COMMUN. MATH. SCI.
Vol. 22, No. 1, pp. 167–179

A NEW PRIORI ERROR ESTIMATION OF NONCONFORMING ELEMENT FOR TWO-DIMENSIONAL LINEARLY ELASTIC SHALLOW SHELL EQUATIONS*

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Abstract. In this paper, we mainly propose a new priori error estimation for the two-dimensional linearly elastic shallow shell equations, which rely on a family of Kirchhoff-Love theories. As the displacement components of the points on the middle surface have different regularities, the non-conforming element for the discretization shallow shell equations is analysed. Then, relying on the enriching operator, a new error estimate of energy norm is given under the regularity assumption $\vec{\zeta}_H \times \zeta_3 \in (H^{1+m}(\omega))^2 \times H^{2+m}(\omega)$ with any $m>0$. Compared with the classic error analysis in other shell literature, convergence order of numerical solution can be controlled by its corresponding approximation error with an arbitrarily high order term, which fills the gap in the computational shell theory. Finally, numerical results for the saddle shell and cylindrical shell confirm the theoretical prediction.

Keywords. Nonconforming elements; enriching operator; error estimation.

AMS subject classifications. 65N15; 65N30.

1. Introduction

The shallow shell is a thin elastic shell with very small thickness and belongs to the family of Kirchhoff-Love theories. Moreover, the equations describing such shells are generally fourth-order partial differential equations with variable coefficients. The finite element method is an effective numerical method for solving these kinds of equations. For a conforming element, it requires piecewise polynomials to be C^1 continuous, which can be difficult to construct in discrete space, particularly in more than two-dimensional space.

In contrast to the conforming elements [1, 2, 3, 4], the continuity requirements of the nonconforming finite element were lower than those of their conforming counterparts. Consequently, nonconforming finite element method has become an important numerical scheme to solve high order elliptic problems, such as Stokes problem, Reissner-Mindlin plate bending problem, and shell problem. Since approximate spaces of nonconforming elements are not subspaces of the original problem solution, the second Strang lemma [5] is needed to be analysed in error estimation, rather than the Céa lemma of the conforming elements. However, in the second Strang lemma, there is another error term in addition to the approximation error, which is often known as a consistency error or nonconforming error.

In the analysis of consistency error estimates, most of the literature relied on the Green formulation and trace theorem, and the regularity of the solution required $\zeta_3 \in H^{2+m}(\omega)$, $m>\frac{1}{2}$. If the integral region or boundary conditions were complex, or the equation has discontinuous coefficients [6] and singular perturbation problems [7], the classical method may have the potential difficulty for low regularity of solutions. For

*Received: July 19, 2022; Accepted (in revised form): May 19, 2023. Communicated by Shi Jin.
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instance, in the fourth-order problem, if solution $\zeta_3 \in H^{2+m}(\omega)$, $0<m\leq\frac{1}{2}$, the trace of the second-order derivative on the element boundary does not necessarily belong to L^2 space. To avoid such regularity, Gudi [8] conducted a posteriori error analysis of the elliptic boundary value problem by the discontinuous Galerkin method, as well as [9, 10] and [11] analyzed a priori error of the nonconforming Morley and Crouzeix-Raviart elements, where the regularity of the solution was claimed to be H^m.

In this paper, we prove a new error estimate of the nonconforming Morley element method for two-dimensional linearly elastic shallow shell equations. The highlight of this paper is that the bilinear form of the equation is different from that of biharmonic equation. And, motivated by the literature [12], when the solution $\zeta_3\in H^m(\omega)$ with $m\geq 3$, the standard error estimate holds if the region is convex, while this postulate is usually invalid if the region is non-convex [13]. Therefore, we will use an enriching operator to prove that the error order of the solution of the shallow shell equations can be of arbitrary order.

The rest of this paper is arranged as follows. In Section 2, the boundary value problem of shallow shell equations and its variational problem are introduced, and the nonconforming Morley element is employed to discretize the third displacement variable. In Section 3, the consistency error of nonconforming element is proved bounded with respect to the energy norm by the enriching operator, which is controlled by the oscillation of L. In Section 4, we have carried out some numerical experiments, and the results verify the convergence order of the energy norm under the minimum assumption.

2. Equations and nonconforming element approximation

To begin with, we introduce some notations [17]. Throughout this paper, Latin indices i, j, k,... take their values in the set $\{1,2,3\}$ while Greek indices α,β,ρ take their values in the set $\{1,2\}$. The notation $\delta_{\alpha\beta}$ designates the Kronecker symbol, $\partial_{\alpha\beta}=\frac{\partial^2}{\partial x_\alpha \partial x_\beta}$, $\Delta=\partial_{\alpha\alpha}$ and ∇ denote the Laplacian operator and gradient operator, respectively. We adopt the standard conventions for Sobolev spaces $H^m(\omega)$, where the corresponding norms and semi-norms of a function v are defined on an open set ω [15]:

$$\|v\|_{H^m}^2=\sum_{k=0}^m |v|_{H^k}^2,\ |v|_{H^k}^2=\int_\omega \sum_{|\alpha|=k}|\nabla^\alpha v|^2 \mathrm{d}x.$$

In particular, the convention $H^0=L^2$ is adopted. The notation P_l designates Lagrange triangular element (the space of all polynomials of degree $\leq l$ on each element). The positive constant C is independent of the diameter h_T.

