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EXISTENCE AND DECAY OF GLOBAL STRONG SOLUTIONS TO THE NONHOMOGENEOUS INCOMPRESSIBLE LIQUID CRYSTAL SYSTEM WITH VACUUM AND DENSITY-DEPENDENT VISCOSITY*

XIA YE† AND MINGXUAN ZHU‡

Abstract. This paper is concerned with the initial value problem of the three-dimensional nonhomogeneous incompressible liquid crystal system with vacuum and density-dependent viscosity. We prove the existence of global strong solution on $\mathbb{R}^3\times(0,\infty)$ under the initial norm $\|u_0\|_{\dot{H}^\alpha}+\|\nabla d_0\|_{\dot{H}^\alpha}(1/2<\alpha\leq 1)$ being suitably small. In addition, the algebraic decay rate estimates of the global strong solution are obtained.

Keywords. Nonhomogeneous incompressible liquid crystal system; Density-dependent viscosity; Global existence; Vacuum; Decay.

AMS subject classifications. 35Q35; 76A15; 76D03.

1. Introduction

In this paper, we consider the nonhomogeneous incompressible liquid crystal system with density-dependent viscosity in $\mathbb{R}^3$

$$\begin{cases} \rho_t+\operatorname{div}(\rho u)=0, \\ \rho u_t+\rho u\cdot\nabla u+\nabla P=\operatorname{div}(\mu(\rho)\nabla u)-\operatorname{div}(\nabla d\odot\nabla d), \\ d_t+u\cdot\nabla d=\Delta d+|\nabla d|^2 d, \\ \operatorname{div}u=0, \quad |d|^2=1, \end{cases} \tag{1.1}$$

where the unknowns are the vector fields $u=(u_1,u_2,u_3)$, ρ and P representing the fluid velocity, density and pressure, respectively. And $d:\mathbb{R}^3\to S^2$, it belongs to the unit sphere. In addition, $\nabla d\odot\nabla d=(\partial_i d\cdot\partial_j d)_{3\times 3}$. The viscosity coefficient $\mu=\mu(\rho)$ is a function of density, and satisfies

$$0<\mu(\rho), \quad \mu(\rho)\in C^1[0,\infty). \tag{1.2}$$

System (1.1), which is a simplification of the original Ericksen-Leslie model [8, 18], can be used to describe the evolutionary behavior of nematic liquid crystal flows. We refer to the monographs [2, 5] for a detailed presentation of the physical foundations of continuum theories of liquid crystals. Gao, Tao and Yao [9] studied the local well-posedness of strong solutions. Liu [29] established the global strong solutions with vacuum, provided that $\|\nabla u_0\|_{L^2}+\|\Delta d_0\|_{L^2}$ is suitably small. Later, Liu [30] established the global strong solutions with vacuum, provided that $\bar{\rho}+\|\nabla d_0\|_{L^3}$ is suitably small.

When the viscosity coefficient $\mu(\rho)$ is a constant, it is the classical nonhomogeneous incompressible liquid crystal system. Wen and Ding [40] established the local well-posedness of strong solutions to the three dimensional liquid crystal system. Li

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†School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, 330022, P.R. China (yexia@jxnu.edu.cn).

‡Corresponding author. School of Mathematical Sciences, Qufu Normal University, Qufu, 273100, P.R. China; College of Mathematics and Physics, Wenzhou University, Wenzhou, 325035, P.R. China (mxzhu@qfnu.edu.cn).

and Wang [21] obtained the local well-posedness of this system without the initial value compatibility condition. Global well-posedness was studied under some smallness assumption on the initial data and the initial density away from vacuum. Li and Wang [22] studied the initial-boundary value problem for the density-dependent incompressible flow of liquid crystals in a three-dimensional bounded smooth domain. For the initial density away from vacuum, local solution and global small solution were obtained. Hu and Liu [13] studied the global existence and uniqueness of solution to this system with variable density in the framework of Besov spaces. Ding, Huang and Xia [7] proved global existence and uniqueness of the strong solutions with vacuum and small initial data in three dimensions. In two dimensions, Liu and Zhang [27] obtained global well-posedness with vacuum with the initial data satisfying some smallness condition. They also present a Serrins-type criterion depending only on ∇d, for the breakdown of local strong solutions. Li [19] proved the global existence and uniqueness of strong solutions to this system with vacuum. More precisely, for the two dimensional case, he requires that the basic energy $\|\sqrt{\rho_0}u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2$ is small, while for the three dimensional case, he needed the smallness of $(\|\sqrt{\rho_0}u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2)(\|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2)$. De Anna [4] established the global existence of solutions for small initial data by assuming that the initial density is bounded and kept away far from vacuum, while the initial velocity and the gradient of the initial director field belong to certain critical Besov spaces. Liu and Zhong [31] obtained global well-posedness of strong solutions with large velocity and vacuum provided that $\|\rho_0\|_{L^\infty}+\|\nabla d_0\|_{L^3}$ is suitably small. The large-time behavior of the solution was also obtained. Zhong [44] proved that there exists a unique global strong solution provided that the initial orientation satisfies a geometric condition in dimension two. Liu et al. [28] proved that the 2D nonhomogeneous incompressible nematic liquid crystal flows admit a unique global strong solution provided that the initial data density and the gradient of orientation decay are not too slow at infinity, and the initial orientation satisfies a geometric condition.

When the density ρ is a constant, it is the homogeneous incompressible liquid crystal system. In dimension two, Lin, Lin and Wang [23] established both interior and boundary regularity theorems for such a flow under smallness conditions. Hong [12] proved global existence of solutions to the Ericksen-Leslie system, where the solutions are regular except for at a finite number of singular times. Xu and Zhang [42] proved the global existence and regularity of weak solution for the liquid crystal flows with the large initial velocity. The uniqueness of weak solution was also proved by using the Littlewood-Paley analysis in [42]. In dimension three, Lin and Wang [26] studied the global existence of weak solutions. In the N-dimensional whole space, Schonbek and Shibata [35] proved the global well-posedness of strong solutions for small initial data by combining the maximal L_p-L_q regularities and L_p-L_q decay properties of solutions for the Stokes equations and heat equations. Several works on the mathematical analysis for general liquid crystal system can been found in [1,14,17,20,24,25,34,36–39,41] and references therein.

Further more, if $d=d_1$ for some constant vector $d_1\in S^2$, system (1.1) reduces to the nonhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity. Desjardins [6] established the global weak solution with more regularity provided that $\mu(\rho)$ is a small perturbation of a positive constant in dimension two. If vacuum does not appear, Gui and Zhang [10] obtained the global well-posedness in the case when the initial density ρ_0 is a small perturbation around a positive state in H^s with $s\geq 2$. Local existence of strong solutions with compatibility condition was established by Cho and Kim [3]. Huang and Wang [15] obtained the global strong solutions under the assumption that $\|\nabla\mu(\rho_0)\|_{L^q},(q>2)$ is small in dimension two. Zhang [43](see also Huang and Wang [16]) proved the global strong solutions under the assumption that $\|\nabla u_0\|_{L^2}$ is small in dimension three. Then, He-Li-Lv [11] established the global strong solutions with smallness in $\|u_0\|_{\dot H^\alpha}(1/2<\alpha\le 1)$ in the whole 3D space.

In this paper, we consider the Cauchy problem of (1.1) with the initial data

$$\rho(x,0)=\rho_0(x),\quad u(x,0)=u_0(x),\quad d(x,0)=d_0(x),\ \text{ and }\ |d_0(x)|=1,\qquad x\in\mathbb{R}^3 \tag{1.3}$$

and the far-field behavior

$$\rho(x,t)\to 0,\ \ u(x,t)\to 0,\ \ |d(x,t)|\to 1,\ \ as\ |x|\to+\infty,\ \ \text{for}\ \ t>0. \tag{1.4}$$

Theorem 1.1. *For given numbers $\bar\rho\ge 0$, $p>3$, and $\alpha\in(1/2,1]$, assume the initial data (ρ_0,u_0,d_0) satisfies*

$$0\le\rho_0\le\bar\rho,\quad \rho_0\in L^{3/2}\cap H^1,\quad \nabla\mu(\rho_0)\in L^p,\quad u_0\in H_0^1\cap\dot H^\alpha,\quad \nabla d_0\in L^{3/2}\cap\dot H^\alpha\cap H^1. \tag{1.5}$$

Then there exists a positive constant ε_0 depending on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$, such that if

$$\Pi_0\triangleq\|u_0\|_{\dot H^\alpha}+\|\nabla d_0\|_{\dot H^\alpha}\le\varepsilon_0,$$

the Cauchy problem (1.1)-(1.5) has a unique global strong solution (ρ,u,d) satisfying that for any $0<\tau<T$,

$$\begin{cases}0\le\rho\in C([0,T];L^{3/2}\cap L^\infty),\quad \nabla\mu(\rho)\in C([0,T];L^p),\\ P\in L^\infty([\tau,T];L^2\cap H^1),\ P_t\in L^2([\tau,T];L^2),\\ (\nabla u,\nabla^2 d)\in L^\infty([0,T];L^2)\cap L^2([0,T];L^2),\\ (\sqrt{\rho}u_t,\nabla d_t)\in L^\infty([\tau,T];L^2)\cap L^2([0,T];L^2),\\ (\nabla u_t,\nabla^2 d_t)\in L^2([\tau,T];L^2)\cap L^\infty([\tau,T];L^2),\\ (\sqrt{\rho}u_{tt},\nabla d_{tt})\in L^2([\tau,T];L^2).\end{cases} \tag{1.6}$$

For all $t>1$, there exist these global decay estimates:

$$\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla^2 d(\cdot,t)\|_{L^2}^2+\|P(\cdot,t)\|_{L^2}^2\le Ct^{-1}, \tag{1.7}$$

$$\|\sqrt{\rho}u_t(\cdot,t)\|_{L^2}^2+\|\nabla d_t(\cdot,t)\|_{L^2}^2+\|\nabla P(\cdot,t)\|_{L^2}^2\le Ct^{-2}, \tag{1.8}$$

here C depends only on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$ and Π_0.

In addition, we also have

$$\|\nabla u_t(\cdot,t)\|_{L^2}^2+\|\nabla^2 d_t(\cdot,t)\|_{L^2}^2\le Ct^{-2}, \tag{1.9}$$

here C depends only on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$, Π_0, $\|\nabla u_0\|_{L^2},\|\nabla\rho_0\|_{L^2}$ and $\|\nabla^2 d_0\|_{L^2}$.

Remark 1.1. Inspired by Reference [11], we obtain a weighted estimate of the solution and the existence of a global strong solution without the initial value compatibility condition. Moreover, comparing with [29], our smallness condition can be seen as an improvement.

REMARK 1.2. The constant C in (1.9) depends on the initial data $\|\nabla u_0\|_{L^2}$, $\|\nabla\rho_0\|_{L^2}$ and $\|\nabla^2 d_0\|_{L^2}$ while the constant in (1.7) and (1.8) does not depend on these initial values. Moreover, if the initial density $\rho_0\in H^2$, we can prove the existence of the global classical strong solution for (1.1)-(1.5) without initial compatibility condition.

We now make some comments on the analysis in this paper. Based on the local strong solution, we need some global a priori estimates to extend the local solution to globally in time. First, we give some a priori hypothesis (Proposition 3.1). Under the a priori assumptions, we can obtain some uniform global estimates. The key point is to close a priori hypothesis and obtain higher-order decay estimates of strong solution. To this end, we consider dividing the time into segments on time $[0,\sigma(T)]$ and $[\sigma(T),T]$, and linearize the velocity equation to overcome the difficulty of strong coupling of equations. Then, thanks to the definition of $\sigma(T)$, the construction of equations and some time-weighted estimates of solution (Lemmas 3.2-3.4), using Sobolev inequality, we can obtain the bound on $L^1([0,\sigma(T)];L^\infty)$ of ∇u which can be controlled by Π_0 (see (3.69)). The difficulty lies in the bound for $L^1[\sigma(T),T);L^\infty)$ of ∇u which can be controlled by Π_0. Using interpolation inequality $\|\nabla d_0\|_{L^2}^2\le C\|\nabla d_0\|_{L^{3/2}}^{4\alpha/(1+2\alpha)}\|\nabla d_0\|_{\dot H^\alpha}^{2/(1+2\alpha)}$ to get the bound on $L^\infty[\sigma(T),T);L^2)$ of $t(\sqrt{\rho}u_t,\nabla d_t)$ and $t^{1/2}(\nabla u,\nabla^2 d)$ which can be controlled by $\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{2/(1+2\alpha)}$, respectively, (see (3.9), (3.18), (3.51)). However, the time-weighted power is still insufficient, utilizing the estimation of solution to elliptic equation to obtain $\|\nabla u\|_{L^2}^2\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^4+\|\nabla^3 d\|_{L^2}^2\right)$, and to increase the time-weighted power of $\|\nabla u\|_{L^2}^2$ and using Sobolev inequality to get the bound of $t^{6/5}\|\nabla^2 d\|_{L^2}^2$, which is the key to prove bound on the $L^1([\sigma(T),T);L^\infty)$ of ∇u. Finally, we use these a priori estimates to close the priori hypothesis (Proposition 3.1) and take full advantage of the structure of equations to obtain these higher-order decay estimates for strong solution.

