COMMUN. MATH. SCI.
Vol. 22, No. 1, pp. 227–255

A VARIATIONAL APPROACH FOR PRICE FORMATION MODELS IN ONE DIMENSION*

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Abstract. In this paper, we study a class of first-order mean-field games (MFGs) that model price formation. Using Poincaré lemma, we eliminate one of the equations of the MFGs system and obtain a variational problem for a single function. We prove the uniqueness of the solutions to the variational problem and address the existence of solutions by applying relaxation arguments. Moreover, we establish a correspondence between solutions of the MFGs system and the variational problem. Based on this correspondence, we introduce an alternative numerical approach for the solution of the original MFGs problem. We end the paper with numerical results for a linear-quadratic model.

Keywords. Mean Field Games; Price formation; Potential Function, Lagrange multiplier.

AMS subject classifications. 35A15; 49N80; 49Q22.

1. Introduction

Here, we consider the numerical solution of the first-order mean-field games (MFGs) system introduced in [34] to model price formation. The solution to this system determines the price ϖ of a commodity with supply Q when a large group of rational agents trades that commodity. The original price problem reads as follows:

PROBLEM 1.1. *Suppose that $m_0\in\mathcal{P}(\mathbb{R})$, $H\in C^1(\mathbb{R})$, and Q, V, and u_T are continuous. Assume further that H is uniformly convex. Find $u,m:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $\varpi:[0,T]\to\mathbb{R}$ satisfying $m\geqslant 0$,*

$$\begin{cases}-u_t+H(\varpi+u_x)+V(x)=0 & [0,T]\times\mathbb{R},\\ m_t-(H'(\varpi+u_x)m)_x=0 & [0,T]\times\mathbb{R},\\ -\int_{\mathbb{R}}H'(\varpi+u_x)m\mathrm{d}x=Q(t) & [0,T],\end{cases}\tag{1.1}$$

and

$$\begin{cases}m(0,x)=m_0(x)\\ u(T,x)=u_T(x)\end{cases}\quad x\in\mathbb{R}.\tag{1.2}$$

We work under assumptions similar to the ones introduced in [34] to guarantee the existence and uniqueness of (u,m,ϖ) solving (1.1) and (1.2). The first two assumptions require standard growth and convexity properties for H.

*Received: May 30, 2022; Accepted (in revised form): June 08, 2023. Communicated by Pierre-Emmanuel Jabin.

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Assumption 1.1. *There exist constants $c>0$ and $p>1$, such that the Legendre-Fenchel transform of H, the function L in (1.3), satisfies*

$$L(v) \geqslant c|v|^p.$$

Assumption 1.2. *For all $x \in \mathbb{R}$, the map $p \mapsto H(p)$ is uniformly convex; that is, there exists a constant $\kappa>0$ such that $H''(p) \geqslant \kappa$ for all $p \in \mathbb{R}$. Moreover, there exists a positive constant, C, such that $|H'''| \leqslant C$.*

For the supply, to simplify, we assume it is a smooth function of time.

Assumption 1.3. *The supply function, Q, is $C^\infty([0,T])$.*

The following assumption is technical and was used in [34] to get bounds for the price.

Assumption 1.4. *The potential V, the terminal cost u_T, and the initial density function m_0 are $C^2(\mathbb{R})$, and V, u_T are globally Lipschitz. Furthermore, there exists a constant $C>0$ such that*

$$|V''| \leqslant C, \quad |u_T''| \leqslant C, \quad |m_0''| \leqslant C.$$

The following condition guarantees the uniqueness of solutions of (1.1) and (1.2).

Assumption 1.5. *The potential V and the terminal cost u_T are convex.*

Finally, because we are interested in problems where the agent's assets are bounded, we require the following assumption on m_0. This assumption further simplifies some technical points in the presentation.

Assumption 1.6. *The initial density function m_0 has compact support; that is, there exists $R_0>0$ such that* $\operatorname{supp}(m_0) \subset [-R_0, R_0]$.

The existence of solutions (u, m, ϖ) to Problem 1.1 under Assumptions 1.2, 1.1, 1.4, and 1.5 was proved in [34]. The first equation is solved in the viscosity sense by the value function of a typical player $u \in C([0,T]\times\mathbb{R})$. The second equation is solved in the distributional sense by the probability distribution of the agents $m \in C([0,T], \mathcal{P}(\mathbb{R}))$. The price ϖ is a continuous function on $[0,T]$.

Problem 1.1 is derived by considering price formation on a market where agents interact through the price function alone. Based on the price, they optimize the cost criteria

$$\int_0^T (L(v(s)) + \varpi(s)v(s) - V(x(s)))\,\mathrm{d}t + u_T(x(T))$$

by choosing v to control their dynamics $x'(s) = v(s)$. The optimal selection is characterized by the first equation in (1.1). The Hamiltonian H is the Legendre transform of a Lagrangian L, as we introduce below. Under the optimal selection, the evolution of m_0 in (1.2) is determined by the second equation in (1.1), while the third equation imposes an equilibrium condition between the aggregated trading rate and the supply Q. The price arises as the Lagrange multiplier of the equilibrium constraint, being the only function for which agents minimize their costs while satisfying the equilibrium constraint. Remarkably, in our approach, we obtain the rate of change of the price, $\dot{\varpi}$, as the Lagrange multiplier of a constraint imposed on a new variational problem (see (4.20)).

Price formation models offer a load-adaptive pricing strategy relevant in energy markets. For instance, [6] and [7] modeled intraday electricity markets, obtaining a price from the solution of forward-backward equations. In [27] and [31] authors studied the effects of a major player in the market. The latest paper considered N-agent setting. A deterministic N-agent price model was studied in [8]. A MFG model of homogeneous agents for the electricity markets was considered in [28]. In [4], the price equilibrium is obtained for a finite number of agents who optimally control their production and trading rates to satisfy a demand subjected to common noise. Stackelberg games for price formation under revenue optimization were proposed in [13] and [43], and Cournot models in [23]. A MFG of optimal switching was presented in [5] to model the transition to renewable energies. [24] studied the convergence of a finite-population game to a MFG. In their model, a market clearing condition matched an aggregated inventory trade with noise to a demand/supply rate. Other works incorporating market-clearing conditions are [44] and [30], the former specializing in Solar Renewable Energy Certificate Markets and the latter in exchange markets. The stochastic supply case was studied in [32], where authors obtained a price from a Lagrange multiplier rule for the balance constraint.

The standard MFG system exhibits a coupling of two partial differential equations with initial and terminal conditions (see for example [16]). Several numerical methods have been proposed to solve these MFG systems. Finite difference schemes and Newton-based methods were introduced in [1] and [2]. A recent survey can be found in [3]. Optimization methods and Fourier series approximations were proposed in [38]. Machine learning methods have been studied in [17, 18, 42], and [37]. However, the MFG system (1.1)-(1.2) not only couples a forward equation for m with a backward equation for u but also determines the coupling term ϖ through an integral constraint, which is the third equation in (1.1). Therefore, the numerical approximation of the solution (u,m,ϖ) of Problem 1.1 is challenging, and the main application of our methods is a novel numerical scheme for Problem 1.1.

The word Potential in MFGs is used in two unrelated contexts. Potential MFGs ([16, 36, 39]) are MFG systems given by the first-order optimality conditions of a minimization problem. Previously, standard optimization techniques were used for its numerical solution ([14]). In contrast, our potential approach relies on the structure of the continuity equation and Poincaré lemma ([19], Theorem 1.22). We introduce a potential functional that integrates the transport equation in (1.1).

Poincaré lemma was used for the continuity equation in [12] for the MFG planning problem. The authors obtained a variational problem for a potential function by eliminating one of the equations in the MFG system. Moreover, the solution (u,m) of the planning MFG can be recovered using only the solution of the variational problem. The structure of the MFG planning problem differs from that in Problem 1.1 in two critical aspects: the initial-terminal conditions and the way the constraint couples the equations.

In Section 2, we use the existence result for Problem 1.1 provided in [34] to formally obtain a potential function, φ. In Proposition 2.1, we show that (1.1) corresponds to the Euler-Lagrange equation of a constrained variational problem depending on φ. To introduce this problem, let L be the Legendre transform of H; that is,

$$L(y) = \sup_{p\in\mathbb{R}} [py - H(p)], \quad y \in \mathbb{R}, \tag{1.3}$$

and let $\mathbb{L}:\mathbb{R}\times\mathbb{R}_0^+\to\mathbb{R}_0^+\cup\{+\infty\}$ be given by

$$\mathbb{L}(z,y)=\begin{cases} L\left(\frac{z}{y}\right)y, & (z,y)\in\mathbb{R}\times\mathbb{R}^+,\\ +\infty, & z\neq 0, y=0,\\ 0, & z=0, y=0.\end{cases} \tag{1.4}$$

The constrained variational problem is

PROBLEM 1.2. *Suppose that $m_0\in\mathcal{P}(\mathbb{R})$, H is uniformly convex, and Q, V, and u_T are continuous. Find $\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$ that minimizes the functional*

$$\varphi\mapsto\int_0^T\int_{\mathbb{R}}\mathbb{L}(\varphi_t,\varphi_x)-V(x)\varphi_x-u_T'(x)\varphi_t\,\mathrm{d}x\mathrm{d}t,$$

over the set of functions such that $\varphi_x(t,\cdot)$ is a probability density on $\mathbb{R}$ for $t\in[0,T]$, $\varphi_x(0,\cdot)=m_0(\cdot)$, and satisfying

$$\int_{\mathbb{R}}\varphi(t,x)-M_0(x)\mathrm{d}x=-\int_0^t Q(s)\mathrm{d}s,\quad t\in[0,T],$$

where $M_0(x)=\int_{-\infty}^x m_0(y)\mathrm{d}y$.

We rigorously study Problem 1.2 in Section 3 by considering a variational problem and a relaxed version. We use Proposition A.1 and Corollary A.1 in the Appendix to show that the formulation of Problem 1.2 is independent of the solution (u,m,ϖ) of Problem 1.1, and relies only on problem data. In Section 3.1, relying on the technical results of Proposition 3.1 and Corollary 3.1, Proposition 3.2 shows uniqueness of solutions to Problem 1.2. Moreover, Proposition 3.3 offers additional bounds on the minimization problem. However, the limit of minimizing sequences may not be admissible. Therefore, in Section 3.2, we use Proposition 3.4, Lemma 3.1, and Theorem 3.1 to identify a relaxed variational problem that is lower semi-continuous. Theorem 3.2 obtains existence of minimizers in the relaxed setting. Next, we address the relation between solutions of Problems 1.1 and 1.2 in Section 4. In Proposition 4.1, we show that the price ϖ in (1.1) is given by a Lagrange multiplier associated to the variational setting. The improvement of our approach consists in obtaining the solution of Problem 1.1 from a simpler problem, Problem 1.2, which is a convex minimization problem with constraints. This is the content of our main result, proved in Section 4:

THEOREM 1.1. *Suppose that $\varphi\in C^2([0,T]\times\mathbb{R})$ solves Problem 1.2 . Then, the solution (u,m,ϖ) of Problem 1.1 admits the representation*

$$\begin{cases} u(t,x)=u_T(x)-\int_t^T H\left(L'\left(\frac{\varphi_t(s,x)}{\varphi_x(s,x)}\right)\right)\mathrm{d}s-(T-t)V(x), & (t,x)\in[0,T]\times\mathbb{R}\\ m(t,x)=\varphi_x(t,x), & (t,x)\in[0,T]\times\mathbb{R}\\ \varpi(t)=w_T-\int_t^T w(s)\mathrm{d}s, & t\in[0,T],\end{cases}$$

where $w_T=\int_{\mathbb{R}}(\mathbb{L}_z^*(T,y)-u_T'(y))\varphi_x(T,y)\mathrm{d}y$,

$$w(s)=\int_{\mathbb{R}}\left((\mathbb{L}_z^*(s,y))_t+\left(\mathbb{L}_y^*(s,y)-V(y)\right)_x\right)\varphi_x(s,y)\mathrm{d}y,\quad s\in[0,T],$$

and $\mathbb{L}_z^*$ and $\mathbb{L}_y^*$ are defined in (4.10) and (4.11), respectively.