Define the space

$$\vec{V}_H(\omega)=\left\{\vec{\eta}_H=(\eta_1,\eta_2)\in H^1(\omega)\times H^1(\omega);\ \vec{\eta}_H=\vec{0} \text{ on } \gamma_0\right\},$$

$$V_3(\omega)=\left\{\eta_3\in H^2(\omega);\ \eta_3=\partial_\nu\eta_3=0 \text{ on } \gamma_0\right\},$$

where $\partial_\nu:=\sum_{i=1}^{2}\nu_i\partial x_i$ denotes the normal derivative of η_3 along γ_0 (γ_0 is measurable subset of γ with length $\gamma_0\geq 0$, γ is boundary of the set ω).

Let ω be a bounded open subset of R^2 with a Lipschitz-continuous boundary γ. Now, we consider the boundary value problem of two-dimensional shallow shell equations [16, 14] as follows:

$$-\partial_{\alpha\beta}m_{\alpha\beta}-\partial_\beta(n^\theta_{\alpha\beta}\partial_\alpha\theta)=p_3+\partial_\alpha s_\alpha \text{ in } \omega,$$

$$
\begin{aligned}
&-\partial_\beta n^\theta_{\alpha\beta} = p_\alpha \text{ in } \omega,\\
&\zeta_\alpha = 0 \text{ and } \zeta_3 = \partial_\nu \zeta_3 = 0 \text{ on } \gamma_0,\\
&(\partial_\alpha m_{\alpha\beta})\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) + n^\theta_{\alpha\beta}(\partial_\alpha\zeta_3)\nu_\beta = -s_\alpha\nu_\alpha \text{ on } \gamma_1 = \gamma - \gamma_0,\\
&m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_1,\\
&n^\theta_{\alpha\beta}\nu_\beta = 0 \text{ on } \gamma_1,
\end{aligned}
$$

where $\theta \in \mathcal{C}^3(\overline{\omega})$, ν_α and $\tau_\alpha = \nu_\beta + \frac{\pi}{2}$ represent the shape of the middle surface of the shallow shell, the direction of normal and tangential derivatives along γ, respectively,

$$
\begin{aligned}
&m_{\alpha\beta} := -\left\{ \frac{4\lambda\mu}{3(\lambda+2\mu)} \Delta\zeta_3\delta_{\alpha\beta} + \frac{4}{3}\mu\partial_{\alpha\beta}\zeta_3 \right\},\\
&n^\theta_{\alpha\beta} := \left\{ \frac{4\lambda\mu}{\lambda+2\mu} e^\theta_{\rho\rho}(\vec{\zeta})\delta_{\alpha\beta} + 4\mu e^\theta_{\alpha\beta}(\vec{\zeta}) \right\},\\
&e^\theta_{\alpha\beta}(\vec{\zeta}) := \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3),\\
&p_i := \int_{-1}^{1} f_i \mathrm{d}y_3 + g_i^+ + g_i^-,\ f_i \in L^2(\omega\times(-1,1)),\ g_i \in L^2((\omega\times 1)\cup(\omega\times -1)),\\
&s_\alpha := \int_{-1}^{1} y_3 f_\alpha \mathrm{d}y_3 + g_\alpha^+ - g_\alpha^-.
\end{aligned}
$$

By Green's formulas, the variational problem of shallow shell equations is to find $\vec{\zeta} = (\vec{\zeta}_H, \zeta_3) \in \vec{V}(\omega) := (\vec{V}_H(\omega), V_3(\omega))$ such that

$$
B(\vec{\zeta},\vec{\eta}) = (L,\vec{\eta}),\ \forall \vec{\eta} \in \vec{V}(\omega), \tag{2.1}
$$

where $B(\cdot,\cdot)$ is a bilinear form, and $(\cdot,\cdot)$ denotes the L^2 inner product,

$$
\begin{aligned}
B(\vec{\zeta},\vec{\eta}) :=& \int_\omega \left(-m_{\alpha\beta}\zeta_3\partial_{\alpha\beta}\eta_3 + n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\alpha\theta\partial_\beta\eta_3 + n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\beta\eta_\alpha \right) \mathrm{d}\omega\\
=& \int_\omega \left(\frac{4\lambda\mu}{\lambda+2\mu} \left(\frac{1}{3}(\Delta\zeta_3\Delta\eta_3 + e^\theta_{\rho\rho}(\vec{\zeta}) e^\theta_{\sigma\sigma}(\vec{\zeta})) \right) \right.\\
&\left. + 4\mu \left(\frac{1}{3}\partial_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}(\eta_3) + e^\theta_{\alpha\beta}(\vec{\zeta}) e^\theta_{\alpha\beta}(\vec{\eta}) \right) \right) \mathrm{d}\omega,
\end{aligned}
$$

$$
(L,\vec{\eta}) := \int_\omega (p_i\eta_i - s_\alpha\partial_\alpha\eta_3)\mathrm{d}\omega.
$$

We assume that the bilinear form is bounded and elliptic so that the system of Equations (2.1) has a unique solution (Ref. [16, 18]).

