_2^{1/2}\|w\|^{1/2}\|w\|\\ &\le\frac{\nu}{4}\|w\|^2+\frac{1}{6\nu}\|v\|_2^2\|v\|^2\|w\|_2^2,\end{aligned} \tag{4.68}$$

and choose appropriate ε such that $C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\le\frac{\nu}{4}$, which implies

$$|b(w,\psi,w)|\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|w\|^2\le\frac{\nu}{4}\|w\|^2. \tag{4.69}$$

Denoting $L\le\frac{1}{6\nu}\|v\|_2^2\|v\|^2$ which will be determined later by Grashof number, (4.67)-(4.69) yields

$$\frac{d}{dt}\|w(t)\|_2^2+\Big[\nu\lambda_{m+1}-\frac{1}{3\nu}\|v\|_2^2\|v\|^2\Big]\|w(t)\|^2\le 0,$$

which implies

$$\|w(t)\|_2^2\le e^{-\eta(t-s)}\|w(s)\|_2^2 \tag{4.70}$$

for $s<t$ and some $\eta>0$ provided that the hypothesis in Lemma 2.2 are satisfied.

Repeating the procedure in Section 4.1 for autonomous case, we see that

$$\begin{aligned}\int_{t_0}^{t}\|v(s)\|^2ds&\le\frac{1}{\nu}\|v(t_0)\|_2^2+\frac{2}{\nu^2\lambda_1}\|f\|_2^2(t-t_0)\\ &\quad+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big](t-t_0)\end{aligned} \tag{4.71}$$

holds, which implies

$$\frac{1}{T}\int_t^{t+T}\|v(s)\|^2ds\le\frac{4}{\nu^2\lambda_1}\|f\|_2^2+\frac{1}{\nu}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big] \tag{4.72}$$

for sufficiently large $T>0$.

By the existence of global attractor $\mathcal{A}$ in H, we can set the radius of bounded absorbing ball as $\|v\|_2\le\rho$. By using Lemma 2.2, we only need

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\Big(\nu\lambda_{m+1}-\frac{\|v\|_2^2}{3\nu}\|v(\tau)\|^2\Big)d\tau\\ &\ge\nu\lambda_{m+1}-\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{\rho^2}{3\nu}\|v(\tau)\|^2d\tau\\ &\ge\nu\lambda_{m+1}-\Big(\frac{4\rho^2\|f\|_2^2}{3\nu^3\lambda_1}+\frac{\rho^2}{3\nu^2}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big]\Big).\end{aligned} \tag{4.73}$$

Then, (2.7) and (2.8) hold for m sufficiently large provided that

$$\lambda_{m+1}>\frac{4\rho^2\|f\|_2^2}{3\nu^4\lambda_1}+\frac{\rho^2}{3\nu^3}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|_{L^\infty(\partial\Omega)}^2+\frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^4\Big]. \tag{4.74}$$

Hence, $\|w(t)\|_2^2 \to 0$ as $t \to \infty$ for trajectories inside $\mathcal{A}$. In terms of $\lambda_{m+1} \sim c'\lambda_1 m$ with some non-dimensional constant c' for $m \to \infty$, the model is asymptotically determining if

$$m \geq \frac{4\rho^2}{3C'} Gr^2 + \frac{\rho^2}{3C'\nu^3\lambda_1}\Big[\frac{C|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon|\partial\Omega|}{\nu}\|\varphi\|^4_{L^\infty(\partial\Omega)}\Big], \tag{4.75}$$

where c' is dependent on the shape of the domain Ω only.

Therefore, the proof is completed. □

• **Proof of Theorem 3.9.**

Proof.

Step 1: The estimate of $v_-(t) \in C_b([0,M];H)$. From the well-posedness of autonomous system, we can see that $v(t) \in C([0,M];H)$ and $v_+(t) \in C_b(\mathbb{R}_+;H)$ which also leads to the existence of solution $v_-(t) \in C([0,M];H)$ for the following problem considered as

$$\begin{cases} \dfrac{d}{dt}v_-(t) + \nu A(v_-(t)) + Q_m\tilde{F}(v_+(t)+v_-(t)) = g_-, \\ v_-(t)|_{t=0} = 0 \end{cases} \tag{4.76}$$

for fixed $M > 0$.