Because the previous main result holds under regularity assumptions, further study of the properties of solutions of Problem 1.2 is required. However, some classes of problems admit regular solutions, as in the linear-quadratic setting, which we illustrate in Section 5. In this case, Theorem 1.1 shows that existence of solutions of Problem 1.1 is equivalent to existence of solutions of Problem 1.2, as well as for its relaxed formulation. Moreover, because Problem 1.2 is a convex minimization problem, we approximate its solution φ using standard optimization methods. Furthermore, using the approximations for φ and Theorem 1.1, we obtain efficient approximation methods for the solution to Problem 1.1. In the linear-quadratic setting, we use the explicit formulas provided in [34] as benchmarks. For all these benchmarks, our numerical method provides accurate approximations.

2. Derivation of the variational problem

In this section, we present a formal derivation of the variational problem for the potential function using the solution of the MFGs system. In Proposition 2.1, we show that (1.1) corresponds to the Euler-Lagrange equation of a variational problem. The rigorous statement of the variational problem is given in Section 3, where we no longer rely on the solution of the MFGs system.

Recalling (1.4), let us consider the functional

$$\int_0^T \int_{\mathbb{R}} \mathbb{L}(\varphi_t,\varphi_x) - V\varphi_x \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}} u_T'(x)(\varphi(T,x)-\varphi(0,x))\ \mathrm{d}x \tag{2.1}$$

subject to

$$\int_{\mathbb{R}} -(\varphi_t + Q\varphi_x)\mathrm{d}x = 0, \quad \text{on} \quad [0,T], \tag{2.2}$$

and with initial condition

$$\varphi(0,x) = \int_{-\infty}^x m_0(y)\mathrm{d}y,$$

where $\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$. We consider the augmented functional associated with the constraint (2.2); that is, we introduce a Lagrange multiplier $\varpi:[0,T]\to\mathbb{R}$ and define

$$\tilde{I}[\varphi,\varpi] := \int_0^T \int_{\mathbb{R}} \mathbb{L}(\varphi_t,\varphi_x) - \varpi(\varphi_t + Q\varphi_x\ ) - V\varphi_x - u_T'(x)\varphi_t\ \mathrm{d}x\mathrm{d}t \tag{2.3}$$

with initial condition $\varphi(0,x) = \int_{-\infty}^x m_0(y)\mathrm{d}y$. In Section 4, we address the existence of the price ϖ as a Lagrange multiplier associated with a minimizer of (2.1). By considering critical points (φ,ϖ) of the functional in (2.3), we obtain the following.

Proposition 2.1. *Let (φ,ϖ) be a critical point of the functional (2.3) over $C^2([0,T]\times\mathbb{R})\times C^1([0,T])$ satisfying $\varphi(0,x) = \int_{-\infty}^x m_0(y)\mathrm{d}y$. Assume further that $\varphi_x > 0$. Then, the corresponding Euler-Lagrange equation is equivalent to*

$$\begin{cases} -\left(L'\left(\frac{\varphi_t}{\varphi_x}\right) - \varpi\right)_t + \left(H\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)\right)_x + V' = 0 & [0,T]\times\mathbb{R}, \\ -\int_{\mathbb{R}} \varphi_t + Q\varphi_x\ \mathrm{d}x = 0 & t\in[0,T], \end{cases} \tag{2.4}$$

with terminal condition

$$L'\left(\frac{\varphi_t(T,x)}{\varphi_x(T,x)}\right) - \varpi(T) = u_T'(x) \quad x\in\mathbb{R}. \tag{2.5}$$

The function φ obtained as follows allows to identify the variational problem whose Euler-Lagrange Equation (2.4) and (2.5) is precisely (1.1) and (1.2). Let (u,m,ϖ) solve Problem 1.1 with $m>0$. Then, the second equation in (1.1) can be written as

$$\operatorname{div}_{(t,x)}(m,-H'(\varpi+u_x)m)=0, \quad [0,T]\times\mathbb{R}. \tag{2.6}$$

The previous equation combined with Poincaré lemma (see [19], Theorem 1.22) gives the existence of a function (the potential) $\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} m=\varphi_x, \\ H'(\varpi+u_x)m=\varphi_t. \end{cases} \tag{2.7}$$

Because H is uniformly convex, H' is strictly monotone. Therefore, by (1.3), we have

$$L'(y)=(H')^{-1}(y). \tag{2.8}$$

Hence, from the second equation in (2.7), we deduce that

$$u_x=L'\left(\frac{\varphi_t}{\varphi_x}\right)-\varpi. \tag{2.9}$$

If $V\in C^1(\mathbb{R})$, and u is twice differentiable, we differentiate the Hamilton-Jacobi equation in (1.1) with respect to x to obtain

$$-(u_x)_t+(H(\varpi+u_x))_x+V'=0.$$

Using the previous equation and (2.9), we obtain (2.4) and (2.5).

REMARK 2.1. Notice that the initial condition implies that $\varphi_x(0,x)=m_0(x)$, $x\in\mathbb{R}$, which is the first equation in (1.2). Moreover, we have the following explicit formula for φ in terms of the solution (u,m,ϖ) of (1.1) and (1.2)

$$\varphi(t,x)=\int_{-\infty}^{x} m_0(y)\mathrm{d}y+\int_0^t H'(\varpi(s)+u_x(s,x))m(s,x)\mathrm{d}s, \quad (t,x)\in[0,T]\times\mathbb{R}. \tag{2.10}$$

Therefore, the potential function φ, which in principle has a closed formula arising from the solution of (1.1) and (1.2), can be characterized using the initial condition with m_0, (2.4) and (2.5), which depend only, up to ϖ, on problem data.

REMARK 2.2. Notice that the first equation in (2.4) shows that the expression

$$-\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)_t+\left(H\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)\right)_x+V'$$

is independent of $x\in\mathbb{R}$, so it is a function of time only and equal to $\dot{\varpi}$. Similarly, (2.5) shows that

$$L'\left(\frac{\varphi_t(T,x)}{\varphi_x(T,x)}\right)-u_T'(x)$$

is independent of $x\in\mathbb{R}$, and equal to the constant $\varpi(T)$. Because any numerical method to compute φ provides an approximation of the value $\varphi(t,x)$, we can not expect the numerical approximation to be independent of x in (2.4) and (2.5). Therefore, we can not rely on these formulas to recover ϖ using an approximation of φ. In Section 4, we provide a formula approximating ϖ that averages the dependence on x, and thus, can be implemented with any approximation of the potential.

Remark 2.3. To understand the role that the uni-dimensionality of (1.1) plays in the use of Poincaré lemma, let us obtain (2.7) from (2.6) using differential forms. Consider the 1-form $\omega_1 = m\,\mathrm{d}x + mH'(\varpi + u_x)\,\mathrm{d}t$. Then,

$$\mathrm{d}\omega_1 = \frac{\partial(m)}{\partial t}\,\mathrm{d}t \wedge \mathrm{d}x + \frac{\partial(mH'(\varpi+u_x))}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}t = \left(\frac{\partial(m)}{\partial t} - \frac{\partial(mH'(\varpi+u_x))}{\partial x}\right)\mathrm{d}t \wedge \mathrm{d}x = 0.$$

Thus, ω_1 is a closed form, and, by Poincaré lemma, it is exact; that is, $\omega_1 = \mathrm{d}\omega_0$ for a 0-form ω_0, which can be identified with a function, $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$. Then, (2.7) follows from writing $\mathrm{d}\omega_0 = \mathrm{d}\varphi = \frac{\partial \varphi}{\partial t}\,\mathrm{d}t + \frac{\partial \varphi}{\partial x}\,\mathrm{d}x$ and using the expression for ω_1.

Now, let us consider (2.6) in dimension $d > 1$. In this setting, the state variable $x \in \mathbb{R}$, as presented in Problem 1.1, is replaced by $\mathbf{x} \in \mathbb{R}^d$, the supply function $Q : [0,T] \to \mathbb{R}$ is replaced by $\mathbf{Q} = (Q_1, \ldots, Q_d) : [0,T] \to \mathbb{R}^d$, and $H, V : \mathbb{R}^d \to \mathbb{R}$. The corresponding MFG system is

$$\begin{cases} -u_t + H(\varpi + Du) + V(\mathbf{x}) = 0 & [0,T] \times \mathbb{R}^d, \\ m_t - \operatorname{div}(mDH(\varpi + Du)) = 0 & [0,T] \times \mathbb{R}^d, \\ -\int_{\mathbb{R}^d} mH_{p_i}(\varpi + Du)\mathrm{d}\mathbf{x} = Q_i(t) & [0,T],\, i \in \{1,\ldots,d\}, \end{cases}$$

and

$$\begin{cases} m(0,\mathbf{x}) = m_0(\mathbf{x}) \\ u(T,\mathbf{x}) = u_T(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \mathbb{R}^d.$$

The previous system is a natural extension of (1.1) and (1.2) for a multi-dimensional commodity with supply $\mathbf{Q}$. However, the existence and uniqueness results we rely on were presented in [34] for the one-dimensional model. Nonetheless, consider the d-form

$$\begin{aligned} \omega_d = {} & m\,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d + mH_{p_1}\,\mathrm{d}t \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_d \\ & + mH_{p_2}\,\mathrm{d}x_1 \wedge \mathrm{d}t \wedge \mathrm{d}x_3 \wedge \ldots \wedge \mathrm{d}x_d + \ldots + mH_{p_d}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_{d-1} \wedge \mathrm{d}t. \end{aligned}$$

Then,

$$\mathrm{d}\omega_d = \left(\frac{\partial m}{\partial t} - \frac{\partial(mH_{p_1})}{\partial x_1} - \ldots - \frac{\partial(mH_{p_d})}{\partial x_d}\right)\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d = 0.$$

As before, by Poincaré lemma, ω_d, being closed is exact. Thus, $\omega_d = \mathrm{d}\omega_{d-1}$ for a $(d-1)$-form ω_{d-1}. However, unlike in the one-dimensional setting, ω_{d-1} can not be identified with a function on $[0,T] \times \mathbb{R}^d$. Yet, it is possible to obtain a variational problem in the context of differential forms, similar to those introduced in [21] and [20], and obtain existence and uniqueness of minimizers in a weak sense. While most of the tools developed in [21] and [20] could be used in our setting, we have chosen to limit the scope of the present work to the one-dimensional case, which allows us to illustrate the main features of our approach and rely on the results presented in [34].

Proof. **(Proof of Proposition 2.1.)** Let (φ, ϖ) be a critical point of (2.3). Taking $(\beta^1, \beta^2) \in C_c^1((0,T] \times \mathbb{R}) \times C([0,T])$, we have

$$\frac{d}{d\varepsilon}\tilde{I}[(\varphi,\varpi) + \varepsilon(\beta^1,\beta^2)]\bigg|_{\varepsilon=0} = 0. \tag{2.11}$$

The previous identity implies that

$$-(\mathbb{L}_z(\varphi_t,\varphi_x)-\varpi)_t-(\mathbb{L}_y(\varphi_t,\varphi_x))_x+V'=0, \tag{2.12}$$

and

$$\mathbb{L}_z(\varphi_t(T,x),\varphi_x(T,x))-\varpi(T)-u_T'(x)=0 \tag{2.13}$$

on $[0,T]\times\mathbb{R}$. Because $\varphi_x>0$, (1.4) gives

$$\begin{aligned}
&\mathbb{L}_z(\varphi_t,\varphi_x)=L'\left(\frac{\varphi_t}{\varphi_x}\right),\\
&(\mathbb{L}_y(\varphi_t,\varphi_x))_x=\left(-\frac{\varphi_t}{\varphi_x}L'\left(\frac{\varphi_t}{\varphi_x}\right)+L\left(\frac{\varphi_t}{\varphi_x}\right)\right)_x=-\frac{\varphi_t}{\varphi_x}\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)_x.
\end{aligned} \tag{2.14}$$

Notice that, by (2.8), we have

$$\left(H\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)\right)_x=\frac{\varphi_t}{\varphi_x}\left(L'\left(\frac{\varphi_t}{\varphi_x}\right)\right)_x. \tag{2.15}$$

Combining the identities in (2.14) with (2.15) and using (2.12), we deduce the first equation in (2.4). Using the first identity of (2.14) in (2.13), we obtain (2.5).