Then, let $\bar{\omega}$ be a polygon domain, $\mathcal{T}_h$ be a family of shape-regular partitions with the mesh size $h \to 0$. For a given $T \in \mathcal{T}_h$, T is a triangle element, $h_T = diam(T)$, $h = \max_{T\in\mathcal{T}_h} h_T$, $\rho_T = \underset{S\subset T}{superior}\, diam(S)$, where S is a ball contained in T. A triangular partition $\mathcal{T}_h$ satisfies the regular assumption that there exists a constant $\sigma > 0$ which does not depend on h such that for all $T\in\mathcal{T}_h$, $\frac{h_T}{\rho_T}<\sigma$. Moreover, a triangular partition $\mathcal{T}_h$ satisfies the inverse assumption that there exists a constant $\kappa>0$ such that for all $T\in\mathcal{T}_h$, $\frac{h}{h_T}\le\kappa$.

Because the displacement components $(\vec{\zeta}_H,\zeta_3)$ belong to different Sobolev spaces, we use the linear triangular element

$$V_{\alpha h}=\{\eta_h\in C^0(\omega):\eta_h|_T\in P_1(T),\forall T\in\mathcal{T}_h,\eta_h=0 \text{ on } \gamma_0\},\ \alpha=1,2,$$

to discretize "horizontal" components, and use the nonconforming element to discretize "vertical" component. The definition of the associated nonconforming finite element space V_{3h} over $\mathcal{T}_h$ is as follows:

The Morley element [19, 20]. Space is defined as $\forall e$, $A_i\in\eta_h$, $i=1,2,3$, $T\in\mathcal{T}_h$,

$$V_{3h}=\left\{\eta_h\in L^2(\omega):\eta_h|_T\in P_2(T),\eta_h \text{ is continuous at } A_i,\eta_h(A_i)=\int_e\left[\frac{\partial\eta_h}{\partial\nu}\right]_e(p)\mathrm{d}s=0\right\},$$

and the interpolant $\Pi_{3h}:H^2(\omega)\to V_{3h}$ is defined by $\Pi_{3h}|_T=\Pi_T$ with

$$\begin{cases}\Pi_T\zeta_3(A_i)=\zeta_3(A_i),\\ \displaystyle\int_e\frac{\partial\Pi_T\zeta_3}{\partial\nu}(p)\mathrm{d}s=\int_e\frac{\partial\zeta_3}{\partial\nu}(p)\mathrm{d}s,\ \forall e\in\partial T,\end{cases}$$

where e is all edges, A_i and p are internal vertices and midpoints of $\mathcal{T}_h$ here and in the rest of the article.

And the nonconforming element meets the following conditions [21, 22]:

- Function $\eta_h\in V_{3h}$ is continuous at vertices A_i of T and vanishes lying on γ_0;
- For every polynomial $q\in P_2$, the integral $\int_\omega p\eta_h\mathrm{d}\omega$ is continuous over each inter-element side F and vanishes when $F\in\gamma_0$;
- For every $q\in P_2$, the integral $\int_\omega p\frac{\eta_h}{\partial\nu}\mathrm{d}\omega$ is continuous at the midpoints of each inter-element side F and vanishes at the midpoints when $F\in\gamma_0$.

We set norm on the discretization space $\vec{V}_h(\omega):=(V_{\alpha h}(\omega))^2\times V_{3h}(\omega),\alpha=1,2$ as

$$\|\vec{\eta}_h\|_h:=\sum_\alpha\|\eta_{\alpha h}\|_{H^1}+\|\eta_{3h}\|_{H^2},$$

where

$$\|\cdot\|_{H^\alpha}=\left(\sum_{T\in\mathcal{T}_h}\|\cdot\|^2_{\alpha,T}\right)^{\frac{1}{2}}.$$

Through the interpolation theory [5, 23], the error of Morley element is, for any $\zeta_3\in H^{2+m}(\omega)$ with $m\ge1$, there exists C such that

$$\|\zeta_3-\Pi\zeta_3\|_{L^2}+h\|\zeta_3-\Pi\zeta_3\|_{H^1}+h^2\|\zeta_3-\Pi\zeta_3\|_{H^2}\le Ch^{2+m}\|\zeta_3\|_{H^{2+m}}.$$

Now, let's consider that the nonconforming approximation form of Equations (2.1) is finding $\vec{\eta}_h=(\eta_{1h},\eta_{2h},\eta_{3h})\in\vec{V}_h(\omega)$ such that

$$B_h(\vec{\zeta}_h,\vec{\eta}_h)=(L,\vec{\eta}_h),\ \forall\vec{\eta}_h\in\vec{V}_h(\omega),\tag{2.2}$$

where

$$B_h(\vec{\zeta}_h,\eta_{\alpha h})=\sum_{T\in\mathcal{T}_h}\int_T n^\theta_{\alpha\beta}(\vec{\zeta}_h)\partial_\beta\eta_{\alpha h}\mathrm{d}x,\ \forall\eta_{\alpha h}\in V_{\alpha h}(\omega),$$

$$B_h(\vec{\zeta}_h,\eta_{3h})=\sum_{T\in\mathcal{T}_h}\int_T\Big(-m_{\alpha\beta}\zeta_{3h}\partial_{\alpha\beta}\eta_{3h}+n^\theta_{\alpha\beta}(\vec{\zeta}_h)\partial_\alpha\theta\partial_\beta\eta_{3h}\Big)\mathrm{d}x,\ \forall\eta_{3h}\in V_{3h}(\omega).$$

