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FLOCKING BEHAVIOR OF THE CUCKER-SMALE MODEL UNDER A GENERAL DIGRAPH ON THE INFINITE CYLINDER*

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Abstract. In this paper, we generalize the Cucker-Smale model under a general digraph on the infinite cylinder with the help of the Lie group structure of the infinite cylinder and study the flocking behavior of this model. We show that for $0 \le \beta < 1/(2h)$ unconditional flocking occurs, where h is the shortest height of the spanning trees of the digraph, and conditional flocking occurs for $\beta \ge 1/(2h)$ under some conditions depending only on the initial data.

Keywords. Cucker-Smale model; flocking behavior; digraph; infinite cylinder.

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1. Introduction

Flocking is a form of collective behavior of a large number of interacting agents with a common group objective. Flocking phenomena are ubiquitous in nature and man-made complex systems, e.g., flocking of birds, schooling of fish, and application to unmanned aerial vehicles such as aircraft or satellites. Among many flocking models, the model proposed by Cucker and Smale [6] has attracted much attention [5,11]. In the original Cucker-Smale (in short C-S) model, the interaction topology is an undirected complete graph. It is well known that the interaction topology plays an important role in the analysis of flocking behavior. Hence, many researchers focus on studying the C-S model under different interaction topologies, such as the hierarchical leadership structure [18], the rooted leadership structure [14], and the general digraph with a spanning tree [7]. Besides, the authors in [15, 16] modified the original C-S model, named as Motsch-Tadmor model, with asymmetric interaction strength.

The above work studied the C-S model in Euclidean space. In recent years, many researchers have generalized the C-S model from Euclidean space to nonlinear spaces, (e.g., the Riemannian manifolds and Lie groups) and investigated its dynamic behavior. The main problem in generalizing the C-S model on the nonlinear spaces is as follows. For the C-S model in Euclidean space, the differences of the velocities of agents are used to adjust the accelerations. But in most nonlinear spaces, the velocities of different agents can not be compared with each other directly, since they belong to different tangent spaces. To the best of our knowledge, there are two methods that can be used to overcome this problem. The work [10] has generalized the C-S model on the Riemannian manifolds by using parallel transport, depending on the Riemannian metric, to transform the velocities of agents to a common tangent space. Further research about the C-S model on the hyperboloid and unit sphere are studied in [1,4]. For a Lie group, the velocities of all agents are first transformed to the tangent space of the unit element of the Lie group by using the left translation, which depends on the product structure

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of the Lie group, and then they can be compared [17]. However, in the aforementioned literature, each agent is communicated by an undirected complete graph and the weight function is assumed to have a positive lower bound.

Since the infinite cylinder is an important kind of Lie group, which is also a Riemannian manifold, the collective behavior of the multi-agent systems on the infinite cylinder has attracted much interest [8, 9]. Hence, in this article, we first generalize the C-S model under a general digraph on the infinite cylinder and then study its flocking behavior. When generalizing the C-S model on the infinite cylinder, we also meet the problem that the velocities of different agents cannot be compared directly, since the infinite cylinder is a Lie group. Hence, in this article, the comparisons of the velocities of agents are made with the help of the left translation on the infinite cylinder.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries of the Lie group and algebraic graph theory. In Section 3, we generalize the C-S model under a general digraph on the infinite cylinder and study its flocking behavior. Finally, Section 4 draws some conclusions.

2. Mathematical preliminaries

In the following two subsections, we introduce some definitions and properties of the Lie group and directed graph.

2.1. Lie group.

In this subsection, we review some definitions and notations of the Lie group. For an introduction to the Lie group theory, we refer to [12].

DEFINITION 2.1. *A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m: G\times G\to G$ and inverse map $i: G\to G$, given by*

$$m(g,f)=g\cdot f \quad \textit{and} \quad i(g)=g^{-1}$$

are smooth.

DEFINITION 2.2. *Let G be a Lie group and g be an element of G. Then, the left translation $L_g: G\to G$ is defined by*

$$L_g(f)=g\cdot f, \quad f\in G.$$

Furthermore, for any $f\in G$ and $X_f\in T_fG$, the left translation L_g induces a differential map $(L_g)_*: T_fG\to T_{g\cdot f}G$ by

$$(L_g)_*(X_f)=\frac{d}{d\tau}\left[g\cdot\gamma(\tau)\right]\big|_{\tau=0}. \tag{2.1}$$

Here T_fG is the tangent space of G at f, and $\gamma:(-\epsilon,\epsilon)\to G$ is a smooth curve on G satisfying $\gamma(0)=f$ and $\dot\gamma(0)=X_f$, where $(-\epsilon,\epsilon)$ is a sub-interval of $\mathbb{R}$.

