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GLOBAL MILD SOLUTIONS OF THE NON-CUTOFF VLASOV-POISSON-BOLTZMANN SYSTEM*

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Abstract. This paper is concerned with the Cauchy problem on the Vlasov-Poisson-Boltzmann system in the torus domain. The Boltzmann collision kernel is assumed to be angular non-cutoff with $0\le\gamma<1$ and $1/2\le s<1$, where γ,s are two parameters describing the kinetic and angular singularities, respectively. We obtain the global-in-time unique mild solutions, and prove that the solutions converge to the global Maxwellian with the large-time decay rate of $O(e^{-\lambda t})$ in the $L^1_kL^2_v$-norm for some $\lambda>0$. Furthermore, we justify the property of propagation of regularity of solutions in the spatial variable.

Keywords. The Vlasov-Poisson-Boltzmann system; Non-cutoff; Mild regularity; Global existence; Time decay.

AMS subject classifications. 35Q20; 35Q83; 35B35.

1. Introduction

The Vlasov-Poisson-Boltzmann system is a physical model describing the time evolution of dilute charged particles (e.g. electrons) in the absence of an external magnetic field [6]. When the constant background charge density is normalized to be unity, the Vlasov-Poisson-Boltzmann system reads

$$\partial_t F+v\cdot\nabla_x F+\nabla_x\phi\cdot\nabla_v F=Q(F,F), \tag{1.1}$$

$$\Delta_x\phi=\int_{\mathbb{R}^3}Fdv-1 \tag{1.2}$$

with prescribed initial data

$$F(0,x,v)=F_0(x,v). \tag{1.3}$$

Here, $F(t,x,v)\ge 0$ represents the density function at time $t\ge 0$, with spatial coordinate $x=(x_1,x_2,x_3)\in\mathbb{T}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The electric potential $\phi=\phi(t,x)$ generating the self-consistent electric field $\nabla_x\phi$ in (1.1) is coupled with $F(t,x,v)$ through the Poisson Equation (1.2). The bilinear collision operator $Q(F,G)$ on the right-hand side of (1.1) is defined by

$$Q(F,G)(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\sigma)[F(u')G(v')-F(u)G(v)]d\sigma du,$$

where (v,u) and (v',u'), denoting velocities of two particles before and after their collsions respectively, satisfy

$$v'=\frac{v+u}{2}+\frac{|v-u|}{2}\sigma,\quad u'=\frac{v+u}{2}-\frac{|v-u|}{2}\sigma.$$

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The non-negative Boltzmann collision kernel $B(v-u,\sigma)$ depends only on the ralative velocity $|v-u|$ and on the deviation angle θ given by $\cos\theta=\langle\sigma,(v-u)/|v-u|\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual dot product in $\mathbb{R}^3$. Without loss of generality, we suppose that $B(v-u,\sigma)$ is supported on $\cos\theta\geq 0$, c.f. [22]. Throughout the paper we further assume that the collision kernel $B(v-u,\sigma)$ takes the product form as follows:

$$B(v-u,\sigma)=C_B|v-u|^\gamma b(\cos\theta),$$

for a constant $C_B>0$, where $|v-u|^\gamma$ is called the kinetic factor with $\gamma>-3$, and $b(\cos\theta)$ is called the angular part satisfying that there are $C_b>0,\ 0<s<1$ such that

$$\frac{1}{C_b\theta^{1+2s}}\leq\sin\theta b(\cos\theta)\leq\frac{C_b}{\theta^{1+2s}},\quad\forall\ \theta\in(0,\frac{\pi}{2}].$$

Recall $\gamma=0$ is the Maxwellian molecules case and meanwhile the case of $-3<\gamma<0$ and $0<\gamma<1$ are called, respectively, soft potential and hard potential. In the rest of this paper we are concerned about the case of Maxwellian molecules and hard potential, i.e., $0\leq\gamma<1$.

We will consider the Cauchy problem (1.1)–(1.3) around a normalized global Maxwellian

$$\mu(v)=(2\pi)^{-3/2}e^{-|v|^2/2}.$$

Set the perturbation $f=f(t,x,v)$ by

$$F=\mu+\sqrt{\mu}f.$$

Then f and ϕ satisfy the perturbed system:

$$\partial_t f+v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f-\frac{1}{2}v\cdot\nabla_x\phi f-\nabla_x\phi\cdot v\sqrt{\mu}+Lf=\Gamma(f,f),\tag{1.4}$$

$$\Delta_x\phi=\int_{\mathbb{R}^3}\sqrt{\mu}f\,dv,\tag{1.5}$$

$$f(0,x,v)=f_0(x,v).\tag{1.6}$$

Here, the linearized collision operator L and the nonlinear collision operator Γ are respectively given by

$$Lf=-\mu^{-1/2}Q(\mu,\sqrt{\mu}f)-\mu^{-1/2}Q(\sqrt{\mu}f,\mu),$$
$$\Gamma(f,g)=\mu^{-1/2}Q(\sqrt{\mu}f,\sqrt{\mu}g).$$

Recalling (1.5), we can rewrite $\phi(t,x)$ in terms of $f(t,x,v)$ as

$$\phi(t,x)=-\frac{1}{4\pi|x|}*_x\int_{\mathbb{R}^3}\mu^{1/2}f\,dv,\tag{1.7}$$

where $*_x$ denotes the convolution with respect to the x variable.

Now we start to state the main results of this paper. As in [14], we introduce the function space X_T with $0<T\leq\infty$. Define

$$X_T:=L^1_kL^\infty_TL^2_v$$

with corresponding norm

$$\|f\|_{X_T} := \int_{\mathbb{Z}^3} \sup_{0\le t\le T} |\hat{f}(t,k,\cdot)|_{L^2_v} d\Sigma(k) < +\infty.$$

Here, the Fourier transform of $f(t,x,v)$ with respect to $x\in\mathbb{T}^3$ is defined by

$$\hat{f}(t,k,v) = \mathscr{F}_x f(t,k,v) = \int_{\mathbb{T}^3} e^{-ix\cdot k} f(t,x,v)dx, \quad x\cdot k = \sum_{j=1}^{3} x_j k_j,$$

for $k\in\mathbb{Z}^3$, where $i=\sqrt{-1}\in\mathbb{C}$. For any given $t\ge 0$, we define the following norms in x and v:

$$\|f(t)\|_{L^1_k L^2_v} = \int_{\mathbb{Z}^3} |\hat{f}(t,k)|_{L^2_v} d\Sigma(k),$$

and

$$\|\nabla_x\phi(t)\|_{L^1_k} = \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(t,k)| d\Sigma(k).$$

For any integer $m\ge 0$, the corresponding high-order norms are defined by

$$\|f(t)\|_{L^1_{k,m} L^2_v} = \int_{\mathbb{Z}^3} \langle k\rangle^m |\hat{f}(t,k)|_{L^2_v} d\Sigma(k),$$

and

$$\|\nabla_x\phi(t)\|_{L^1_{k,m}} = \int_{\mathbb{Z}^3} \langle k\rangle^m |\widehat{\nabla_x\phi}(t,k)| d\Sigma(k).$$

As in [2], we define

$$\begin{aligned} |f|_D^2 = & \int\int\int B(v-u,\sigma)\mu(u)(f(v')-f(v))\,\overline{(f(v')-f(v))} \\ & + \int\int\int B(v-u,\sigma)f(u)\,\overline{f(u)}\,(\sqrt{\mu'}-\sqrt{\mu})^2, \end{aligned} \tag{1.8}$$

where the integration is over $\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma$. Here, we use $\overline{f}$ to denote the standard complex conjugate of f. We can also refer to the work of Gressman-Strain [22] for another equivalent definition of this norm.

For given function $f(t,x,v)$ with corresponding $\phi(t,x)$ in (1.7), we define the energy functional and energy dissipation rate functional respectively as

$$\begin{aligned} \mathscr{E}_T(f) = & \|f\|_{L^1_k L^\infty_T L^2_v} + \|\nabla_x\phi\|_{L^1_k L^\infty_T} \\ = & \int_{\mathbb{Z}^3} \sup_{0\le t\le T} |\hat{f}(t,k,\cdot)|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \sup_{0\le t\le T} |\widehat{\nabla_x\phi}(t,k)| d\Sigma(k) \end{aligned} \tag{1.9}$$

and

$$\mathscr{D}_T(f) = \|f\|_{L^1_k L^2_T L^2_{v,D}} = \int_{\mathbb{Z}^3} \left(\int_0^T |\hat{f}(t,k,\cdot)|_D^2 \, dt \right)^{1/2} d\Sigma(k). \tag{1.10}$$

THEOREM 1.1. *Let $0\le\gamma<1$, $1/2\le s<1$. Assume that $F_0(x,v)=\mu+\mu^{1/2}f_0(x,v)\ge 0$. If*

$$\epsilon_0=\|f_0\|_{L^1_kL^2_v}+\|\nabla_x\phi_0\|_{L^1_k},$$

is sufficiently small, by the Poisson Equation (1.5),

$$\nabla_x\phi_0(x)=\nabla_x\phi(0,x)=\nabla_x\Delta_x^{-1}(\sqrt{\mu},f)_{L^2_v},$$

then there exists a unique global mild solution $f(t,x,v)$ to the Cauchy problem (1.4)–(1.6) of the Vlasov-Poisson-Boltzmann system such that $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\ge 0$ and

$$\|f(t)\|_{L^1_kL^2_v}+\|\nabla_x\phi(t)\|_{L^1_k}\lesssim e^{-\lambda t}\epsilon_0, \tag{1.11}$$

for some $\lambda>0$, and any $t\ge 0$.