It is easy to show that the discrete bilinear form $B_h(\cdot,\cdot)$ satisfies the boundedness and coercivity conditions with respect to the norm $\|\cdot\|_h$, i.e., there exist constants $C_1>0$, $C_2>0$ such that

$$B_h(\vec{\zeta}_h,\vec{\eta}_h)\le C_1\|\vec{\zeta}_h\|_h\|\vec{\eta}_h\|_h,\ \forall\vec{\zeta}_h,\vec{\eta}_h\in\vec{V}_h(\omega),$$

$$B_h(\vec{\eta}_h,\vec{\eta}_h)\ge C_2\|\vec{\eta}_h\|_h^2,\ \forall\vec{\eta}_h\in\vec{V}_h(\omega).$$

3. A priori error estimation

By the Céa lemma and Strang's lemma [5, 24], we have the following error estimate for the discretization problem (2.2)

$$\|\zeta_\alpha-\zeta_{\alpha h}\|_{H^1}\le\inf_{\eta_{\alpha h}\in V_{\alpha h}}||\zeta_\alpha-\eta_{\alpha h}||_{H^1},\tag{3.1a}$$

$$\|\zeta_3-\zeta_{3h}\|_{H^2}\le C\left(\inf_{\eta_{3h}\in V_{3h}}\|\zeta_3-\eta_{3h}\|_{H^2}+\sup_{w_{3h}\in V_{3h}}\frac{|B_h(\zeta_3,w_{3h})-(L,w_{3h})|}{||w_{3h}||_{H^2}}\right),\tag{3.1b}$$

where $w_{3h}=\zeta_{3h}-\eta_{3h}\neq0$.

We make a detailed analysis for the nonconforming error of (3.1b). If we use the traditional way of proof [25, 26], integration by parts, the well known result is as follows

$$|B_h(\zeta_3,w_{3h})-(L,w_{3h})|\le Ch(|\zeta_3|+h\|f\|_{0,\omega})\|w_{3h}\|_{H^2},\ \forall\zeta_3\in H^2(\omega)\cap H^3(\omega).$$

To improve the priori error estimates for the convergence of the numerical schemes presented in Section 2, we use the enriching operators. Enriching operators occupy an important position in the study of obstacle problems for clamped plates [27], which is also exploited to study the convergence properties in [8, 28, 29, 30].

Thus, we define the enriching operator $E_h:V_{3h}\to W_{3h}$ on the triangular finite element space

$$\begin{aligned}&[\nabla^n(E_h\eta_{3h})](A_i)\\=&\frac{1}{|\mathcal{T}_p|}\sum_{T\in\mathcal{T}_p}(\nabla^n\eta_{3h}|_T)(A_i),\ \forall A_i \text{ be a vertex},\ n=0,1,2,\ i=1,2,3,\end{aligned}\tag{3.2a}$$

$$\forall e\in F_h,\ \int_e E_h\eta_{3h}q\mathrm{d}s=\int_e\eta_{3h}q\mathrm{d}s,\ \forall q\in P_{l-6},\tag{3.2b}$$

$$\forall e\in F_h,\ \int_e\frac{\partial E_h\eta_{3h}}{\partial\nu}q\mathrm{d}s=\int_e\frac{\partial\eta_{3h}}{\partial\nu}q\mathrm{d}s,\ \forall q\in P_{l-5},\tag{3.2c}$$

$$[3mm]\forall T\in\mathcal{T}_h,\ \int_T E_h\eta_{3h}q\mathrm{d}x=\int_T\eta_{3h}q\mathrm{d}x,\ \forall q\in P_{l-6},\tag{3.2d}$$

where W_{3h} be the associated $l \geq 5$ order conforming Argyris (P_l) finite element space, it is corresponding $(l-4)$ order nonconforming triangular element, $\mathcal{T}_p$ is the set of triangles that share common vertices p. The case $l=6$ matches the sextic (P_6) Argyris element, refer to Figure 3.1, where solid dot $\bullet$ denotes the value of the vertex, the value of the midpoint along the edge and the average value of the function over the element T; the small circle $\circ$ and larger circle $\bigcirc$ represent the values of the first order and second order derivatives at the vertex, respectively; arrow $\uparrow$ denotes the value of the normal derivative along the edge. The case $l=7$ matches the septic (P_7) Argyris element, refer to Figure 3.2.

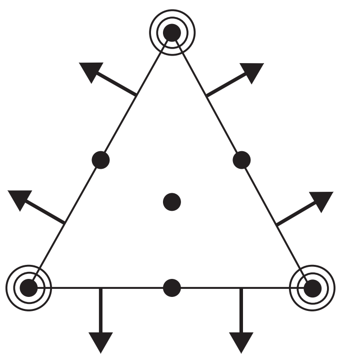


Fig. 3.1: *P_6 Argyris element*

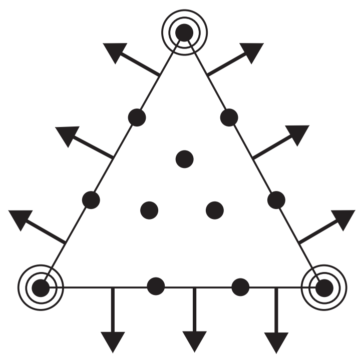


Fig. 3.2: *P_7 Argyris element*

In addition, the enriching operators of the Morley element can also be the HCT macro-element (values of the vertex, values of the first-order partial derivatives at vertices, and the values of the outer normal derivative at the midpoint of the edge of the element) [31]. However, this macro element can only be used for quadratic and cubic Lagrange elements in two dimensions.