2.2. A directed graph.

A digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ consists of a vertex set $\mathcal{V}=\{1,2,...,N\}$ and an arc set $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$, where an arc is an ordered pair of vertices of $\mathcal{V}$. If $(j,i)\in\mathcal{E}$, we say that vertex j is a neighbor of i. For the case that $(i,i)\in\mathcal{E}$, we understand that $\mathcal{G}$ has a self-loop at vertex i. The neighborhood set of vertex i is $\mathcal{N}_i=\{j|(j,i)\in\mathcal{E}\}$. Let $\bar N$ be the maximal cardinality of the neighbor sets $\mathcal{N}_i$. We say that a matrix $A=(a_{ij})_{N\times N}$ is an adjacency matrix of the given graph $\mathcal{G}$ with $a_{ij}>0$ if $(j,i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The distance $d(i,j)$ from vertices i to j is the length of a shortest path from i to j.

We now consider the case that $\mathcal{G}$ has a spanning tree. For each root r of $\mathcal{G}$, define

$$h_r = \max_{i \in \mathcal{V}} d(r,i)$$

which can be viewed as the height of the spanning tree with root r and define

$$h = \min\{h_r | r \text{ is a root}\}.$$

Let $\mathcal{R}$ be the set of roots r such that $h_r = h$ and the cardinality of $\mathcal{R}$ is denoted by n_r.

A matrix $A = (a_{ij})_{N\times N}$ is nonnegative, denoted by $A \geq 0$, if each entry $a_{ij} \geq 0$. The nonnegative matrix A is *stochastic* if each of its row-sum is equal to 1. A is said to be scrambling if for each pair of indexes i and j there exists an index k such that $a_{ik} > 0$ and $a_{jk} > 0$. The *ergodicity coefficient* of A is defined by

$$\chi(A) = \min_{i,j} \sum_{k=1}^{N} \min\{a_{ik}, a_{jk}\}.$$

It is easy to see that A is scrambling if and only if $\chi(A) > 0$.

The following two lemmas [7] will play a crucial role when we study the flocking behavior of the C-S model under a general digraph on the infinite cylinder.

LEMMA 2.1. *Let $A \geq 0$ and all diagonal entries be positive. Assume that $\mathcal{G}(A)$ has a spanning tree. Let h be the shortest height of all trees of $\mathcal{G}(A)$ and $\underline{a} = \min\{a_{ij} : a_{ij} > 0\}$. Then, $\chi(A^h) \geq n_r \underline{a}^h$.*

LEMMA 2.2. *Assume that $A = (a_{ij})_{N\times N}$ is stochastic. For all $v = [v_1, v_2, \ldots, v_N]^\top \in \mathbb{R}^N$, we have*

$$\max_{i,j} |u_i - u_j| \leq (1 - \chi(A)) \max_{i,j} |v_i - v_j|, \text{ where } u = Av = [u_1, u_2, \ldots, u_N]^\top.$$

3. The Cucker-Smale model on the infinite cylinder

In this section, we generalize the C-S model under a directed graph on the infinite cylinder with the help of the left translation on the Lie group. The infinite cylinder $\mathbb{S}^1 \times \mathbb{R} = \{(x_1, x_2, z) \in \mathbb{R}^3 | x_1^2 + x_2^2 = 1\}$ with the topology induced from the Euclidean space $\mathbb{R}^3$ is a Riemannian manifold. The following lemma shows the Riemannian distance between any two points on the infinite cylinder [3].

LEMMA 3.1. *Let $\boldsymbol{x} = (x_1, x_2, z_1)$ and $\boldsymbol{y} = (y_1, y_2, z_2)$ be any two points on $\mathbb{S}^1 \times \mathbb{R}$. Then the Riemannian distance $d(\boldsymbol{x}, \boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by*

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\arccos^2(x_1 y_1 + x_2 y_2) + (z_2 - z_1)^2}.$$

Besides, the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ is also a communicative Lie group. Notice that every element $\boldsymbol{x} \in \mathbb{S}^1 \times \mathbb{R}$ can be represented in the form of $\boldsymbol{x} = (\cos\theta, \sin\theta, z)$. Then the multiply operation and inverse operation are given by the following definition.

DEFINITION 3.1. *For any two elements $\boldsymbol{x} = (\cos\theta, \sin\theta, z_1)$ and $\boldsymbol{y} = (\cos\psi, \sin\psi, z_2)$ on $\mathbb{S}^1 \times \mathbb{R}$, the multiply and inverse operations are given by*

$$\boldsymbol{x} \cdot \boldsymbol{y} = (\cos(\theta + \psi), \sin(\theta + \psi), z_1 + z_2) \quad \text{and} \quad \boldsymbol{x}^{-1} = (\cos(-\theta), \sin(-\theta), -z_1).$$

Furthermore, the unit element $\boldsymbol{e} \in \mathbb{S}^1 \times \mathbb{R}$ is $\boldsymbol{e} = (1, 0, 0)$.