THEOREM 1.2. *Under the assumptions of Theorem 1.1, for any integer $m\ge 0$, if*

$$\varepsilon_0=\|f_0\|_{L^1_{k,m}L^2_v}+\|\nabla_x\phi_0\|_{L^1_{k,m}},$$

is sufficiently small, then the solution $f(t,x,v)$ to the Cauchy problem (1.4)–(1.6) established in Theorem 1.1 satisfies

$$\begin{aligned}&\int_{\mathbb{Z}^3}\langle k\rangle^m\sup_{0\le t\le T}|\hat{f}(t,k)|_{L^2_v}d\Sigma(k)+\int_{\mathbb{Z}^3}\langle k\rangle^m\sup_{0\le t\le T}|\widehat{\nabla_x\phi}(t,k)|d\Sigma(k)\\&\quad+\int_{\mathbb{Z}^3}\langle k\rangle^m\left(\int_0^T|\hat{f}(t,k)|^2_D\,dt\right)^{1/2}d\Sigma(k)\lesssim\|f_0\|_{L^1_{k,m}L^2_v}+\|\nabla_x\phi_0\|_{L^1_{k,m}},\end{aligned} \tag{1.12}$$

for any $T>0$.

First of all, we recall some known results on the Boltzmann equations. For global solutions to the renormalized equation with large initial data, we mention the classical works by Diperna and Lions [8], Lions [32], Desvillettes-Villani [13], Alexandre-Villani [1]. When considering the Boltzmann equation with cutoff either in the whole space or a torus domain, we mention Guo [24,25], Liu-Yang-Yu [30,31]. For the non-cutoff cases, the global-in-time existence theory in the perturbation framework for the Boltzmann equation has been well established in smooth Sobolev space. Gressman-Strain [22] first constructed the global small-amplitude classical solution in a periodic box for hard potential case $\gamma+2s\ge 0$ and for the general soft potential case $\gamma+2s<0$, then Strain [35] extended these results to the whole space. Similar results were also independently obtained by AMUXY in their series of works [2–5] in the whole space. A key point in those well-known works is to characterize the dissipation property in the L^2 norm in v for the linearized Boltzmann collision operator and further carry out the energy estimates by controlling the trilinear term in an appropriate way. Recently, Duan-Liu-Sakamoto-Strain [14] proved the existence of small-amplitude global-in-time unique mild solutions to both the Landau equation and the Boltzmann equation without angular cutoff, they created a new function space with low regularity in the spatial variable to treat the problem for the case when the spatial domain is either a torus, or a finite channel with boundary, and they also obtained the large-time behavior of solutions for both hard and soft potentials. When considering the Cauchy problem for the non-cutoff Boltzmann equation on the torus, Duan-Li-Liu [15] established the global-in-time Gevrey smoothness in velocity and space variables for a class of low-regularity mild solutions near Maxwellians with the Gevrey index depending only on the angular singularity. Readers are referred to [7, 9–11, 28, 29, 34] for more information.

Back to the Vlasov-Poisson-Boltzmann system, there have been some investigations about the dynamical problems. Global-in-time renormalized solutions with large initial data to the Vlasov-Poisson-Boltzmann system were constructed by Lions [32] and this result was later extended to the case with boundary in [33]. On the other hand, the global classical solutions to the Vlasov-Poisson-Boltzmann system near Maxwellian was firstly established in [26] in periodic box. Since then, there have been extensive works on the global solutions to this system in the whole space $\mathbb{R}^3_x$. For the hard-sphere model, the global existence of solutions to the Vlasov-Poisson-Boltzmann system was proved in [38] and [17] in different function spaces, and the corresponding large-time behavior of solutions was obtained in [39] and [16], respectively. For the case of hard potentials and soft pontentials, the authors obtained the global classical solutions and the optimal time rate in [18] and [21]. These works listed above are all under Grad's angular cut-off assumption, we can refer to [12, 19, 23, 27] for more information. For the Vlasov-Poisson-Boltzmann systen with non-cutoff case, when initial data is near Maxwellians, Duan-Liu [20] established the global existence and convergence rates of classical solutions to the Cauchy problem without angular cutoff for soft potentials $-3<\gamma<-2s$ with strong angular singularity $1/2\le s<1$. And Xiao-Xiong-Zhao [37] obtained the globally smooth solutions near a given global Maxwellian to the Cauchy problem for hard potentials $\gamma+2s\ge 0$ with weak angular singularity $0<s<1/2$.

Finally we sketch the main ideas used in deducing our results. Our main ideas are inspired by the work by Duan-Liu-Sakamoto-Strain [14], they use the L^1_k norm to replace the L^∞_x norm when studying the non-cutoff Boltzmann equation, since L^1_k norm has the Banach algebra property. In our paper, we consider the Cauchy problem of the Vlasov-Poisson-Boltzmann system for hard potentials $\gamma\ge 0$ with strong angular singularity $1/2\le s<1$ because of the existence of the nonlinear terms $\widehat{\nabla_x\phi\cdot\nabla_v f}$, $v\cdot\widehat{\nabla_x\phi f}$ in (2.5). In fact, when making estimates for the nonlinear terms above, due to the Fourier transform in x, the following L^2_v inner product

$$(\widehat{\nabla_x\phi\cdot\nabla_v f},\hat f)_{L^2_v} \tag{1.13}$$

does not disappear. We can regard the first-order velocity differentiation as $\nabla_v=(\nabla_v)^{1/2}(\nabla_v)^{1/2}$ in a rough way, so that (1.13) can be bounded by

$$\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\;|\langle v\rangle^{\gamma/2}(1-\Delta_v)^{s/2}\hat f(l)|_{L^2_v}|\langle v\rangle^{\gamma/2}(1-\Delta_v)^{s/2}\hat f(k)|_{L^2_v}d\Sigma(l),$$

where $s\ge 1/2$ and $\gamma\ge 0$ are used. Notice that

$$|\langle v\rangle^{\gamma/2}(1-\Delta_v)^{s/2}\hat f|^2_{L^2_v}+|\hat f|^2_{L^2_{s+\gamma/2}}\lesssim|\hat f|^2_D.$$

Thus (1.13) can be bounded by the dissipation norm

$$\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\;|\hat f(l)|_D|\hat f(k)|_D d\Sigma(l).$$

For the term $(v\cdot\widehat{\nabla_x\phi f},\hat f)_{L^2_v}$, we use the following inequality, for $2s+\gamma\ge 1$,

$$|v|\le\langle v\rangle^{s+\gamma/2}\langle v\rangle^{s+\gamma/2}$$

to make an estimate for it. For more specific details, one can refer to Lemmas 2.5 and 2.6 in the next section. When making macroscopic estimates, we construct a time-frequency interactive functional to deduce the desired estimates, which is different from Duan-Liu-Sakamoto-Strain [14], by using dual argument. And finally we point out that, throughout the paper we do not need the conservation laws:

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\sqrt{\mu}f(t,x,v)dvdx=0,$$

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}v_i\sqrt{\mu}f(t,x,v)dvdx=0,\qquad i=1,2,3,$$

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}(|v|^2\sqrt{\mu}f(t,x,v)+|\nabla_x\phi|^2)dvdx=0.$$

The rest of the paper will be arranged as follows. In Section 2, we list basic lemmas concerning the properties of L and Γ, and give the estimates on all the nonlinear terms in Section 3. Section 4 is concerned with the estimates on the macroscopic dissipation. In Section 5, we complete the proof of Theorems 1.1 and 1.2.

Notations. In this paper, we let C stand for some positive (generally large) inessential constant and λ denote some positive (generally small) inessential constant, where both C and λ may change values from line to line. Furthermore $A\lesssim B$ means $A\leq CB$, and $A\gtrsim B$ means $B\lesssim A$. In addition, $A\sim B$ means $A\lesssim B$ and $B\lesssim A$.

2. Preliminaries

In this section, we need to make some preparations for the rest of this paper. Firstly, we introduce the following macro-micro decomposition. As we know, the null space of the operator L is given by

$$\mathcal{N}=KerL=\text{span}\{\sqrt{\mu},v_1\sqrt{\mu},v_2\sqrt{\mu},v_3\sqrt{\mu},|v|^2\sqrt{\mu}\}.$$

We further define $\mathbf{P}$ the orthogonal projection from $L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$, then for any given function f, one can write

$$\begin{cases}\mathbf{P}f=\{a(t,x)+b(t,x)\cdot v+c(t,x)(|v|^2-3)\}\mu^{1/2},\\ a=\displaystyle\int_{\mathbb{R}^3}\mu^{1/2}fdv,\\ b=\displaystyle\int_{\mathbb{R}^3}v\mu^{1/2}fdv,\\ c=\displaystyle\frac{1}{6}\int_{\mathbb{R}^3}(|v|^2-3)\mu^{1/2}fdv.\end{cases}\tag{2.1}$$

Thus, we have the macro-micro decomposition introduced in [25],

$$f(t,x,v)=\mathbf{P}f(t,x,v)+\{\mathbf{I}-\mathbf{P}\}f(t,x,v),$$

where $\mathbf{P}f$ and $\{\mathbf{I}-\mathbf{P}\}f$ are called the macroscopic component and the microscopic component of $f(t,x,v)$, respectively.