Then, for $\forall \eta_{3h} \in V_{3h}$, we have

$$\|\eta_{3h}-E_h\eta_{3h}\|_{L^2}+h\|\eta_{3h}-E_h\eta_{3h}\|_{H^1}+h^2\|\eta_{3h}-E_h\eta_{3h}\|_{H^2} \leq Ch^2\|\eta_{3h}\|_{H^2}. \tag{3.3}$$

Lemma 3.1. *Let ζ_3 and ζ_{3h} be the solution and numerical solution for the third component of the displacement, respectively. Suppose that $\zeta_3 \in H^{2+m}(\omega)\cap H^2(\omega)$, $m \geq 0$, $p_i \in L^2(\omega)$, $s_\alpha \in L^2(\omega)$, $L \in L^2(\omega)$, the function $\theta \in \mathcal{C}^3(\overline{\omega})$, then we have*

$$|B_h(\zeta_3,w_{3h})-(L,w_{3h})| \leq C\left\{\inf_{\eta_{3h}\in V_{3h}}\|\zeta_3-\eta_{3h}\|_{H^2}+\|\vec{\eta}_h\|_{\vec{G}}+Osc(L)\right\}\|w_{3h}\|_{H^2} \tag{3.4}$$

and

$$\|\zeta_3-\zeta_{3h}\|_{H^2} \leq Ch^m\|\zeta_3\|_{H^{2+m}}+\|\vec{\zeta}\|_{\vec{G}}+Osc(L), \tag{3.5}$$

where

$$\vec{G}=H^2(\omega)\times H^2(\omega)\times H^3(\omega),$$

$$Osc(L)=\left(\sum_{T\in\mathcal{T}_h} h^4 \inf_{q\in P_{l-6}} \|L-q\|_{0,T}^2\right)^{\frac{1}{2}}, \ T \text{ is a triangular element.}$$

Proof. By the nonconforming discretization problem (2.2) and the variational problem (2.1) of two-dimensional linearly elastic shallow shell equations, for $\forall \eta_{3h}\in V_{3h}$, $\forall E_h w_{3h}\in H^2(\omega)$ and $B_h(\zeta_3,E_hw_{3h})-(L,E_hw_{3h})=0$, we can derive

$$\begin{aligned}
B_h(\zeta_3,w_{3h})-(L,w_{3h}) =& B_h(\zeta_3,w_{3h}-E_hw_{3h})-(L,w_{3h}-E_hw_{3h})\\
=& B_h(\zeta_3-\eta_{3h},w_{3h}-E_hw_{3h})+B_h(\eta_{3h},w_{3h}-E_hw_{3h})\\
&-(L,w_{3h}-E_hw_{3h})\\
=& I_1+I_2+I_3.
\end{aligned}$$

First, we can get by Cauchy-Schwarz inequality

$$I_1\le \|\zeta_3-\eta_{3h}\|_{H^2}\|w_{3h}-E_hw_{3h}\|_{H^2}\le C\|\zeta_3-\eta_{3h}\|_{H^2}\|w_{3h}\|_{H^2}.$$

Next, using Green's formula and for any $T\in\mathcal{T}_h$, $\eta_{3h}|_T\in P_2(T)$, we can obtain

$$\begin{aligned}
I_2=&\sum_{T\in\mathcal{T}_h}\int_T \left(-m_{\alpha\beta}\partial_{\alpha\beta}(w_{3h}-E_hw_{3h})+n^\theta_{\alpha\beta}\partial_\alpha\theta\partial_\beta(w_{3h}-E_hw_{3h})\right)\mathrm{d}x_1\mathrm{d}x_2\\
=&\sum_{T\in\mathcal{T}_h}\int_T \left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}\Delta(w_{3h}-E_hw_{3h})+\frac{4\mu}{3}\partial_{\alpha\beta}\eta_{3h}\partial_{\alpha\beta}(w_{3h}-E_hw_{3h})\right)\mathrm{d}x_1\mathrm{d}x_2\\
&+\sum_{T\in\mathcal{T}_h}\int_T n^\theta_{\alpha\beta}\partial_\alpha\theta\partial_\beta(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2\\
=&\sum_{T\in\mathcal{T}_h}\int_T \frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta^2\eta_{3h}(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2\\
&-\sum_{T\in\mathcal{T}_h}\int_T \partial_\beta(n^\theta_{\alpha\beta}\partial_\alpha\theta)(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2\\
&-\sum_{T\in\mathcal{T}_h}\int_{\partial T}(w_{3h}-E_hw_{3h})\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\frac{\partial\Delta\eta_{3h}}{\partial\nu}-\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu\partial\tau}\partial\tau\right)\mathrm{d}s\\
&+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial(w_{3h}-E_hw_{3h})}{\partial\nu}\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}+\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu^2}\right)\mathrm{d}s\\
&+\sum_{T\in\mathcal{T}_h}\int_{\partial T} n^\theta_{\alpha\beta}\partial_\alpha\theta(w_{3h}-E_hw_{3h})\nu_\beta\mathrm{d}s,
\end{aligned}$$