2. Preliminaries

In this section, we list the preliminary lemmas, which will be used.

First, using the method of reference [32], we can similarly obtain the following local existence of strong solutions to the problem (1.1)-(1.5).

LEMMA 2.1. *Assume that (ρ_0,u_0,d_0) satisfies (1.5). Then there exist a small time $T_1>0$ and a unique strong solution (ρ,u,d) to the problem (1.1)-(1.5) in $\mathbb{R}^3\times(0,T_1)$ satisfying (1.6).*

LEMMA 2.2 ([11]). *Let $F\in L^{6/5}\cap L^r$ with $r\in[2p/(p+2),p]$, assume (ρ,u) be a smooth solution to the following problem*

$$\begin{cases}-\operatorname{div}(\mu(\rho)\nabla u)+\nabla P=F, & x\in\mathbb{R}^3,\\ \operatorname{div}u=0, & x\in\mathbb{R}^3,\\ u(x)\to 0, & |x|\to\infty,\end{cases}\tag{2.1}$$

then, there exists a positive constant C depending on μ,r,p such that (ρ,u,P) satisfies

$$\|\nabla u\|_{L^2}+\|P\|_{L^2}\le C\|F\|_{L^{6/5}},\tag{2.2}$$

$$\|\nabla^2 u\|_{L^r}+\|\nabla P\|_{L^r}\le C\left(\|F\|_{L^r}+\|\nabla\mu(\rho)\|_{L^p}^{\frac{p(5r-6)}{2r(p-3)}}\|F\|_{L^{6/5}}\right).\tag{2.3}$$

Moreover, if $F=\operatorname{div}g_1+g_2$ with $g_1\in L^2\cap L^{\bar r}$, $g_2\in L^{6/5}\cap L^{\frac{3\bar r}{\bar r+3}}$ for some $\bar r\in(6p/(p+$

6),p), then

$$\|\nabla u\|_{L^2\cap L^{\bar r}}+\|P\|_{L^2\cap L^{\bar r}}\le C\left(\|g_1\|_{L^2\cap L^{\bar r}}+\|\nabla\mu(\rho)\|_{L^p}^{\frac{3p(\bar r-2)}{2\bar r(p-3)}}\|g_1\|_{L^2}+\|g_2\|_{L^{6/5}\cap L^{\frac{3\bar r}{\bar r+3}}}\right). \tag{2.4}$$

LEMMA 2.3 ([33]). *Let $f:\mathbb{R}^n\to\mathbb{R}$, if $f\in L^q(\mathbb{R}^n)$ and $D^m f\in L^r$, $1\le q,r\le\infty$, for the $0\le j\le m$, the following inequality holds*

$$\|D^j u\|_{L^p}\le C\|D^m f\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{2.5}$$

where

$$\frac{1}{p}=\frac{j}{m}+\alpha\Big(\frac{1}{r}-\frac{m}{n}\Big)+(1-\alpha)\frac{1}{q},$$

for all α in the interval

$$\frac{j}{m}\le\alpha\le 1,$$

with the following exceptional cases

(1) *If $j=0, rm<n, q=\infty$, then we make the additional assumption that either f tends to zero at infinity or $f\in L^{\tilde q}$ for finite $\tilde q>0$.*
(2) *If $1<r<\infty$ and $m-j-n/r$ is a non-negative integer, then (2.5) holds only for α satisfying $j/m\le\alpha<1$.*

3. Time-independent estimates

In this section, we give the following key a priori estimates.

PROPOSITION 3.1. *Assume (ρ,u,d) be a smooth solution of (1.1)-(1.5), then there exists a positive constant ε_0 depending only on $p,\mu(\rho),\bar\rho,\alpha,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}},\|\nabla\mu(\rho_0)\|_{L^p}$, such that (ρ,u,d) satisfies*

$$\begin{cases}\sup\limits_{0\le t\le T}\|\nabla\mu(\rho)\|_{L^p}\le 2\|\nabla\mu(\rho_0)\|_{L^p}, & \sup\limits_{0\le t\le T}\|\nabla d\|_{L^3}^3\le 2\|\nabla d_0\|_{\dot H^\alpha}^{\frac{4}{2\alpha+1}},\\ \displaystyle\int_0^T(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4)\,dt\le 2\Pi_0, & \displaystyle\int_0^T\|\nabla u\|_{L^\infty}dt\le 2,\end{cases} \tag{3.1}$$

then the following estimates hold

$$\begin{cases}\sup\limits_{0\le t\le T}\|\nabla\mu(\rho)\|_{L^p}\le 3/2\|\nabla\mu(\rho_0)\|_{L^p}, & \sup\limits_{0\le t\le T}\|\nabla d\|_{L^3}^3\le \|\nabla d_0\|_{\dot H^\alpha}^{\frac{4}{2\alpha+1}},\\ \displaystyle\int_0^T(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4)\,dt\le \Pi_0, & \displaystyle\int_0^T\|\nabla u\|_{L^\infty}dt\le 1,\end{cases} \tag{3.2}$$

provided that the initial data $\Pi_0\le\varepsilon_0$.

LEMMA 3.1. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5) satisfying (3.1), then*

$$0\le\rho(x,t)\le\bar\rho,\quad \text{for any}\quad (x,t)\in\mathbb{R}^3\times[0,T),\quad 0<\underline{\mu}\le\mu(\rho)\le\bar\mu, \tag{3.3}$$

$$\sup_{0\le t\le T}(\|\rho\|_{L^{3/2}}+\|\nabla d\|_{L^{3/2}})\le C,\qquad \sup_{0\le t\le T}\|\nabla d\|_{L^2}^2+\int_0^T\|\nabla^2 d\|_{L^2}^2dt\le C\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}} \tag{3.4}$$

and

$$\sup_{0\le t\le T}\|\sqrt{\rho}u\|_{L^2}^2+\int_0^T\|\nabla u\|_{L^2}^2dt\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right), \tag{3.5}$$

here these constants $\underline{\mu}$ *and* $\bar{\mu}$ *depend only on* $\bar{\rho}$*, the constant* C *depends on* $\bar{\rho},p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$ *and* Π_0*.*

Proof. First, multiplying $(1.1)_1$ by $|\rho|^{m-2}\rho$, and integrating it by parts, we get

$$\sup_{0\le t\le T}\|\rho\|_{L^m}=\|\rho_0\|_{L^m}.$$

Let $m\to\infty$, we get

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le C,$$

due to $\mu(\rho)\in C'[0,\infty)$ and $\mu(\rho)>0$, it is easy to deduce that (3.3) holds. Multiplying $(1.1)_3$ by Δd, by integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla d\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2&\le C\left(\|u\cdot\nabla d\|_{L^2}^2+\|\nabla d\|_{L^4}^4\right)\\&\le C\left(\|u\|_{L^6}^2\|\nabla d\|_{L^3}^2+\|\nabla d\|_{L^6}^2\|\nabla d\|_{L^3}^2\right)\\&\le C\left(\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}+\|\nabla^2 d\|_{L^2}^2\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}\right)\\&\le C\left(\|\nabla u\|_{L^2}^4+\|\Delta d\|_{L^2}^4\right)\|\nabla d\|_{L^2}^2+1/2\|\nabla^2 d\|_{L^2}^2,\end{aligned} \tag{3.6}$$

here, we have used $\|f\|_{L^3}\le C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2}$. It follows from (3.1), (3.6) and Gronwall's inequality that

$$\begin{aligned}&\sup_{0\le t\le T}\|\nabla d\|_{L^2}^2+\int_0^T\|\nabla^2 d\|_{L^2}^2dt\le C\|\nabla d_0\|_{L^2}^2\\&\le C\|\nabla d_0\|_{L^{3/2}}^{\frac{4\alpha}{1+2\alpha}}\|\nabla d_0\|_{L^{6/(3-2\alpha)}}^{\frac{2}{1+2\alpha}}\\&\le C\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}},\end{aligned} \tag{3.7}$$

where we have used Lemma 2.3 and the inequality $\|f\|_{L^{6/(3-2\alpha)}}\le C\|f\|_{\dot H^\alpha}$.

Secondly, in order to estimate $\|\nabla d\|_{L^{3/2}}\le C$, let $\varphi_\eta(\theta)=\left(\eta^2+\theta^2\right)^{3/4}$, then $|\varphi_\eta'(\theta)|\le 3/2|\theta|^{1/2}$ and $\varphi_\eta''(\theta)>0$. Applying operator ∇ to $(1.1)_3$, and multiplying it by $\varphi_\eta'(\nabla d)$, yields

$$\begin{aligned}&\frac{d}{dt}\int\varphi_\eta(\nabla d)dx\le C\left(|\int\nabla(u\cdot\nabla d)\varphi_\eta'(\nabla d)dx|+|\int\nabla(|\nabla d|^2 d)\varphi_\eta'(\nabla d)dx|\right)\\&\le C\left(\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^{3/2}}^{3/2}+\|\nabla u\|_{L^2}\|\Delta d\|_{L^2}\|\nabla d\|_{L^{3/2}}^{1/2}+\|\nabla d\|_{L^{7/2}}^{7/2}+\|\nabla d\|_{L^3}^{3/2}\|\nabla^2 d\|_{L^2}\right)\\&\le C\left(\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^{3/2}}^{3/2}+\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^4+\|\Delta d\|_{L^2}^2\|\nabla d\|_{L^{3/2}}^{3/2}\right)\end{aligned}$$

which, by Gronwall's inequality, and letting $\eta\to0$, gives

$$\sup_{0\le t\le T}\|\nabla d\|_{L^{3/2}}\le C.$$

To estimate (3.5), multiplying $(1.1)_2$ by u in L^2 and by integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla u\|_{L^2}^2 &\le C\|\nabla d\|_{L^6}^2\|\nabla d\|_{L^3}^2\\ &\le C\left(\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2+\|\Delta d\|_{L^2}^4\right).\end{aligned}$$

Integrating it over [0,T), using $\alpha\le 1$ and (3.1), it is easy to deduce that

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{\rho}u\|_{L^2}^2+\int_0^T\|\nabla u\|_{L^2}^2dt\\ &\le\|\sqrt{\rho_0}u_0\|_{L^2}^2+C\left(\sup_{0\le t\le T}\|\nabla d\|_{L^2}^2\int_0^T\|\nabla^2 d\|_{L^2}^2dt+\int_0^T\|\nabla^2 d\|_{L^2}^4dt\right)\\ &\le C\left(\|\sqrt{\rho_0}u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2+\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right)\\ &\le C\left(\|\rho_0\|_{L^{3/(2\alpha)}}\|u_0\|_{L^{6/(3-2\alpha)}}^2+\|\nabla d_0\|_{L^{3/2}}^{\frac{4\alpha}{1+2\alpha}}\|\nabla d_0\|_{L^{6/(3-2\alpha)}}^{\frac{2}{1+2\alpha}}+\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right)\\ &\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right).\end{aligned}$$

Hence, we complete the proof of Lemma 3.1. □

LEMMA 3.2. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5) satisfying (3.1), then*

$$\sup_{0\le t\le T}t^{1-\alpha}\|\nabla^2 d\|_{L^2}^2+\int_0^T t^{1-\alpha}\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le C\|\nabla d_0\|_{\dot H^\alpha}^2 \tag{3.8}$$

and

$$\sup_{0\le t\le T}t\|\nabla^2 d\|_{L^2}^2+\int_0^T t\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le C\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}, \tag{3.9}$$

here C depends only on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$ and Π_0.