Next, we will generalize the Cucker-Smale model under a general digraph from Euclidean space to the infinite cylinder. We first recall the C-S model under a directed graph on the Euclidean space. Let $\mathrm{x}_i \in \mathbb{R}^n$ and $\mathrm{v}_i \in \mathbb{R}^n$ be the position and velocity of the i-th agent on the Euclidean space $\mathbb{R}^n$. The C-S model under the directed graph $\mathcal{G}$ on $\mathbb{R}^n$ reads as follows.

$$\begin{cases} \dfrac{d\mathrm{x}_i}{dt} = \mathrm{v}_i, \qquad i=1,2,\dots,N, \\ \dfrac{d\mathrm{v}_i}{dt} = \displaystyle\sum_{i\in\mathcal{N}_j} \varphi(\|\mathrm{x}_i-\mathrm{x}_j\|)(\mathrm{v}_j-\mathrm{v}_i). \end{cases} \tag{3.1}$$

Let us introduce the C-S model under the digraph $\mathcal{G}$ on the infinite cylinder. Let $\{\boldsymbol{x}_i(t),\boldsymbol{v}_i(t)\}_{i=1}^N$ be the positions and velocities of N agents on $\mathbb{S}^1\times\mathbb{R}$. From system (3.1), we can see that, in Euclidean space, the velocity of agent i is adjusted by the differences in the velocities of agents. However, for any two agents i and j on $\mathbb{S}^1\times\mathbb{R}$, the velocities $\boldsymbol{v}_i\in T_{\boldsymbol{x}_i}(\mathbb{S}^1\times\mathbb{R})$ and $\boldsymbol{v}_j\in T_{\boldsymbol{x}_j}(\mathbb{S}^1\times\mathbb{R})$ cannot be directly compared when $\boldsymbol{x}_i\neq\boldsymbol{x}_j$, since they belong to different tangent spaces. Inspired by the work of [17], we first transform each velocity $\boldsymbol{v}_i\in T_{\boldsymbol{x}_i}(\mathbb{S}^1\times\mathbb{R})$ to the tangent space $T_{\boldsymbol{e}}(\mathbb{S}^1\times\mathbb{R})$ of the unit element $\boldsymbol{e}$ by using the differential map $(L_{\boldsymbol{x}_i^{-1}(t)})_*: T_{\boldsymbol{x}_i(t)}(\mathbb{S}^1\times\mathbb{R})\to T_{\boldsymbol{e}}(\mathbb{S}^1\times\mathbb{R})$, defined in equality (2.1). Then, the velocity of each agent can be compared with others after transformation. Denote $\boldsymbol{\xi}_i(t)=(L_{\boldsymbol{x}_i^{-1}(t)})_*(\boldsymbol{v}_i(t))$ and the generalized C-S model under the directed graph $\mathcal{G}$ on the infinite cylinder is given by following:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \qquad i=1,2,\dots,N, \\ \dfrac{d\boldsymbol{\xi}_i}{dt} = \displaystyle\sum_{i\in\mathcal{N}_j} \varphi(d(\boldsymbol{x}_i,\boldsymbol{x}_j))(\boldsymbol{\xi}_j-\boldsymbol{\xi}_i). \end{cases} \tag{3.2}$$

Here the weight function $\varphi(s)=\kappa/(1+s^2)^\beta$, in which $\kappa>0$ and $\beta\geq 0$ are system parameters, quantifies the way the agents influence each other. The definition of flocking on the infinite cylinder is adapted from [17].

DEFINITION 3.2. *System (3.2) has time-asymptotic flocking on the infinite cylinder if and only if a solution $\{\boldsymbol{x}_i(t),\boldsymbol{\xi}_i(t)\}_{i=1}^N$ of system (3.2) satisfies, for $i,j=1,2,\dots,N$,*

$$\lim_{t\to\infty}\|\boldsymbol{\xi}_i(t)-\boldsymbol{\xi}_j(t)\|=0, \quad \sup_{0\leq t<\infty} d(\boldsymbol{x}_i(t),\boldsymbol{x}_j(t))<\infty,$$

where $\|\cdot\|$ is the norm on $T_{\boldsymbol{e}}(\mathbb{S}^1\times\mathbb{R})$.

In the remaining part of this section, we first present the concrete form of the C-S model (3.2) on $\mathbb{S}^1\times\mathbb{R}$ when the position of each agent $\boldsymbol{x}_i(t)$ on $\mathbb{S}^1\times\mathbb{R}$ is represented by $\boldsymbol{x}_i=(\cos\theta_i,\sin\theta_i,z_i)$ and then study the flocking behavior of this model. The velocity of the i-th agent is $\boldsymbol{v}_i=(-\dot\theta_i\sin\theta_i,\dot\theta_i\cos\theta_i,\dot z_i)$. According to equality (2.1) and Definition 3.1, we have

$$\begin{aligned} \boldsymbol{\xi}_i(t) &= \left(L_{\boldsymbol{x}_i^{-1}(t)}\right)_*(\boldsymbol{v}_i(t)) = \frac{d}{d\tau}\left[\boldsymbol{x}_i^{-1}(t)\cdot\boldsymbol{x}_i(t+\tau)\right]\Big|_{\tau=0} \\ &= \frac{d}{d\tau}\left[(\cos(\theta_i(t+\tau)-\theta_i(t)),\sin(\theta_i(t+\tau)-\theta_i(t)),z_i(t+\tau)-z_i(t))\right]\Big|_{\tau=0} \\ &= (0,\dot\theta_i(t),\dot z_i(t)). \end{aligned} \tag{3.3}$$