Finally, taking $\beta^1\equiv 0$ in (2.11), we obtain

$$\int_0^T\left(\int_{\mathbb{R}}-(\varphi_t+Q\varphi_x)\mathrm{d}x\right)\beta^2\,\mathrm{d}t=0,$$

where β^2 is arbitrary. Thus, the second equation in (2.4) holds. □

3. The variational approach

Here, we examine a variational problem associated with the MFG system (1.1)-(1.2). This problem is obtained by minimizing the functional (2.1) in a suitable class of admissible functions. We study the existence and uniqueness of solutions to this variational problem. Relying on the results in the Appendix, we show that, contrary to the formal derivation in Section 2, we can formulate the variational problem regardless of the solution (u,m,ϖ) of (1.1). In Section 3.1, we obtain uniqueness of solutions to the variational problem. We introduced a relaxed formulation in Section 3.2 to obtain existence. The results of this section allow us to establish a formula representing the solution to the MFG system (1.1)-(1.2), in terms of the solution to this variational problem, as we prove in Section 4.

First, we recall that, under Assumptions 1.1-1.5, Theorem 1 in [34] gives existence and uniqueness of solutions (u,m,ϖ) to Problem 1.1, where $m\in\mathcal{P}(\mathbb{R})\cap C([0,T]\times\mathbb{R})$. Moreover, u is a viscosity solution to the first equation in (1.1), Lipschitz continuous and semi-concave in x, and u_x, u_{xx}, m are bounded. Furthermore, by the results in [10], ϖ is Lipschitz continuous.

Next, to formulate our variational problem, we use Corollary A.1 in the **Appendix** to motivate the choice of the function spaces.

Remark 3.1. Consider the potential φ associated with the solution (u,m,ϖ) of the MFG system (1.1)-(1.2), as given by (2.7). By Corollary A.1, we deduce that the gradient of the potential φ has compact support; that is, $\operatorname{supp}(\varphi_t(t,\cdot)),\operatorname{supp}(\varphi_x(t,\cdot))\subseteq[-R_m,R_m]$ for all $t\in[0,T]$. Thus, (A.11) shows that, by selecting

$$R>(R_0+C_0T)\left(1+C_0Te^{C_0T}\right),$$

we obtain a compact set $[-R,R]$ that depends only on problem data and which contains the support of the gradient of φ when (2.10) holds. Thus, using this compact set, we can formulate our variational problem independently of the solution (u,m,ϖ) of the MFGs system (1.1) and (1.2). Notice that (2.10) already suggests a candidate for a minimizer. However, we study the existence of solutions to the variational problem independently of solutions to the MFGs system. Moreover, if uniqueness holds and we have existence for both problems, then (2.10) is the unique minimizer.

3.1. Statement of the variational problem. In this subsection, we present our variational approach rigorously using only problem data. We start with the notations and the definition of admissible functions, deriving technical properties in Proposition 3.1 and Corollary 3.1. Then, we formulate our variational problem, obtaining uniqueness in Proposition 3.2 and bounds in Proposition 3.3.

Let R_0 be given by Assumption 1.6 and let

$$R>\max\left\{(R_0+C_0T)\left(1+C_0Te^{C_0T}\right),R_0+\|Q\|_{L^1([0,T])}\right\}. \tag{3.1}$$

Notice that, by (A.11), R is an upper bound for R_m, as required, according to Remark 3.1. The additional requirement $R>R_0+\|Q\|_{L^1([0,T])}$ guarantees that the set of admissible functions we define below is not empty. Set

$$\Omega_R=[0,T]\times[-R,R],\quad \Omega=[0,T]\times\mathbb{R}.$$

We denote by $\mathcal{M}(\Omega_R)$ ($\mathcal{M}(\Omega)$) the set of Radon measures on $\Omega_R\subset\mathbb{R}^2$ ($\Omega\subset\mathbb{R}^2$) and by $BV(\Omega_R)$ ($BV(\Omega)$) the set of functions with bounded variation on Ω_R (Ω) (see [9,26]).

To define the admissible set for our variational problem, we rewrite the balance condition, the second equation in (2.4). Recall that $\operatorname{supp}(m_0)\subset[-R_0,R_0]$ and $R>R_0$. Let

$$M_0(x)=\int_{-\infty}^{x}m_0(y)\mathrm{d}y=\int_{-R}^{x}m_0(y)\mathrm{d}y,\quad x\in\mathbb{R}, \tag{3.2}$$

be the cumulative density function of m_0. Note that after integrating the balance condition over $[0,t]$, and requiring that $\int_{\mathbb{R}}\varphi_x(t,x)\mathrm{d}x=1$ for $t\in[0,T]$ (which follows in case that (2.7) holds), we get

$$\int_0^t\int_{\mathbb{R}}\varphi_t\ \mathrm{d}x\mathrm{d}s=-\int_0^t Q(s)\mathrm{d}s,\quad t\in[0,T].$$

Therefore, we write the balance condition as

$$\int_{\mathbb{R}}\varphi(t,x)-M_0(x)\ \mathrm{d}x=-\int_0^t Q(s)\mathrm{d}s,\quad t\in[0,T]. \tag{3.3}$$

Relying on (3.3) and taking into account the discussion in Remark 3.1, for any set $A\subset\mathbb{R}^2$ satisfying $[0,T]\times[-R,R]\subseteq A$, we denote

$$\begin{aligned}\mathcal{B}_R(A)=\{&\varphi:[0,T]\times[-R,R]:\ (\varphi-M_0)\in W^{1,1}(A),\\ &\operatorname{supp}(\varphi_t(t,\cdot)),\operatorname{supp}(\varphi_x(t,\cdot))\subseteq(-R,R),\ t\in[0,T]\},\end{aligned}$$

$$\begin{aligned}\mathcal{B}(A)=\Bigg\{&\varphi\in\mathcal{B}_R(A):\ \varphi_x\geqslant 0,\ \varphi(0,x)=\int_{-R}^{x}m_0(y)\mathrm{d}y,\ x\in\mathbb{R}\\ &\int_{\mathbb{R}}\varphi(t,x)-M_0(x)\ \mathrm{d}x=-\int_0^t Q(s)\ \mathrm{d}s,\ \int_{-R}^{R}\varphi_x(t,x)\mathrm{d}x=1,\ t\in[0,T]\Bigg\},\end{aligned}$$

which are convex sets. Before proceeding, we prove a crucial property of the set $\mathcal{B}(\Omega)$.

PROPOSITION 3.1. *For any function $\varphi\in\mathcal{B}(\Omega)$, we have, for $t\in[0,T]$,*

$$\varphi(t,x)=\begin{cases}0 & x\in(-\infty,-R],\\ \varphi(t,x) & x\in(-R,R),\\ 1 & x\in[R,+\infty).\end{cases}$$

Finally, the set of admissible functions for our variational problem is given by

$$\mathcal{A}(\Omega_R)=\{\varphi\in\mathcal{B}(\Omega_R):\ \varphi(t,-R)=0\}. \tag{3.4}$$

As a result of Proposition 3.1, we obtain the following relation between the admissible set $\mathcal{A}(\Omega_R)$ and the set $\mathcal{B}(\Omega)$.

COROLLARY 3.1. *For any function $\varphi\in\mathcal{B}(\Omega)$ there exists a function $\tilde\varphi\in\mathcal{A}(\Omega_R)$ such that $\varphi\equiv\tilde\varphi$ in Ω_R. The opposite is also true.*

Under Assumption 1.4, we have

$$\left|\int_{\Omega_R}u_T'(x)\varphi_t\ \mathrm{d}x\mathrm{d}t\right|\leqslant \mathrm{Lip}(u_T)\int_{\Omega_R}|\varphi_t|\ \mathrm{d}x\mathrm{d}t, \tag{3.5}$$

where $\mathrm{Lip}(u_T)$ is the Lipschitz constant of u_T. Relying on the previous inequality, we consider the following variational problem

$$\inf_{\varphi\in\mathcal{B}(\Omega)}\int_{\Omega_R}\mathbb{L}(\varphi_t,\varphi_x)-V\varphi_x-u_T'(x)\varphi_t\ \mathrm{d}x\mathrm{d}t,$$

which, by Corollary 3.1, coincides with the following (see (2.1))

$$\inf_{\varphi\in\mathcal{A}(\Omega_R)}I[\varphi], \tag{3.6}$$

where

$$I[\varphi]:=\int_{\Omega_R}\mathbb{L}(\varphi_t,\varphi_x)-V\varphi_x-u_T'(x)\varphi_t\ \mathrm{d}x\mathrm{d}t.$$

As anticipated in Remark 3.1, (3.1) guarantees that the previous variational problem does not rely on the solution (u,m,ϖ) to (1.1)-(1.2) but only on the data of Problem 1.1. Moreover, the next result shows that the infimum in (3.6) can be attained by at most one function.

PROPOSITION 3.2. *Suppose that Assumptions 1.1-1.6 hold. Then, at most, one function attains the infimum in (3.6).*

Next, we prove that the infimum in (3.6) is bounded.

PROPOSITION 3.3. *Assume that Assumptions 1.1-1.6 hold. Then, there exist positive constants, C_1 and C_2, depending only on the problem data such that*

$$-C_2\leqslant\inf_{\varphi\in\mathcal{A}(\Omega_R)}I[\varphi]\leqslant C_1. \tag{3.7}$$

Furthermore, there exists a positive constant, C, depending only on problem data, such that for every minimizing sequence $\{\varphi^n\}_{n\in\mathbb{N}}$ of the variational problem (3.6), we have

$$\int_{\Omega_R}\tfrac{|\varphi_t^n|^p}{(\varphi_x^n)^{p-1}}\mathrm{d}x\mathrm{d}t\leqslant C,\quad \|\varphi_t^n\|_{L^1(\Omega_R)}\leqslant C. \tag{3.8}$$

Thus, for all minimizing sequences of the variational problem (3.6), we obtain uniform bounds in $W^{1,1}(\Omega_R)$. Therefore, any minimizing sequence has a weakly convergent sub-sequence in $BV(\Omega_R)$ ([26], Chapter 5). However, it is not guaranteed that the infimum in (3.6) is attained in $\mathcal{A}(\Omega_R)$. Therefore, we enlarge the set of admissible functions by relaxing the conditions defining $\mathcal{A}(\Omega_R)$, as we present in the next section.

Proof. **(Proof of Proposition 3.1.)** Because $(\varphi - M_0) \in W^{1,1}(\Omega)$ and $\lim_{x\to+\infty} M_0(x) = 1$, for each $t \in [0,T]$, there exists a sequence x_k such that $x_k \to \infty$ and $\lim_{k\to+\infty} \varphi(t,x_k) = 1$. On the other hand, recalling that

$$\operatorname{supp}(\varphi_t(t,\cdot)), \operatorname{supp}(\varphi_x(t,\cdot)) \subset (-R,R),$$

we have that φ is constant on $\Omega \setminus [0,T] \times (-R,R)$. Consequently, $\varphi(t,x) = 1$ for $x \in [R,+\infty)$. Similarly, we can prove that $\varphi(t,x) = 0$, $x \in (-\infty,-R]$. □

Proof. **(Proof of Proposition 3.2.)** Let φ^1 and φ^2 attain the infimum in (3.6). By Proposition 3.3, we denote

$$\ell = \min_{\varphi \in \mathcal{A}(\Omega_R)} I[\varphi] \in \mathbb{R}.$$

Thus, $I[\varphi^1] = I[\varphi^2] = \ell$. Set $\bar\varphi = \frac{1}{2}(\varphi^1 + \varphi^2)$. Due to the properties of the Legendre-Fenchel transform (1.3), L is convex, which in turn implies the convexity of L (see [12], Lemma 8.1). Thus, we obtain

$$\ell \leqslant I[\bar\varphi] \leqslant \tfrac{1}{2} I[\varphi^1] + \tfrac{1}{2} I[\varphi^2] = \ell. \tag{3.9}$$

Hence, $\bar\varphi$ is also minimizer of (3.6). Let $\tilde\varphi = \frac{\varphi^1 + \bar\varphi}{2}$ and

$$\begin{aligned} &\mathcal{U}_1 = \{(t,x) \in \Omega_R : \varphi^1_x > 0\}, \quad \mathcal{U}_2 = \{(t,x) \in \Omega_R : \varphi^2_x > 0\}, \\ &\tilde{\mathcal{U}} = \{(t,x) \in \Omega_R : \tilde\varphi_x > 0\} = \bar{\mathcal{U}} = \{(t,x) \in \Omega_R : \bar\varphi_x > 0\} = \mathcal{U}_1 \cup \mathcal{U}_2. \end{aligned}$$