Proof. Multiplying $(1.1)_3$ by Δd_t in L^2, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^2 d\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2&=\int u\cdot\nabla d\cdot\Delta d_t dx-\int|\nabla d|^2 d\Delta d_t dx\\ &\le C\left(\|\nabla(u\cdot\nabla d)\|_{L^2}^2+\|\nabla(|\nabla d|^2 d)\|_{L^2}^2\right)+\frac{1}{2}\|\nabla d_t\|_{L^2}^2.\end{aligned} \tag{3.10}$$

In the following, using Hölder's inequality and Sobolev's inequality to estimate the right-hand side of (3.10).

$$\begin{aligned}\|\nabla(u\cdot\nabla d)\|_{L^2}^2&\le C\left(\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^\infty}^2+\|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2\right)\\ &\le C\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2+\varepsilon\|\nabla^3 d\|_{L^2}^2,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|\nabla(|\nabla d|^2 d)\|_{L^2}^2&\le C\left(\|\nabla d\nabla^2 d\|_{L^2}^2+\|\nabla d\|_{L^6}^6\right)\\ &\le C\left(\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla^2 d\|_{L^2}^2+\|\nabla d\|_{L^6}^6\right)\\ &\le C\|\nabla^2 d\|_{L^2}^6+\varepsilon\|\nabla^3 d\|_{L^2}^2,\end{aligned} \tag{3.12}$$

here we have used $\|f\|_{L^\infty}\le C\|\nabla f\|_{L^2}^{1/2}\|\nabla^2 f\|_{L^2}^{1/2}$. According to $(1.1)_3$, a direct calculation gives that

$$\begin{aligned}
&\|\nabla^3 d\|_{L^2}^2\\
\le& C\left(\|\nabla d_t\|_{L^2}^2+\|\nabla(u\cdot\nabla d)\|_{L^2}^2+\|\nabla(|\nabla d|^2 d)\|_{L^2}^2\right)\\
\le& C\left(\|\nabla d_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^\infty}^2+\|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2+\|\nabla d\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2+\|\nabla d\|_{L^6}^6\right)\\
\le& C\left(\|\nabla d_t\|_{L^2}^2+\|\nabla u\|_{L^2}^4\|\nabla^2 d\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^6\right)+\frac{1}{2}\|\nabla^3 d\|_{L^2}^2. \qquad (3.13)
\end{aligned}$$

Inserting (3.11)-(3.13) into (3.10), choosing ε sufficiently small, we obtain

$$\frac{d}{dt}\|\nabla^2 d\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|\nabla^2 d\|_{L^2}^2, \qquad (3.14)$$

which, by using Gronwall's inequality, (3.1) and (3.13), gives

$$\sup_{0\le t\le T}\|\nabla^2 d\|_{L^2}^2+\int_0^T\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le C\|\nabla^2 d_0\|_{L^2}^2. \qquad (3.15)$$

Multiplying (3.14) by t, we have

$$\frac{d}{dt}(t\|\nabla^2 d\|_{L^2}^2)+t\|\nabla d_t\|_{L^2}^2\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)t\|\nabla^2 d\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2,$$

which, by Gronwall's inequality, (3.1), (3.7) and (3.13), also gives

$$\begin{aligned}
\sup_{0\le t\le T}t\|\nabla^2 d\|_{L^2}^2+\int_0^T t\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le& C\|\nabla d_0\|_{L^2}^2\\
\le& C\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}. \qquad (3.16)
\end{aligned}$$

Combining with (3.15) and (3.16), the standard interpolation arguments imply that, for any $\alpha\in(1/2,1]$

$$\sup_{0\le t\le T}t^{1-\alpha}\|\nabla^2 d\|_{L^2}^2+\int_0^T t^{1-\alpha}\left(\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)dt\le C(\alpha)\|\nabla d_0\|_{\dot H^\alpha}^2.$$

□

LEMMA 3.3. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5) satisfying (3.1), then*

$$\sup_{0\le t\le T}t^{1-\alpha}\|\nabla u\|_{L^2}^2+\int_0^T t^{1-\alpha}\|\sqrt{\rho}u_t\|_{L^2}^2dt\le C\Pi_0, \qquad (3.17)$$

$$\sup_{0\le t\le T}t\|\nabla u\|_{L^2}^2+\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2dt+\int_0^T t\|\nabla u\|_{L^2}^2dt\le C\Pi_0^{\frac{1}{1+2\alpha}}, \qquad (3.18)$$

here C depends only on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$ and Π_0.

Proof. To overcome the difficulty of strong coupling of equations, for fixed (ρ,u,d), the initial value problem $(1.1)_2$ with the initial data $u(x,0)=u_0(x)$ is divided into the following two linear initial value problems:

$$\begin{cases}\rho w_t+\rho u\cdot\nabla w+\nabla P-\operatorname{div}(\mu(\rho)\nabla w)=-\operatorname{div}(\nabla d\odot\nabla d),\\ \operatorname{div}w=0,\qquad w|_{t=0}=0\end{cases} \qquad (3.19)$$

and

$$\begin{cases}\rho v_t+\rho u\cdot\nabla v+\nabla P-\operatorname{div}(\mu(\rho)\nabla v)=0,\\ \operatorname{div}v=0,\qquad v|_{t=0}=u_0.\end{cases}\tag{3.20}$$

For the problem of (3.19) with homogeneous boundary conditions, we will prove the estimate of w can be controlled by d, and for the problem of (3.20) with inhomogeneous boundary conditions, we will prove the estimate of v can be controlled by the initial data of v.

First, we consider the initial value problem (3.19). Now, from Lemma 2.2, (3.3) and (3.4), we give the following inequality which will be used later.

$$\begin{aligned}\|\nabla^2 w\|_{L^2}&\le C\left(\|\rho w_t+\rho u\cdot\nabla w+\operatorname{div}(\nabla d\odot\nabla d)\|_{L^2}+\|\rho w_t+\rho u\cdot\nabla w+\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right)\\ &\le C\Big(\|\sqrt{\rho}w_t\|_{L^2}+\|\rho\|_{L^{3/2}}^{1/2}\|\sqrt{\rho}w_t\|_{L^2}+\|u\|_{L^6}\|\nabla w\|_{L^3}+\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}\\ &\quad+\|\nabla d\|_{L^{3/2}}\|\nabla^2 d\|_{L^6}\Big)\\ &\le C\left(\|\sqrt{\rho}w_t\|_{L^2}+\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}+\|\nabla^2 d\|_{L^2}^3+\|\nabla^3 d\|_{L^2}\right)+\frac{1}{2}\|\nabla^2 w\|_{L^2},\end{aligned}$$

it yields that

$$\|\nabla^2 w\|_{L^2}\le C\left(\|\sqrt{\rho}w_t\|_{L^2}+\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}+\|\nabla^2 d\|_{L^2}^3+\|\nabla^3 d\|_{L^2}\right).\tag{3.21}$$

Multiplying $(3.19)_1$ by w_t in L^2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\mu(\rho)|\nabla w|^2dx+\int\rho|w_t|^2dx\\ &=-\int\rho u\cdot\nabla w\cdot w_t dx+\frac{1}{2}\int\mu_t(\rho)|\nabla w|^2dx+\int\nabla d\odot\nabla d\cdot\nabla w_t dx\triangleq\sum_{i=1}^{3}M_i.\end{aligned}\tag{3.22}$$

To estimate M_i, applying Sobolev's inequality, (3.3) and (3.21), the estimate deduces

$$\begin{aligned}|M_1|&\le C\|\sqrt{\rho}w_t\|_{L^2}\|u\|_{L^6}\|\nabla w\|_{L^3}\le C\|\sqrt{\rho}w_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\|\nabla^2 w\|_{L^2}^{1/2}\\ &\le\frac{1}{4}\|\sqrt{\rho}w_t\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^6+\|\nabla^3 d\|_{L^2}^2\right).\end{aligned}\tag{3.23}$$

By (3.3), (3.21) and using the fact $\mu_t(\rho)=-\operatorname{div}(\mu(\rho)u)$ and (3.21), we get

$$\begin{aligned}|M_2|&\le\|u\|_{L^6}\|\nabla w\|_{L^3}\|\nabla^2 w\|_{L^2}\le C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\|\nabla^2 w\|_{L^2}^{3/2}\\ &\le C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{1/2}\left(\|\sqrt{\rho}w_t\|_{L^2}^{3/2}+\|\nabla u\|_{L^2}^3\|\nabla w\|_{L^2}^{3/2}+\|\nabla^2 d\|_{L^2}^{9/2}+\|\nabla^3 d\|_{L^2}^{3/2}\right)\\ &\le\frac{1}{8}\|\sqrt{\rho}w_t\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^6+\|\nabla^3 d\|_{L^2}^2\right),\end{aligned}\tag{3.24}$$

$$\begin{aligned}M_3&=\frac{d}{dt}\int\nabla d\odot\nabla d\cdot\nabla w dx-2\int\nabla d\odot\nabla d_t\cdot\nabla w dx\\ &\le\frac{d}{dt}\int\nabla d\odot\nabla d\cdot\nabla w dx+C\|\nabla^2 d\|_{L^2}^2\|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}+C\|\nabla d_t\|_{L^2}^2\\ &\le\frac{d}{dt}\int\nabla d\odot\nabla d\cdot\nabla w dx+\frac{1}{2}\|\sqrt{\rho}w_t\|_{L^2}^2+C\left(\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}^4\right)\|\nabla w\|_{L^2}^2\\ &\quad+C\left(\|\nabla^2 d\|_{L^2}^6+\|\nabla^3 d\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right).\end{aligned}\tag{3.25}$$

Substituting (3.23)-(3.25) into (3.22), we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla w\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|\nabla w\|_{L^2}^2\\&\quad+C\left(\|\nabla^2 d\|_{L^2}^6+\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+\frac{d}{dt}\int\nabla d\odot\nabla d\cdot\nabla w dx\right)\end{aligned}\tag{3.26}$$

which together with Gronwall's inequality, (3.1) and (3.15), implies that

$$\sup_{0\le t\le T}\|\nabla w\|_{L^2}^2+\int_0^T\|\sqrt{\rho}w_t\|_{L^2}^2dt\le C\|\Delta d_0\|_{L^2}^2+\frac{1}{2}\|\nabla w\|_{L^2}^2,\tag{3.27}$$

where we have used (3.15) to get

$$\begin{aligned}\||\nabla d|^2\nabla w\|_{L^1}&\le C\|\nabla d\|_{L^3}\|\nabla d\|_{L^6}\|\nabla w\|_{L^2}\\&\le C\|\nabla d\|_{L^2}^{1-\alpha/2}\|\nabla d\|_{L^{6/(3-2\alpha)}}^{\alpha/2}\|\nabla^2 d\|_{L^2}^2+\frac{1}{2}\|\nabla w\|_{L^2}^2\\&\le C\|\nabla d_0\|_{H^\alpha}^{\alpha/2}\|\nabla^2 d\|_{L^2}^2+\frac{1}{2}\|\nabla w\|_{L^2}^2\\&\le C\|\Delta d_0\|_{L^2}^2+\frac{1}{2}\|\nabla w\|_{L^2}^2.\end{aligned}$$

It follows from (3.27) that

$$\sup_{0\le t\le T}\|\nabla w\|_{L^2}^2+\int_0^T\|\sqrt{\rho}w_t\|_{L^2}^2dt\le C\|\Delta d_0\|_{L^2}^2.\tag{3.28}$$

Multiplying (3.26) by t, using (3.16), gives

$$\sup_{0\le t\le T}t\left(\|\nabla w\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)+\int_0^T t\|\sqrt{\rho}w_t\|_{L^2}^2dt\le C\|\nabla d_0\|_{L^2}^2.\tag{3.29}$$

Collecting (3.28) and (3.29), deduce that

$$\sup_{0\le t\le T}t^{1-\alpha}\|\nabla w\|_{L^2}^2+\int_0^T t^{1-\alpha}\|\sqrt{\rho}w_t\|_{L^2}^2dt\le C\|\nabla d_0\|_{\dot H^\alpha}^2.\tag{3.30}$$

Now, we consider the Cauchy problem of (3.20), multiplying $(3.20)_1$ by v_t in L^2, in view of (3.3), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\mu(\rho)|\nabla v|^2dx+\int\rho|v_t|^2dx=-\int\rho u\cdot\nabla v\cdot v_tdx+\frac{1}{2}\int\mu_t(\rho)|\nabla v|^2dx\\&\le C\left(\|u\|_{L^6}^2\|\nabla v\|_{L^3}^2+\|u\|_{L^6}\|\nabla v\|_{L^3}\|\nabla^2 v\|_{L^2}\right)+\frac{1}{2}\|\sqrt{\rho}v_t\|_{L^2}^2\\&\le C\|\nabla u\|_{L^2}^4\|\nabla v\|_{L^2}^2+\frac{1}{2}\|\sqrt{\rho}v_t\|_{L^2}^2+\varepsilon\|\nabla^2 v\|_{L^2}^2.\end{aligned}\tag{3.31}$$