On the other hand, we are concerned about the estimate on the linearized Boltzmann operator L, that is the following lemma:

Lemma 2.1 (Proposition 2.1 in [2]). *Let $0<s<1$ and $\gamma>-3$. It holds that*

$$(Lf,f)_{L^2_v}\gtrsim|\{\mathbf{I}-\mathbf{P}\}f|^2_D.\tag{2.2}$$

For this term $|f|_D$ above, we refer to [2] to obtain that,

LEMMA 2.2. *Let $0<s<1$ and $\gamma>-3$. Then there exist two generic constants $C_1,C_2>0$ such that*

$$C_1\{|f|^2_{H^s_{\gamma/2}}+|f|^2_{L^2_{s+\gamma/2}}\}\le|f|^2_D\le C_2|f|^2_{H^s_{s+\gamma/2}}. \tag{2.3}$$

Here the weighted fractional Sobolev norm $|f(v)|^2_{H^s_\ell}=|\langle v\rangle^\ell f(v)|^2_{H^s}$ is given by

$$|f|^2_{H^s_\ell}=|\langle v\rangle^\ell f|^2_{L^2_v}+\int_{\mathbb{R}^3}dv\int_{\mathbb{R}^3}du\frac{[\langle v\rangle^\ell f(v)-\langle u\rangle^\ell f(u)]^2}{|v-u|^{3+2s}}\mathcal{X}_{|v-u|\le1}.$$

The following lemma concerns on the estimate on the nonlinear Boltzmann collision operator Γ, which can be found in [5, Theorem 1.2].

LEMMA 2.3. *Let $0<s<1$ and $\gamma>\max\{-3,-\frac{3}{2}-2s\}$. It holds that*

$$|(\Gamma(f,g),h)_{L^2_v}|\lesssim|f|_{L^2_v}\;|g|_D|h|_D. \tag{2.4}$$

Now we will make the estimates for the nonlinear terms in the following Equation (2.5), wherein the Fourier transform has been taken in variable x. Let $f=f(t,x,v)$, $0\le t\le T$, $x\in\mathbb{T}^3$, $v\in\mathbb{R}^3$, be a smooth solution to

$$\partial_t f+v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f-\frac{1}{2}v\cdot\nabla_x\phi f-\nabla_x\phi\cdot v\sqrt{\mu}+Lf=\Gamma(f,f).$$

Taking the Fourier transform in x, we conclude

$$\begin{aligned}&\partial_t\hat{f}(t,k,v)+iv\cdot k\hat{f}(t,k,v)-\sqrt{\mu}v\cdot\widehat{\nabla_x\phi}(t,k)+L\hat{f}(t,k,v)\\=&\widehat{\Gamma(f,f)}(t,k,v)-\widehat{\nabla_x\phi\cdot\nabla_v}f(t,k,v)+\frac{1}{2}v\cdot\widehat{\nabla_x\phi f}(t,k,v).\end{aligned} \tag{2.5}$$

To the end we always use the notation

$$\widehat{\Gamma(f,g)}(k,v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\sigma)\mu^{1/2}(u)\big([\hat{f}(u')*\hat{g}(v')](k)-[\hat{f}(u)*\hat{g}(v)](k)\big)d\sigma du,$$

where the convolutions are taken with respect to k:

$$\begin{aligned}[\hat{f}(u')*\hat{g}(v')](k)&=\int_{\mathbb{Z}^3}\hat{f}(k-l,u')\hat{g}(l,v')d\Sigma(l),\\ [\hat{f}(u)*\hat{g}(v)](k)&=\int_{\mathbb{Z}^3}\hat{f}(k-l,u)\hat{g}(l,v)d\Sigma(l).\end{aligned}$$

For the other two terms, we have

$$\begin{aligned}\widehat{\nabla_x\phi\cdot\nabla_v}f(k,v)&=\int_{\mathbb{Z}^3}\widehat{\nabla_x\phi}(k-l)\cdot\nabla_v\hat{f}(l,v)d\Sigma(l),\\ v\cdot\widehat{\nabla_x\phi f}(k,v)&=v\cdot\int_{\mathbb{Z}^3}\widehat{\nabla_x\phi}(k-l)\hat{f}(l,v)d\Sigma(l).\end{aligned}$$

With the above information provided, we first give the estimate on the nonlinear term $\widehat{\Gamma(f,g)}$ in the following lemma whose proof can be found in [14].

LEMMA 2.4. *Let* $\gamma > \max\{-3, -2s - \frac{3}{2}\}$. *Then we have*

$$\left|\left(\widehat{\Gamma(f,g)}(k), \hat{h}(k)\right)_{L^2_v}\right| \lesssim \int_{\mathbb{Z}^3} |\hat{f}(k-l)|_{L^2_v} \ |\hat{g}(l)|_D \ |\hat{h}(k)|_D \ d\Sigma(l). \tag{2.6}$$

The following lemma concerns the estimate on $\widehat{\nabla_x\phi\cdot\nabla_v f}$.

LEMMA 2.5. *Let* $0 \le \gamma < 1$, $1/2 < s < 1$. *It holds that*

$$\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f}, \hat{h}\right)_{L^2_v}\right| \lesssim \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ |\hat{f}(l)|_D \ |\hat{h}(k)|_D \ d\Sigma(l). \tag{2.7}$$

Proof. Noticing $1/2 < s < 1$, by Fubini's theorem and Parseval indentity, one has

$$\begin{aligned}
\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f}, \hat{h}\right)_{L^2_v}\right| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{Z}^3} \widehat{\nabla_x\phi}(k-l)\cdot\nabla_v\hat{f}(l,v) \ \bar{\hat{h}}(k,v) d\Sigma(l) dv\right| \\
&= \left|\int_{\mathbb{Z}^3}\int_{\mathbb{R}^3} \widehat{\nabla_x\phi}(k-l)\cdot\nabla_v\hat{f}(l,v) \ \bar{\hat{h}}(k,v) dv d\Sigma(l)\right| \\
&= \left|\int_{\mathbb{Z}^3} \widehat{\nabla_x\phi}(k-l)\cdot\int_{\mathbb{R}^3} i\xi \ \mathscr{F}_v[\hat{f}(l)] \ \overline{\mathscr{F}_v[\hat{h}(k)]} \ d\xi dl\right| \\
&\le \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ \left|\langle\xi\rangle^{1/2}\mathscr{F}_v[\hat{f}(l)]\right|_{L^2_\xi} \ \left|\langle\xi\rangle^{1/2}\mathscr{F}_v[\hat{h}(k)]\right|_{L^2_\xi} d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ |\hat{f}(l)|_{H^s} \ |\hat{h}(k)|_{H^s} d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ |\hat{f}(l)|_{H^s_{\gamma/2}} \ |\hat{h}(k)|_{H^s_{\gamma/2}} d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ |\hat{f}(l)|_D \ |\hat{h}(k)|_D \ d\Sigma(l)
\end{aligned}$$

where $\mathscr{F}_v$ is the Fourier transform with respect to v-variable, ξ denotes the corresponding frequency variable. Here we have used (2.3) and the fact that: for $\gamma \ge 0$ and suitable function g,

$$|g|_{H^s} \le |g|_{H^s_{\gamma/2}}.$$

This completes the proof of Lemma 2.5. ◻

The following lemma concerns the estimate on $v\cdot\widehat{\nabla_x\phi f}$.