Arguing as in (3.9), we have

$$\ell \leqslant I[\tilde\varphi] \leqslant \tfrac{1}{2} I[\varphi^1] + \tfrac{1}{2} I[\bar\varphi] = \ell. \tag{3.10}$$

This with (3.9), yields that

$$\mathbb{L}(\varphi^1_t, \varphi^1_x), \mathbb{L}(\varphi^2_t, \varphi^2_x), \mathbb{L}(\bar\varphi_t, \bar\varphi_x), \mathbb{L}(\tilde\varphi_t, \tilde\varphi_x) < +\infty \ \text{ a.e. in } \Omega_R.$$

Therefore,

$$\begin{aligned} \varphi^1_t &= 0 \quad \text{a.e. in } \Omega_R \setminus \mathcal{U}_1, \\ \varphi^2_t &= 0 \quad \text{a.e. in } \Omega_R \setminus \mathcal{U}_2, \\ \bar\varphi_t &= 0 \quad \text{a.e. in } \Omega_R \setminus \bar{\mathcal{U}} \\ \tilde\varphi_t &= 0 \quad \text{a.e. in } \Omega_R \setminus \tilde{\mathcal{U}}. \end{aligned} \tag{3.11}$$

Furthermore, (3.10) implies

$$\int_{\Omega_R} \Big(\tfrac{1}{2} \mathbb{L}(\varphi^1_t, \varphi^1_x) + \tfrac{1}{2} \mathbb{L}(\bar\varphi_t, \bar\varphi_x) - \mathbb{L}(\tilde\varphi_t, \tilde\varphi_x) \Big)\, dx dt = 0. \tag{3.12}$$

The convexity of L and (3.12) implies

$$\tfrac{1}{2}\mathbb{L}(\varphi_t^1,\varphi_x^1)+\tfrac{1}{2}\mathbb{L}(\bar{\varphi}_t,\bar{\varphi}_x)-\mathbb{L}(\tilde{\varphi}_t,\tilde{\varphi}_x)=0, \quad \text{a.e. in } \Omega_R.$$

Consequently, the following also holds

$$\tfrac{1}{2}\mathbb{L}(\varphi_t^1,\varphi_x^1)+\tfrac{1}{2}\mathbb{L}(\bar{\varphi}_t,\bar{\varphi}_x)-\mathbb{L}(\tilde{\varphi}_t,\tilde{\varphi}_x)=0, \quad \text{a.e. in} \quad \mathcal{U}_1\cap\bar{\mathcal{U}}\cap\tilde{\mathcal{U}}. \tag{3.13}$$

Because $\mathbb{L}$ is strictly convex in $\mathbb{R}\times\mathbb{R}^+$ and $\mathcal{U}_1\cap\bar{\mathcal{U}}\cap\tilde{\mathcal{U}}=\mathcal{U}_1\subset\mathbb{R}\times\mathbb{R}^+$, we obtain from (3.13) that

$$\begin{cases}\varphi_t^1=\bar{\varphi}_t\\ \varphi_x^1=\bar{\varphi}_x.\end{cases} \quad \text{a.e. in} \quad \mathcal{U}_1$$

Hence,

$$\begin{cases}\varphi_t^1=\varphi_t^2\\ \varphi_x^1=\varphi_x^2.\end{cases} \quad \text{a.e. in} \quad \mathcal{U}_1 \tag{3.14}$$

Taking φ^2 instead of φ^1 in (3.10) and arguing as before, we obtain

$$\begin{cases}\varphi_t^1=\varphi_t^2\\ \varphi_x^1=\varphi_x^2.\end{cases} \quad \text{a.e. in} \quad \mathcal{U}_2 \tag{3.15}$$

Combing (3.11), (3.14) and (3.15), we conclude that $\varphi^1=\varphi^2$. □

Proof. **(Proof of Proposition 3.3.)** First, we prove the upper bound in (3.7). Let

$$\varphi^0(t,x)=M_0(x-q(t)),$$

where M_0 is defined by (3.2) and $q(t)=\int_0^t Q(\tau)\ d\tau$. Therefore, since $q(0)=0$, we have

$$\varphi^0(0,x)=\int_{-R}^{x} m_0(y)\mathrm{d}y, \quad \varphi_x^0=m_0(x-q(t)), \quad \varphi_t^0=-m_0(x-q(t))Q(t). \tag{3.16}$$

Thus, recalling (3.1),

$$\varphi^0\in\mathcal{A}(\Omega_R). \tag{3.17}$$

Taking into account (3.16), we have

$$\inf_{\mathcal{A}(\Omega)} I[\varphi]\leqslant I[\varphi^0]\leqslant\int_{\Omega_R}\mathbb{L}(\varphi_t^0,\varphi_x^0)+\|V\|_{L^\infty([-R,R])}\varphi_x^0+\|u_T'\|_{L^\infty(\Omega_R)}|\varphi_t^0|\ \mathrm{d}x\mathrm{d}t=:C_1. \tag{3.18}$$

Next, relying on this bound, we prove (3.8), implying the lower bound in (3.7). By Assumption 1.1 , $L\geqslant 0$. Thus, for all $\varphi\in\mathcal{A}(\Omega_R)$, we have

$$-\int_{\Omega_R} V\varphi_x+u_T'\varphi_t\mathrm{d}x\mathrm{d}t\leqslant I[\varphi]. \tag{3.19}$$

Recalling that V is continuous and taking into account Assumption 1.4 by (3.5) and (3.19), we get

$$-C(u_T,R,V)-\mathrm{Lip}(u_T)\int_{\Omega_R}|\varphi_t(t,x)|\mathrm{d}x\mathrm{d}t\leqslant I[\varphi]. \tag{3.20}$$

From (3.18) follows that for any minimizing sequence $\{\varphi^n\}_{n\in\mathbb{N}}$, there exists N such that $n\geqslant N$ implies $I[\varphi^n]\leqslant C_1+1$. Consequently, recalling the definition of $\mathbb{L}$ in (1.4), by Assumption 1.1 and (3.5), we deduce that

$$\begin{aligned} c\int_{\Omega_R}\frac{|\varphi_t^n|^p}{(\varphi_x^n)^{p-1}}\mathrm{d}x\mathrm{d}t &\leqslant C_1+1+||V||_{L^\infty([-R,R])}\int_{\Omega_R}\varphi_x^n\mathrm{d}x\mathrm{d}t+\mathrm{Lip}(u_T)\int_{\Omega_R}|\varphi_t^n(t,x)|\,\mathrm{d}x\mathrm{d}t\\ &\leqslant C+\mathrm{Lip}(u_T)\int_{\Omega_R}|\varphi_t^n(t,x)|\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.21}$$

for all $n\in\mathbb{N}$. On the other hand, by Young's inequality, we have

$$\int_{\Omega_R}|\varphi_t^n|\,\mathrm{d}x\mathrm{d}t=\int_{\Omega_R}\frac{|\varphi_t^n|}{(\varphi_x^n)^{\frac{p-1}{p}}}(\varphi_x^n)^{\frac{p-1}{p}}\mathrm{d}x\mathrm{d}t\leqslant\varepsilon\int_{\Omega_R}\frac{|\varphi_t^n|^p}{(\varphi_x^n)^{p-1}}\mathrm{d}x\mathrm{d}t+C(\varepsilon)\int_{\Omega_R}\varphi_x^n\mathrm{d}x\mathrm{d}t, \tag{3.22}$$

where $\varepsilon=\frac{c}{2\mathrm{Lip}(u_T)}$. Recalling that $\int_{-R}^R\varphi_x(\cdot,x)\mathrm{d}x=1$, the preceding inequality and (3.21) imply (3.8). Finally, (3.8) and (3.20) yield the lower bound in (3.7). □

3.2. Relaxed variational problem. Here, we relax the variational problem (3.6) to ensure the existence of minimizers in the set of admissible functions. Using Proposition 3.4, Lemma 3.1, and Theorem 3.1, we identify the appropriate relaxed problem for which lower semi-continuity holds, and we obtain existence of minimizers in Theorem 3.2.

First, we extend the functional in (3.6) to the convex set

$$BV_0^+(\Omega_R)=\{\psi\in BV(\Omega_R):\psi_x\geqslant 0\}.$$

For that, let $W:BV_0^+(\Omega_R)\to\mathbb{R}\cup\{+\infty\}$ be given by

$$W[\varphi]=\begin{cases}\int_{\Omega_R}\mathbb{L}(\varphi_t,\varphi_x)-V\varphi_x-u_T'(x)\varphi_t\ \mathrm{d}x\mathrm{d}t, & \varphi\in W^{1,1}(\Omega_R)\cap BV_0^+(\Omega_R)\\ +\infty, & \text{otherwise.}\end{cases}$$

In $BV(\Omega_R)$, we consider the intermediate convergence; that is, $(\varphi_k)_{k\in\mathbb{N}}\subset BV(\Omega_R)$ converges to $\varphi\in BV(\Omega_R)$ in the intermediate (or strict) sense if

$$\varphi_k\to\varphi\ \text{in}\ L^1(\Omega_R)\quad\text{and}\quad\|D\varphi_k\|(\Omega_R)\to\|D\varphi\|(\Omega_R),$$

where $\|D\varphi\|(\Omega_R)$ is the total variation of the measure $D\varphi$ on Ω_R (see [9]). We recall that $W^{1,1}(\Omega_R)$ is dense in $BV(\Omega_R)$ with respect to the intermediate convergence (see Theorem 10.1.2 in [11]). We aim to define a functional $\mathcal{W}$, the sequential lower semicontinuous envelope of W w.r.t. intermediate convergence on BV (Chapter 3, [22]); that is

$$\begin{aligned}\mathcal{W}[\varphi]=\sup\{G[\varphi]:\ &G\leqslant W,\ G \text{ is sequentially lower semicontinuous on } BV(\Omega_R)\\ &\text{w.r.t. intermediate convergence}\},\end{aligned}$$

which is the greatest functional below W that is sequentially lower semi-continuous w.r.t intermediate convergence in $BV(\Omega_R)$. Let

$$\mathcal{J}[\varphi]=\inf\left\{\liminf_{n\to\infty}I[\varphi^n]:\{\varphi^n\}\subset W^{1,1}(\Omega_R)\cap BV_0^+(\Omega_R),\right.$$

$$\left.\varphi^n \to \varphi \text{ in the sense of the intermediate convergence in } BV(\Omega_R)\right\}.$$

Next, we prove that actually $\mathcal{W}=\mathcal{J}$ and obtain an explicit expression for $\mathcal{J}$.

Assuming that Assumption 1.1 holds for some $c>0$, $p>1$ and arguing as in (3.22) by using Young's inequality, we obtain

$$cp(|v_1|+v_2)\leqslant \mathbb{L}(v_1,v_2)+c(2p-1)v_2, \quad (v_1,v_2)\in\mathbb{R}\times\mathbb{R}_0^+. \tag{3.23}$$

Let

$$f(v_1,v_2)=\mathbb{L}(v_1,v_2)-u_T'v_1-Vv_2=f_N(v_1,v_2)+f_L(v_1,v_2), \tag{3.24}$$

where

$$f_N(v_1,v_2)=\mathbb{L}(v_1,v_2)+c(2p-1)v_2, \quad f_L(v_1,v_2)=-u_T'v_1-(V+c(2p-1))v_2.$$

According to Lemmas 8.1 and 8.3 in [12], if Assumption 1.2 holds, the function $\mathbb{L}$ defined in (1.4) is convex and lower semicontinuous in $\mathbb{R}\times\mathbb{R}_0^+$; therefore, f_N is convex and lower semicontinuous.