since $\Delta^2\eta_{3h}=0$, we get

$$\begin{aligned}
I_2=&-\sum_{T\in\mathcal{T}_h}\int_T \partial_\beta(n^\theta_{\alpha\beta}\partial_\alpha\theta)(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2+\sum_{T\in\mathcal{T}_h}\int_{\partial T} n^\theta_{\alpha\beta}\partial_\alpha\theta(w_{3h}-E_hw_{3h})\nu_\beta\mathrm{d}s\\
&-\sum_{T\in\mathcal{T}_h}\int_{\partial T}(w_{3h}-E_hw_{3h})\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\frac{\partial\Delta\eta_{3h}}{\partial\nu}-\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu\partial\tau}\partial_\tau\right)\mathrm{d}s\\
&+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial(w_{3h}-E_hw_{3h})}{\partial\nu}\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}+\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu^2}\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
=&-\sum_{T\in\mathcal{T}_h}\int_T \partial_\beta(n^\theta_{\alpha\beta}\partial_\alpha\theta)(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2+\sum_{T\in\mathcal{T}_h}\int_{\partial T} n^\theta_{\alpha\beta}\partial_\alpha\theta(w_{3h}-E_hw_{3h})\nu_\beta\mathrm{d}s\\
&-\sum_{T\in\mathcal{T}_h}\int_{\partial T}[(w_{3h}-E_hw_{3h})]\left\{\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\frac{\partial\Delta\eta_{3h}}{\partial\nu}-\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu\partial\tau}\partial_\tau\right)\right\}\mathrm{d}s\\
&-\sum_{T\in\mathcal{T}_h}\int_{\partial T}\{(w_{3h}-E_hw_{3h})\}\left[\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\frac{\partial\Delta\eta_{3h}}{\partial\nu}-\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu\partial\tau}\partial_\tau\right)\right]\mathrm{d}s\\
&+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left[\frac{\partial(w_{3h}-E_hw_{3h})}{\partial\nu}\right]\left\{\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}+\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu^2}\right)\right\}\mathrm{d}s\\
&+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left\{\frac{\partial(w_{3h}-E_hw_{3h})}{\partial\nu}\right\}\left[\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}+\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu^2}\right)\right]\mathrm{d}s,
\end{aligned}
$$

where $[\eta]=\eta|_{T_1}-\eta|_{T_2}$ and $\{\eta\}|_F=\frac{1}{2}(\eta|_{T_1}+\eta|_{T_2})$ denote the jumps and averages of function, respectively. When V_{3h} denotes Morley element, we know

$$
\frac{\partial\Delta\eta_{3h}}{\partial\nu}=0,\ \ \frac{\partial^2\eta_{3h}}{\partial\nu\partial\tau}=0,
$$

and $\Delta\eta_{3h}$, $\frac{\partial^2\eta_{3h}}{\partial\nu^2}$ are constant. By the constraint (3.2c), we obtain

$$
\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial(w_{3h}-E_hw_{3h})}{\partial\nu}\left[\left(\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\eta_{3h}+\frac{4\mu}{3}\frac{\partial^2\eta_{3h}}{\partial\nu^2}\right)\right]\mathrm{d}s=0.
$$

Thus,

$$
\begin{aligned}
I_2=&-\sum_{T\in\mathcal{T}_h}\int_T \partial_\beta(n^\theta_{\alpha\beta}\partial_\alpha\theta)(w_{3h}-E_hw_{3h})\mathrm{d}x_1\mathrm{d}x_2+\sum_{T\in\mathcal{T}_h}\int_{\partial T} n^\theta_{\alpha\beta}\partial_\alpha\theta(w_{3h}-E_hw_{3h})\nu_\beta\mathrm{d}s\\
\le&\|n^\theta_{\alpha\beta}\partial_\alpha\theta\|_{H^1}\cdot\|w_{3h}-E_hw_{3h}\|_{L^2}+\|n^\theta_{\alpha\beta}\partial_\alpha\theta\|_{L^2(e)}\cdot\|(w_{3h}-E_hw_{3h})\nu_\beta\|_{L^2(e)}\\
\le&\|n^\theta_{\alpha\beta}(\vec{\eta}_h)\partial_\alpha\theta-n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\alpha\theta\|_{H^1}\cdot\|w_{3h}-E_hw_{3h}\|_{L^2}\\
&+\|n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\alpha\theta\|_{H^1}\cdot\|w_{3h}-E_hw_{3h}\|_{L^2}\\
&+\|n^\theta_{\alpha\beta}(\vec{\eta}_h)\partial_\alpha\theta-n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\alpha\theta\|_{L^2(e)}\cdot\|w_{3h}-E_hw_{3h}\|_{L^2(e)}\\
&+\|n^\theta_{\alpha\beta}(\vec{\zeta})\partial_\alpha\theta\|_{L^2(e)}\cdot\|w_{3h}-E_hw_{3h}\|_{L^2(e)}\\
\le&Ch\|\vec{\zeta}\|_{\vec{G}}\|w_{3h}\|_{H^2}.
\end{aligned}
$$

In this inequality we used the Cauchy-Schwarz inequality, the trace theorem with scaling, triangle inequality, standard interpolation and inverse estimates and (3.3).

Finally, we obtain by (3.2d)

$$
\begin{aligned}
I_3=&\sum_{T\in\mathcal{T}_h}\int_T(L,w_{3h}-E_hw_{3h})\mathrm{d}x\\
=&\sum_{T\in\mathcal{T}_h}\int_T(L-q)(w_{3h}-E_hw_{3h})\mathrm{d}x\\
\le&\left(\sum_{T\in\mathcal{T}_h}h^4\|L-q\|^2_{0,T}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h^{-4}\|w_{3h}-E_hw_{3h}\|^2_{0,T}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C\left(\sum_{T\in\mathcal{T}_h} h^4\|L-q\|_{0,T}^2\right)^{\frac{1}{2}}\|w_{3h}\|_{H^2},\ \forall q\in P_{l-6}(T).$$

A combination of I_1, I_2, I_3 yields (3.4).