LEMMA 2.6. *Let* $0 \le \gamma < 1$, $1/2 < s < 1$. *It holds that*

$$\left|\left(v\cdot\widehat{\nabla_x\phi f}, \hat{h}\right)_{L^2_v}\right| \lesssim \int_{\mathbb{Z}^3} |\widehat{\nabla_x\phi}(k-l)| \ |\hat{f}(l)|_D \ |\hat{h}(k)|_D \ d\Sigma(l). \tag{2.8}$$

Proof. We use Fubini's theorem and Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
\left|\left(v\cdot\widehat{\nabla_x\phi f},\hat{h}\right)_{L^2_v}\right| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{Z}^3} v\cdot\widehat{\nabla_x\phi}(k-l)\hat{f}(l,v)\ \bar{\hat{h}}(k,v)d\Sigma(l)dv\right| \\
&= \left|\int_{\mathbb{Z}^3}\int_{\mathbb{R}^3} v\cdot\widehat{\nabla_x\phi}(k-l)\hat{f}(l,v)\ \bar{\hat{h}}(k,v)dvd\Sigma(l)\right| \\
&\le \int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\left(\int_{\mathbb{R}^3}|\langle v\rangle^{1/2}\hat{f}(l,v)|\ |\langle v\rangle^{1/2}\hat{h}(k,v)|\ dv\right)d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\ |\langle v\rangle^{1/2}\hat{f}(l,v)|_{L^2_v}\ |\langle v\rangle^{1/2}\hat{h}(k,v)|_{L^2_v}d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\ |\hat{f}(l,v)|_{L^2_{s+\gamma/2}}\ |\hat{h}(k,v)|_{L^2_{s+\gamma/2}}d\Sigma(l) \\
&\le \int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\ |\hat{f}(l)|_D\ |\hat{h}(k)|_D\ d\Sigma(l).
\end{aligned}
$$

Here, we have used the fact that: Since $\gamma\ge 0, 1/2<s<1$, we have $s+\gamma/2\ge 1/2$, and by Lemma 2.2, one has

$$|\langle v\rangle^{1/2}g|_{L^2_v}\le |g|_{L^2_{s+\gamma/2}}\lesssim |g|_D,$$

for suitable function g. And this completes the proof of Lemma 2.6. ∎

3. Nonlinear estimates

The goal of this section is to make the energy estimates on those nonlinear terms in (2.5). We always suppose $0\le\gamma<1$, $1/2\le s<1$ in the sequel. The first lemma concerns the estimates on the nonlinear term $\widehat{\Gamma(f,f)}$, the proof of the estimate (2.6) will be used, which has been proved in [14].

LEMMA 3.1. *It holds that*

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\Gamma(f,g)},\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k) \\
\le&\eta\|h\|_{L^1_kL^2_TL^2_{v,D}}+C_\eta\|f\|_{L^1_kL^\infty_TL^2_v}\|g\|_{L^1_kL^2_TL^2_{v,D}},
\end{aligned}
\tag{3.1}
$$

where $\eta>0$ is an arbitrary small constant, and C_η is a universal large constant depending only on η.

For the estimate on $\widehat{\nabla_x\phi\cdot\nabla_v f}$, we have:

LEMMA 3.2. *It holds that*

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f},\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k) \\
\le&\eta\|h\|_{L^1_kL^2_TL^2_{v,D}}+C_\eta\|\nabla_x\phi\|_{L^1_kL^\infty_T}\|f\|_{L^1_kL^2_TL^2_{v,D}},
\end{aligned}
\tag{3.2}
$$

where $\eta>0$ is an arbitrary small constant, and C_η is a universal large constant depending only on η.

Proof. Firstly, we use (2.7) to yield

$$\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f},\hat{h}\right)_{L^2_v}\right|\le\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\ |\hat{f}(l)|_D\ |\hat{h}(k)|_D\ d\Sigma(l).$$

Then, as in [14], we apply Cauchy-Schwarz inequality with respect to $\int_0^T(\cdot)dt$ and further use Young inequality to yield

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f},\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\
\leq&\int_{\mathbb{Z}^3}\left(\int_0^T\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\,|\hat{f}(t,l)|_D\,|\hat{h}(t,k)|_D\,d\Sigma(l)\,dt\right)^{1/2}d\Sigma(k)\\
\leq&\int_{\mathbb{Z}^3}\left(\int_0^T\big(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\,|\hat{f}(t,l)|_D\,d\Sigma(l)\big)^2dt\right)^{1/4}\times\big(\int_0^T|\hat{h}(t,k)|^2_D dt\big)^{1/4}d\Sigma(k)\\
\leq&\eta\int_{\mathbb{Z}^3}\big(\int_0^T|\hat{h}(t,k)|^2_D\,dt\big)^{1/2}d\Sigma(k)\\
&\qquad+C_\eta\int_{\mathbb{Z}^3}\left(\int_0^T\big(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\,|\hat{f}(t,l)|_D\,d\Sigma(l)\big)^2dt\right)^{1/2}d\Sigma(k),
\end{aligned}
$$

where $\eta>0$ is an arbitrary small constant. For the second term on the right-hand side of the above estimate, by using Minkowski inequality, one has

$$
\begin{aligned}
&\left(\int_0^T\big(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\,|\hat{f}(t,l)|_D\,d\Sigma(l)\big)^2dt\right)^{1/2}\\
\leq&\int_{\mathbb{Z}^3}\left(\int_0^T|\widehat{\nabla_x\phi}(t,k-l)|^2\;|\hat{f}(t,l)|^2_D\,dt\right)^{1/2}d\Sigma(l).
\end{aligned}
$$

Therefore we can get

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\left(\int_0^T\big(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\,|\hat{f}(t,l)|_D\,d\Sigma(l)\big)^2dt\right)^{1/2}dk\\
\leq&\int_{\mathbb{Z}^3}\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\widehat{\nabla_x\phi}(t,k-l)|\left(\int_0^T|\hat{f}(t,l)|^2_D\,dt\right)^{1/2}d\Sigma(l)d\Sigma(k).
\end{aligned}
$$

Subsequently, we use Fubini's theorem and translation invariance to compute

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\widehat{\nabla_x\phi}(t,k-l)|\left(\int_0^T|\hat{f}(t,l)|^2_D\,dt\right)^{1/2}d\Sigma(l)d\Sigma(k)\\
=&\int_{\mathbb{Z}^3}\left(\int_0^T|\hat{f}(t,l)|^2_D\,dt\right)^{1/2}\left(\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\widehat{\nabla_x\phi}(t,k-l)|d\Sigma(k)\right)d\Sigma(l)\\
=&\|\nabla_x\phi\|_{L^1_kL^\infty_T}\int_{\mathbb{Z}^3}\left(\int_0^T|\hat{f}(t,l)|^2_D\,dt\right)^{1/2}d\Sigma(l).
\end{aligned}
$$

By collecting all the estimates above, we can get the desired estimate (3.2). ∎

For the estimate on $v\cdot\widehat{\nabla_x\phi f}$, we have:

LEMMA 3.3. *It holds that*

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(v\cdot\widehat{\nabla_x\phi f},\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\
\leq&\eta\|h\|_{L^1_kL^2_TL^2_{v,D}}+C_\eta\|\nabla_x\phi\|_{L^1_kL^\infty_T}\|f\|_{L^1_kL^2_TL^2_{v,D}},
\end{aligned}
\tag{3.3}
$$

where $\eta>0$ is an arbitrary small constant, and C_η is a universal large constant depending only on η.

Proof. We use (2.8) to obtain

$$\begin{aligned}&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(v\cdot\widehat{\nabla_x\phi f},\hat{h}\right)_{L^2_v}\right|dt\right)^{1/2}dk\\ &\le\int_{\mathbb{Z}^3}\left(\int_0^T\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(k-l)|\;|\hat{f}(l)|_D\;|\hat{h}(k)|_D\;d\Sigma(l)dt\right)^{1/2}d\Sigma(k).\end{aligned}$$

The remaining process of the proof is similar to that of Lemma 3.2 above, we omit it for brevity of presentation. ∎

The following lemmas will be useful for dealing with the nonlinear terms arising from the macroscopic estimates in the next section.

Lemma 3.4 ([14] Lemma 4.4). *Assume that $\zeta(v)$ depends only on v and decays rapidly at infinity. Then we have*

$$\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\Gamma(f,f)},\zeta(v)\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k)\lesssim\|f\|_{L^1_kL^\infty_TL^2_v}\|f\|_{L^1_kL^2_TL^2_{v,D}}. \tag{3.4}$$

Lemma 3.5. *Assume that $\zeta(v)$ depends only on v and decays rapidly at infinity. Then we have*

$$\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\nabla_x\phi\cdot\nabla_vf},\zeta(v)\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k)\lesssim\|\nabla_x\phi\|_{L^1_kL^\infty_T}\|f\|_{L^1_kL^2_TL^2_{v,D}} \tag{3.5}$$

and

$$\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(v\cdot\widehat{\nabla_x\phi f},\zeta(v)\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k)\lesssim\|\nabla_x\phi\|_{L^1_kL^\infty_T}\|f\|_{L^1_kL^2_TL^2_{v,D}}. \tag{3.6}$$

Proof. We just need to prove (3.5), since the proof of (3.6) is similar. By using Lemma 2.5, one has

$$\begin{aligned}&\left(\int_0^T\left|\left(\widehat{\nabla_x\phi\cdot\nabla_vf},\zeta(v)\right)_{L^2_v}\right|^2dt\right)^{1/2}\\ &\lesssim\left(\int_0^T\left(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\;|\hat{f}(t,l)|_D\;d\Sigma(l)\right)^2dt\right)^{1/2}.\end{aligned}$$

Then we use the Minkowski inequality

$$\big|\big|\|\cdot\|_{L^1_l}\big|\big|_{L^2_t}\le\big|\big|\|\cdot\|_{L^2_t}\big|\big|_{L^1_l}$$

to obtain

$$\begin{aligned}&\left(\int_0^T\left(\int_{\mathbb{Z}^3}|\widehat{\nabla_x\phi}(t,k-l)|\;|\hat{f}(t,l)|_D\;d\Sigma(l)\right)^2dt\right)^{1/2}\\ &\lesssim\int_{\mathbb{Z}^3}\left(\int_0^T|\widehat{\nabla_x\phi}(t,k-l)|^2\;|\hat{f}(t,l)|_D^2\;dt\right)^{1/2}d\Sigma(l).\end{aligned}$$