For the next result, we compute the recession function, $\bar{f}_N$, of f_N, which is given by

$$\bar{f}_N(z,y):=\sup\{f_N(w_1+z,w_2+y)-f_N(w_1,w_2):\ (w_1,w_2)\in\mathrm{dom}_e(f_N)\},$$

for $(z,y)\in\mathbb{R}\times\mathbb{R}_0^+$, where

$$\mathrm{dom}_e(f_N):=\left\{(w_1,w_2)\in\mathbb{R}\times\mathbb{R}_0^+:\ f_N(w_1,w_2)<+\infty\right\}.$$

Because f_N is convex, from Theorem 4.70 in [29], we have

$$\bar{f}_N(z,y)=\lim_{t\to\infty}\frac{f_N((z,y)t+(w_1,w_2))-f_N(w_1,w_2)}{t},$$

for any $(w_1,w_2)\in\mathbb{R}\times\mathbb{R}_0^+$. Taking $(w_1,w_2)=(0,0)$ in the preceding equation and considering (1.4), we deduce that f_N is equal to its recession function; that is,

$$\bar{f}_N(z,y)=f_N(z,y)=\begin{cases} L\left(\frac{z}{y}\right)y+c(2p-1)y & (z,y)\in\mathbb{R}\times\mathbb{R}^+,\\ +\infty & z\neq 0, y=0,\\ 0 & z=0, y=0,\end{cases}$$

where the constants c and p are given by Assumption 1.1. Using the preceding observation, we prove that the first integrand of the functional in (2.1) is sequentially lower semi-continuous w.r.t. the weak -$*$ convergence of measures.

PROPOSITION 3.4. *Suppose that Assumption 1.2 holds. Let* $(v_1^n,v_2^n)\in L^1(\Omega_R)\times L^1(\Omega_R;\mathbb{R}_0^+)$ *and* $\mu=(\mu_1,\mu_2)\in\mathcal{M}(\Omega_R)\times\mathcal{M}(\Omega_R;\mathbb{R}_0^+)$ *be such that*

$$(v_1^n,v_2^n)\mathcal{L}^2\lfloor\Omega_R\overset{*}{\rightharpoonup}(\mu_1,\mu_2), \quad \text{in } \mathcal{M}(\Omega_R)\times\mathcal{M}(\Omega_R;\mathbb{R}_0^+).$$

Then,

$$\liminf_{n\to\infty}\int_{\Omega_R} f_N(v_1^n,v_2^n)\mathrm{d}x\mathrm{d}t$$

$$\geqslant \int_{\Omega_R} f_N\left(\frac{d\mu}{d\mathcal{L}^2}(t,x)\right) \mathrm{d}x\mathrm{d}t + \int_{\Omega_R} f_N\left(\frac{d\mu_s}{d||\mu_s||}(t,x)\right) d||\mu_s||(t,x),$$

where $\mu = \frac{d\mu}{d\mathcal{L}^2}\lfloor\Omega_R + \mu_s$ *is the Radon-Nikodym decomposition of* μ *w.r.t. the two-dimensional Lebesgue measure* $\mathcal{L}^2$*, and* $\frac{d\mu_s}{d||\mu_s||}$ *is the Radon-Nikodym derivative of* μ_s *w.r.t its total variation.*

Proof. By Assumption 1.2, f_N is convex and lower semi-continuous in $\mathbb{R}\times\mathbb{R}_0^+$ and $\bar{f}_N = f_N$. Hence, the proof follows from Theorem 5.19 in [29]. □

LEMMA 3.1. *Suppose that Assumption 1.4 holds. Let* $v_1^n \to v_1$ *and* $v_2^n \to v_2$ *weakly in* $\mathcal{M}(\Omega_R)$*. Then,*

$$\lim_{n\to\infty}\left(-\int_{\Omega_R}(V+C)v_1^n + u_T' v_2^n \mathrm{d}x\mathrm{d}t\right) = -\int_{\Omega_R}(V+C)v_1 + u_T'(x)v_2 \;\mathrm{d}x\mathrm{d}t,$$

for any $C\in\mathbb{R}$*.*

Proof. It is enough to notice that the functions V and $u_T'(x)$ are continuous. □

Now, we are ready to prove that $\mathcal{W} = \mathcal{J}$.

THEOREM 3.1. *Suppose that Assumptions 1.2 and 1.4 hold for some* $c>0$ *and* $p>1$*. Then, for every* $\varphi\in BV_0^+(\Omega_R)$

$$\mathcal{W}[\varphi] = \mathcal{J}[\varphi] = \int_{\Omega_R} f_N\left(\frac{d(D_{t,x}\varphi)}{d\mathcal{L}^2}(t,x)\right) \mathrm{d}x\mathrm{d}t - \int_{\Omega_R}(V + c(2p-1))\varphi_x - \int_{\Omega_R} u_T'(x)\varphi_t + \int_{\Omega_R} f_N\left(\frac{d(D_{t,x}\varphi)_s}{d||(D_{t,x}\varphi)_s||}(t,x)\right) d||(D_{t,x}\varphi)_s||(t,x).$$

Next, relying on Theorem 3.1, we state the relaxed variational problem. We set

$$\begin{aligned}
&\mathcal{K}_0(\Omega_R) = \left\{\varphi\in BV_0^+(\Omega_R) : \operatorname{supp}(\varphi_t(t,\cdot)), \operatorname{supp}(\varphi_x(t,\cdot)) \subseteq (-R,R)\right\},\\
&\mathcal{K}(\Omega_R) = \Bigg\{\varphi\in\mathcal{K}_0(\Omega_R) : \varphi(0,x) = \int_{-R}^x m_0(y)\mathrm{d}y,\ \varphi(t,-R) = 0,\\
&\qquad \int_{-R}^R \varphi_x(t,x) = 1,\ \int_{-R}^R \varphi(t,x) - M_0(x)\;\mathrm{d}x = -\int_0^t Q(s)\;\mathrm{d}s,\ t\in[0,T]\Bigg\}.
\end{aligned}$$

Note that $\mathcal{A}(\Omega_R)\subset\mathcal{B}(\Omega_R)\subset\mathcal{K}(\Omega_R)$, so (3.17) guarantees that $\mathcal{K}(\Omega_R)$ is a nonempty convex set. Our relaxed variational problem is

$$\min_{\varphi\in\mathcal{K}(\Omega_R)} \mathcal{I}[\varphi], \tag{3.25}$$

where

$$\mathcal{I}[\varphi] = \int_{\Omega_R} f_N\left(\frac{d(D_{t,x}\varphi)}{d\mathcal{L}^2}(t,x)\right) \mathrm{d}x\mathrm{d}t - \int_{\Omega_R}(V + c(2p-1))\varphi_x - \int_{\Omega_R} u_T'(x)\varphi_t + \int_{\Omega_R} f_N\left(\frac{d(D_{t,x}\varphi)_s}{d||(D_{t,x}\varphi)_s||}(t,x)\right) d||(D_{t,x}\varphi)_s||(t,x).$$

The next theorem proves the existence of solutions to the preceding variational problem.

THEOREM 3.2. *Suppose that Assumptions 1.1-1.5 hold. Then, there exists $\varphi\in\mathcal{K}(\Omega_R)$ such that*

$$\mathcal{I}[\varphi]=\min_{\psi\in\mathcal{K}(\Omega_R)}\mathcal{I}[\psi].$$

REMARK 3.2. Under Assumptions 1.2-1.5, Theorem 1 in [34] implies that φ given by (2.10) belongs to $\mathcal{K}(\Omega_R)$ and is a minimizer of (3.25). Thus, under Assumptions 1.1-1.5, if we have uniqueness, φ given by Theorem 3.2, and φ given by (2.10) coincide.

Proof. **(Proof of Theorem 3.1.)** Taking into account (3.23) and the definition of f_N, (3.24), we have

$$cp(|v_1|+v_2)\leqslant f_N(v_1,v_2),\quad (v_1,v_2)\in\mathbb{R}\times\mathbb{R}_0^+.$$

Recalling that f_N is convex and using the preceding estimate, the proof follows from Remark 5.37 in [29] and Proposition 3.1. □

Proof. **(Proof of Theorem 3.2.)** We recall that $W^{1,1}(\Omega_R)$ is dense in $BV(\Omega_R)$ with respect to the intermediate convergence (see Theorem 10.1.2 in [11]). Accordingly, we can take a minimizing sequence, $\{\varphi^n\}_{n=1}^{\infty}$, such that $\varphi^n\in W^{1,1}(\Omega_R)$. Therefore,

$$\min_{\psi\in\mathcal{K}(\Omega_R)}\mathcal{I}[\psi]=\lim_{n\to\infty}\mathcal{I}[\varphi^n]=\liminf_{n\to\infty}I[\varphi^n],$$

where I, $\mathcal{I}$ are defined by (3.6) and (3.25), respectively. Note that because

$$\varphi_x^n\in\mathcal{P}(-R,R)\cap L^1(-R,R),$$

where $\mathcal{P}(-R,R)$ denotes the set of probability measures on $(-R,R)$, there exists $\mu\in\mathcal{P}(-R,R)$, such that

$$||\varphi_x^n||_{L^1(\Omega_R)}\leqslant C,\quad \varphi_x^n\rightharpoonup\mu \text{ weakly in } \mathcal{M}(\Omega_R). \tag{3.26}$$

Combining these estimates with the argument in Proposition 3.3, we deduce that

$$||\varphi_t^n||_{L^1(\Omega_R)}\leqslant C. \tag{3.27}$$

Consequently, because Ω_R is bounded, Prohorov lemma (see Theorem 2.29 in [35]) gives the existence of $\nu\in\mathcal{M}(\Omega_R)$ such that $\varphi_t^n\rightharpoonup\nu$ weakly in $\mathcal{M}(\Omega_R)$. On the other hand, because $\int_{-R}^{R}\varphi(t,x)-M_0(x)\ \mathrm{d}x=-\int_0^t Q(s)\ \mathrm{d}s$, we have that $\left|\int_{-R}^{R}\varphi^n\mathrm{d}x\right|\leqslant C$, where C does not depend on φ. Hence, by Poincaré inequality (see Theorem 1 in Section 5.8.1 in [25]) from (3.26) and (3.27), we get

$$||\varphi^n||_{L^1(\Omega_R)}\leqslant||\varphi_x^n||_{L^1(\Omega_R)}+||\varphi_t^n||_{L^1(\Omega_R)}+C\leqslant C.$$

Therefore, $||\varphi^n||_{W^{1,1}(\Omega_R)}\leqslant C$. Consequently, Rellich-Kondrachov theorem (see Theorem 1, Section 5.7 in [25]) implies that there exists $\varphi\in L^\alpha(\Omega_R)$ for $\alpha\in[1,2)$, such that φ^n converges to φ strongly in $L^\alpha(\Omega_R)$. In particular, φ^n converges to φ strongly in $L^1(\Omega_R)$. This convergence, combined with (3.26) and (3.27), implies that there exists $\varphi\in BV(\Omega_R)$, such that $\varphi^n\to\varphi$ in the sense of intermediate convergence in $BV(\Omega_R)$. Finally, relying on this and recalling that $\varphi^n\in\mathcal{K}(\Omega_R)\cap W^{1,1}(\Omega_R)$ from [11, Theorem 10.2.2], we deduce that $\varphi\in\mathcal{K}(\Omega_R)$. Moreover, recalling the definition of f_N and using Propositions 3.4 and 3.1, we get

$$\min_{\psi\in\mathcal{K}(\Omega_R)}\mathcal{I}[\psi]=\lim_{n\to\infty}\mathcal{I}[\varphi^n]=\liminf_{n\to\infty}I[\varphi^n]\geqslant\mathcal{I}[\varphi]=\min_{\psi\in\mathcal{K}(\Omega_R)}\mathcal{I}[\psi],$$

which completes the proof. □

4. Price as Lagrange multiplier

In this section, we provide a representation formula for the price ϖ using the minimizer φ of (3.6). This formula shows that the Lagrange multiplier associated with the balance constraint (3.3) characterizes the price.