Using Lemma 2.2, (3.3), (3.4) and the properties of solution to elliptic equation, we obtain

$$\begin{aligned}\|\nabla^2 v\|_{L^2}^2&\le C\left(\|\sqrt{\rho}v_t+\rho u\cdot\nabla v\|_{L^2}^2+\|\sqrt{\rho}v_t+\rho u\cdot\nabla v\|_{L^{6/5}}^2\right)\\&\le C\left(\|\sqrt{\rho}v_t\|_{L^2}^2+\|\rho\|_{L^{3/2}}\|\sqrt{\rho}v_t\|_{L^2}^2+\|u\|_{L^6}^2\|\nabla v\|_{L^3}^2\right)\\&\le C\|\nabla u\|_{L^2}^4\|\nabla v\|_{L^2}^2+C\|\sqrt{\rho}v_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2 v\|_{L^2}^2\end{aligned}$$

which, inserting into (3.31), and choosing ε small enough, gives

$$\frac{d}{dt}\int |\nabla v|^2 dx + \int \rho |v_t|^2 dx \le C\|\nabla u\|_{L^2}^4 \|\nabla v\|_{L^2}^2. \tag{3.32}$$

Combining this with Gronwall's inequality and (3.1) implies that

$$\sup_{0\le t\le T} \|\nabla v\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} v_t\|_{L^2}^2 dt \le C\|\nabla v_0\|_{L^2}^2. \tag{3.33}$$

Multiplying $(3.20)_1$ by v in L^2, thanks to (3.3), we obtian

$$\sup_{0\le t\le T} \|\sqrt{\rho} v\|_{L^2}^2 + \int_0^T \|\nabla v\|_{L^2}^2 dt \le \|\sqrt{\rho_0} v_0\|_{L^2}^2. \tag{3.34}$$

Multiplying (3.32) by t and using (3.34), we deduce

$$\sup_{0\le t\le T} t\|\nabla v\|_{L^2}^2 + \int_0^T t\|\sqrt{\rho} v_t\|_{L^2}^2 dt \le C\|\sqrt{\rho_0} v_0\|_{L^2}^2. \tag{3.35}$$

Combining this and (3.33), gives

$$\sup_{0\le t\le T} t^{1-\alpha}\|\nabla v\|_{L^2}^2 + \int_0^T t^{1-\alpha}\|\sqrt{\rho} v_t\|_{L^2}^2 dt \le C\|v_0\|_{\dot{H}^\alpha}^2. \tag{3.36}$$

According to the additivity of the solution for the linear equation, we have $u = v + w$, combining with (3.30) and (3.36), gives (3.17).

And from (3.29) and (3.35), we have

$$\begin{aligned}\sup_{0\le t\le T} t\|\nabla u\|_{L^2}^2 + \int_0^T t\|\sqrt{\rho} u_t\|_{L^2}^2 dt &\le C\left(\|\sqrt{\rho_0} u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\\ &\le C\left(\|u_0\|_{\dot{H}^\alpha}^2 + \|\nabla d_0\|_{\dot{H}^\alpha}^{\frac{2}{1+2\alpha}}\right).\end{aligned} \tag{3.37}$$

It deduces from Lemma 2.2 and (3.3) that

$$\begin{aligned}\|P\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &\le C\left(\|\rho u_t\|_{L^{6/5}}^2 + \|\rho u\cdot\nabla u\|_{L^{6/5}}^2 + \|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}^2\right)\\ &\le C\left(\|\sqrt{\rho}\|_{L^3}^2\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\rho\|_{L^6}^2\|u\|_{L^6}^2\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^{3/2}}\|\nabla^2 d\|_{L^6}^2\right)\\ &\le C\left(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla^3 d\|_{L^2}^2\right),\end{aligned} \tag{3.38}$$

hence, from (3.5), (3.9), (3.37) and (3.38), it is easy to find that

$$\begin{aligned}&\int_0^T t\|\nabla u\|_{L^2}^2 dt\\ &\le C\left(\int_0^T t\|\sqrt{\rho} u_t\|_{L^2}^2 dt + \sup_{0\le t\le T} t\|\nabla u\|_{L^2}^2 \int_0^T \|\nabla u\|_{L^2}^2 dt + \int_0^T t\|\nabla^3 d\|_{L^2}^2 dt\right)\\ &\le C\left(\|u_0\|_{\dot{H}^\alpha}^2 + \|\nabla d_0\|_{\dot{H}^\alpha}^{\frac{2}{1+2\alpha}}\right) \le \Pi_0^{\frac{1}{1+2\alpha}},\end{aligned}$$

which together with (3.37), gives (3.18). Therefore, we complete the proof of Lemma 3.3. ∎

LEMMA 3.4. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5) satisfying (3.1), then*

$$\sup_{0\le t\le\sigma(T)} t^{2-\alpha}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right) + \int_0^{\sigma(T)} t^{2-\alpha}\left(\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\right)dt \le C\Pi_0, \tag{3.39}$$

$$\begin{aligned}&\sup_{\sigma(T)\le t\le T} t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)\\ &\quad+\int_{\sigma(T)}^T t^2\left(\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2+\|\nabla^4 d\|_{L^2}^2\right)dt\\ &\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right),\end{aligned} \tag{3.40}$$

here $\sigma(T)\triangleq\min\{1,T\}$, C depends only on $\bar\rho,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$ and Π_0.

Proof. Operating ∂_t to $(1.1)_2$, and multiplying it by u_t in L^2, we obtain after integration by parts and $\mu_t(\rho)+\nabla\mu(\rho)\cdot u=0$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\mu(\rho)}\nabla u_t\|_{L^2}^2=-2\int\rho u\cdot\nabla u_t\cdot u_t dx-\int\rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx\\ &\quad-\int\rho u_t\cdot\nabla u\cdot u_t dx+\int(u\cdot\nabla\mu(\rho))\nabla u\cdot\nabla u_t dx+\int[\mathrm{div}(\nabla d\odot\nabla d)]_t\cdot u_t dx\\ &\triangleq\sum_{i=1}^5 M_i.\end{aligned} \tag{3.41}$$

M_i can be dealt as follows, we get by direct calculation and (3.3) that

$$\begin{aligned}&|M_1|+|M_3|\le C(\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}+\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6})\\ &\le C\|\nabla u\|_{L^2}^4\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{1}{8}\|\nabla u_t\|_{L^2}^2,\end{aligned} \tag{3.42}$$

$$\begin{aligned}&M_2\le C\left(\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^3}^2+\|u\|_{L^6}^2\|u_t\|_{L^6}\|\nabla^2 u\|_{L^2}+\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\right)\\ &\le C\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2+\frac{1}{8}\|\nabla u_t\|_{L^2}^2.\end{aligned} \tag{3.43}$$

From Lemma 2.2, (3.13) and similar to (3.21), we have

$$\|\nabla P\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla d_t\|_{L^2}^2\right). \tag{3.44}$$

Inserting (3.44) into (3.43), we get

$$M_2\le C\|\nabla u\|_{L^2}^4\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla d_t\|_{L^2}^2\right)+\frac{1}{8}\|\nabla u_t\|_{L^2}^2. \tag{3.45}$$

Using (3.1), we have

$$\begin{aligned}M_4&\le C\|\nabla\mu(\rho)\|_{L^p}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^{2p/(p-2)}}\\ &\le C\|u\|_{L^6}^{1/2}\|\nabla u\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{(p-3)/p}\|\nabla^2 u\|_{L^2}^{3/p}\\ &\le C\left(\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3+\|\nabla u\|_{L^2}^4\right)+\frac{1}{8}\|\nabla u_t\|_{L^2}^2,\end{aligned} \tag{3.46}$$

it follows from (3.44) that

$$M_4 \le C\|\nabla u\|_{L^2}\left(\|\sqrt{\rho}u_t\|_{L^2}^3+\|\nabla u\|_{L^2}^9+\|\nabla^2 d\|_{L^2}^9 +\|\nabla d_t\|_{L^2}^3+\|\nabla u\|_{L^2}^3\right)+\frac{1}{8}\|\nabla u_t\|_{L^2}^2, \tag{3.47}$$

$$M_5 \le C\|\nabla^2 d\|_{L^2}^4\|\nabla d_t\|_{L^2}^2+\frac{1}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right). \tag{3.48}$$

Substituting (3.42)-(3.48) into (3.41), we have

$$\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2 \le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla d_t\|_{L^2}\right) \times\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+C\left(\|\nabla u\|_{L^2}^{10}+\|\nabla^2 d\|_{L^2}^{10}+\|\nabla u\|_{L^2}^4\right)+\frac{1}{8}\|\nabla^2 d_t\|_{L^2}^2. \tag{3.49}$$

Operating ∂_t to $(1.1)_3$, and multiplying it by d_{tt}, by integrating the resulting equality by parts, (3.1) and (3.4), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla d_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2=-\int (u\cdot\nabla d)_t\cdot d_{tt}dx+\int (|\nabla d|^2 d)_t\cdot d_{tt}dx\\
&\le C\left(\|u_t\cdot\nabla d\|_{L^2}^2+\|u\cdot\nabla d_t\|_{L^2}^2+\||\nabla d|^2 d_t\|_{L^2}^2+\|\nabla d\odot\nabla d_t\|_{L^2}^2\right)+\frac{1}{2}\|d_{tt}\|_{L^2}^2\\
&\le C\left(\|u_t\|_{L^6}^2\|\nabla d\|_{L^3}^2+\|u\|_{L^6}^2\|\nabla d_t\|_{L^3}^2+\|\nabla d\|_{L^6}^4\|d_t\|_{L^6}^2+\|\nabla d\|_{L^6}^2\|\nabla d_t\|_{L^3}^2\right)+\frac{1}{2}\|d_{tt}\|_{L^2}^2\\
&\le C_1\|\nabla u_t\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|\nabla d_t\|_{L^2}^2+\varepsilon\|\nabla^2 d_t\|_{L^2}^2+\frac{1}{2}\|d_{tt}\|_{L^2}^2
\end{aligned}\tag{3.50}$$

and by $(1.1)_3$ to get

$$\begin{aligned}
&\|\nabla^2 d_t\|_{L^2}^2\\
&\le C\left(\|d_{tt}\|_{L^2}^2+\|u_t\cdot\nabla d\|_{L^2}^2+\|u\cdot\nabla d_t\|_{L^2}^2+\||\nabla d\odot\nabla d_t\|_{L^2}^2+\|(\nabla d)^2 d_t\|_{L^2}^2\right)\\
&\le C\left(\|d_{tt}\|_{L^2}^2+\|u_t\|_{L^6}^2\|\nabla d\|_{L^3}^2+\|u\|_{L^6}^2\|\nabla d_t\|_{L^3}^2+\|\nabla d\|_{L^6}^2\|\nabla d_t\|_{L^3}^2+\|\nabla d\|_{L^6}^4\|d_t\|_{L^6}^2\right)\\
&\le C\|d_{tt}\|_{L^2}^2+C_1\|\nabla u_t\|_{L^2}^2+\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|\nabla d_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2 d_t\|_{L^2}^2.
\end{aligned}\tag{3.51}$$

Adding $4C_1\times$(3.49) with (3.50), by (3.51) and choosing ε sufficiently small, gives

$$\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+C_1\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\\
&\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla d_t\|_{L^2}\right)\\
&\quad\times\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+\left(\|\nabla u\|_{L^2}^{10}+\|\nabla^2 d\|_{L^2}^{10}+\|\nabla u\|_{L^2}^4\right),
\end{aligned}\tag{3.52}$$

which multiplied by $t^{2-\alpha}$, we have

$$\begin{aligned}
&\frac{d}{dt}t^{2-\alpha}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+C_1t^{2-\alpha}\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\\
&\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla d_t\|_{L^2}\right)\\
&\quad\times t^{2-\alpha}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+Ct^{2-\alpha}\left(\|\nabla u\|_{L^2}^{10}+\|\nabla^2 d\|_{L^2}^{10}+\|\nabla u\|_{L^2}^4\right).
\end{aligned}\tag{3.53}$$

According to the definition of $\sigma(T)$, $\sigma(T)\le 1$, $1/2<\alpha\le 1$, $\sup\limits_{0\le t\le\sigma(T)}\|\nabla u\|_{L^2}^2\le Ct^{\alpha-1}$,