We further apply Fubini's theorem and translation invariance to get

$$\begin{aligned}
&\int_{\mathbb{Z}^3}\int_{\mathbb{Z}^3}\left(\int_0^T |\widehat{\nabla_x\phi}(t,k-l)|^2\ |\hat{f}(t,l)|_D^2\ dt\right)^{1/2} d\Sigma(l)d\Sigma(k)\\
\lesssim&\int_{\mathbb{Z}^3}\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\widehat{\nabla_x\phi}(t,k-l)|\left(\int_0^T |\hat{f}(t,l)|_D^2\ dt\right)^{1/2} d\Sigma(l)d\Sigma(k)\\
\lesssim&\int_{\mathbb{Z}^3}\left(\int_0^T |\hat{f}(t,l)|_D^2\ dt\right)^{1/2}\left(\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\widehat{\nabla_x\phi}(t,k-l)|d\Sigma(k)\right)d\Sigma(l)\\
\lesssim&\|\nabla_x\phi\|_{L^1_kL^\infty_T}\|f\|_{L^1_kL^2_TL^2_{v,D}}.
\end{aligned}$$

This completes the proof of Lemma 3.5. ∎

4. Macroscopic estimates

In this section we shall derive the uniform a priori estimates for the macroscopic part of a solution to the linearized system with a nonhomogeneous source $h(t,x,v)$:

$$\begin{cases}\partial_t f+v\cdot\nabla_x f-\nabla_x\phi\cdot v\sqrt{\mu}+Lf=h,\\ \Delta_x\phi=\int_{\mathbb{R}^3}\sqrt{\mu}f dv,\\ f(0,x,v)=f_0(x,v).\end{cases}\tag{4.1}$$

Notice that for the nonlinear Vlasov-Poisson-Boltzmann system (1.4) and (1.5), the nonhomogeneous source takes the form of

$$h=-\nabla_x\phi\cdot\nabla_v f+\frac{1}{2}v\cdot\nabla_x\phi f+\Gamma(f,f),$$

which satisfies the mass conservation law

$$(h,\sqrt{\mu})_{L^2_v}=0.\tag{4.2}$$

Here we emphasize that: throughout this section, $T>0$ is an arbitrary fixed constant, and the universal constant $C>0$ is independent of T.

THEOREM 4.1. *Let f be a solution to the linearized system (4.1). If h satisfies (4.2), then we have*

$$\begin{aligned}\|[a,b,c]\|_{L^1_kL^2_T}\lesssim&\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^1_kL^2_TL^2_{v,D}}+\|f\|_{L^1_kL^\infty_TL^2_v}+\|f_0\|_{L^1_kL^2_v}\\ &+\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\hat{h}(t,k),\mu^{1/4}\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k),\end{aligned}\tag{4.3}$$

where $[\cdot,\cdot,\cdot]$ represents a vector.

The rest of this section is devoted to the proof of Theorem 4.1. Firstly, we give the macroscopic equations of the linearized system (4.1). Define the moment functions $\Theta=(\Theta_{jm}(\cdot))_{3\times3}$ and $\Lambda=(\Lambda_j(\cdot))_{1\le j\le 3}$ by

$$\Theta_{jm}(f)=\int_{\mathbb{R}^3}(v_jv_m-1)\mu^{1/2}fdv,\qquad \Lambda_j(f)=\frac{1}{10}\int_{\mathbb{R}^3}(|v|^2-5)v_j\mu^{1/2}fdv.$$

Then, as in [20], we can deduce a fluid-type system of equations

$$\begin{cases}\partial_t a+\nabla_x\cdot b=0,\\ \partial_t b_j+\partial_j(a+2c)+\sum_{m=1}^{3}\partial_m\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}f)-\partial_j\phi=(h,v_j\sqrt{\mu})_{L^2_v},\\ \partial_t c+\frac{1}{3}\nabla_x\cdot b+\frac{5}{3}\sum_{j=1}^{3}\partial_j\Lambda_j(\{\mathbf{I}-\mathbf{P}\}f)=\frac{1}{6}(h,(|v|^2-3)\sqrt{\mu})_{L^2_v},\\ \Delta_x\phi=a,\\ \partial_t\Lambda_j(\{\mathbf{I}-\mathbf{P}\}f)+\partial_j c=\Lambda_j(r+h),\\ \partial_t\{\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}f)+2c\delta_{jm}\}+\partial_j b_m+\partial_m b_j=\Theta_{jm}(r+h),\end{cases}\tag{4.4}$$

with

$$r=-v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+Lf.$$

As in [36], we construct a time-frequency interactive functional to deduce the following estimate. In order to express clearly, some notations are given as follows. For two complex vectors $z_1,\ z_2\in\mathbb{C}^3$, $(z_1|z_2)=z_1\cdot\overline{z_2}$ denotes the dot product in the complex field $\mathbb{C}^3$, where $\overline{z_2}$ is the complex conjugate of z_2. And we usually use $\mathscr{R}$ to denote the real part of a complex number.

Lemma 4.1. *For any $t\geq 0$ and $k\in\mathbb{T}^3$, there are two suitable constants $0<\kappa_2\ll\kappa_1$ such that the time-frequency interactive functional $\mathcal{E}_{int}(\hat{f})(t,k)$ defined by*

$$\begin{aligned}\mathcal{E}_{int}(\hat{f})(t,k)=&\frac{1}{1+|k|^2}\sum_{j=1}^{3}\left(ik_j\hat{c}|\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})\right)\\ &+\frac{\kappa_1}{1+|k|^2}\sum_{j,m=1}^{3}\left(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\right)\\ &+\frac{\kappa_2}{1+|k|^2}\sum_{j=1}^{3}\left(ik_j\hat{a}|\widehat{b_j}\right)\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\partial_t\mathscr{R}\mathcal{E}_{int}(\hat{f})(t,k)+\frac{\lambda|k|^2}{1+|k|^2}(|\hat{b}|^2+|\hat{c}|^2)+\lambda|\hat{a}|^2\\ \lesssim&|\{\mathbf{I}-\mathbf{P}\}\hat{f}|_D^2+|(\hat{h},\mu^{1/4})_{L^2_v}|^2.\end{aligned}\tag{4.6}$$

Proof. We shall make estimates on $\hat{b},\hat{c},\hat{a}$ individually and then take the proper linear combination to deduce the desired energy inequality (4.6).

Estimate of $\hat{b}$. For any $0<\eta<1$, one has

$$\begin{aligned}&\partial_t\mathscr{R}\sum_{j,m=1}^{3}\left(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\{\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\}\right)+|k|^2|\hat{b}|^2\\ \leq&\eta(1+|k|^2)|\hat{a}|^2+C_\eta|k|^2|\hat{c}|^2+C_\eta(1+|k|^2)|\{\mathbf{I}-\mathbf{P}\}\hat{f}|_D^2+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2.\end{aligned}\tag{4.7}$$

In fact, observe the identity

$$\sum_{j,m=1}^{3}|ik_j\widehat{b_m}+ik_m\widehat{b_j}|^2=2|k|^2|\hat{b}|^2+2|k\cdot\hat{b}|^2.$$

On the other hand, compute from $(4.4)_6$ that

$$\begin{aligned}&\sum_{j,m=1}^{3}|ik_j\widehat{b_m}+ik_m\widehat{b_j}|^2\\&=\sum_{j,m=1}^{3}\Big(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|-\partial_t\{\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\}+\Theta_{jm}(\hat{r}+\hat{h})\Big)\\&=-\partial_t\sum_{j,m=1}^{3}\Big(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\{\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\}\Big)\\&\quad+\sum_{j,m=1}^{3}\Big(ik_j\partial_t\widehat{b_m}+ik_m\partial_t\widehat{b_j}\Big|\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\Big)\\&\quad+\sum_{j,m=1}^{3}\Big(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\Theta_{jm}(\hat{r}+\hat{h})\Big).\end{aligned}$$

We further have

$$\begin{aligned}&\partial_t\sum_{j,m=1}^{3}\Big(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\{\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\}\Big)+2|k|^2|\hat{b}|^2+2|k\cdot\hat{b}|^2\\&=\sum_{j,m=1}^{3}\Big(ik_j\partial_t\widehat{b_m}+ik_m\partial_t\widehat{b_j}\Big|\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2\hat{c}\delta_{jm}\Big)\\&\quad+\sum_{j,m=1}^{3}\Big(ik_j\widehat{b_m}+ik_m\widehat{b_j}\Big|\Theta_{jm}(\hat{r}+\hat{h})\Big)=S_1+S_2.\end{aligned}\tag{4.8}$$