Based on Céa lemma and interpolation error estimation [5], we have

$$\inf_{\eta_{3h}\in V_{3h}}\|\zeta_3-\eta_{3h}\|_{H^2}\leq\|\zeta_3-\Pi\zeta_3\|_{H^2}\leq Ch^m\|\zeta_3\|_{H^{2+m}},\ \zeta_3\in H^{2+m}(\omega),$$

which, together with (3.4), gives (3.5). ◻

THEOREM 3.1. *Let $\vec{\zeta}$ and $\vec{\zeta}_h$ be the solutions of equation(2.1) and equation(2.2), respectively. Suppose that $p_i\in L^2(\omega)$, $s_\alpha\in L^2(\omega)$, $L\in L^2(\omega)$ be Lipschitz continuous, $\theta\in\mathcal{C}^3(\overline{\omega})$. If $\vec{\zeta}\in(H^{1+m}(\omega)\times H^{1+m}(\omega)\times H^{2+m}(\omega))\cap(H^1(\omega)\times H^1(\omega)\times H^2(\omega))$, then there hold*

$$\|\vec{\zeta}-\vec{\zeta}_h\|_h\leq Ch^m(\|\zeta_\alpha\|_{H^{1+m}}+2\|\zeta_3\|_{H^{2+m}})+\|\vec{\zeta}\|_{\vec{G}}+Osc(L).$$

Proof. Based on Céa lemma and interpolation error estimation, for the first two components of the displacement we have

$$\inf_{\eta_{\alpha h}\in V_{\alpha h}}\|\zeta_\alpha-\eta_{\alpha h}\|_{H^1}\leq\|\zeta_\alpha-\Pi\zeta_{\alpha h}\|_{H^1}\leq Ch^m\|\zeta_\alpha\|_{H^{1+m}},\ \zeta_\alpha\in H^{1+m}(\omega),$$

which, together with the estimation (3.5) of the third component of the displacement, gives the expected result. ◻

4. Numerical experiments

In this section, we mainly conduct numerical experiments on saddle shell and cylindrical shell to verify the theoretical results.

4.1. Saddle shell. The equation of the saddle shell S at the point (x_1,x_2) is as follows:

$$\theta(x_1,x_2)=-0.8\left(\frac{x_1^2}{4}-\frac{x_1}{2}\right)+0.8\left(\frac{x_2^2}{4}-\frac{x_2}{2}\right).$$

The domain ω on S is defined by

$$\omega:=\left\{(x_1,x_2)\in\mathbb{R}^2;\ 0\leq x_1\leq 2,\ 0\leq x_2\leq 2\right\},$$

and

$$\gamma_0=\left\{(x_1,x_2)\in\mathbb{R}^2;\ x_1=0,\ x_1=2,\ 0\leq x_2\leq 2\right\}$$

is the clamped boundary.

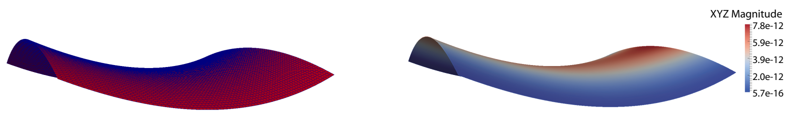
On the basis of Ref. [32, 33], we take the Young's modulus $E=2\times10^{11}$Pa and the Poisson ratio $\nu=0.3$. Then, combined with the following calculation formula of Lamé constant,

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)},\ \mu=\frac{E}{2(1+\nu)},\ \lambda\geq0,\ \mu\geq0, \tag{4.1}$$

we get

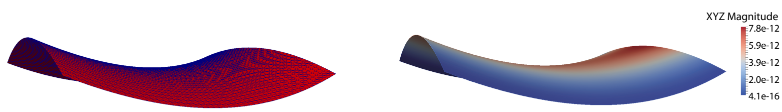
$$\lambda=1.15\times10^{11}\text{Pa},\ \mu=7.69\times10^{10}\text{Pa}.$$

Suppose that the applied force at the saddle shell be $p_1 = p_2 = 0$, and $p_3 = 20$Pa. Due to the lack of an exact solution, we use the extremely fine mesh (96800) calculated under the minimum assumption $m = 1$, $H^2(\omega) \times H^2(\omega) \times H^3(\omega)$ as the exact solution, as shown in Figure 4.1. And Figure 4.2 shows the extremely fine mesh (57800) under the minimum assumption of the solution $m = 2$, $H^3(\omega) \times H^3(\omega) \times H^4(\omega)$, we use its finite element solution [34] as the exact solution.



(a) Mesh dividing (b) Deformation

Fig. 4.1: *Numerical result of exact solution: the mesh is* 96800.



(a) Mesh dividing (b) Deformation

Fig. 4.2: *Numerical result of exact solution: the mesh is* 57800.

Then, the result under the P_1, P_1, Morley elements as the numerical solution, and the error under the energy norm is calculated, as shown in Table 4.1.