$\sup\limits_{0\le t\le\sigma(T)}\|\nabla^2 d\|_{L^2}^2\le Ct^{\alpha-1}$, we can get the following inequalities

$$\begin{aligned}&\int_0^{\sigma(T)}\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}dt\le C\int_0^{\sigma(T)}t^{\frac{\alpha-1}{2}}\|\sqrt{\rho}u_t\|_{L^2}dt\\ \le& C\int_0^{\sigma(T)}t^{1-\alpha}\|\sqrt{\rho}u_t\|_{L^2}^2dt\int_0^{\sigma(T)}t^{2\alpha-2}dt\le C,\end{aligned}$$

$$\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^{10}dt\le Ct^{2\alpha-1}\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^4dt\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right).$$

Inserting the above inequalities into (3.53) and (3.44), it is easy to deduce that

$$\begin{aligned}&\sup_{0\le t\le\sigma(T)}t^{2-\alpha}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+\int_0^{\sigma(T)}t^{2-\alpha}\left(\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\right)dt\\ \le& C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right).\end{aligned}\tag{3.54}$$

On the other hand, with the help of (3.18), we give the following inequality

$$\int_{\sigma(T)}^T t^2\|\nabla u\|_{L^2}^4dt\le\sup_{\sigma(T)\le t\le T}t\|\nabla u\|_{L^2}^2\int_{\sigma(T)}^T t\|\nabla u\|_{L^2}^2dt\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right)^2.\tag{3.55}$$

Multiplying (3.52) by t^2, and according to the definition of $\sigma(T)$, (3.13), (3.18), (3.44) and (3.55), by Gronwall's inequality we obtain

$$\begin{aligned}&\sup_{\sigma(T)\le t\le T}t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)\\ &+\int_{\sigma(T)}^T t^2\left(\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\right)dt\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right),\end{aligned}\tag{3.56}$$

where we used $\int_{\sigma(T)}^T t^2\|f\|_{L^2}^{10}dt\le\int_{\sigma(T)}^T t^5\|f\|_{L^2}^{10}dt\le t^4\sup_{\sigma(T)\le t\le T}\|f\|_{L^2}^8\int_{\sigma(T)}^T t\|f\|_{L_2}^2dt$. Applying Hölder's inequality, we arrive at

$$\begin{aligned}&\|\nabla^4 d\|_{L^2}^2\le C\left(\|\nabla^2 d_t\|_{L^2}^2+\|\nabla^2(u\cdot\nabla d)\|_{L^2}+\|\nabla^2(|\nabla d|^2d)\|_{L^2}^2\right)\\ \le& C\left(\|\nabla^2 d_t\|_{L^2}^2+\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^3 d\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^3\|\nabla^3 d\|_{L^2}+\|\nabla^2 d\|_{L^2}^4\|\nabla^3 d\|_{L^2}^2\right)\end{aligned}$$

and

$$\begin{aligned}\|\nabla^2 d_t\|_{L^2}^2\le& C\left(\|d_{tt}\|_{L^2}^2+\|(u\cdot\nabla d)_t\|_{L^2}^2+\|(|\nabla d|^2d)\|_{L^2}^2\right)\\ \le& C\left(\|d_{tt}\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}+\|\nabla u\|_{L^2}^2\|\nabla d_t\|_{L^2}\|\nabla^2 d_t\|_{L^2}\right.\\ &\left.+\|\nabla^2 d\|_{L^2}^4\|\nabla d_t\|_{L^2}^2\right).\end{aligned}$$

Combining with above inequalities, (3.18), (3.38) and (3.56), we get

$$\sup_{\sigma(T)\le t\le T}t^2\|\nabla u\|_{L^2}^2+\int_{\sigma(T)}^T t^2\left(\|\nabla^2 d_t\|_{L^2}^2+\|\nabla^4 d\|_{L^2}^2\right)dt\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^{\frac{2}{1+2\alpha}}\right),$$

this together with (3.54) and (3.56), finishes the proof of Lemma 3.4. □

Remark 3.1. In order to prove the bound on $L^1([0,T];L^\infty)$ of ∇u can be controlled by Π_0, we need to obtain different powers of time-weighted estimates on $[0,\sigma(T)]$ and $[\sigma(T),T]$.

In addition, we need the following Lemma 3.5 to obtain the result of Proposition 3.1.

Lemma 3.5. *Assume that (ρ,u,d) be a smooth solution to (1.1) satisfying (3.1), then*

$$\sup_{\sigma(T)\le t\le T} t^{6/5}\|\nabla^2 d\|_{L^2}^2 \le C\Pi_0^{\frac{3}{5(1+2\alpha)}}, \tag{3.57}$$

$$\int_0^T \|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}dt \le C\Pi_0^{\frac{31}{70(1+2\alpha)}}, \tag{3.58}$$

here C depends only on $\bar\rho, p, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla d_0\|_{L^{3/2}}$ and Π_0.

Proof. First, it is easily deduced from (3.4) and (3.40) that

$$\sup_{\sigma(T)\le t\le T} t^{3/5}\|\nabla^2 d\|_{L^2} \le C\sup_{\sigma(T)\le t\le T}\|\nabla d\|_{L^{3/2}}^{2/5}\sup_{\sigma(T)\le t\le T}\left(t^{3/5}\|\nabla^3 d\|_{L^2}^{3/5}\right)\le C\Pi_0^{\frac{3}{10(1+2\alpha)}}.$$

Next, along with (3.8) and (3.40), we give

$$\begin{aligned}
&\int_0^T \|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}dt\\
&\le C\sup_{0\le t\le\sigma(T)}\|\nabla d\|_{L^{3/2}}\int_0^{\sigma(T)}\|\nabla^2 d\|_{L^6}dt + C\int_{\sigma(T)}^T \|\nabla^2 d\|_{L^2}\|\nabla d\|_{L^{3/2}}^{5/7}\|\nabla^4 d\|_{L^2}^{2/7}dt\\
&\le C\int_0^{\sigma(T)} t^{\frac{1-\alpha}{2}}\|\nabla^3 d\|_{L^2}t^{\frac{\alpha-1}{2}}dt + C\sup_{\sigma(T)\le t\le T}t^{3/5}\|\nabla^2 d\|_{L^2}\int_{\sigma(T)}^T t^{-31/35}t^{2/7}\|\nabla^4 d\|_{L^2}^{2/7}dt\\
&\le C\left(\int_0^{\sigma(T)} t^{\alpha-1}dt\right)^{1/2}\|\nabla d_0\|_{\dot H^\alpha} + C\Pi_0^{\frac{31}{70(1+2\alpha)}}\left(\int_{\sigma(T)}^T t^{-31/30}dt\right)^{6/7}\\
&\le C\Pi_0^{\frac{31}{70(1+2\alpha)}},
\end{aligned}$$

here, we used Lemma 2.3 to get the inequality $\|\nabla^2 f\|_{L^{3/2}}\le C\|\nabla f\|_{L^{3/2}}^{5/7}\|\nabla^4 f\|_{L^2}^{2/7}$. □

Proof. **(Proof of Proposition 3.1.)** To estimate $\int_0^T\|\nabla u\|_{L^\infty}dt\le 1$, choosing

$$r\in\left(3,\min\{6,p,\frac{\alpha-2}{\alpha-1},\frac{6}{3-2\alpha}\}\right). \tag{3.59}$$

Thanks to Gagliardo-Nirenberg inequality

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C\int_0^T\left(\|\nabla u\|_{L^2}+\|\nabla^2 u\|_{L^r}\right)dt. \tag{3.60}$$

Next, to prove the upper bound of $\int_0^T\|\nabla^2 u\|_{L^r}dt$. With the help of (3.3), we estimate the following inequality

$$\begin{aligned}
&\|\rho u_t+\rho u\cdot\nabla u+\mathrm{div}(\nabla d\odot\nabla d)\|_{L^r}\\
&\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^{\frac{6r}{6-r}}}+\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^{\frac{6r}{6-r}}}\right)
\end{aligned}$$

$$\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^2 u\|_{L^r}^{\frac{4r-6}{5r-6}}+\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^3 d\|_{L^r}^{\frac{4r-6}{5r-6}}\right)$$

$$\le C\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+\|\nabla^2 d\|_{L^2}^{\frac{6r-6}{r}}\right)+\varepsilon\left(\|\nabla^2 u\|_{L^r}+\|\nabla^3 d\|_{L^r}\right), \tag{3.61}$$

where, we used $\|f\|_{L^{6r/(6-r)}}\le C\|\nabla f\|_{L^2}^{r/(5r-6)}\|\nabla^2 f\|_{L^r}^{(4r-6)/(5r-6)}$.

$$\begin{aligned}&\|\rho u_t+\rho u\cdot\nabla u+\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\\ \le& C\left(\|\sqrt{\rho}\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}+\|\rho\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^2}+\|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right)\\ \le& C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right).\end{aligned} \tag{3.62}$$

Using (3.61), (3.62) and taking ε suitably small, it deduces from Lemma 2.2 that

$$\begin{aligned}&\|\nabla^2 u\|_{L^r}+\|\nabla P\|_{L^r}\\ \le& C\left(\|\rho u_t+\rho u\cdot\nabla u+\mathrm{div}(\nabla d\odot\nabla d)\|_{L^r}+\|\rho u_t+\rho u\cdot\nabla u+\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right)\\ \le& C\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+\|\nabla^2 d\|_{L^2}^{\frac{6r-6}{r}}\right)\\ &+C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\mathrm{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right)+\varepsilon\|\nabla^3 d\|_{L^r}.\end{aligned} \tag{3.63}$$

To estimate $\|\nabla^3 d\|_{L^r}$, according to the properties of elliptic equations, we give

$$\|\nabla^3 d\|_{L^r}\le C\left(\|\nabla d_t\|_{L^r}+\|\nabla(u\cdot\nabla d)\|_{L^r}+\|\nabla(|\nabla d|^2 d)\|_{L^r}\right). \tag{3.64}$$

We will use Lemma 2.3 to estimate each term on the right-hand side of (3.64) as follows:

$$\|\nabla d_t\|_{L^r}\le C\|\nabla d_t\|_{L^2}^{(6-r)/(2r)}\|\nabla^2 d_t\|_{L^2}^{(3r-6)/(2r)}, \tag{3.65}$$

$$\begin{aligned}&\|\nabla(u\cdot\nabla d)\|_{L^r}\le C\left(\|\nabla u\|_{L^6}\|\nabla u\|_{L^{\frac{6r}{6-r}}}+\|u\|_{L^6}\|\nabla^2 d\|_{L^{\frac{6r}{6-r}}}\right)\\ \le& C\left(\|\nabla^2 d\|_{L^2}\|\nabla u\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^2 u\|_{L^r}^{\frac{4r-6}{5r-6}}+\|\nabla u\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^3 d\|_{L^r}^{\frac{4r-6}{5r-6}}\right)\\ \le& C\left(\|\nabla^2 d\|_{L^2}^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^{\frac{5r-6}{r}}\|\nabla^2 d\|_{L^2}\right)+\varepsilon\left(\|\nabla^3 d\|_{L^r}+\|\nabla^2 u\|_{L^r}\right)\\ \le& C\left(\|\nabla^2 d\|_{L^2}^{\frac{6r-6}{r}}+\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}\right)+\varepsilon\left(\|\nabla^3 d\|_{L^r}+\|\nabla^2 u\|_{L^r}\right)\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}&\|\nabla(|\nabla d|^2 d)\|_{L^r}\le C\left(\|\nabla d\|_{L^{3r}}^3+\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^{\frac{6r}{6-r}}}\right)\\ \le& C\left(\|\nabla d\|_{L^6}^{\frac{12r-12}{5r-6}}\|\nabla^3 d\|_{L^r}^{\frac{3r-6}{5r-6}}+\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^2}^{\frac{r}{5r-6}}\|\nabla^3 d\|_{L^r}^{\frac{4r-6}{5r-6}}\right)\\ \le& C\left(\|\nabla^2 d\|_{L^2}^{\frac{6r-6}{r}}+\varepsilon\|\nabla^3 d\|_{L^r}\right),\end{aligned} \tag{3.67}$$

here, we have used $\|f\|_{L^{3r}}\le C\|f\|_{L^6}^{(4r-4)/(5r-6)}\|\nabla^2 f\|_{L^r}^{(r-2)/(5r-6)}$ and

$$\|f\|_{L^{\frac{6r}{6-r}}}\le C\|f\|_{L^2}^{\frac{r}{5r-6}}\|\nabla f\|_{L^r}^{\frac{4r-6}{5r-6}}.$$