For S_1, we decompose it as

$$\begin{aligned}S_1=&\sum_{j,m=1}^{3}\Big(-\partial_t\widehat{b_m}\Big|ik_j\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2ik_j\hat{c}\delta_{jm}\Big)\\&+\sum_{j,m=1}^{3}\Big(-\partial_t\widehat{b_j}\Big|ik_m\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})+2ik_m\hat{c}\delta_{jm}\Big).\end{aligned}$$

One can use the Fourier transform of $(4.4)_2$ and $(4.4)_4$:

$$\partial_t\widehat{b_m}+ik_m(\hat{a}+2\hat{c})+\sum_{j=1}^{3}ik_j\Theta_{mj}(\{\mathbf{I}-\mathbf{P}\}\hat{f})-ik_m\hat{\phi}=(\hat{h},v_m\sqrt{\mu})_{L^2_v}.\tag{4.9}$$

$$-|k|^2\hat{\phi}=\hat{a}\tag{4.10}$$

to estimate it as

$$
\begin{aligned}
S_1 \leq & \eta|k|^2(|\hat{a}|^2+|\hat{c}|^2)+\eta|\hat{a}|^2+C_\eta|k|^2|\hat{c}|^2+C_\eta|k|^2\sum_{j,m}|\Theta_{j,m}\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2 \\
\leq & \eta(1+|k|^2)|\hat{a}|^2+C_\eta|k|^2|\hat{c}|^2+C_\eta|k|^2|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2. \qquad (4.11)
\end{aligned}
$$

For S_2, notice that

$$
r=-v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+Lf.
$$

We then use (2.4) to obtain

$$
|\Theta_{jm}(\hat{r})|^2\lesssim(1+|k|^2)|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D.
$$

Therefore, S_2 is bounded by

$$
\begin{aligned}
S_2 \leq & \eta|k|^2|\widehat{b_m}|^2+C_\eta\sum_{jm}(|\Theta_{jm}(\hat{r})|^2+|\Theta_{jm}(\hat{h})|^2) \\
\leq & \eta|k|^2|\widehat{b_m}|^2+C_\eta(1+|k|^2)|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2. \qquad (4.12)
\end{aligned}
$$

Thus, with $\eta>0$ suitably small, (4.7) follows by taking the real part of (4.8), plugging (4.11) and (4.12) into it.
Estimate of $\hat{c}$. For any $0<\eta<1$, one has

$$
\begin{aligned}
&\partial_t\mathscr{R}\sum_{j=1}^{3}(\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})|ik_j\hat{c})+(1-\eta)|k|^2|\hat{c}|^2 \\
\leq & \eta|k|^2|\hat{b}|^2+C_\eta(1+|k|^2)|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2. \qquad (4.13)
\end{aligned}
$$

In fact, the Fourier transform of $(4.4)_5$ gives

$$
\partial_t\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})+ik_j\hat{c}=\Lambda_j(\hat{r}+\hat{h}),
$$

we then take the complex dot product with $ik_j\hat{c}$ to derive

$$
\begin{aligned}
&\partial_t(\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})|ik_j\hat{c})+|k_j|^2|\hat{c}|^2 \\
=&(\Lambda_j(\hat{r}+\hat{h})|ik_j\hat{c})+(\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})|ik_j\partial_t\hat{c}) \\
=&S_3+S_4. \qquad (4.14)
\end{aligned}
$$

S_3 is bounded by

$$
\begin{aligned}
S_3 \leq & \eta|k_j|^2|\hat{c}|^2+C_\eta(|\Lambda_j(\hat{r})|^2+|\Lambda_j(\hat{h})|^2 \\
\leq & \eta|k_j|^2|\hat{c}|^2+C_\eta(1+|k|^2)|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D+C_\eta|(\hat{h},\mu^{1/4})_{L^2_v}|^2. \qquad (4.15)
\end{aligned}
$$

For S_4, taking the Fourier transform of $(4.4)_3$

$$
\partial_t\hat{c}+\frac{1}{3}k\cdot\hat{b}+\frac{5}{3}\sum_{j=1}^{3}ik_j\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat{f})=\frac{1}{6}(\hat{h},(|v|^2-3)\sqrt{\mu})_{L^2_v}
$$

to replace $\partial_t\hat{c}$, one has

$$
S_4\leq\eta|k|^2|\hat{b}|^2+C_\eta|k|^2|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2. \qquad (4.16)
$$

Hence, one can take the real part of (4.14) and apply (4.15) and (4.16), and then take the summation over $\{j=1,2,3\}$ to deduce (4.13).

<u>*Estimate on $\hat{a}$*</u>. For any $0<\eta<1$, one has

$$\begin{aligned}
&\partial_t \mathscr{R}\sum_{j=1}^{3}(\widehat{b_j}|ik_j\hat{a})+(1-\eta)(1+|k|^2)|\hat{a}|^2\\
&\leq |k|^2|\hat{b}|^2+C_\eta|k|^2(|\hat{c}|^2+|\{\mathbf{I}-\mathbf{P}\}\hat{f}|_D^2)+C|(\hat{h},\mu^{1/4})_{L^2_v}|^2.
\end{aligned}\tag{4.17}$$

Indeed, we take the complex inner product with $ik_j\hat{a}$ and (4.9), and then take the summation over $\{j=1,2,3\}$ to get

$$\begin{aligned}
&\partial_t\sum_j(\widehat{b_j}|ik_j\hat{a})+|k|^2|\hat{a}|^2-\sum_j(ik_j\hat{\phi}|ik_j\hat{a})\\
&=-2\sum_j(ik_j\hat{c}|ik_j\hat{a})-\sum_{j,m}(ik_m\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat{f})|ik_j\hat{a})\\
&\quad+\sum_j\big((\hat{h},v_j\sqrt{\mu})_{L^2_v}|ik_j\hat{a}\big)+\sum_j(\widehat{b_j}|ik_j\partial_t\hat{a}).
\end{aligned}\tag{4.18}$$

Using (4.10), one has

$$-\sum_j(ik_j\hat{\phi}|ik_j\hat{a})=\sum_j\left(k_j^2\frac{\hat{a}}{|k|^2}|\hat{a}\right)=|\hat{a}|^2.$$

The first two terms on the right-hand side of (4.18) are bounded by

$$\eta|k|^2|\hat{a}|^2+C_\eta|k|^2(|\hat{c}|^2+|\{\mathbf{I}-\mathbf{P}\}\hat{g}|_D^2),$$

the third term is bounded by

$$\eta|k|^2|\hat{a}|^2+C_\eta|(\hat{h},\mu^{1/4})_{L^2_v}|^2,$$

while for the last term, we have

$$\sum_j(\widehat{b_j}|ik_j\partial_t\hat{a})=\sum_j(\widehat{b_j}|ik_j(-ik\cdot\hat{b}))=|k\cdot\hat{b}|^2\leq|k|^2|\hat{b}|^2.$$

Here we have used the Fourier transform of $(4.4)_1$

$$\partial_t\hat{a}+ik\cdot\hat{b}=0.$$

Then, putting the above estimates into (4.18) and taking the real part yields (4.17) .

Finally, (4.6) follows from the proper linear combination of (4.7), (4.13) and (4.17) by taking $0<\eta<1$ small enough and choosing two suitable constants $0<\kappa_2\ll\kappa_1$. This completes the proof of Lemma 4.1. □

Proof. **(Proof of Theorem 4.1.)** By using (4.10), one has

$$\begin{aligned}
&\frac{|k|^2}{1+|k|^2}(|\hat{b}|^2+|\hat{c}|^2)+|\hat{a}|^2\\
&\geq\frac{|k|^2}{1+|k|^2}\left(|\hat{a}|^2+|\hat{b}|^2+|\hat{c}|^2+\frac{1}{|k|^2}|\hat{a}|^2\right)\\
&\geq\frac{|k|^2}{1+|k|^2}\big(|\hat{a}|^2+|\hat{b}|^2+|\hat{c}|^2+|\widehat{\nabla_x\phi}|^2\big).
\end{aligned}$$

We apply the result in Lemma 4.1 to give

$$\begin{aligned}&\partial_t\mathscr{R}\mathcal{E}_{int}(\hat f)(t,k)+\frac{\lambda|k|^2}{1+|k|^2}\big(|\hat a|^2+|\hat b|^2+|\hat c|^2+|\widehat{\nabla_x\phi}|^2\big)\\ \lesssim&|\{\mathbf{I}-\mathbf{P}\}\hat f|_D^2+|(\hat h,\mu^{1/4})_{L^2_v}|^2.\end{aligned}$$

Due to $k\in\mathbb{Z}^3$ it holds that

$$\begin{aligned}&\partial_t\mathscr{R}\mathcal{E}_{int}(\hat f)(t,k)+\lambda\big(|\hat a|^2+|\hat b|^2+|\hat c|^2+|\widehat{\nabla_x\phi}|^2\big)\\ \lesssim&|\{\mathbf{I}-\mathbf{P}\}\hat f|_D^2+|(\hat h,\mu^{1/4})_{L^2_v}|^2,\end{aligned}$$

we then integrate it over $[0,T]$ to yield

$$\begin{aligned}&\mathscr{R}\mathcal{E}_{int}(\hat f)(T,k)-\mathscr{R}\mathcal{E}_{int}(\hat f)(0,k)+\lambda\int_0^T\big(|\hat a|^2+|\hat b|^2+|\hat c|^2+|\widehat{\nabla_x\phi}|^2\big)dt\\ \lesssim&\int_0^T|\{\mathbf{I}-\mathbf{P}\}\hat f|_D^2dt+\int_0^T|(\hat h,\mu^{1/4})_{L^2_v}|^2dt.\end{aligned}$$