PROPOSITION 4.1. *Suppose that Assumptions 1.2-1.5 hold. Let R satisfy (3.1). Let (u,m,ϖ) solve (1.1) and let $\varphi \in \mathcal{A}(\Omega_R)$ attain the minimum in (3.6). Furthermore, assume that $\varphi \in C^2(\Omega_R)$. Then, ϖ is given by a Lagrange multiplier $w:[0,T]\to\mathbb{R}$ associated with φ.*

The previous result is critical to proving our main result, Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Because φ solves Problem 1.2, we obtain ϖ according to Proposition 4.1. Therefore, (φ,ϖ) minimizes (2.3), and by Proposition 2.1 satisfies an Euler-Lagrange equation equivalent to (2.4)-(2.5). Since the solution (u,m,ϖ) of (1.1) defines a potential function according to (2.10), which satisfies (2.4)-(2.5), the convexity of (2.3) implies that this potential is a minimizer of (2.3). Thus, by Proposition 3.2, we conclude that the potential function defined by (u,m,ϖ) coincides with the minimizer φ. Thus, we can recover u and m using (2.7); that is,

$$u(t,x)=u_T(x)-\int_t^T H\left(L'\left(\frac{\varphi_t(s,x)}{\varphi_x(s,x)}\right)\right)\mathrm{d}s-(T-t)V(x), \quad (t,x)\in[0,T]\times\mathbb{R}, \tag{4.1}$$

where the right-hand side of the previous expression is well defined because φ_x and φ_t have the same compact support, and $m(t,x)=\varphi_x(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}$. ☐

Proof. **(Proof of Proposition 4.1.)** The existence and uniqueness of the solution, (u,m,ϖ), to Problem 1.1 follows from Theorem 1 in [34]. Because L is convex, and recalling Remark 2.1, the solution (u,m,ϖ) to Problem 1.1 defines a minimizer of (3.6) in $\mathcal{A}(\Omega_R)$, which is given by (2.10). By Proposition 3.2, we deduce that φ coincides with (2.10), as it is the unique minimizer of (3.6). Furthermore, by Remark 2.1 and Corollary A.1, it follows that there exists $0<R_1<R$ such that

$$\operatorname{supp}(\varphi_x(t,\cdot))\subseteq[-R_1,R_1]\subseteq(-R,R), \quad t\in[0,T]. \tag{4.2}$$

Because φ is the minimizer of (3.6)

$$\varphi_t(t,x)=0 \quad a.e. \quad \text{in}\{\varphi_x(t,x)=0:(t,x)\in\Omega_R\}. \tag{4.3}$$

Let

$$\overline{x}(t):=\int_{\mathbb{R}} x\varphi_x(t,x)\mathrm{d}x=R-\int_{-R}^{R}\varphi(t,x)\mathrm{d}x, \quad t\in[0,T]. \tag{4.4}$$

Let $\psi\in W^{1,1}(\Omega_R)$ be such that $\psi(t,\cdot)$ is a cumulative distribution function on $(-R,R)$ for $t\in[0,T]$, and satisfies

$$\psi(0,x)=\varphi(0,x), \quad \operatorname{supp}(\psi_x(t,\cdot))\subseteq\operatorname{supp}(\varphi_x(t,\cdot))\subseteq[-R_1,R_1]\subset(-R,R), \quad t\in[0,T], \tag{4.5}$$

and

$$\psi_t(t,x)=0 \quad a.e. \quad \text{in} \quad \{(t,x)\in\Omega_R:\ \psi_x(t,x)=0\}. \tag{4.6}$$

Set

$$\overline{z}(t):=\int_{\mathbb{R}} x\psi_x(t,x)\mathrm{d}x=R-\int_{-R}^{R}\psi(t,x)\mathrm{d}x, \quad t\in[0,T]. \tag{4.7}$$

Notice that $|\overline{x}-\overline{z}|\leqslant 4R$. Let $0<\varepsilon<\min\left\{\frac{R-R_1}{4R},1\right\}$. Thus,

$$[-R_1,R_1]\subset(-R-\varepsilon(\overline{x}(t)-\overline{z}(t)),R-\varepsilon(\overline{x}(t)-\overline{z}(t)))\cap(-R,R),\quad t\in[0,T]. \tag{4.8}$$

Let

$$\varphi^\varepsilon(t,x):=(1-\varepsilon)\varphi(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))+\varepsilon\psi(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t))).$$

We claim that $\varphi^\varepsilon\in\mathcal{A}(\Omega_R)$. Indeed, by (4.5) and (4.8), we have

$$\varphi^\varepsilon_x\geqslant 0,\quad \int_{-R}^{R}\varphi^\varepsilon_x \mathrm{d}x=1,\quad \varphi^\varepsilon(0,x)=M_0(x)\quad x\in[-R,R],$$
$$\varphi^\varepsilon(t,-R)=0,\quad t\in[0,T].$$

It remains to prove that φ^ε satisfies the balance condition; that is,

$$\int_{-R}^{R}\varphi^\varepsilon(t,x)\mathrm{d}x=-\int_0^t Q(s)ds+\int_{-R}^{R}M_0(x)\mathrm{d}x,\quad t\in[0,T]. \tag{4.9}$$

Because $\varphi\in\mathcal{A}(\Omega_R)$, (4.4) shows that to prove (4.9), it is enough to verify that

$$\int_{\mathbb{R}}x\varphi^\varepsilon_x(t,x)\mathrm{d}x=\overline{x}(t),\quad t\in[0,T].$$

Computing the left-hand side of the previous identity, we have

$$\begin{aligned}
&\int_{\mathbb{R}}x\varphi^\varepsilon_x(t,x)\mathrm{d}x\\
&=(1-\varepsilon)\int_{\mathbb{R}}x\varphi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))\mathrm{d}x+\varepsilon\int_{\mathbb{R}}x\psi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))\mathrm{d}x\\
&=(1-\varepsilon)\int_{\mathbb{R}}x\varphi_x(t,x)\mathrm{d}x+\varepsilon(\overline{x}(t)-\overline{z}(t))+\varepsilon\int_{\mathbb{R}}x\psi_x(t,x)\mathrm{d}x+\varepsilon(\overline{x}(t)-\overline{z}(t))\\
&=(1-\varepsilon)\left(\overline{x}(t)+\varepsilon(\overline{x}(t)-\overline{z}(t))\right)+\varepsilon\left(\overline{z}(t)+\varepsilon(\overline{x}(t)-\overline{z}(t))\right)\\
&=\overline{x}(t).
\end{aligned}$$

Therefore, $\varphi^\varepsilon\in\mathcal{A}(\Omega_R)$, and the map $\varepsilon\mapsto I[\varphi^\varepsilon]$ has a minimum at $\varepsilon=0$; that is,

$$\lim_{\varepsilon\to 0^+}\frac{d}{d\varepsilon}I[\varphi^\varepsilon]\geqslant 0.$$

To compute the left-hand side of the previous inequality, we notice that

$$\begin{aligned}
\varphi^\varepsilon_t(t,x)=&(1-\varepsilon)\Big(\varphi_t(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))-\varphi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))\varepsilon\left(\dot{\overline{x}}(t)-\dot{\overline{z}}(t)\right)\Big)\\
&+\varepsilon\Big(\psi_t(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))-\psi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))\varepsilon\left(\dot{\overline{x}}(t)-\dot{\overline{z}}(t)\right)\Big),\\
\varphi^\varepsilon_x(t,x)=&(1-\varepsilon)\varphi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t)))+\varepsilon\psi_x(t,x-\varepsilon(\overline{x}(t)-\overline{z}(t))).
\end{aligned}$$

Note that (4.3) and (4.6) imply

$$\varphi^\varepsilon_t(t,x)=0\quad a.e.\quad \text{in}\quad \{\varphi^\varepsilon_x(t,x)=0:(t,x)\in\Omega_R\}.$$

Furthermore,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{d}{d\varepsilon}\varphi^\varepsilon_t(t,x) &= -\varphi_t(t,x)-\varphi_{tx}(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)-\varphi_x(t,x)\left(\dot{\overline{x}}(t)-\dot{\overline{z}}(t)\right)+\psi_t(t,x)\\ &= \frac{d}{dt}\Big(-\varphi(t,x)+\psi(t,x)-\varphi_x(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)\Big),\\ \lim_{\varepsilon\to 0^+}\frac{d}{d\varepsilon}\varphi^\varepsilon_x(t,x) &= -\varphi_x(t,x)-\varphi_{xx}(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)+\psi_x(t,x)\\ &= \frac{d}{dx}\Big(-\varphi(t,x)+\psi(t,x)-\varphi_x(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)\Big).\end{aligned}$$

For ease of notation, we denote

$$\mathbb{L}^*_z(t,x)=\begin{cases}\mathbb{L}_z(\varphi_t(t,x),\varphi_x(t,x)), & \varphi_x(t,x)>0\\ 0, & \text{otherwise,}\end{cases} \tag{4.10}$$

and

$$\mathbb{L}^*_y(t,x)=\begin{cases}\mathbb{L}_y(\varphi_t(t,x),\varphi_x(t,x)), & \varphi_x(t,x)>0\\ 0, & \text{otherwise.}\end{cases} \tag{4.11}$$

Taking into account (4.3) and (4.6), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{d}{d\varepsilon}I[\varphi^\varepsilon] &= \int_0^T\int_{-R}^R\left(\mathbb{L}^*_z(t,x)-u_T'(x)\right)\frac{d}{dt}\Big(\psi(t,x)-\varphi(t,x)-\varphi_x(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)\Big)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_{-R}^R\left(\mathbb{L}^*_y(t,x)-V(x)\right)\frac{d}{dx}\Big(\psi(t,x)-\varphi(t,x)-\varphi_x(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)\Big)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Integrating by parts in the right-hand side of the previous identity, using that $\overline{x}(0)=\overline{z}(0)$, and recalling (4.2) and (4.5), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{d}{d\varepsilon}I[\varphi^\varepsilon] &= \int_{-R}^R\left(\mathbb{L}^*_z(T,x)-u_T'(x)\right)\Big(\psi(T,x)-\varphi(T,x)-\varphi_x(T,x)\left(\overline{x}(T)-\overline{z}(T)\right)\Big)\mathrm{d}x\\ &-\int_0^T\int_{-R}^R\Big(\left(\mathbb{L}^*_z(t,x)\right)_t+\left(\mathbb{L}^*_y(t,x)-V(x)\right)_x\Big)\\ &\qquad\Big(\psi(t,x)-\varphi(t,x)-\varphi_x(t,x)\left(\overline{x}(t)-\overline{z}(t)\right)\Big)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.12}$$

Recalling (4.4) and (4.7), we have

$$\overline{x}(t)-\overline{z}(t)=\int_{-R}^R\left(\psi(t,x)-\varphi(t,x)\right)\mathrm{d}x,\quad t\in[0,T].$$

Using the previous identity and Fubini's theorem, we write the first term on the right-hand side of (4.12) as follows

$$\begin{aligned}&\int_{-R}^R\left(\mathbb{L}^*_z(T,x)-u_T'(x)\right)\Big(\psi(T,x)-\varphi(T,x)-\varphi_x(T,x)\left(\overline{x}(T)-\overline{z}(T)\right)\Big)\mathrm{d}x\\ &=\int_{-R}^R\left(\mathbb{L}^*_z(T,x)-u_T'(x)-\int_{-R}^R\left(\mathbb{L}^*_z(T,y)-u_T'(y)\right)\varphi_x(T,y)\mathrm{d}y\right)\left(\psi(T,x)-\varphi(T,x)\right)\mathrm{d}x.\end{aligned} \tag{4.13}$$

Similarly, the second term on the right-hand side of (4.12) becomes

$$\begin{aligned}&\int_0^T\int_{-R}^R\Big((\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x\Big)(\psi(t,x)-\varphi(t,x)-\varphi_x(t,x)(\overline{x}(t)-\overline{z}(t)))\,\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{-R}^R\Big((\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x\\&\qquad-\int_{-R}^R\Big((\mathbb{L}_z^*(t,y))_t+\big(\mathbb{L}_y^*(t,y)-V(y)\big)_x\Big)\varphi_x(t,y)\mathrm{d}y\Big)(\psi(t,x)-\varphi(t,x))\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.14}$$

Define the Lagrange multiplier by

$$\begin{aligned}w_T&=\int_{-R}^R(\mathbb{L}_z^*(T,y)-u_T'(y))\varphi_x(T,y)\mathrm{d}y,\\w(t)&=\int_{-R}^R\Big((\mathbb{L}_z^*(t,y))_t+\big(\mathbb{L}_y^*(t,y)-V(y)\big)_x\Big)\varphi_x(t,y)\mathrm{d}y,\quad t\in[0,T].\end{aligned}\tag{4.15}$$

Then, replacing (4.13) and (4.14) in (4.12), we get

$$\begin{aligned}&\int_{-R}^R\Big(\mathbb{L}_z^*(T,x)-u_T'(x)-w_T\Big)(\psi(T,x)-\varphi(T,x))\,\mathrm{d}x\\&\quad+\int_0^T\int_{-R}^R\Big((\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x-w(t)\Big)(\psi(t,x)-\varphi(t,x))\,\mathrm{d}x\mathrm{d}t\geqslant0.\end{aligned}\tag{4.16}$$