Table 4.1: *The convergence results for the saddle shell on four different meshes.*

	Error	1/2	1/4	1/8	1/16
$m = 1$	$\Vert\vec{\zeta} - \vec{\zeta}_h\Vert_h$	1.7687e-10	1.0057e-10	5.3643e-11	2.8070e-11
	Convergence order	0.81	0.91	0.93	\
$m = 2$	$\Vert\vec{\zeta} - \vec{\zeta}_h\Vert_h$	5.7666e-11	1.7933e-11	4.2453e-12	8.6999e-13
	Convergence order	1.69	2.08	2.29	\

4.2. Cylindrical shell. The equation of the cylindrical shell S at the point (x_1, x_2) is as follows:

$$\vec{\theta}(x_1, x_2) = \sqrt{10^2 - x^2}.$$

The domain ω on S is defined by

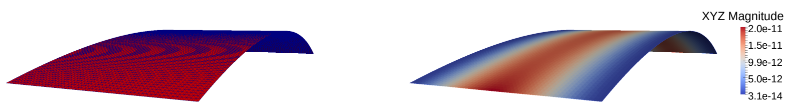
$$\omega := \left\{(x_1, x_2) \in \mathbb{R}^2;\ -\pi \le x_1 \le \pi,\ -1 \le x_2 \le 1\right\},$$

and

$$\gamma_0=\left\{(x_1,x_2)\in\mathbb{R}^2;\ -\pi\le x_1\le\pi,\ x_2=-1,\ x_2=1\right\}$$

is the clamped boundary.

On the basis of [35], we take the Young's modulus $E=9\times10^{10}$Pa and the Poisson ratio $\nu=0.3$. According to (4.1), it can be calculated $\lambda=5.19\times10^{10}$Pa, $\mu=3.46\times10^{10}$Pa, similar to the force we applied in Section 4.1. In Figure 4.3, the extremely fine mesh (110060) under the minimum assumption of the solution $H^2(\omega)\times H^2(\omega)\times H^3(\omega)$, $m=1$ is used as the exact solution. And in Figure 4.4, we show the extremely fine mesh (62546) under the minimum assumption of the solution is $H^3(\omega)\times H^3(\omega)\times H^4(\omega)$, $m=2$, which is used as the exact solution.



(a) Mesh dividing (b) Deformation

Fig. 4.3: *Numerical result of exact solution: the mesh is* 101400.

(a) Mesh dividing (b) Deformation

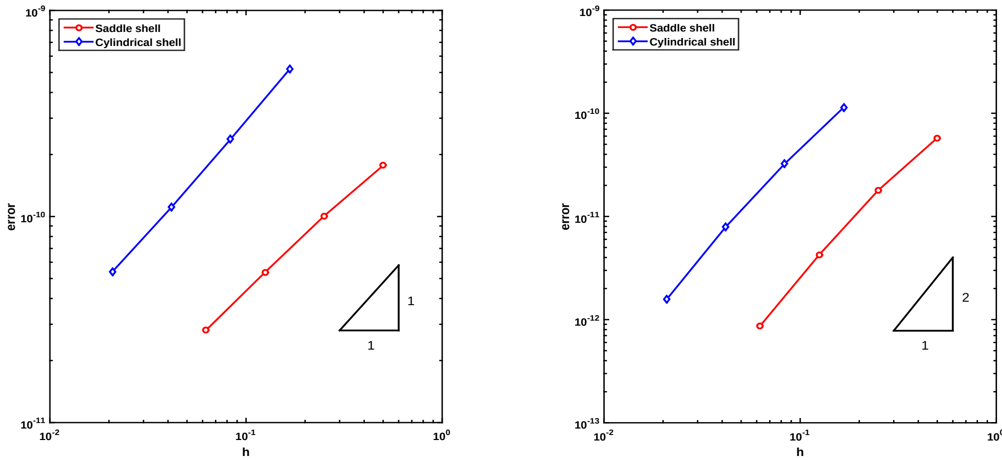
Fig. 4.4: *Numerical result of exact solution: the mesh is* 60000.

Then, we adopt the solution of the nonconforming element (P_1, P_1, Morley elements) as the numerical solution, and the error under the energy norm is calculated, as shown in Table 4.2.

Table 4.2: *The convergence results for the cylindrical shell on four different meshes.*

	Error	1/6	1/12	1/24	1/48
$m=1$	$\\|\vec{\zeta}-\vec{\zeta}_h\\|_h$	5.1802e-10	2.3709e-10	1.1089e-10	5.3974e-11
	Convergence order	1.13	1.10	1.04	\
$m=2$	$\\|\vec{\zeta}-\vec{\zeta}_h\\|_h$	2.5315e-10	7.1923e-011	1.7540e-11	3.4784e-12
	Convergence order	1.82	2.04	2.33	\

In Figure 4.5, we report the convergence rate of the Morley element in the energy norm. Under the least regularity assumption $H^2\times H^2\times H^3$ and $H^3\times H^3\times H^4$, we can find that the convergence rate of the hyperbolic paraboloid shell and cylindrical shell are 1 and 2, respectively, which is consistent with the theoretical analysis.



(a) $m=1$ (b) $m=2$

Fig. 4.5: *Order of convergence with* $\|\vec{\zeta}-\vec{\zeta}_h\|_h$.

5. Conclusion

In this paper, we have proved a new error estimate for two-dimensional linearly elastic shallow shell equations. On the basis that the displacement components belonged to different Sobolev spaces, we analyzed the nonconforming element to discrete displacement variables. Particularly, in the energy norm error estimation, consistency error of these nonconforming elements can be limited to arbitrary order by enriching operator.

Acknowledgements. We are very grateful to Professor P.G. Ciarlet and the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences for their direction and advice. This paper is supported by the National Natural Science Foundation of China (NSFC.11971379) and the Distinguished Youth Foundation of Shaanxi Province (2022JC-01).

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