Applying Lemma 3.1, the Riemannian distance $d(\boldsymbol{x}_i,\boldsymbol{x}_j)$ between agents i and j is as follows:

$$\begin{aligned} d(\boldsymbol{x}_i,\boldsymbol{x}_j) &= \sqrt{\arccos^2(\cos\theta_i\cos\theta_j+\sin\theta_i\sin\theta_j)+(z_i-z_j)^2} \\ &= \sqrt{\arccos^2(\cos(\theta_i-\theta_j))+(z_i-z_j)^2}. \end{aligned} \tag{3.4}$$

By equalities (3.3) and (3.4), the concrete form of the generalized C-S model on the infinite cylinder (3.2) can be equivalently described by following, for $i=1,2,\dots,N$,

$$\begin{cases} \ddot{\theta}_i(t) = \sum\limits_{j\in\mathcal{N}_i} \varphi_{ij}(t)(\dot{\theta}_j(t)-\dot{\theta}_i(t)), \\ \ddot{z}_i(t) = \sum\limits_{j\in\mathcal{N}_i} \varphi_{ij}(t)(\dot{z}_j(t)-\dot{z}_i(t)). \end{cases} \tag{3.5}$$

Here $\varphi_{ij}(t)=\varphi(d(\boldsymbol{x}_i(t),\boldsymbol{x}_j(t)))=\varphi\Big(\sqrt{\arccos^2(\cos(\theta_i(t)-\theta_j(t)))+(z_i(t)-z_j(t))^2}\Big)$.

Let $\dot{\theta}_i=\omega_i$ and $\dot{z}_i=p_i$, then the above dynamical system can be rewritten as

$$\begin{cases} \dot{\theta}_i(t)=\omega_i(t), \qquad \dot{\omega}_i(t)=\sum\limits_{j\in\mathcal{N}_i}\varphi_{ij}(t)(\omega_j(t)-\omega_i(t)), \\ \dot{z}_i(t)=p_i(t), \qquad \dot{p}_i(t)=\sum\limits_{j\in\mathcal{N}_i}\varphi_{ij}(t)(p_j(t)-p_i(t)). \end{cases} \tag{3.6}$$

REMARK 3.1. From equalities (3.3) and (3.4), we can see that flocking behavior given in Definition 3.2 is equivalent to

$$\lim_{t\to\infty}|\dot{\theta}_i(t)-\dot{\theta}_j(t)|=0,\ \lim_{t\to\infty}|\dot{z}_i(t)-\dot{z}_j(t)|=0, \text{ and } \sup_{0\le t<\infty}|z_i(t)-z_j(t)|<\infty. \tag{3.7}$$

Recall that $\boldsymbol{x}_i\in\mathbb{S}^1\times\mathbb{R}$ and $\boldsymbol{v}_i\in T(\mathbb{S}^1\times\mathbb{R})=T\mathbb{S}^1\times T\mathbb{R}$. The first limit in (3.7) means that the diameter of angular velocities converges to 0. The second limit in (3.7) means that the diameter of vertical velocities converges to 0. Hence, the two limits in (3.7) are equivalent to the first limit in Definition 3.2. From (3.4), one can see that the third inequality in (3.7) is equivalent to the second inequality in Definition 3.2.

Next, we will study the flocking behavior of system (3.6). Let us define

$$\Omega(t)=\max_{1\le i,j\le N}|\omega_i(t)-\omega_j(t)|, \quad \Lambda(t)=\max_{1\le i,j\le N}|p_i(t)-p_j(t)|.$$

LEMMA 3.2. *Consider system (3.6). For $t\ge 0$, we have $D^+\Omega(t)\le 0$ and $D^+\Lambda(t)\le 0$, where D^+ denotes the upper-right Dini derivative.*

Proof. We first prove $D^+\Omega(t)\le 0$. Suppose that $\Omega(t)=\omega_i(t)-\omega_j(t)$ at time t; Then, we have

$$\begin{aligned} \frac{d}{dt}(\omega_i-\omega_j)^2 &= 2(\omega_i-\omega_j)(\dot{\omega}_i-\dot{\omega}_j)=2(\omega_i-\omega_j)\Big[\sum_{k\in\mathcal{N}_i}\varphi_{ki}(\omega_k-\omega_i)-\sum_{l\in\mathcal{N}_j}\varphi_{lj}(\omega_l-\omega_j)\Big] \\ &= 2(\omega_i-\omega_j)\Big[\sum_{k\in\mathcal{N}_i}\varphi_{ki}[(\omega_k-\omega_j)-(\omega_i-\omega_j)]-\sum_{l\in\mathcal{N}_j}\varphi_{lj}[(\omega_i-\omega_j)-(\omega_i-\omega_l)]\Big] \\ &\le 0. \end{aligned}$$

Thus $D^+\Omega(t)\le 0$. The proof of $D^+\Lambda(t)\le 0$ is similar. ∎

Now we state the flocking result for system (3.6). For convenience, denote $z_i(t)-z_j(t)$ by $z_{ij}(t)$.