Taking the inner product of (4.76) with $v_-(t)$ in H, we can arrive at

$$\frac{1}{2}\frac{d}{dt}\|v_-(t)\|_2^2 + \nu\|v_-(t)\|^2 \leq |(\tilde{F}(v_- + v_+), v_-)| + (g, v_-),$$

with

$$\begin{aligned} |(\tilde{F}(v_- + v_+), v_-)| \leq & |(B(v_+,v_+),v_-)| + |(B(v_+,v_-),v_-)| + |(B(v_-,v_+),v_-)| \\ & + |(B(v_-,v_-),v_-)| + |(B(v_+,\psi),v_-)| + |(B(v_-,\psi),v_-)| \\ & + |(B(\psi,v_+),v_-)| + |(B(\psi,v_-),v_-)| \\ = & |(B(v_-,v_+),v_-)| + |(B(v_-,\psi),v_-)| + |(B(\psi,v_+),v_-)| \end{aligned} \tag{4.77}$$

and

$$(g, v_-) = (P_L(f(x) + \nu F) - B(\psi), v_-) \tag{4.78}$$

from the property of trilinear operator and orthogonality of v_- and v_+ in H.

By using the analogous technique in Section 4.1, choosing ε small enough such that $C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \leq \frac{\nu}{16}$, then we have the following estimates

$$\begin{aligned} |(B(v_-,v_+),v_-)| & \leq 2^{1/4}\|v_-\|_2^{1/2}\|v_-\|^{1/2}\|v_-\|_2^{1/2}\|v_-\|^{1/2}\|v_+\| \\ & \leq \frac{\nu}{16}\|v_-\|^2 + \frac{4}{\nu}\|v_+\|^2\|v_-\|_2^2 \end{aligned} \tag{4.79}$$

and

$$|(B(v_-,\psi),v_-)| \leq C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|v_-\|^2 \leq \frac{\nu}{16}\|v_-\|^2 \tag{4.80}$$

and

$$|(B(\psi,v_+),v_-)| \leq C\|\varphi\|_{L^\infty(\partial\Omega)}\|v_+\|_2\|v_-\| \leq \frac{\nu}{16}\|v_-\|^2 + \frac{C}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|v_+\|_2^2 \tag{4.81}$$

and

$$|(B(\psi),v_-)|\leq\frac{\nu}{16}\|v_-\|^2+\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\tag{4.82}$$

and

$$\nu|<F,v_->|\leq\frac{\nu}{16}\|v_-\|^2+\frac{\nu C}{\varepsilon}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}\tag{4.83}$$

since $\|\varphi\|_{L^2(\partial\Omega)}\leq C|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}$. Moreover,

$$|(f,v_-)|\leq\frac{\nu}{16}\|v_-\|^2+\frac{2}{\nu\lambda_1}\|f\|_2^2.\tag{4.84}$$

Then we conclude that

$$\begin{aligned}&\frac{d}{dt}\|v_-(t)\|_2^2+(\nu\lambda_{m+1}-\frac{8}{\nu}\|v_+\|^2)\|v_-(t)\|_2^2\\ \leq&\frac{C}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|v_+\|_2^2+\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}+\frac{\nu C}{\varepsilon}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{2}{\nu\lambda_1}\|f\|_2^2.\end{aligned}\tag{4.85}$$

Repeat the procedure in above proof of Theorem 3.9, by using Lemma 2.2, we can derive that there exists a positive constant $\tilde\eta$ such that

$$\begin{aligned}\|v_-(t)\|_2^2\leq&\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}+\frac{\nu C}{\varepsilon}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{2}{\nu\lambda_1}\|f\|_2^2\\&+\frac{C}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\int_{-M}^t e^{-\tilde\eta(t-s)}\|v_+(t)\|_2^2\end{aligned}\tag{4.86}$$

for $0\leq t\leq M$ provided that m sufficiently large as (3.21), which implies $v_-(t)\in C_b([0,M],H)$ which also contains the initial time $t=0$ due to $v_+(t)\in C_b(\mathbb{R}_+,H)$.