Notice that, in the previous inequality, the function $\phi=\psi-\varphi$ can be selected to be strictly positive or negative in any neighborhood of $(0,T)\times(-R,R)$. Therefore, we can infer the nullity of the functions in both integrals in (4.16) as follows. First, select ψ satisfying $\psi(T,\cdot)=\varphi(T,\cdot)$. Then, (4.16) shows that

$$\int_0^T\int_{-R}^R\Big((\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x-w(t)\Big)\phi(t,x)\mathrm{d}x\mathrm{d}t\geqslant0.\tag{4.17}$$

The regularity of

$$(t,x)\mapsto(\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x-w(t)$$

allows the localization of the integral in (4.17) using ϕ, and we conclude that

$$(\mathbb{L}_z^*(t,x))_t+\big(\mathbb{L}_y^*(t,x)-V(x)\big)_x-w(t)=0\quad\text{a.e. }(t,x)\in(0,T)\times(-R,R).\tag{4.18}$$

Then, (4.16) reduces to

$$\int_{-R}^R\Big(\mathbb{L}_z^*(T,x)-u_T'(x)-w_T\Big)(\psi(T,x)-\varphi(T,x))\,\mathrm{d}x\geqslant0,$$

and we proceed as before by localizing the integral using $x\mapsto\phi(T,x)$ to conclude that

$$\mathbb{L}_z^*(T,x)-u_T'(x)-w_T=0\quad\text{a.e. }x\in(-R,R).\tag{4.19}$$

Recalling (2.12), which characterizes ϖ, the identities (4.18) and (4.19) show that the price, ϖ, is given by the Lagrange multiplier (4.15) according to

$$\dot\varpi(t)=w(t),\quad t\in[0,T],\quad\varpi(T)=w_T,\tag{4.20}$$

which completes the proof. ∎

5. Numerical results

In this section, we provide the results of the potential approach applied to the price formation MFG system with quadratic cost and oscillating supply. We use the semi-explicit formulas introduced in [34] to assess the error in our approximation. We use the standard solver for finite-dimensional convex problems provided by the software Mathematica to approximate the potential function in a discrete grid in time and space.

Let $\kappa \in \mathbb{R}$, $\eta \geqslant 0$, and $c>0$. For the quadratic cost configuration, we take

$$H(p)=\frac{1}{2c}p^2, \quad V(x)=-\frac{\eta}{2}(x-\kappa)^2, \quad \text{and} \quad u_T(x)\equiv 0.$$

Thus, $L(v)=\frac{c}{2}v^2$. As shown in [34] and [33], a feature of the quadratic setting is the solvability of the Hamilton-Jacobi equation in (1.1) in the class of quadratic functions of x with time-dependent coefficients

$$u(t,x)=a_0(t)+a_1(t)x+a_2(t)x^2, \quad t\in[0,T],\ x\in\mathbb{R}.$$

The coefficients a_0, a_1, and a_2 solve an ODE system that derives from the Hamilton-Jacobi equation by matching powers of the x variable. Figure 5.1c shows the value function for $(x,t)\in[0,T]\times[-1,1]$. Moreover, the price has the following explicit formula

$$\varpi(t)=\eta(\kappa-\overline{m}_0)(T-t)-\eta\int_t^T\int_0^s Q(r)drds-cQ(t), \quad t\in[0,T],$$

where $\overline{m}_0=\int_{\mathbb{R}} x m_0(x)dx$. The initial condition m_0 is centered at $x=0$ and with compact support $[-0.5,0.5]$ (see Figure 5.1a). The vector-field transporting m_0 is

$$b(t,x)=-\frac{1}{c}(\varpi(t)+a_1(t)+2a_2(t)x), \quad t\in[0,T],\ x\in\mathbb{R},$$

which we use to compute m using the method of characteristics (see Figure 5.2b). Thus, recalling (2.7) and (2.10), we have explicit formulas for φ, φ_x, and φ_t. We use the previous expressions as a benchmark for the approximation obtained using (4.15).

For the discretization of the time variable, we set $T=1$ and $N_t=20$ time steps uniformly spaced. Thus, $h_t=0.05$ is the time step size. To discretize the space variable, the selection of R in (3.1), where $R_0=0.5$, becomes

$$R>\max\{5.57742\ , \quad 0.811579\}.$$

However, to simplify the computational cost, we optimize the selection of R by looking at the support of $m(t,\cdot)$ for $t\in[0,T]$, which we illustrate in Figure 5.2b. Thus, we discretize the space variable in the space domain $[-1,1]$ using $N_x=40$ time steps equally spaced. Thus, $h_x=0.05$ is the step size.

Because in several applications, the supply function satisfies a mean reversion assumption, we assume that it follows the ordinary differential equation

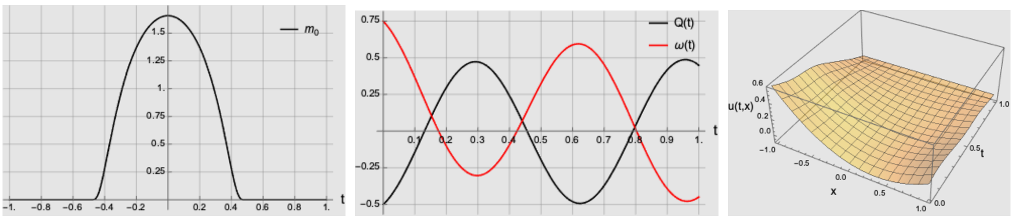
$$\begin{cases}\dot{Q}(t)=\overline{Q}(t)-\alpha Q(t), & t\in[0,T],\\ Q(0)=q_0,\end{cases}$$

where $\overline{Q}:[0,T]\to\mathbb{R}$ represents the average supply over time, $\alpha\in\mathbb{R}$ measures the tendency to go towards the average, and $q_0\in\mathbb{R}$ is the initial supply. For numerical purposes, we select

$$\overline{Q}(t)=5\sin(3\pi t), \quad \alpha=4, \quad q_0=-0.5.$$

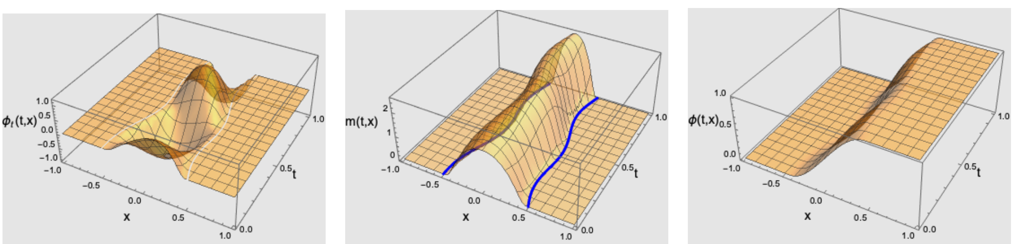
While the particular choice of Q does not change the problem substantially, the preceding choice has oscillatory features, as we want to demonstrate how price changes, and, at the same time, we recover simple analytic expressions. As Figure 5.1b shows, the price inherits the oscillating behavior from the supply.

Using the solution (u,m,ϖ), we get φ_t, φ_x from (2.7) (see Figures 5.2a and 5.2b), and so (2.10) gives φ, illustrated in Figure 5.2c. The value of (3.6) is 0.106525, which we use as an additional benchmark to assess our numerical approximation.



(a) m_0 (b) Price and supply (c) value function u

Fig. 5.1: *Data m_0 and Q, and solutions u and ϖ for $\overline{Q}(t) = 5\sin(3\pi t)$.*



(a) φ_t (b) $\varphi_x = m$ (c) potential φ

Fig. 5.2: *Analytic solution φ and its partial derivatives for $\overline{Q}(t) = 5\sin(3\pi t)$. The blue lines outline the support of m.*

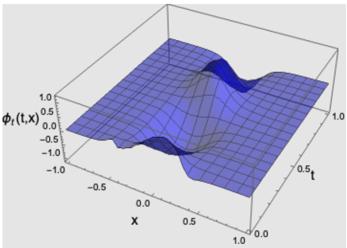
We discretize (3.6) over the time-space grid using finite differences to approximate φ_t and φ_x; that is

$$\varphi_t(t_i,x_j) = \frac{\varphi(t_i+h_t,x_j)-\varphi(t_i,x_j)}{h_t}, \quad \varphi_x(t_i,x_j) = \frac{\varphi(t_i,x_j+h_x)-\varphi(t_i,x_j)}{h_x},$$

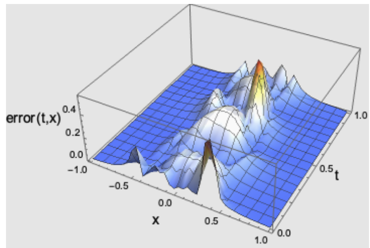
for $i=1,\dots,20$, and $j=1,\dots,40$. We obtain a finite-dimensional convex optimization problem with the following constraints

$$\varphi_x(t_i,x_j) \geqslant 0, \quad \sum_{j=1}^{N_x} (\varphi(t_i,x_j) - M_0(x_j))h_x + \sum_{k=0}^{i} Q(t_k)h_t,$$
$$\varphi(0,x_j) - M_0(x_j) = 0, \quad \varphi(t_i,-1) = 0, \quad \varphi(t_i,1) = 1, \quad i=1,\dots,N_t,\, j=1,\dots,N_x,$$

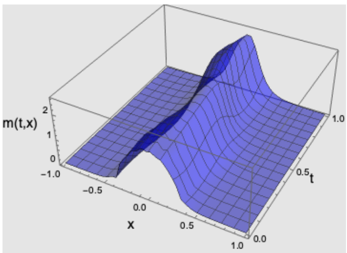
which correspond to the discretization of the admissible set $\mathcal{A}(\Omega_R)$ (see (3.4)). The results are depicted in Figure 5.3. The approximated value of (3.6) is 0.103765, in good agreement with the theoretical value 0.106525.



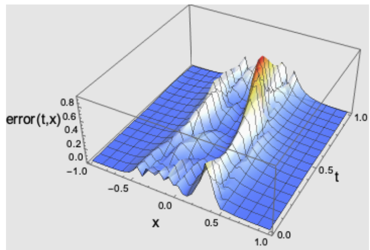
(a) φ_t approximation



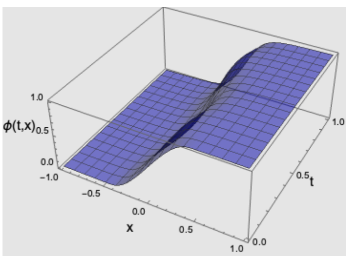
(b) approximation error



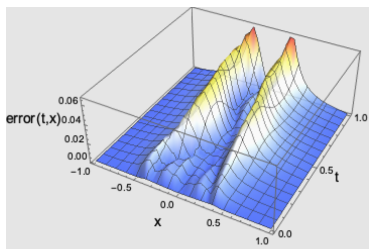
(c) $\varphi_x = m$ approximation



(d) approximation error

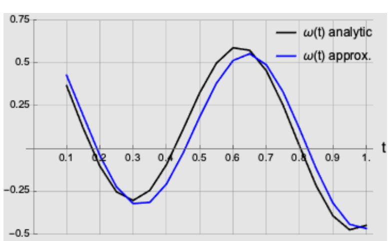


(e) φ approximation

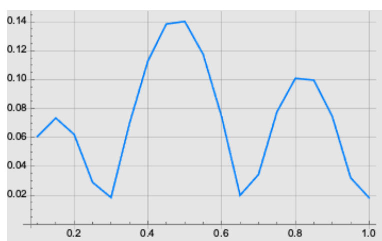


(f) approximation error

Fig. 5.3: *Approximated solution* φ *for* $\overline{Q}(t) = 5\sin(3\pi t)$.



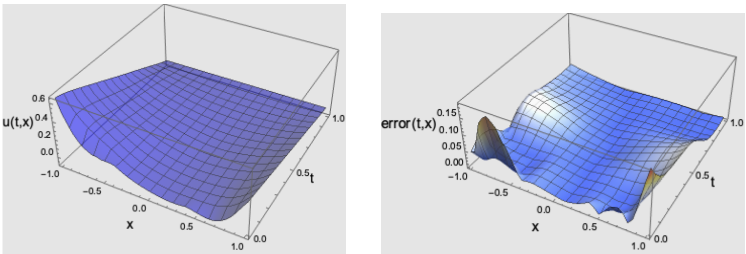
(a) ϖ approximation



(b) approximation error

Fig. 5.4: *Approximated solution* ϖ *for* $\overline{Q}(t) = 5\sin(3\pi t)$.