THEOREM 3.1. *Consider system (3.6). Assume that there exists a constant $\rho>0$ such that*

$$\Lambda(0)\le \frac{\varphi(0)\delta\rho}{h}, \tag{3.8}$$

where

$$\delta=\frac{n_r}{h!}\Big(\frac{h\mu}{\varphi(0)e^{\bar N}}\Big)^h \text{ and } \mu=\min_{(j,i)\in\mathcal{E}}\varphi(|z_{ij}(0)|+\pi+\rho). \tag{3.9}$$

Then, the agents exponentially converge to flocking. Specifically,

$$\lim_{t\to\infty}\Omega(t)=0 \quad \text{and} \quad \lim_{t\to\infty}\Lambda(t)=0.$$

Proof. We first state that δ defined in equality (3.9) satisfies $0<\delta\le 1$. Please see the proof in the Appendix.

Next, let us use the induction method to prove the following two inequalities: For $t\in\mathbb{N}$,

$$\Omega(\hat h t)\le(1-\delta)^t\Omega(0),\qquad \Lambda(\hat h t)\le(1-\delta)^t\Lambda(0), \tag{3.10}$$

where $\hat h=h/\varphi(0)$. It is trivially true for $t=0$. We suppose that inequality (3.10) holds for all t up to some $t^*\ge 0$. By Lemma 3.2, we have that $\Omega(t)$ and $\Lambda(t)$ are non-increasing functions. Thus, for any $0\le t\le t^*$ and $\hat h t\le\tau<\hat h(t+1)$, we deduce $\Omega(\tau)\le\Omega(\hat h t)\le(1-\delta)^t\Lambda(0)$ and $\Lambda(\tau)\le\Lambda(\hat h t)\le(1-\delta)^t\Lambda(0)$. Additionally, for $(j,i)\in\mathcal{E}$, we obtain

$$\begin{aligned}|z_{ij}(\tau)|&\le|z_{ij}(0)|+\int_0^\tau|\dot z_{ij}(s)|ds\le|z_{ij}(0)|+\sum_{t=0}^{t^*}\int_{\hat h t}^{\hat h(t+1)}\Lambda(s)ds\\&\le|z_{ij}(0)|+\hat h\sum_{t=0}^{t^*}(1-\delta)^t\Lambda(0)\le|z_{ij}(0)|+\frac{\hat h\Lambda(0)}{\delta}.\end{aligned} \tag{3.11}$$

Recall that $d(\boldsymbol{x}_i,\boldsymbol{x}_j)=\sqrt{[\arccos(\cos(\theta_i-\theta_j))]^2+(z_i-z_j)^2}$. Then, inequalities (3.8) and (3.11) imply

$$\begin{aligned}d(\boldsymbol{x}_i(\tau),\boldsymbol{x}_j(\tau))&\le\arccos(\cos(\theta_i(\tau)-\theta_j(\tau)))+|z_{ij}(\tau)|\\&\le\pi+|z_{ij}(0)|+\frac{\hat h\Lambda(0)}{\delta}\le|z_{ij}(0)|+\pi+\rho.\end{aligned} \tag{3.12}$$

Consequently, the non-increasing property of the weight function $\varphi(\cdot)$ implies

$$\varphi(d(\boldsymbol{x}_i(\tau),\boldsymbol{x}_j(\tau)))\ge\varphi(|z_{ij}(0)|+\pi+\rho).$$

Define $\omega(t)=[\omega_1(t),\omega_2(t),\dots,\omega_N(t)]^\top$ and $p(t)=[p_1(t),p_2(t),\dots,p_N(t)]^\top$. Then, the second and fourth equations of system (3.6) can be written as

$$\begin{bmatrix}\dot\omega(t)\\ \dot p(t)\end{bmatrix}=\begin{bmatrix}-L(t)&\mathbf{0}\\ \mathbf{0}&-L(t)\end{bmatrix}\begin{bmatrix}\omega(t)\\ p(t)\end{bmatrix}, \tag{3.13}$$

where $L(t)=D(t)-A(t)$ is the Laplacian matrix of $A(t)$. The matrix $A(t)=(a_{ij}(t))_{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with $a_{ij}(t)=\varphi(d(\boldsymbol{x}_i(t)),\boldsymbol{x}_j(t))$ if $j\in\mathcal{N}_i$ and $a_{ij}(t)=0$ otherwise. Besides, $a_{ii}(t)=\varphi(0)$ for $t\geq 0$. And $D(t)=diag(\sigma_1(t),\sigma_2(t),...,\sigma_N(t))$ is defined as $\sigma_i(t)=\sum_{j\in\mathcal{N}_i}\varphi(d(\boldsymbol{x}_i(t),\boldsymbol{x}_j(t)))$. Since the off-diagonal entries of $-L(t)$ are nonnegative, then system (3.13) is positive [2, Theorem 10.6], i.e., its solutions remain nonnegative whenever the initial data are nonnegative. The solution of system (3.13) can be written as