Hence, inserting (3.65)-(3.67) into (3.64), adding (3.63), choosing ε sufficiently small, give

$$\begin{aligned}
&\|\nabla^3 d\|_{L^r}+\|\nabla^2 u\|_{L^r}\\
\leq & C\Big(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+\|\nabla^2 d\|_{L^2}^{\frac{6r-6}{r}}\\
&\quad+\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\Big)\\
\leq & C\Big(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+\|\nabla u\|_{L^2}^{2(r-1)}+\|\nabla^2 d\|_{L^2}^{2(r-1)}\\
&\quad+\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\Big). \qquad (3.68)
\end{aligned}$$

On the one hand, it follows from (3.58), (3.59), (3.68), that

$$\begin{aligned}
&\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}+\|\nabla^2 u\|_{L^r}\right)dt\\
\leq & C\sup_{0\leq t\leq\sigma(T)}\left(t^{\frac{1-\alpha}{2}}\|\nabla u\|_{L^2}\right)\int_0^{\sigma(T)}t^{\frac{\alpha-1}{2}}dt+C\sup_{0\leq t\leq\sigma(T)}t^{\frac{2-\alpha}{2}}\|\sqrt{\rho}u_t\|_{L^2}\int_0^{\sigma(T)}t^{\frac{\alpha-2}{2}}dt\\
&+C\sup_{0\leq t\leq\sigma(T)}((\|\sqrt{\rho}u_t\|_{L^2}t^{\frac{2-\alpha}{2}})^{\frac{6-r}{2r}}\int_0^{\sigma(T)}\left(\|\nabla u_t\|_{L^2}t^{\frac{2-\alpha}{2}}\right)^{\frac{3r-6}{2r}}t^{\frac{\alpha-2}{2}}dt\\
&+C\sup\nolimits_{0\leq t\leq\sigma(T)}(t^{1-\alpha}\|\nabla u\|_{L^2}^2)^{r-1}\int_0^{\sigma(T)}t^{(\alpha-1)(r-1)}dt+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)dt\\
&+C\int_0^{\sigma(T)}\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}dt\\
\leq & C\Pi_0^{\frac{31}{70(1+2\alpha)}}+\|u_0\|_{\dot H^\alpha}^{\frac{6-r}{2r}}\left(\int_0^{\sigma(T)}t^{2-\alpha}\|\nabla u_t\|_{L^2}^2dt\right)^{\frac{3r-6}{4r}}\left(\int_0^{\sigma(T)}t^{\frac{2(\alpha-2)r}{r+6}}dt\right)^{\frac{r+6}{4r}}\\
\leq & C\Pi_0^{\frac{31}{70(1+2\alpha)}}. \qquad (3.69)
\end{aligned}$$

On the other hand, from Lemma 2.2, (3.18) and (3.40), we have

$$\begin{aligned}
\int_{\sigma(T)}^T\|\nabla u\|_{L^2}dt\leq & C\int_{\sigma(T)}^T\left(\|\rho u_t\|_{L^{6/5}}+\|\rho u\cdot\nabla u\|_{L^{6/5}}+\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}\right)dt\\
\leq & C\left(\int_{\sigma(T)}^T t^2\|\nabla u_t\|_{L^2}^2dt\right)^{1/2}\left(\int_{\sigma(T)}^T t^{-2}dt\right)^{1/2}+\int_{\sigma(T)}^T\|\nabla u\|_{L^2}^2dt\\
&+C\int_{\sigma(T)}^T\|\operatorname{div}(\nabla d\odot\nabla d)\|_{L^{6/5}}dt\\
\leq & C\Pi_0^{\frac{31}{70(1+2\alpha)}}, \qquad (3.70)
\end{aligned}$$

$$\begin{aligned}
&\int_{\sigma(T)}^T\|\sqrt{\rho}u_t\|_{L^2}^{(6-r)/(2r)}\|\nabla u_t\|_{L^2}^{(3r-6)/(2r)}\\
\leq & C\left(\int_{\sigma(T)}^T t^{1-\alpha}\|\sqrt{\rho}u_t\|_{L^2}^2dt+\int_{\sigma(T)}^T t^{2-\alpha}\|\nabla u_t\|_{L^2}^2dt\right)\\
\leq & C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right), \qquad (3.71)
\end{aligned}$$

$$\begin{aligned}
&\int_{\sigma(T)}^{T}\left(\|\nabla u\|_{L^2}^{2(r-1)}+\|\nabla^2 d\|_{L^2}^{2(r-1)}+\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}^2\right)dt\\
&\le \sup_{\sigma(T)\le t\le T}\left[t^{(1-\alpha)}\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)\right]^{(r-2)}\int_{\sigma(T)}^{T}\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)dt\\
&\quad+\int_{\sigma(T)}^{T}\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4+\|\nabla u\|_{L^2}^2\right)dt\\
&\le C\left(\|u_0\|_{\dot H^\alpha}^2+\|\nabla d_0\|_{\dot H^\alpha}^2\right),
\end{aligned}\tag{3.72}$$

$$\begin{aligned}
\int_{\sigma(T)}^{T}\|\nabla u_t\|_{L^2}dt &\le C\left(\int_{\sigma(T)}^{T}t^2\|\nabla u_t\|_{L^2}^2dt\right)^{1/2}\left(\int_{\sigma(T)}^{T}t^{-2}dt\right)^{1/2}\\
&\le C\Pi_0^{\frac{1}{2(1+2\alpha)}}.
\end{aligned}\tag{3.73}$$

Combining with (3.70)-(3.73), applying (3.68), (3.60) to show

$$\int_{\sigma(T)}^{T}\|\nabla u\|_{L^\infty}dt\le C\Pi_0^{\frac{31}{70(1+2\alpha)}},$$

which, together with (3.69), choosing Π_0 small enough, deduces that

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le 1.\tag{3.74}$$

Next, to prove (3.2), we rewrite the equation of $\mu(\rho)$ as follows:

$$\mu_t(\rho)+u\cdot\nabla\mu(\rho)=0$$

which, multiplying by $|\nabla\mu(\rho)|^{p-2}\nabla\mu(\rho)$ and integrating it by parts, we find

$$\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^p}\le C\|\nabla u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^p}.$$

This, combined with (3.1), shows that

$$\begin{aligned}
\sup_{0\le t\le T}\|\nabla\mu(\rho)\|_{L^p}&\le\|\nabla\mu(\rho_0)\|_{L^p}\exp\left\{\int_0^T\|\nabla u\|_{L^\infty}dt\right\}\\
&\le\|\nabla\mu(\rho_0)\|_{L^p}\exp\left\{C\Pi_0^{\frac{31}{70(1+2\alpha)}}\right\}\\
&\le 3/2\|\nabla\mu(\rho_0)\|_{L^p},
\end{aligned}\tag{3.75}$$

where we have chosen Π_0 small enough.

Similarly, applying $(1.1)_3$, the inequality $\|f\|_{L^9}^3\le C\|\nabla|f|^{3/2}\|_{L^2}$ and Hölder's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\|\nabla d\|_{L^3}^3+\|\sqrt{|\nabla d|}\nabla^2 d\|_{L^2}^2&\le C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^3}^3+C\|\nabla d\|_{L^5}^5\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^3}^3+C\|\nabla d\|_{L^3}^2\|\nabla d\|_{L^9}^3\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^3}^3+C\|\nabla d\|_{L^3}^2\|\sqrt{|\nabla d|}\nabla^2 d\|_{L^2}^2,
\end{aligned}$$

which together with Gronwall's inequality, by (3.1) and we choose $\|\nabla d_0\|_{\dot H^\alpha}$ small enough to get

$$\begin{aligned}&\sup_{0\le t\le T}\|\nabla d\|_{L^3}^3+\int_0^T\|\sqrt{|\nabla d|}\nabla^2 d\|_{L^2}^2dt\le C\|\nabla d_0\|_{L^3}^3\\ &\le C\|\nabla d_0\|_{L^{3/2}}^{\frac{3(2\alpha-1)}{2\alpha+1}}\|\nabla d_0\|_{L^{6/(3-2\alpha)}}^{\frac{6}{2\alpha+1}}\\ &\le C\|\nabla d_0\|_{\dot H^\alpha}^{\frac{6}{2\alpha+1}}\le\|\nabla d_0\|_{\dot H^\alpha}^{\frac{4}{2\alpha+1}}.\end{aligned}\tag{3.76}$$

To prove $\int_0^T\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)dt\le\Pi_0$, one derives from (3.8) and (3.17) that

$$\begin{aligned}\int_0^T&\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)dt\le\sup_{0\le t\le\sigma(T)}t^{2-2\alpha}\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\int_0^{\sigma(T)}t^{2\alpha-2}dt\\ &+\sup_{\sigma(T)\le t\le T}\left(t^{1-\alpha}\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)\right)\int_{\sigma(T)}^T\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)dt\\ \le& C\Pi_0^{\frac{2+2\alpha}{1+2\alpha}}\le\Pi_0,\end{aligned}$$

here we have chosen Π_0 small enough. This combined with (3.74), (3.75) and (3.76) complete the proof of Proposition 3.1. ∎

LEMMA 3.6. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5), let $\bar p\triangleq\min\{6,p\}$, then*

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla d\|_{H^1}^2+\|\nabla\rho\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|d_t\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)dt\le C,\tag{3.77}$$

$$\sup_{0\le t\le T}\left(t\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)+\int_0^T t\left(\|\nabla u_t\|_{L^2}^2+\|d_{tt}\|_{L^2}^2\right)dt\le C,\tag{3.78}$$

$$\begin{aligned}\|\nabla u_t\|_{L^{\bar p}}+\|P_t\|_{L^{\bar p}}\le& C\|\sqrt{\rho}u_{tt}\|_{L^2}+C\left(1+\|\nabla u\|_{H^1}^{3/2}+\|\nabla^2 d\|_{H^1}^{3/2}\right)\left(\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\right)\\ &+C\left(\|\nabla u\|_{H^1}+\|\nabla^2 d\|_{H^1}\right)\left(\|\nabla u\|_{L^2}^3+\|\nabla^2 d\|_{L^2}^3+\|\nabla^2 u\|_{L^2}+\|\nabla^3 d\|_{L^2}\right),\end{aligned}\tag{3.79}$$

here C depends only on $\bar p,p,\mu,\|\rho_0\|_{L^{3/2}},\|\nabla d_0\|_{L^{3/2}}$, Π_0, $\|\nabla u_0\|_{L^2}$, $\|\nabla\rho_0\|_{L^2}$ and $\|\nabla^2 d_0\|_{L^2}$.

Proof. First, according to (3.4), (3.13), (3.15), (3.27) and (3.33), it is easy to deduce that (3.77) holds. And multiplying (3.52) by t, it follows from (3.9), (3.18) and (3.77) that (3.78) holds.