Step 2: The existence and uniqueness of $v_-(t)$ in $C_b(\mathbb{R}_+,H)$. The uniform boundedness in **Step 1** yields $v_{-,M}(t)$ is bounded with respect to M, which lets us claim that $\{v_{-,M}(t)\}_{M=1}^\infty$ is a Cauchy sequence in $C_b(\mathbb{R}_+,H)$.

Denoting $v_{M_1,M_2}(t)=v_{-,M_1}-v_{-,M_2}$ for $M_1>M_2$, which satisfies

$$\begin{cases}\partial_t v_{M_1,M_2}(t)+\nu Av_{M_1,M_2}=|\tilde F(v_{-,M_1}(t)+v_{+,M_1}(t))-\tilde F(v_{-,M_2}(t)+v_{+,M_2}(t))|,\\ v_{M_1,M_2}|_{t=M_2}=v_{-,M_1}(M_2),\end{cases}\tag{4.87}$$

then multiplying (4.87) with v_{M_1,M_2}, noting that $v_{-,M_1}(M_2)$ is uniformly bounded with respect to M_1 and M_2 in $C_b(\mathbb{R}_+,H)$, using the similar argument in the proof of Theorem 3.9, we can deduce

$$\|v_{M_1,M_2}(t)\|_2^2\leq e^{-\eta(t+M_2)}\|v_{-,M_1}(-M_2)\|_2^2\leq Ce^{-\eta(t+M_2)},$$

which implies that $\{v_{-,M}(t)\}_{M=1}^\infty$ is indeed a Cauchy sequence.

Passing to the limit as $M\to\infty$ for $\{u_{-,M}(t)\}$, the existence and uniqueness of desired solution for $(E2)$ in (3.20) has been obtained with $v_-(t)\in C_b(\mathbb{R}_+,H)$.

Step 3: Reduction. Let $v_-^1(t)$ and $v_-^2(t)$ be the two solutions of (4.76) with corresponding lower frequency functions $v_+^1(t)$ and $v_+^2(t)$, respectively, we denote $v_-(t)=v_-^1(t)-v_-^2(t)$ satisfying

$$\begin{cases}\frac{d}{dt}v_-(t)+\nu A(v_-(t))=Q_m\tilde F(v_+^2(t)+v_-^2(t))-Q_m\tilde F(v_+^1(t)+v_-^1(t)),\\ v_-(t)|_{t=0}=0.\end{cases}\tag{4.88}$$

Multiplying (4.88) by v_- and integrating over Ω, we can derive

$$\frac{1}{2}\frac{d}{dt}\|v_-(t)\|_2^2+\nu\|v_-(t)\|^2\le|(\tilde F(v_+^1(t)+v_-^1(t))-\tilde F(v_+^2(t)+v_-^2(t)),v_-)|, \tag{4.89}$$

with

$$\begin{aligned}&|(\tilde F(v_+^1(t)+v_-^1(t))-\tilde F(v_+^2(t)+v_-^2(t)),v_-)|\\ \le&|(B(v_+,v_+^1),v_-)|+|(B(v_+,v_+^1),v_-)|+|(B(v_-,v_+^1),v_-)|+|(B(v_-,v_-^1),v_-)|\\ &+|(B(v_+^2,v_+),v_-)|+|(B(v_+^2,v_-),v_-)|+|(B(v_-^2,v_+),v_-)|+|(B(v_-^2,v_-),v_-)|\\ &+|(B(v_+,\psi),v_-)|+|(B(v_-,\psi),v_-)|+|(B(\psi,v_+),v_-)|+|(B(\psi,v_-),v_-)|\\ =&|(B(v_-,v^1),v_-)|+|(B(v_-,\psi),v_-)|+|(B(\psi,v_+),v_-)|\end{aligned}$$

from the property of trilinear operator and orthogonality of v_- and v_+ in H such as $|(B(v_-^2,v_+),v_-)|=|(B(v_-^2,v_-),v_+)|=0$.