$$\begin{bmatrix}\omega(t)\\ p(t)\end{bmatrix}=\Psi(t,t_0)\begin{bmatrix}\omega(t_0)\\ p(t_0)\end{bmatrix}=\begin{bmatrix}\Psi_{11}(t,t_0) & \Psi_{12}(t,t_0)\\ \Psi_{21}(t,t_0) & \Psi_{22}(t,t_0)\end{bmatrix}\begin{bmatrix}\omega(t_0)\\ p(t_0)\end{bmatrix}, \tag{3.14}$$

where $\Psi(t,t_0)$ is the state transition matrix. The positivity of system (3.13) implies that $\Psi(t,t_0)$ is nonnegative. Since $\mathbf{1}_{2N}=[1,1,..,1]^\top\in\mathbb{R}^{2N}$ is a constant solution of (3.13), we have that $\mathbf{1}_{2N}=\Psi(t,t_0)\mathbf{1}_{2N}$. As a result, $\Psi(t,t_0)$ is stochastic. Furthermore, $[\mathbf{1}_N,\mathbf{0}_N]=[1,...,1,0,...,0]^\top\in\mathbb{R}^N\times\mathbb{R}^N$ is also a constant solution of (3.13). Then, we have $[\mathbf{1}_N,\mathbf{0}_N]=\Psi(t,t_0)[\mathbf{1}_N,\mathbf{0}_N]$, which implies that $\Psi_{11}(t,t_0)$ is stochastic and $\Psi_{12}(t,t_0)=0$. Similiarly, we can deduce that $\Psi_{22}(t,t_0)$ is stochastic and $\Psi_{21}(t,t_0)=0$. Consequently, the solution of system (3.13) is

$$\begin{bmatrix}\omega(t)\\ p(t)\end{bmatrix}=\begin{bmatrix}\Psi_{11}(t,t_0) & \mathbf{0}\\ \mathbf{0} & \Psi_{22}(t,t_0)\end{bmatrix}\begin{bmatrix}\omega(t_0)\\ p(t_0)\end{bmatrix}. \tag{3.15}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned}\Omega(\hat{h}(t^*+1))&\leq\big(1-\chi(\Psi_{11}(\hat{h}(t^*+1),\hat{h}t^*))\big)\Omega(\hat{h}t^*),\\ \Lambda(\hat{h}(t^*+1))&\leq\big(1-\chi(\Psi_{22}(\hat{h}(t^*+1),\hat{h}t^*))\big)\Lambda(\hat{h}t^*).\end{aligned} \tag{3.16}$$

Let us estimate $\chi(\Psi_{11}(\hat{h}(t^*+1),\hat{h}t^*))$ and $\chi(\Psi_{22}(\hat{h}(t^*+1),\hat{h}t^*))$. Consider the auxiliary system

$$\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{r}}\end{bmatrix}=\begin{bmatrix}C-\bar{N}\varphi(0)I_N & \mathbf{0}\\ \mathbf{0} & C-\bar{N}\varphi(0)I_N\end{bmatrix}\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{r}\end{bmatrix}, \tag{3.17}$$

where $C=(c_{ij})_{N\times N}$ is given by $c_{ij}=\varphi(|z_{ij}(0)|+\pi+\rho)$ if $j\in\mathcal{N}_i$ and $c_{ij}=0$ otherwise, I_N is the unit matrix, and $\bar{N}$ is the maximal cardinality of the neighbor sets $\mathcal{N}_i$. We see that the auxiliary system (3.17) is also positive. For any nonnegative vector $[\boldsymbol{u}_0^\top,\boldsymbol{v}_0^\top]^\top\in\mathbb{R}^{2N}$, the inequality $\varphi(d(\boldsymbol{x}_i(t),\boldsymbol{x}_j(t)))\geq\varphi(|z_{ij}(0)|+\pi+\rho)$ implies that, for $\hat{h}t^*\leq t\leq\hat{h}(t^*+1)$,

$$(A(t)-D(t))\boldsymbol{u}_0\geq(C-\bar{N}\varphi(0)I_N)\boldsymbol{u}_0 \text{ and } A(t)-D(t))\boldsymbol{v}_0\geq(C-\bar{N}\varphi(0)I_N)\boldsymbol{v}_0.$$

Then, by the comparison principle [13, Corollary 1.6], we obtain

$$\Psi_{11}(t,t_0)\boldsymbol{u}_0\geq e^{C-\bar{N}\varphi(0)I_N}\boldsymbol{u}_0 \text{ and } \Psi_{22}(t,t_0)\boldsymbol{v}_0\geq e^{C-\bar{N}\varphi(0)I_N}\boldsymbol{v}_0.$$

It follows that

$$\Psi_{kk}(\hat{h}(t^*+1),\hat{h}t^*)\geq e^{(C-\bar{N}\varphi(0)I_N)\hat{h}}=e^{-h\bar{N}}e^{\hat{h}C}\geq e^{-h\bar{N}}\frac{\hat{h}^h}{h!}C^h,\ k=1,2. \tag{3.18}$$