Next, differentiating $(1.1)_2$ with respect to t, we obtain

$$\begin{aligned}-\operatorname{div}(\mu(\rho)\nabla u_t)+\nabla P_t&=(-\rho u_t-\rho u\cdot\nabla u)_t+\operatorname{div}(\mu(\rho)_t\nabla u-(\nabla d\odot\nabla d)_t)\\ &=A_1+\operatorname{div}A_2.\end{aligned}$$

It follows from Lemma 2.2 that

$$\|\nabla u_t\|_{L^{\bar p}}+\|P_t\|_{L^{\bar p}}\le C\left(\|A_1\|_{L^{\frac{3\bar p}{3+\bar p}}}+\|A_2\|_{L^{\bar p}}\right).\tag{3.80}$$

To estimate term by term in (3.80), we first get, by applying (3.3) and (3.77), that

$$
\begin{aligned}
&\|A_1\|_{L^{\frac{3\bar p}{3+\bar p}}}\\
&\le C\left(\|\rho u_{tt}\|_{L^{\frac{3\bar p}{3+\bar p}}}+\|\rho_t u_t\|_{L^{\frac{3\bar p}{3+\bar p}}}+\|\rho_t u\cdot\nabla u\|_{L^{\frac{3\bar p}{3+\bar p}}}+\|\rho u_t\cdot\nabla u\|_{L^{\frac{3\bar p}{3+\bar p}}}+\|\rho u\cdot\nabla u_t\|_{L^{\frac{3\bar p}{3+\bar p}}}\right)\\
&\le C\left(\|\sqrt{\rho}u_{tt}\|_{L^2}+\|\rho_t\|_{L^2}\left(\|u_t\|_{L^{\frac{6\bar p}{6-\bar p}}}+\|\nabla u\|_{L^\infty}\|u\|_{L^{\frac{6\bar p}{6-\bar p}}}\right)+\|\nabla u\|_{L^2}\|u_t\|_{L^{\frac{6\bar p}{6-\bar p}}}\right)\\
&\quad +C\|\nabla u_t\|_{L^2}\|u\|_{L^{\frac{6\bar p}{6-\bar p}}}\\
&\le C\|\sqrt{\rho}u_{tt}\|_{L^2}+\frac{1}{4}\|\nabla u_t\|_{L^{\bar p}}+C\left(1+\|\nabla u\|_{H^1}^{3/2}\right)\left(\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\right)\\
&\quad +\|\nabla u\|_{H^1}\left(\|\nabla u\|_{L^2}^3+\|\nabla^2 d\|_{L^2}^3+\|\nabla^2 u\|_{L^2}+\|\nabla^3 d\|_{L^2}\right),
\end{aligned}
\tag{3.81}
$$

here we use (2.5) to have

$$
\|u\|_{L^{\frac{6\bar p}{6-\bar p}}}\le C\|\nabla u\|_{H^1},\qquad \|u_t\|_{L^{\frac{6\bar p}{6-\bar p}}}\le C\|u_t\|_{L^6}^{\frac{\bar p}{3\bar p-6}}\|\nabla u_t\|_{L^{\bar p}}^{\frac{2\bar p-6}{3\bar p-6}},
$$

use (3.77) to get

$$
\|\rho_t\|_{L^2}\le\|\nabla\rho\cdot u\|_{L^2}\le C\|\nabla\rho\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\le C\|\nabla^2 u\|_{L^2}^{1/2},\tag{3.82}
$$

and use (3.68) and (3.77) to give

$$
\|\nabla u\|_{L^\infty}\le C\left(\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}+\|\nabla u\|_{L^2}^3+\|\nabla^2 d\|_{L^2}^3+\|\nabla^2 u\|_{L^2}+\|\nabla^3 d\|_{L^2}\right).\tag{3.83}
$$

We deduce from (3.1), (3.3) and (3.83) that

$$
\begin{aligned}
\|A_2\|_{L^{\bar p}}&\le C\left(\|\nabla\mu(\rho)\|_{L^p}\|u\|_{L^{\bar p p/(p-\bar p)}}\|\nabla u\|_{L^\infty}+\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^{\bar p}}\right)\\
&\le C\left(\|\nabla u\|_{H^1}\|\nabla u\|_{L^\infty}+\|\nabla^2 d\|_{H^1}\left(\|\nabla d_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\right)\right)\\
&\le C\left(\|\nabla u\|_{H^1}+\|\nabla^2 d\|_{H^1}\right)\left(\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\right)+C\|\nabla u\|_{H^1}\left(\|\nabla u\|_{L^2}^3\right.\\
&\quad\left.+\|\nabla^2 d\|_{L^2}^3+\|\nabla^2 u\|_{L^2}+\|\nabla^3 d\|_{L^2}\right)+C\|\nabla^2 d\|_{H^1}\|\nabla d_t\|_{L^2}\\
&\le C\left(\|\nabla u\|_{H^1}+\|\nabla^2 d\|_{H^1}\right)\left(\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\right)+C\left(\|\nabla u\|_{H^1}\right.\\
&\quad\left.+\|\nabla^2 d\|_{H^1}\right)\left(\|\nabla u\|_{L^2}^3+\|\nabla^2 d\|_{L^2}^3+\|\nabla^2 u\|_{L^2}+\|\nabla^3 d\|_{L^2}\right)
\end{aligned}
\tag{3.84}
$$

where, we have used

$$
\begin{aligned}
\|\nabla d_t\|_{L^2}&\le C\left(\|\nabla(u\cdot\nabla d)\|_{L^2}+\|\nabla^3 d\|_{L^2}+\|\nabla(|\nabla d|^2 d)\|_{L^2}\right)\\
&\le C\left(\|\nabla u\|_{L^2}^3+\|\nabla^2 d\|_{L^2}^3+\|\nabla^3 d\|_{L^2}\right).
\end{aligned}
\tag{3.85}
$$

By inserting the estimates (3.81), (3.84) into (3.80), we obtain (3.79). ◻

LEMMA 3.7. *Assume that (ρ,u,d) be a smooth solution to (1.1)-(1.5), let $\beta=\max\{5/2,(5\bar p p-4\bar p-10p)/(4\bar p p-4\bar p-8p))\}$, then*

$$
\begin{aligned}
&\sup_{0\le t\le\sigma(T)} t^\beta\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+\int_0^{\sigma(T)} t^\beta\left(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2+\|P_t\|_{L^2}^2\right)dt\le C,\\
&\sup_{\sigma(T)\le t\le T} t^2\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+\int_{\sigma(T)}^{T} t^2\left(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2+\|P_t\|_{L^2}^2\right)dt\le C,
\end{aligned}
\tag{3.86}
$$

here C depends only on $\bar{\rho}, p, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla d_0\|_{L^{3/2}}$, Π_0, $\|\nabla u_0\|_{L^2}^2$, $\|\nabla \rho_0\|_{L^2}$ and $\|\nabla^2 d_0\|_{L^2}^2$.

Proof. Operate ∂_t to $(1.1)_2$, and multiplying it by u_{tt} in L^2, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \mu(\rho)|\nabla u_t|^2dx+\int \rho|u_{tt}|^2dx=&\int \mathrm{div}\,[\mu(\rho)_t\nabla u]\,u_{tt}dx+\frac{1}{2}\int \mu(\rho)_t|\nabla u_t|^2dx\\&-\int \rho_t u_t\cdot u_{tt}dx-\int (\rho u\cdot\nabla u)_t\cdot u_{tt}dx-\int \mathrm{div}(\nabla d\odot\nabla d)_t\cdot u_{tt}dx\\&\triangleq\sum_{i=1}^{5}B_i.\end{aligned}\tag{3.87}$$

Next, we gives the estimates of B_i one by one. By some direct calculation, get

$$\begin{aligned}B_1=&\frac{d}{dt}\int \partial_i(\mu(\rho)_t\partial_i u^j)u_t^j dx-\int \partial_i(\mu(\rho)\partial_i u^j)_t u^k\cdot\partial_k u_t^j dx\\&-\int \partial_i(\mu(\rho)\partial_i u^j)u_t^k\partial_k u_t^j dx+\int [\mu(\rho)\partial_i u^j\partial_i u^k]_t\partial_k u_t^j dx+\int [\mu(\rho)u^k\partial_k\partial_i u^j]_t\partial_i u_t^j dx\\=&-\frac{d}{dt}\int \mu(\rho)_t\partial_i u^j\partial_i u_t^j dx+\sum_{i=1}^{4}B_{1i},\end{aligned}\tag{3.88}$$

we estimate the left-hand side term in (3.88). Applying $(1.1)_2$, (3.3), (3.77), (3.79), (3.82) and (3.83), we have

$$\begin{aligned}B_{11}=&-\int(\rho u_t^j)_t u^k\partial_k u_t^j dx-\int(\rho u\cdot\nabla u^j)_t u^k\partial_k u_t^j dx\\&-\int \partial_j P_t u^k\partial_k u_t^j dx-\int \mathrm{div}(\nabla d\odot\nabla d)_t^j u^k\partial_k u_t^j dx\\\le& C\,(\|\rho_t\|_{L^2}\|u_t\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\rho u_{tt}\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\\&+\|\rho_t\|_{L^2}\|u\|_{L^\infty}^2\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\rho u_t\|_{L^6}\|\nabla u\|_{L^6}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\\&+\|u\|_{L^\infty}^2\|\nabla u_t\|_{L^2}^2+\|P_t\|_{L^3}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\&+\|\nabla^2 d_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}\|u\|_{L^\infty}+\|\nabla^2 d\|_{L^6}\|\nabla d_t\|_{L^6}\|u\|_6\|\nabla u_t\|_{L^2})\\\le& C\Bigg(\|u_t\|_{L^6}^{1-\frac{\bar p}{3\bar p-6}}\|\nabla u_t\|_{L^{\bar p}}^{\frac{\bar p}{3\bar p-6}}\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}+\|\sqrt{\rho}u_{tt}\|_{L^2}\|\nabla u\|_{H^1}^{1/2}\|\nabla u_t\|_{L^2}\\&+\|\nabla u\|_{H^1}^{3/2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}^2\\&+\|P_t\|_{L^2}^{1-\frac{\bar p}{3(\bar p-2)}}\|P_t\|_{L^{\bar p}}^{\frac{\bar p}{3(\bar p-2)}}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}+\|\nabla^2 d_t\|_{L^2}\|\nabla^2 d\|_{H^1}^{1/2}\|\nabla u\|_{H^1}^{1/2}\|\nabla u_t\|_{L^2}\Bigg)\\\le&\frac{1}{8}\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\left(1+\|\nabla u\|_{H^1}^3+\|\nabla^2 d\|_{H^1}^3+\|\sqrt{\rho}u_t\|_{L^2}^3\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\&+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right),\end{aligned}\tag{3.89}$$

here we use the inequality $\|f\|_{L^6}\le C\|\nabla f\|_{L^2}$, $(1.1)_1$, (3.3), (3.38) and (3.77) to get

$$\begin{aligned}&\|P_t\|_{L^2}\le C\,(\|(\rho u_t)_t\|_{L^{6/5}}+\|(\rho u\cdot\nabla u)_t\|_{L^{6/5}}+\|\mu_t(\rho)\nabla u\|_{L^2}+\|\nabla d\odot\nabla d_t\|_{L^2})\\&\le C\,(\|\rho u_{tt}\|_{L^{6/5}}+\|\rho_t u_t\|_{L^{6/5}}+\|\rho_t u\cdot\nabla u\|_{L^{6/5}}+\|\rho u_t\cdot\nabla u\|_{L^{6/5}}+\|\rho u\cdot\nabla u_t\|_{L^{6/5}}\\&\quad+\|\mu_t(\rho)\nabla u\|_{L^2}+\|\nabla d\odot\nabla d_t\|_{L^2})\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\sqrt{\rho}u_{tt}\|_{L^2}+\|\nabla\rho\|_{L^2}\|u\|_{L^6}\|u_t\|_{L^6}+\|\nabla\rho\|_{L^2}\|u\|_{L^6}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\right.\\
&\qquad\left.+\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^6}+\|u\|_{L^6}\|\nabla u_t\|_{L^2}+\|\nabla\mu(\rho)\|_{L^p}\|u\|_{L^6}\|\nabla u\|_{H^1}+\|\nabla d\|_{L^3}\|\nabla^2 d_t\|_{L^2}\right)\\
&\le C\Big(\|\sqrt{\rho}u_{tt}\|_{L^2}+\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}^{3/2}\|\nabla u\|_{L^2}+\|\nabla\mu(\rho)\|_{L^p}\|u\|_{L^6}\|\nabla u\|_{H^1}\\
&\qquad+\|\nabla d\|_{L^3}\|\nabla^2 d_t\|_{L^2}\Big). \qquad (3.90)
\end{aligned}
$$