By using the analogous technique in Section 4.1, choosing ε small enough such that $C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\le\frac{\nu}{6}$, then we have the following estimates

$$|(B(v_-,v^1),v_-)|\le\frac{\nu}{6}\|v_-\|^2+\frac{3}{2\nu}\|v^1\|^2\|v_-\|_2^2 \tag{4.90}$$

and

$$|(B(v_-,\psi,v_-))|\le C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|v_-\|^2\le\frac{\nu}{6}\|v_-\|^2 \tag{4.91}$$

and

$$|(B(\psi,v_+),v_-)|\le C\|\varphi\|_{L^\infty(\partial\Omega)}\|v_+\|_2\|v_-\|\le\frac{\nu}{6}\|v_-\|^2+\frac{C}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v_+\|_2^2. \tag{4.92}$$

Combining (4.89)-(4.92), we derive that

$$\frac{d}{dt}\|v_-(t)\|_2^2+(\nu\lambda_{m+1}-\frac{3}{2\nu}\|v^1\|^2)\|v_-(t)\|_2^2\le\frac{C}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v_+\|_2^2. \tag{4.93}$$

Since the hypothesis (3.21) ensures that there exists a constant $\bar\eta>0$ such that

$$\frac{d}{dt}\|v_-(t)\|_2^2+\beta\|v_-(t)\|_2^2\le\frac{C}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v_+\|_2^2, \tag{4.94}$$

then Gronwall lemma implies that

$$\|v_-(t)\|_2^2\le e^{-\bar\eta(t+M)}\|v_-^1(0)-v_-^2(0))\|_2^2+\frac{C}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2\int_0^t e^{-\bar\eta(t-s)}\|v_+^1(s)-v_+^2(s)\|_2^2ds.$$

Passing to the limit as $M\to\infty$, we conclude that

$$\|v_-(t)\|_2^2\le\frac{C}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2 e^{-\bar\eta(t-s)/2}\sup_{s\in\mathbb{R}_+}\|v_+^1(s)-v_+^2(s)\|_2^2,$$

which means the results hold. ∎

5. Further research

For the two dimensional incompressible Navier-Stokes equation, the determining nodes have been obtained [3,6–10]. However, if the domain is Lischitz, we will give the definition of determining nodes and a useful lemma as following. Then we shall state the main difficulty for achieving the determining nodes for (1.1) or (3.1), which is our further research in future.

DEFINITION 5.1. *Consider the set $\mathcal{E}=\{x_1,x_2,\cdots,x_N\}$ as collected measurable points in Ω, we call the set $\mathcal{E}$ is a set of determining N-nodes if for every two solvers $\tilde{v}$ and v of (3.1) defined by (3.8) and (3.9), respectively satisfying*

$$\lim_{t\to\infty}\|f(t,x)-g(t,x)\|_2=0 \tag{5.1}$$

and

$$\lim_{t\to\infty}\sup_{i=1,\cdots,N}|v(x^i,t)-\tilde{v}(x^i,t)|=0, \tag{5.2}$$

then we have

$$\lim_{t\to\infty}\|v(t)-\tilde{v}(t)\|_2=0, \tag{5.3}$$

where $f(t,x)$ and $g(t,x)$ are given external forces in $L^\infty_{loc}(\mathbb{R}^+;H)$.

LEMMA 5.1 ([7, 10]). *Assume that Ω is covered by N-identical squares. Let $\mathcal{E}=\{x_1,x_2,\cdots,x_N\}$ be points in Ω, distributed one in each square. Then, for each vector field w in $D(A)$, the following inequalities hold:*

$$\begin{aligned}
\|w\|_2^2 &\le \frac{c}{\lambda_1}\eta(w)^2+\frac{c}{\lambda_1^2N^2}\|Aw\|_2^2,\\
\|w\|^2 &\le cN\eta(w)^2+\frac{c}{\lambda_1^2N}\|Aw\|_2^2,\\
\|w\|_{L^\infty(\Omega)}^2 &\le cN\eta(w)^2+\frac{c}{\lambda_1^2N}\|Aw\|_2^2,
\end{aligned}$$

where c depends on the shape of Ω only.

REMARK 5.1. Setting $w=v(t)-\tilde{v}(t)$ and $\eta(w)=\max_{i=1,\cdots,N}\|w(x^i,t)\|_2$, we want to show (5.3) if (5.1) and $\eta(w)\to 0$ hold as $t\to\infty$, which needs a new revised version of the above lemma for two dimensional Navier-Stokes equations in bounded domain with C^2-boundary. By the definition of trace in Lipschitz domain from [19,23], the above Lemma (especially the integration by parts in proof) does not hold for problem (2.7), which is the main difficulty and our objective in future.

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