Since the interaction topology $\mathcal{G}(C)=\mathcal{G}$ has a spanning tree and all diagonal entries $c_{ii}=\varphi(\pi+\rho)$ are positive, inequality (3.18) and Lemma 2.1 implies that $\chi(C^h)\geq n_r\mu^h$. Then, inequality (3.18) implies

$$\chi(\Psi_{kk}(\hat{h}(t^*+1),\hat{h}t^*))\geq e^{-h\bar{N}}\frac{\hat{h}^h}{h!}\chi(C^h)\geq\frac{n_r}{h!}\Big(\frac{\hat{h}\mu}{e^{\bar{N}}}\Big)^h=\delta,\quad k=1,2. \tag{3.19}$$

Then, inequality (3.16) deduces

$$\begin{aligned}\Omega(\hat{h}(t^*+1)) &\le (1-\delta)\Omega(\hat{h}t^*) \le (1-\delta)^{(t^*+1)}\Omega(0),\\ \Lambda(\hat{h}(t^*+1)) &\le (1-\delta)\Lambda(\hat{h}t^*) \le (1-\delta)^{(t^*+1)}\Lambda(0).\end{aligned} \tag{3.20}$$

Hence, inequality (3.10) is proved by induction. The non-increasing properties of $\Omega(t)$ and $\Lambda(t)$ give

$$\Omega(t) \le (1-\delta)^{\lfloor \frac{t}{\hat{h}} \rfloor}\Omega(0),\ \Lambda(t) \le (1-\delta)^{\lfloor \frac{t}{\hat{h}} \rfloor}\Lambda(0),\quad t\in[0,\infty).$$

By the similar method that we used to obtain inequality (3.12), the above inequalities imply $d(\boldsymbol{x}_i(t),\boldsymbol{x}_j(t)) \le |z_{ij}(0)|+\pi+\rho$ for $t\in[0,\infty)$. This completes the proof. ∎

THEOREM 3.2. *Consider system (3.6). If one of the following three hypotheses holds:*

(i) $0\le\beta<1/(2h)$,

(ii) $\beta=1/(2h)$ *and* $\Lambda(0)<\Xi$,

(iii) $\beta>1/(2h)$ *and*

$$\Lambda(0) < \frac{\Xi(2\alpha-1)^{2\alpha-1}[(\mathcal{Z}(0)+\pi)(1-\alpha)+\Pi]}{[(2\alpha-1)^2+(\alpha(\mathcal{Z}(0)+\pi)+\Pi)^2]^\alpha}, \tag{3.21}$$

where $\mathcal{Z}(0)=\max\limits_{1\le i,j\le N}|z_{ij}(0)|$, $\Xi=\frac{n_r\kappa h^{h-1}}{h!e^{h\bar{N}}}$, $\alpha=h\beta$, *and* $\Pi=\sqrt{\alpha^2(\mathcal{Z}(0)+\pi)^2+2\alpha-1}$. *Then, the agents exponentially converge to flocking.*

Proof. By the definition of μ given in equality (3.9), we have $\mu\ge\kappa/(1+(\mathcal{Z}(0)+\pi+\rho)^2)^\beta$. Hence, a sufficient condition for (3.8) is given by

$$\Lambda(0) \le \frac{\Xi\rho}{(1+(\mathcal{Z}(0)+\pi+\rho)^2)^\alpha} =: f(\rho). \tag{3.22}$$

(i) If $\beta\le1/(2h)$, then $2\alpha<1$. By the definition of $f(\rho)$, we have $f(\rho)\to\infty$ as $\rho\to\infty$. Therefore, for any initial data $(\theta_i(0),z_i(0))$, inequality (3.22) holds for appropriately large $\rho>0$. This statement holds by Theorem 3.1.

(ii) If $\beta=1/(2h)$, then $\alpha=1/2$. It can be checked directly that $f(\rho)$ is an increasing function of ρ. Furthermore, $f(\rho)\to\Xi$ as $\rho\to\infty$. Thus, if $\Lambda(0)<\Xi$, then inequality (3.22) holds for appropriately large $\rho>0$. Now, the statement follows from Theorem 3.1.

(iii) Finally, if $\beta>1/(2h)$, then $\alpha>1/2$. By straightforward computations, we have that the function $f(\rho)$ attains its maximum value at $\rho^*=((\mathcal{Z}(0)+\pi)(1-\alpha)+\Pi)/(2\alpha-1)$. In addition,

$$f(\rho^*) = \frac{\Xi(2\alpha-1)^{2\alpha-1}[(\mathcal{Z}(0)+\pi)(1-\alpha)+\Pi]}{[(2\alpha-1)^2+(\alpha(\mathcal{Z}(0)+\pi)+\Pi)^2]^\alpha}.$$

Then inequality (3.21) implies that $\Lambda(0)<f(\rho^*)$. Consequently, condition (3.8) in Theorem 3.1 holds. Now, statement (iii) is proved. ∎

REMARK 3.2. Since the unit circle $\mathbb{S}^1$ is a Lie subgroup of the infinite cylinder $\mathbb{S}^1\times\mathbb{R}$, Theorem 3.1 can be applied to obtain the flocking behavior of the C-S model under a general digraph on the unit circle, that is $z_i\equiv0$ for all agents. More precisely, for each agent $\boldsymbol{x}_i(t)=(\cos\theta_i(t),\sin\theta_i(t))\in\mathbb{S}^1$, the C-S model under a general digraph is $\dot{\theta}_i=\omega_i$, $\dot{\omega}_i=\sum_{j\in\mathcal{N}_i}\varphi(\arccos(\cos(\theta_i-\theta_j)))(\omega_j-\omega_i)$. Meanwhile, by the similar method used in Theorem 3.1, it can be proved that the flocking behavior on the unit circle occurs unconditionally.