Using (4.15), we obtain the corresponding approximation of ϖ, illustrated in Figure 5.4. Because of the implementation of finite differences, we can compute the price on the time horizon $[2h_t, T]$. The plots show good agreement between the values of our



(a) u approximation (b) approximation error

Fig. 5.5: *Approximated solution u for $\overline{Q}(t) = 5\sin(3\pi t)$.*

numerical results with a small discrepancy that improves as the grid size increases (here, we show the results for the finest grid we used).

As the last benchmark, we consider the value function u. To compute u, we round the approximation to avoid indeterminate expressions, and we use (4.1). The result is depicted in Figure 5.5. Again, we obtain a good agreement with the exact solution.

6. Conclusions and further directions

In this paper, we presented a variational approach based on Poincaré lemma, reducing one variable in the MFG price formation model. We studied the variational approach independently of the MFG problem. We obtained existence for a relaxed formulation using bounded variation functions, and we proved uniqueness of the potential function. We showed that price existence follows a Lagrange multiplier rule associated with the balance constrained, an integral equation for the MFG model depending on a supply function. For the price problem, the variational formulation allows an efficient computation without solving the backward-forward coupled problem with integral constraints. The convexity of the variational approach allows the use of standard optimization tools to solve its discrete formulation. Our numerical method shows promising results and good agreement with the explicit solutions. We consider that we can apply a similar approach to the price formation model with common noise, which corresponds to the case of a stochastic supply function. One challenge is the dependence of the variational problem formulation on the supply, requiring the discretization of time, state variables, and the common noise. We plan to investigate this case in future works.

Acknowledgements. The authors were supported by King Abdullah University of Science and Technology (KAUST) baseline funds and KAUST OSR-CRG2021-4674.

Appendix. Preliminary results for the continuity equation. Here, we prove a general result for the continuity equation (the second equation in (1.1)).

We recall the following result from [15] about the existence and uniqueness of solutions to the continuity equation. Let $\mu_0 \in C^1(\mathbb{R})$, $b \in L^1([0,T]; W^{1,1}_{loc}(\mathbb{R})) \cap C([0,T] \times \mathbb{R})$, and $b_x \in L^\infty([0,T] \times \mathbb{R})$. Then, the continuity equation

$$\begin{cases} \mu_t + (b\mu)_x = 0 & [0,T] \times \mathbb{R}, \\ \mu(0,x) = \mu_0(x) & x \in \mathbb{R} \end{cases} \tag{A.1}$$

has a unique solution $\mu \in L^\infty([0,T] \times \mathbb{R})$ in the distributional sense. The existence result follows from Theorem 1.1 in [15], which addresses the existence and uniqueness of distributional solutions to the continuity Equation (A.1) for a vector field b satisfying weaker conditions.

Now, we prove that if the initial condition μ_0 of the continuity equation is compactly supported, the solution μ is also compactly supported.

PROPOSITION A.1. *Let $\mu_0 \in C^1(\mathbb{R})$, $b \in L^1([0,T];W^{1,1}_{loc}(\mathbb{R})) \cap C([0,T]\times\mathbb{R})$, and $b_x \in L^\infty([0,T]\times\mathbb{R})$. Assume further that $\mu_0 \in C^1_c(\mathbb{R})$. Then, the unique solution to the continuity Equation (A.1) has compact support; that is, $\mu \in L^\infty_c([0,T]\times\mathbb{R})$.*

Proof. From the results in [15], it follows that there exists a unique, $\mu \in L^\infty([0,T]\times\mathbb{R})$ solving (A.1) in the distributional sense. Let b^ε be a sequence of functions in $C^\infty([0,T]\times\mathbb{R})$ satisfying:

- b^ε is Lipschitz continuous w.r.t. x, and its Lipschitz constant satisfies $\operatorname{Lip}(b^\varepsilon) \leqslant \operatorname{Lip}(b)$,
- $b^\varepsilon \to b$ uniformly on every compact set of $[0,T]\times\mathbb{R}$.

We can obtain such a sequence b^ε by considering the convolution with standard mollifiers in x and a partition of unity construction in t. Next, we consider the continuity equation with the vector field b^ε

$$\begin{cases} \mu^\varepsilon_t + (b^\varepsilon \mu^\varepsilon)_x = 0 & [0,T]\times\mathbb{R}, \\ \mu^\varepsilon(0,x) = \mu_0(x) & x \in \mathbb{R}. \end{cases} \tag{A.2}$$

Because $b^\varepsilon, b^\varepsilon_x \in C^1([0,T]\times\mathbb{R})$, by Theorem 6.3 in [41], (A.2) has a unique solution $\mu^\varepsilon \in C^1([0,T]\times\mathbb{R})$ given by

$$\mu^\varepsilon(t, X^\varepsilon(t;y)) J(t;y) = \mu_0(y), \quad (t,y) \in [0,T]\times\mathbb{R}, \tag{A.3}$$

where

$$J(t;y) = \exp\left(\int_0^t b^\varepsilon_x(s, X^\varepsilon(s;y))\, \mathrm{d}s \right), \tag{A.4}$$

and X^ε solves the following initial value problem

$$\begin{cases} \dot{X}^\varepsilon(t;y) = b^\varepsilon(t, X^\varepsilon(t;y)) & (t,y) \in (0,T]\times\mathbb{R}, \\ X^\varepsilon(0;y) = y & y \in \mathbb{R}. \end{cases} \tag{A.5}$$

Because $b^\varepsilon \in C^1([0,T]\times\mathbb{R})$, the map $y \mapsto X^\varepsilon(t;y)$ is a diffeomorphism (see [40], Chapter 3). Moreover, because $b^\varepsilon \in C^1([0,T]\times\mathbb{R})$ is Lipschitz continuous w.r.t. x, we have

$$|b^\varepsilon(t,x)| \leqslant C_\varepsilon (1 + |x|), \tag{A.6}$$

where, by the uniform convergence of b^ε to b on compact sets, we have

$$C_\varepsilon \leqslant \max\{1 + \|b(\cdot,0)\|_{L^\infty([0,T])}, \operatorname{Lip}(b)\}, \tag{A.7}$$

for $0 < \varepsilon \ll 1$. Integrating (A.5) on $[0,t]$, for $0 \leqslant t \leqslant T$, and using A.6, we have

$$\begin{aligned} |X^\varepsilon(t;y)| &\leqslant |y| + \int_0^t |b^\varepsilon(s, X^\varepsilon(s;y))| \mathrm{d}s \\ &\leqslant |y| + C_\varepsilon t + C_\varepsilon \int_0^t |X^\varepsilon(s;y)| \mathrm{d}s \end{aligned}$$

$$\leqslant |y| + C_\varepsilon T + C_\varepsilon \int_0^t |X^\varepsilon(s;y)| \mathrm{d}s;$$

that is, for $y \in \mathbb{R}$, the non-negative function

$$\eta(t) := |X^\varepsilon(t;y)|, \quad t \in [0,T],$$

satisfies

$$\eta(t) \leqslant C_1 \int_0^t \eta(s) \mathrm{d}s + C_2,$$

for $C_1 = C_\varepsilon$ and $C_2 = |y| + C_\varepsilon T$. Therefore, by the integral Grönwall's inequality (see [25], Appendix B), we obtain

$$|X^\varepsilon(t;y)| \leqslant (|y| + C_\varepsilon T)\left(1 + C_\varepsilon T e^{C_\varepsilon T}\right). \tag{A.8}$$

Because $\mu_0 \in C_c^1(\mathbb{R})$, there exists $R_0 > 0$ such that $\mu_0(y) = 0$ for all $|y| > R_0$. This, with (A.3), implies that μ^ε may have non-zero values only for those y satisfying $|y| \leqslant R_0$. Hence, (A.8) yields

$$|X^\varepsilon(t;y)| \leqslant (R_0 + C_\varepsilon T)\left(1 + C_\varepsilon T e^{C_\varepsilon T}\right) =: R_\varepsilon.$$

Thus, $\operatorname{supp}(\mu^\varepsilon) \subseteq [0,T] \times [-R_\varepsilon, R_\varepsilon]$, which, by (A.7), provides the existence of $R > 0$, depending on T and $\operatorname{Lip}(b)$, such that $\operatorname{supp} \mu^\varepsilon \subseteq [0,T] \times [-R,R]$ for every $1 \gg \varepsilon > 0$. Furthermore, (A.3) and (A.4) imply that there exists $C \geqslant 0$, depending on T, $\operatorname{Lip}(b)$ and μ_0, such that $||\mu^\varepsilon||_{L^\infty([0,T]\times\mathbb{R})} \leqslant C$ for all $1 \gg \varepsilon > 0$. Therefore, by Banach-Alaoglu theorem, there exists $\bar{\mu} \in L^\infty([0,T] \times \mathbb{R})$ such that

$$\mu^\varepsilon \overset{*}{\rightharpoonup} \bar{\mu} \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^\infty([0,T] \times \mathbb{R}).$$

Consequently, $\operatorname{supp} \bar{\mu} \subseteq [0,T] \times [-R,R]$ as well. On the other hand, μ^ε also solves (A.2) in the sense of distributions; that is,

$$-\int_0^T \int_{\mathbb{R}} \mu^\varepsilon (\phi_t - b^\varepsilon \phi_x) \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \mu_0 \phi \ \mathrm{d}x, \tag{A.9}$$

for any $\phi \in C_c^1([0,T) \times \mathbb{R})$.

Thus, given $\phi \in C_c^1([0,T) \times \mathbb{R})$, we write

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}} \bar{\mu} (\phi_t - b\phi_x) \mathrm{d}x\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}} \mu^\varepsilon (\phi_t - b^\varepsilon \phi_x) + (\mu^\varepsilon - \bar{\mu})(\phi_t - b\phi_x) + \mu^\varepsilon \phi_x (b - b^\varepsilon) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.10}$$

Because ϕ_t, ϕ_x, $b\phi_x \in L^1([0,T] \times \mathbb{R})$, the second term on the right-hand side of (A.10) vanishes as $\varepsilon \to 0$. Furthermore, using the uniform bound for μ^ε, and because b^ε converges uniformly to b in the compact support of ϕ, we obtain that the third term on the right-hand side of (A.10) also vanishes as $\varepsilon \to 0$. Thus, using (A.9), we get

$$-\int_0^T \int_{\mathbb{R}} \bar{\mu} (\phi_t - b\phi_x) \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \mu_0 \phi \ \mathrm{d}x,$$

and since ϕ is arbitrary, we conclude that $\bar{\mu}$ is a solution to (A.1) in the distributional sense.

To conclude the proof, it is enough to recall that the results in [15] provide uniqueness for the initial value problem in (A.1) in the sense of the distributions. ∎

Applying the previous result to the MFG system (1.1)-(1.2), we obtain the following.

COROLLARY A.1. *Suppose that Assumptions 1.1-1.5 hold. Let* (u,m,ϖ) *be the solution to* (1.1). *Assume further that Assumption 1.6 holds with* $R_0>0$. *Then,* m *is compactly supported; that is, there exists a constant* $R_m \geqslant R_0$*, such that* $\operatorname{supp} m(t,\cdot) \subseteq [-R_m, R_m]$ *for* $t\in[0,T]$. *Moreover,* R_m *is bounded by a constant that depends only on the problem data.*

Proof. Let $b(t,x) = H'(\varpi(t)+u_x(t,x))$ denote the vector field of the continuity equation in (1.1). By Proposition 8 in [34], $|u_{xx}| \leqslant C(T,V,u_T)$, which implies that u_x is Lipschitz w.r.t. x. By Assumptions 1.1 and 1.2, for $p>2$, $|H''| \leqslant C(L)$, for some $C(L)>0$. Thus, $b(t,\cdot)\in C^1(\mathbb{R})$ is Lipschitz continuous in $\mathbb{R}$ uniformly with respect to t. Furthermore, the Lipchitz constant satisfies

$$\operatorname{Lip}(b(t,\cdot)) \leqslant C_0,$$

where $C_0 = C_0(T,V,L,u_T)$. Therefore, Proposition A.1 implies the first part of the result. Moreover, (A.8) shows that

$$R_m \leqslant (R_0 + C_0 T)\left(1 + C_0 T e^{C_0 T}\right), \tag{A.11}$$

which concludes the proof. ∎

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