From (4.5), one has

$$\begin{aligned}|\mathcal{E}_{int}(\hat f)(t,k)|\lesssim&(|\hat a|^2+|\hat b|^2+|\hat c|^2)+\sum_{j,m=1}^3\big(|\Theta_{jm}(\{\mathbf{I}-\mathbf{P}\}\hat f)|^2+|\Lambda_j(\{\mathbf{I}-\mathbf{P}\}\hat f)|^2\big)\\ \lesssim&|\mathbf{P}\hat f|_{L^2_v}^2+|\{\mathbf{I}-\mathbf{P}\}\hat f|_{L^2_v}^2\lesssim|\hat f(t,k)|_{L^2_v}^2.\end{aligned}$$

We further have

$$\begin{aligned}&\int_0^T\big(|\hat a|^2+|\hat b|^2+|\hat c|^2+|\widehat{\nabla_x\phi}|^2\big)dt\\ \lesssim&|\hat f(T,k)|_{L^2_v}^2+|\hat f(0,k)|_{L^2_v}^2+\int_0^T|\{\mathbf{I}-\mathbf{P}\}\hat f|_D^2dt+\int_0^T|(\hat h,\mu^{1/4})_{L^2_v}|^2dt.\end{aligned}$$

Thus, we can use the above inequality to yield the desired estimate (4.3). This completes the proof of Theorem 4.1. □

5. Proof of the main results

In this section we give the proof of Theorems 1.1 and 1.2. Recall (1.9) and (1.10) for $\mathscr{E}_T(f)$ and $\mathscr{D}_T(f)$, respectively.

Proof. **(Proof of Theorem 1.1.)** Firstly, we will deduce the uniform a priori estimate on the solution to the Cauchy problem (1.4)-(1.6). The Fourier transform in x of (1.4) gives

$$\begin{aligned}&\partial_t\hat f(t,k,v)+iv\cdot k\hat f(t,k,v)-\sqrt{\mu}v\cdot\widehat{\nabla_x\phi}(t,k)+L\hat f(t,k,v)\\ =&\widehat{\Gamma(f,f)}(t,k,v)-\widehat{\nabla_x\phi\cdot\nabla_v f}(t,k,v)+\frac12 v\cdot\widehat{\nabla_x\phi f}(t,k,v)=\sum_{j=1}^3H_j.\end{aligned}\tag{5.1}$$

Taking the product of (5.1) with the complex conjugate of $\hat f(t,k,v)$ and further taking the real part of the resulting equation, one has

$$\frac12\frac{d}{dt}|\hat f(t,k,v)|^2-\mathscr{R}(ik\cdot v\hat\phi\sqrt{\mu}|\hat f)+\mathscr{R}(L\hat f|\hat f)=\sum_{j=1}^3\mathscr{R}(\widehat{H_j}|\hat f),$$

where $(\cdot|\cdot)$ denotes the complex inner product over the complex field. Integrating the above identity with respect to v, one has

$$\frac{1}{2}\frac{d}{dt}|\hat{f}(t,k)|^2_{L^2_v} - \mathscr{R}(ik\cdot v\hat{\phi}\sqrt{\mu},\hat{f})_{L^2_v} + \mathscr{R}(L\hat{f},\hat{f})_{L^2_v} = \sum_{j=1}^{3}\mathscr{R}(\widehat{H_j},\hat{f})_{L^2_v}, \tag{5.2}$$

Notice that by (2.2),

$$\mathscr{R}(L\hat{f},\hat{f})_{L^2_v} \geq \lambda|\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D.$$

Applying $(2.1)_3$, $(4.4)_1$ and $(4.4)_4$, one has

$$\begin{aligned}-\mathscr{R}(ik\cdot v\hat{\phi}\sqrt{\mu},\hat{f})_{L^2_v} &= -\mathscr{R}(i\hat{\phi}|k\cdot\hat{b}) = -\mathscr{R}(i\hat{\phi}|i\partial_t\hat{a})\\ &= -\mathscr{R}(i\hat{\phi}|-i|k|^2\hat{\phi}) = \frac{1}{2}\frac{d}{dt}|\widehat{\nabla_x\phi}(t,k)|^2.\end{aligned}$$

With the above estimatess, we now integrate the identity (5.2) with respect to t to yield

$$\begin{aligned}&\frac{1}{2}|\hat{f}(t,k)|^2_{L^2_v} + \frac{1}{2}|\widehat{\nabla_x\phi}(t,k)|^2 + \lambda\int_0^t |\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D d\tau\\ =&\frac{1}{2}|\hat{f}_0(k)|^2_{L^2_v} + \frac{1}{2}|\widehat{\nabla_x\phi_0}(k)|^2 + \sum_{j=1}^{3}\int_0^t \mathscr{R}(\widehat{H_j},\hat{f})_{L^2_v}d\tau.\end{aligned}$$

We further have

$$\begin{aligned}&|\hat{f}(t,k)|_{L^2_v} + |\widehat{\nabla_x\phi}(t,k)| + \left(\int_0^t |\{\mathbf{I}-\mathbf{P}\}\hat{f}|^2_D d\tau\right)^{1/2}\\ \leq& C_0\left\{|\hat{f}_0(k)|_{L^2_v} + |\widehat{\nabla_x\phi_0}(k)| + \sum_{j=1}^{3}\left(\int_0^t |(\widehat{H_j},\hat{f})_{L^2_v}|d\tau\right)^{1/2}\right\},\end{aligned}$$

where $C_0>0$ is a suitable constant. Moreover, we take $\sup_{0\leq t\leq T}$ on both sides of the above estimate, and then integrate the resulting inequality with respect to k to deduce

$$\begin{aligned}&\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\hat{f}(t,k)|_{L^2_v}d\Sigma(k) + \int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\widehat{\nabla_x\phi}(t,k)|d\Sigma(k)\\ &\qquad + \int_{\mathbb{Z}^3}\left(\int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|^2_D dt\right)^{1/2} d\Sigma(k)\\ \leq& C_0\left\{\|f_0\|_{L^1_kL^2_v} + \|\nabla_x\phi_0\|_{L^1_k} + \sum_{j=1}^{3}\int_{\mathbb{Z}^3}\left(\int_0^T |(\widehat{H_j},\hat{f})_{L^2_v}|dt\right)^{1/2} d\Sigma(k)\right\}.\end{aligned} \tag{5.3}$$

Recall that

$$\mathscr{E}_T(f) = \|f\|_{L^1_kL^\infty_TL^2_v} + \|\nabla_x\phi\|_{L^1_kL^\infty_T}$$

and

$$\mathscr{D}_T(f) = \|f\|_{L^1_kL^2_TL^2_{v,D}}, \qquad \epsilon_0 = \|f_0\|_{L^1_kL^2_v} + \|\nabla_x\phi_0\|_{L^1_k}.$$

We then apply (3.1), (3.2), (3.3) and (5.3) to deduce

$$\begin{aligned}
&\mathscr{E}_T(f)+\int_{\mathbb{Z}^3}\left(\int_0^T|\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|_D^2dt\right)^{1/2}d\Sigma(k)\\
\le& C_0\epsilon_0+\eta\|f\|_{L^1_kL^2_TL^2_{v,D}}+C_\eta\left\{\|\nabla_x\phi\|_{L^1_kL^\infty_T}+\|f\|_{L^1_kL^\infty_TL^2_v}\right\}\|f\|_{L^1_kL^2_TL^2_{v,D}}\\
\le& C_0\epsilon_0+\eta\|f\|_{L^1_kL^2_TL^2_{v,D}}+C_\eta\mathscr{E}_T(f)\mathscr{D}_T(f).
\end{aligned}\tag{5.4}$$

By using Theorem 4.1 and (3.4)–(3.6), one has the following macroscopic dissipation estimate:

$$\begin{aligned}
\|[a,b,c]\|_{L^1_kL^2_T}\lesssim&\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^1_kL^2_TL^2_{v,D}}+\|f\|_{L^1_kL^\infty_TL^2_v}+\|f_0\|_{L^1_kL^2_v}\\
&+\left\{\|\nabla_x\phi\|_{L^1_kL^\infty_T}+\|f\|_{L^1_kL^\infty_TL^2_v}\right\}\|f\|_{L^1_kL^2_TL^2_{v,D}}.
\end{aligned}\tag{5.5}$$

Consequently, a suitable linear combination of the estimate (5.4) and the above estimate (5.5) gives

$$\mathscr{E}_T(f)+\mathscr{D}_T(f)\lesssim\epsilon_0+\mathscr{E}_T(f)\mathscr{D}_T(f),\tag{5.6}$$

for suitably small $\eta>0$. Therefore, under the smallness assumption on ϵ_0, we can obtain the uniform a priori estimate:

$$\mathscr{E}_T(f)+\mathscr{D}_T(f)\lesssim\epsilon_0.$$

The rest is to prove the local existence and uniqueness of solutions and the non-negativity of $F=\mu+\sqrt{\mu}f$, and the details of the proof are omitted for brevity; see also [14] and [4]. Therefore, the global existence of mild solutions follows with the help of the continuity argument.