4. Conclusion

In this paper, we first generalize the Cucker-Smale model under a general digraph from Euclidean space to the infinite cylinder with the help of the communicative Lie group structure on the infinite cylinder. Then, we provide a unified condition on the initial states in which the exponential convergence to the flocking state would occur. Furthermore, for the Cucker-Smale weight function, a critical exponent below which unconditional flocking holds is given, depending only on the interaction topology.

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Appendix. In this appendix, we will prove that δ defined in equality (3.9) satisfies $0<\delta\le 1$. From the definition of δ, it is clear that $\delta>0$.

Consider the following system

$$\dot{\boldsymbol{y}}(t)=\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N\Big)\boldsymbol{y}(t), \tag{A.1}$$

where $C=(c_{ij})_{N\times N}$ and $c_{ij}=\varphi(|z_{ij}(0)|+\pi+\rho)$ if $j\in\mathcal{N}_i$ and $c_{ij}=0$ otherwise. By the definition of C, the off-diagonal entries of $(hC/\varphi(0)-h\bar{N}I_N)$ are nonnegative. Thus, the system (A.1) is positive. Since $\varphi(\cdot)$ is non-increasing, then, for $i=1,2,\ldots,N$,

$$\Delta_i:=\sum_{j=1}^{N}\frac{c_{ij}}{\varphi(0)}-\bar{N}\le\bar{N}-\bar{N}=0.$$

Let $\Delta=diag(\Delta_1,\Delta_2,\ldots,\Delta_N)$. Consider the following auxiliary system:

$$\dot{\boldsymbol{u}}(t)=\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N-h\Delta\Big)\boldsymbol{u}(t). \tag{A.2}$$

Clearly, the above system (A.2) is also a positive system and the solution is

$$\boldsymbol{u}(t)=\exp\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N-h\Delta\Big)\boldsymbol{u}(0).$$

By the definition of Δ, we have that, for $\mathbf{1}=[1,1,\ldots,1]^\top\in\mathbb{R}^N$,

$$\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N-h\Delta\Big)\mathbf{1}=0.$$

Thus $\mathbf{1}$ is a constant solution of system (A.2). Hence,

$$\exp\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N-h\Delta\Big)\mathbf{1}=\mathbf{1}.$$

It means that $\exp(hC/\varphi(0)-h\bar{N}I_N-h\Delta)$ is nonnegative and stochastic. Recall that $\Delta_i\le 0$ for all $i\in[N]$, then, for any nonnegative vector $\boldsymbol{u}_0$ we have

$$\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N\Big)\boldsymbol{u}_0\le\Big(\frac{hC}{\varphi(0)}-h\bar{N}I_N-h\Delta\Big)\boldsymbol{u}_0.$$

By the comparison principle [13, Corollary 1.6], we deduce

$$\exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N\Big) \le \exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N - h\Delta\Big).$$

By the definition of matrix exponential, we have

$$\exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N\Big) = \exp(-h\bar{N} I_N)\exp\Big(\frac{hC}{\varphi(0)}\Big) = e^{-h\bar{N}} \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{hC}{\varphi(0)}\Big)^k.$$

It follows that

$$\frac{e^{-h\bar{N}}}{h!}\Big(\frac{hC}{\varphi(0)}\Big)^h \le \exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N\Big) \le \exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N - h\Delta\Big). \tag{A.3}$$

Recalling that $\hat{h} = h/\varphi(0)$; Then the above inequality (A.3) implies

$$\frac{C^h}{h!}\Big(\frac{\hat{h}}{e^{\bar{N}}}\Big)^h \le \exp\Big(\frac{hC}{\varphi(0)} - h\bar{N} I_N - h\Delta\Big).$$

Since $\exp(hC/\varphi(0) - h\bar{N} I_N - h\Delta)$ is nonnegative and stochastic, then we have,

$$\frac{\chi(C^h)}{h!}\Big(\frac{\hat{h}}{e^{\bar{N}}}\Big)^h \le \chi(\exp(hC/\varphi(0) - h\bar{N} I_N - h\Delta)) \le 1. \tag{A.4}$$

Finally, Lemma 2.1 and inequality (A.4) imply

$$\delta = \frac{n_r}{h!}\Big(\frac{\hat{h}\mu}{e^{\bar{N}}}\Big)^h \le 1.$$

We complete the proof.

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