Next, we consider the rate of convergence of the obtained solutions. Set

$$w=e^{\lambda t}f,\qquad\varphi=e^{\lambda t}\phi\tag{5.7}$$

with $\lambda>0$ determined later. Since f and ϕ satisfy (1.4), then

$$\begin{aligned}
&\partial_t\hat{w}+iv\cdot k\hat{w}-\sqrt{\mu}v\cdot\widehat{\nabla_x\varphi}+L\hat{w}\\
&\qquad=e^{-\lambda t}\widehat{\Gamma(w,w)}-e^{-\lambda t}\widehat{\nabla_x\varphi\cdot\nabla_vw}+\frac{1}{2}e^{-\lambda t}v\cdot\widehat{\nabla_x\varphi w}+\lambda\hat{w},\\
&\widehat{\Delta_x\varphi}=\int_{\mathbb{R}^3}\sqrt{\mu}\hat{w}dv,
\end{aligned}$$

with initial data

$$\hat{w}(0,k,v)=\hat{w}_0(k,v).$$

Then we repeat the processes used to derive (5.6) to deduce

$$\begin{aligned}
&\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\hat{w}(t,k)|_{L^2_v}d\Sigma(k)+\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\widehat{\nabla_x\varphi}(t,k)|d\Sigma(k)\\
&\qquad+\int_{\mathbb{Z}^3}\left(\int_0^T|\hat{w}(t,k)|_D^2dt\right)^{1/2}d\Sigma(k)\\
\lesssim&\|f_0\|_{L^1_kL^2_v}+\|\nabla_x\phi_0\|_{L^1_k}+\sqrt{\lambda}\int_{\mathbb{Z}^3}\left(\int_0^T|\hat{w}|_{L^2_v}^2dt\right)^{1/2}d\Sigma(k).
\end{aligned}$$

Since $r+2s\geq 1$, we use (2.3) to get

$$|\hat{w}|_{L^2_v}\leq|\hat{w}|_D.$$

By choosing $\lambda>0$ sufficiently small, one then has

$$\begin{aligned}&\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\hat{w}(t,k)|_{L^2_v}d\Sigma(k)+\int_{\mathbb{Z}^3}\sup_{0\leq t\leq T}|\widehat{\nabla_x\varphi}(t,k)|d\Sigma(k)\\&\qquad+\int_{\mathbb{Z}^3}\left(\int_0^T|\hat{w}(t,k)|_D^2dt\right)^{1/2}d\Sigma(k)\\&\lesssim\|f_0\|_{L^1_kL^2_v}+\|\nabla_x\phi_0\|_{L^1_k}.\end{aligned}$$

Subsequently, we use the Minkowski inequality

$$\big|\|\|\cdot\|_{L^1_k}\big|\big|_{L^\infty_T}\leq\big|\|\|\cdot\|_{L^\infty_T}\big|\big|_{L^1_k}$$

and (5.7) to obtain the time decay estimate (1.11). This completes the proof of Theorem 1.1. ◻

Now let's give the proof of Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** As in [14], we can use similar arguments as those in Lemma 3.2 to deduce,

$$\begin{aligned}&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\Gamma(f,f)},\langle k\rangle^{2m}\hat{f}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\&\leq\eta\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}}+C_\eta\|f\|_{L^1_{k,m}L^\infty_TL^2_v}\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}},\end{aligned}\tag{5.8}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\widehat{\nabla_x\phi\cdot\nabla_v f},\langle k\rangle^{2m}\hat{f}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\&\qquad\leq\eta\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}}+C_\eta\|\nabla_x\phi\|_{L^1_{k,m}L^\infty_T}\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}},\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(v\cdot\widehat{\nabla_x\phi f},\langle k\rangle^{2m}\hat{f}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\&\qquad\leq\eta\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}}+C_\eta\|\nabla_x\phi\|_{L^1_{k,m}L^\infty_T}\|f\|_{L^1_{k,m}L^2_TL^2_{v,D}}.\end{aligned}\tag{5.10}$$

We can also use similar arguments as those in Theorem 4.1 to deduce

$$\begin{aligned}\|[a,b,c]\|_{L^1_{k,m}L^2_T}&\lesssim\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^1_{k,m}L^2_TL^2_{v,D}}+\|f\|_{L^1_{k,m}L^\infty_TL^2_v}+\|f_0\|_{L^1_{k,m}L^2_v}\\&\qquad+\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\langle k\rangle^m\hat{h}(t,k),\mu^{1/4}\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k).\end{aligned}\tag{5.11}$$

Note that

$$\hat{h}(t,k)=-\widehat{\nabla_x\phi\cdot\nabla_v f}+\frac{1}{2}v\cdot\widehat{\nabla_x\phi f}+\widehat{\Gamma(f,f)}.$$

By using similar arguments as those in Lemma 3.5, we further deduce that

$$\begin{aligned}&\int_{\mathbb{Z}^3}\left(\int_0^T\left|\left(\langle k\rangle^m\hat{h}(t,k),\mu^{1/4}\right)_{L^2_v}\right|^2dt\right)^{1/2}d\Sigma(k)\\ \lesssim&\{\|f\|_{L^1_{k,m}L^\infty_T L^2_v}+\|\nabla_x\phi\|_{L^1_{k,m}L^\infty_T}\}\|f\|_{L^1_{k,m}L^2_T L^2_{v,D}}.\end{aligned}\tag{5.12}$$

We take the L^2_v inner product of the Fourier transform in x of (1.4) and $\langle k\rangle^{2m}\hat{f}$ to yield

$$\begin{aligned}&(\partial_t\hat{f},\langle k\rangle^{2m}\hat{f})_{L^2_v}+(iv\cdot k\hat{f},\langle k\rangle^{2m}\hat{f})_{L^2_v}-(\sqrt{\mu}v\cdot\widehat{\nabla_x\phi},\langle k\rangle^{2m}\hat{f})_{L^2_v}+(L\hat{f},\langle k\rangle^{2m}\hat{f})_{L^2_v}\\ =&(\widehat{\Gamma(f,f)},\langle k\rangle^{2m}\hat{f})_{L^2_v}-(\widehat{\nabla_x\phi\cdot\nabla_v f},\langle k\rangle^{2m}\hat{f})_{L^2_v}+\frac{1}{2}(v\cdot\widehat{\nabla_x\phi f},\langle k\rangle^{2m}\hat{f})_{L^2_v}.\end{aligned}$$

Then by using the same arguments that were used to derive (5.3), we have

$$\begin{aligned}&\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\langle k\rangle^m\hat{f}(t,k)|_{L^2_v}dk+\int_{\mathbb{Z}^3}\sup_{0\le t\le T}|\langle k\rangle^m\widehat{\nabla_x\phi}(t,k)|d\Sigma(k)\\ &\quad+\int_{\mathbb{Z}^3}\left(\int_0^T|\langle k\rangle^m\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|_D^2dt\right)^{1/2}d\Sigma(k)\\ \le&C_0\left\{\|f_0\|_{L^1_{k,m}L^2_v}+\|\nabla_x\phi_0\|_{L^1_{k,m}}+\int_{\mathbb{Z}^3}\left(\int_0^T\left|(\widehat{\Gamma(f,f)},\langle k\rangle^{2m}\hat{f})_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\right.\\ &\quad+\int_{\mathbb{Z}^3}\left(\int_0^T\left|(\widehat{\nabla_x\phi\cdot\nabla_v f},\langle k\rangle^{2m}\hat{f})_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\ &\quad\left.+\int_{\mathbb{Z}^3}\left(\int_0^T\left|(v\cdot\widehat{\nabla_x\phi f},\langle k\rangle^{2m}\hat{f})_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\right\}.\end{aligned}\tag{5.13}$$

Thus, under the smallness assumption on ε_0, a combination of the estimates (5.8), (5.9), (5.10), (5.11), (5.12) and (5.13) gives

$$\begin{aligned}&\int_{\mathbb{Z}^3}\langle k\rangle^m\sup_{0\le t\le T}\|\hat{f}(t,k,\cdot)\|_{L^2_v}d\Sigma(k)+\int_{\mathbb{Z}^3}\langle k\rangle^m\sup_{0\le t\le T}|\widehat{\nabla_x\phi}(t,k)|d\Sigma(k)\\ &\quad+\int_{\mathbb{Z}^3}\left(\int_0^T|\langle k\rangle^m\hat{f}|_D^2\,dt\right)^{1/2}d\Sigma(k)\lesssim\|f_0\|_{L^1_{k,m}L^2_v}+\|\nabla_x\phi_0\|_{L^1_{k,m}},\end{aligned}$$

which implies (1.12). This completes the proof of Theorem 1.2. □

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