Since $\mu_t(\rho)+\nabla\mu(\rho)\cdot u=0$, we use (3.1), (3.77) and (3.79) to obtain

$$
\begin{aligned}
B_{12}&\le C\left(\|\nabla\mu(\rho)\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}}+\|\nabla^2 u\|_{L^2}\right)\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u\|_{H^1}\|u_t\|_{L^6}^{1-\frac{\bar p}{3\bar p-6}}\|\nabla u_t\|_{L^{\bar p}}^{\frac{\bar p}{3\bar p-6}}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{8}\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\left(1+\|\nabla^2 d\|_{H^1}^3+\|\nabla u\|_{H^1}^3\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&\quad+C\left(\|\nabla^2 d\|_{H^1}+\|\nabla u\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right), \qquad (3.91)
\end{aligned}
$$

using (3.1) and (3.83), we also get

$$
\begin{aligned}
B_{13}&\le C\left(\|\nabla\mu(\rho)\|_{L^p}\|u\|_{L^\infty}\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^{\frac{2p}{p-2}}}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}^2\right)\\
&\le C\left(1+\|\nabla u\|_{H^1}^3+\|\nabla u\|_{L^\infty}\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+C\|\nabla u\|_{H^1}\left(\|\nabla u\|_{L^2}^6\right.\\
&\quad\left.+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right) \qquad (3.92)
\end{aligned}
$$

and

$$
\begin{aligned}
B_{14}&=\int\mu_t(\rho)u^k\partial_k\partial_i u^j\partial_i u_t^j+\mu(\rho)u_t^k\partial_k\partial_i u^i\partial_i u_t^j dx+\frac{1}{2}\int\partial_k\mu(\rho)u^k(\partial_i u_t^j)^2dx\\
&\le C\|\nabla\mu(\rho)\|_{L^p}\left(\|u\|_{L^\infty}^2\|\nabla^2 u\|_{L^{\frac{2p}{p-2}}}\|\nabla u_t\|_{L^2}+\|u\|_{L^\infty}\|\nabla u_t\|_{L^{\bar p}}^{\frac{2\bar p}{p(\bar p-2)}}\|\nabla u_t\|_{L^2}^{\frac{2(p\bar p-\bar p-2p)}{p(\bar p-2)}}\right)\\
&\quad+C\|u_t\|_{L^\infty}\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le C\left(\|\nabla^2 u\|_{L^2}\|\nabla^2 u\|_{L^{\frac{2p}{p-2}}}\|\nabla u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\|\nabla u_t\|_{L^2}^2\right)+\varepsilon\|\nabla u_t\|_{L^{\bar p}}^2\\
&\quad+C(\|\nabla u_t\|_{L^2}+\|\nabla u_t\|_{L^{\bar p}})\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le \tfrac{1}{8}\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\left(1+\|\nabla^2 d\|_{H^1}^3+\|\nabla u\|_{H^1}^3+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&\quad+C\|\nabla u\|_{H^1}^2\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right), \qquad (3.93)
\end{aligned}
$$

where choose ε small enough, and use (3.68) to get

$$
\begin{aligned}
\|\nabla^2 u\|_{L^{\frac{2p}{p-2}}}&\le C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla^3 d\|_{L^2}\right)\\
&\le C\left(\|\sqrt{\rho}\|_{L^3}\|u_t\|_{L^6}+\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla^3 d\|_{L^2}\right)\\
&\le C\left(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2+\|\nabla^3 d\|_{L^2}\right). \qquad (3.94)
\end{aligned}
$$

Substituting (3.89)-(3.93) into (3.88), we get

$$
\begin{aligned}
B_1&\le -\frac{d}{dt}\int\mu(\rho)_t\partial_i u^j\partial_i u_t^j dx+\frac{1}{4}\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\Big(1+\|\nabla u\|_{H^1}^3+\|\nabla^2 d\|_{H^1}^3\\
&\quad+\|\nabla u\|_{L^\infty}+\|\sqrt{\rho}u_t\|_{L^2}^3+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\Big)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&\quad+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right).
\end{aligned}
$$

Applying (3.79), similar to B_{14}, choosing ε suitably small, we deduce

$$\begin{aligned}
B_2 \le & \varepsilon\|\nabla u_t\|_{L^{\bar p}}^2 + C\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\|\nabla u_t\|_{L^2}^2\\
\le & \frac18\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)\\
&+C\left(1+\|\nabla u\|_{H^1}^3+\|\nabla^2 d\|_{H^1}^3+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)
\end{aligned}$$

and

$$\begin{aligned}
B_3 = & \frac12\frac{d}{dt}\int\rho_t|u_t|^2dx + \int(\rho u^j)_t u_t^j\partial_j u_t^i dx\\
\le & \frac12\frac{d}{dt}\int\rho_t|u_t|^2dx + C\left((\|\nabla\rho\|_{L^p}\|u\|^2_{L^{\frac{4p}{p-2}}}+\|\sqrt{\rho}u_t\|_{L^2})\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\right)\\
\le & \frac12\frac{d}{dt}\int\rho_t|u_t|^2dx + C\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_{L^2}\right)\left(\|\nabla u_t\|_{L^2}+\|\nabla u_t\|_{L^{\bar p}}\right)\|\nabla u_t\|_{L^2}\\
\le & \frac12\frac{d}{dt}\int\rho_t|u_t|^2dx + C\left(1+\|\nabla^2 d\|_{L^2}^3+\|\nabla u\|_{H^1}^3+\|\sqrt{\rho}u_t\|_{L^2}^2\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^3 d\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2\right)+\frac18\|\sqrt{\rho}u_{tt}\|_{L^2}^2.
\end{aligned}$$

By some direct calculation, in view of (3.3), (3.4) and (3.77), we get

$$\begin{aligned}
B_4 = & -\frac{d}{dt}\int\rho_t u\cdot\nabla u\cdot u_t dx + \int(\rho u)_t\cdot\nabla(u\cdot\nabla u\cdot u_t)dx\\
&+\int\rho_t u_t\cdot\nabla u\cdot u_t dx + \int\rho_t u\cdot\nabla u_t\cdot u_t dx - \int\rho u_t\cdot\nabla u\cdot u_{tt}dx\\
&-\int\rho u\cdot\nabla u_t\cdot u_{tt}dx\\
\le & -\frac{d}{dt}\int\rho_t u\cdot\nabla u\cdot u_t dx + C\|\rho_t\|_{L^2}\|\nabla u\|_{H^1}\left(\|\nabla^2 u\|_{L^2}^{3/2}\|\nabla u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}\|u_t\|_{L^\infty}\right.\\
&\left.+\|\nabla u\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla u_t\|_{L^2}^2+\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\right)+C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2\\
&+\frac18\|\sqrt{\rho}u_{tt}\|_{L^2}^2\\
\le & -\frac{d}{dt}\int\rho_t u\cdot\nabla u\cdot u_t dx + C\left(1+\|\nabla^2 d\|_{H^1}^3+\|\nabla u\|_{H^1}^3\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)+\frac18\|\sqrt{\rho}u_{tt}\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
B_5 = & \frac{d}{dt}\int\operatorname{div}(\nabla d\odot\nabla d)_t\cdot u_t dx + \int(\nabla d\odot\nabla d)_{tt}\cdot\nabla u_t dx\\
\le & \frac{d}{dt}\int\operatorname{div}(\nabla d\odot\nabla d)_t\cdot u_t dx + C\|\nabla d_t\|_{L^2}^{1/2}\|\nabla^2 d_t\|_{L^2}^{3/2}\|\nabla u_t\|_{L^2}+\frac18\|\nabla d_{tt}\|_{L^2}^2\\
&+C\|\nabla d\|_{L^\infty}^2\|\nabla u_t\|_{L^2}^2\\
\le & \frac{d}{dt}\int\operatorname{div}(\nabla d\odot\nabla d)_t\cdot u_t dx + C\left(1+\|\nabla d_t\|_{L^2}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&+\frac18\|\nabla d_{tt}\|_{L^2}^2.
\end{aligned}$$

In the end, putting $B_1 - B_5$ into (3.87), we find

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u_t\|_{L^2}^2+\|\sqrt{\rho}u_{tt}\|_{L^2}^2 \le C\big(1+\|\sqrt{\rho}u_t\|_{L^2}^3+\|\nabla u\|_{H^1}^3+\|\nabla^2 d\|_{H^1}^3+\|\nabla u\|_{L^\infty}\\
&\quad+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}+\|\nabla d_t\|_{L^2}^2\Big)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&\quad+\frac{d}{dt}\int\Big(\partial_i(\mu(\rho)_t\partial_i u^j)u_t^j dx+\rho_t|u_t|^2+\rho_t u\cdot\nabla u\cdot u_t+\operatorname{div}(\nabla d\odot\nabla d)_t\cdot u_t\Big)dx\\
&\quad+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{H^1}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right)\\
&\quad+\frac{1}{8}\|\nabla d_{tt}\|_{L^2}^2. \qquad (3.95)
\end{aligned}$$

Let

$$\varphi(t)\triangleq\int\Big(\partial_i(\mu(\rho)_t\partial_i u^j)u_t^j dx+\rho_t|u_t|^2+\rho_t u\cdot\nabla u\cdot u_t+\operatorname{div}(\nabla d\odot\nabla d)_t\cdot u_t\Big)dx,$$

it follows from (3.1) that

$$\begin{aligned}
|\varphi(t)|&\le C\big(\|\nabla\mu(\rho)\|_{L^p}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}^2+\|\rho u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}\\
&\qquad+\|\rho_t\|_{L^2}\|\nabla u\|_{H^1}^2\|u_t\|_{L^6}+\|\nabla d\|_{L^3}\|\nabla d_t\|_{L^6}\|\nabla u_t\|_{L^2}\big)\\
&\le\frac{1}{4}\|\nabla u_t\|_{L^2}^2+C_2\|\nabla^2 d_t\|_{L^2}^2+C\left(\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2+\|\nabla u\|_{H^1}^4\right). \qquad (3.96)
\end{aligned}$$

Operating ∂_t to $(1.1)_3$, multiplying it by Δd_{tt}, due to the Sobolev inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 d_t\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2\\
\le&C\left(\|\nabla(u\cdot\nabla d)_t\|_{L^2}^2+\|\nabla(|\nabla d|^2 d)_t\|_{L^2}^2\right)+\frac{1}{2}\|\nabla d_{tt}\|_{L^2}^2\\
\le&C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)\\
&\quad+C\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2+\frac{1}{2}\|\nabla d_{tt}\|_{L^2}^2. \qquad (3.97)
\end{aligned}$$

Then (3.95)+$2C_2$(3.97), by (3.85), we get that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u_t\|_{L^2}^2+2C_2\|\nabla^2 d_t\|_{L^2}^2\right)+\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2\\
&\le C\Big(1+\|\sqrt{\rho}u_t\|_{L^2}^3+\|\nabla u\|_{H^1}^3+\|\nabla^2 d\|_{H^1}^3+\|\nabla u\|_{L^\infty}+\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\\
&\quad+\|\nabla d_t\|_{L^2}^2\big)\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+\varphi'(t)+C\left(\|\nabla u\|_{H^1}^2+\|\nabla^2 d\|_{H^1}^2\right)\\
&\quad\times\left(\|\nabla u\|_{L^2}^6+\|\nabla^2 d\|_{L^2}^6+\|\nabla^2 u\|_{L^2}^2+\|\nabla^3 d\|_{L^2}^2\right). \qquad (3.98)
\end{aligned}$$

To prove the first inequality of Lemma 3.7, multiplying (3.98) by t^β, we apply (3.17), (3.39), (3.44), (3.78) and $\beta=\max\{5/2,(5\bar p p-4\bar p-10p)/(4\bar p p-4\bar p-8p)\}$, to have

$$\begin{aligned}
&\int_0^{\sigma(T)}t^\beta\|\nabla u\|_{H^1}^3\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)dt\\
&\le\sup_{0\le t\le\sigma(T)}t^{3/2}\|\nabla u\|_{H^1}^3\int_0^{\sigma(T)}t\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)dt\le C \qquad (3.99)
\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\sigma(T)} t^{\beta}\|\nabla^2 u\|_{L^2}^{\frac{p(\bar p-2)}{2(p\bar p-\bar p-2p)}}\|\nabla u_t\|_{L^2}^2 dt\\ \le& C\sup_{0\le t\le\sigma(T)}(t\|\nabla^2 u\|_{L^2}^2)^{\frac{p(\bar p-2)}{4(p\bar p-\bar p-2p)}}\int_0^{\sigma(T)} t\|\nabla u_t\|_{L^2}^2 dt\le C.\end{aligned} \tag{3.100}$$

We get after Gronwall's inequality, by (3.77), (3.90), (3.96), (3.99) and (3.100) to obtain

$$\sup_{0\le t\le\sigma(T)} t^{\beta}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+\int_0^{\sigma(T)} t^{\beta}\left(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2\right)\le C. \tag{3.101}$$

To prove the second inequality of Lemma 3.7, multiplying (3.98) by t^2, from (3.1), (3.40), (3.44), (3.74), (3.77), (3.78) and (3.96) and the definition of $\sigma(T)$, we have

$$\sup_{\sigma(T)\le t\le T} t^2\left(\|\nabla u_t\|_{L^2}^2+\|\nabla^2 d_t\|_{L^2}^2\right)+\int_{\sigma(T)}^{T} t^2\left(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|\nabla d_{tt}\|_{L^2}^2\right)dt\le C,$$

which together with (3.18) and (3.101), completes the proof of Lemma 3.7. □

4. Proof of Theorem 1.1.

We devote this section to proving the main result of this paper. First, there exists a T_1 such that the problem of (1.1)-(1.5) has a local strong solution on $\mathbb{R}^3\times(0,T_1]$ (see Lemma 2.1). Hence, there exists a $T_0\in(0,T_1]$ such that (3.1) holds for $T=T_0$.

Set

$$T^*\triangleq\sup\{(\rho,u,P,d)\text{ is a strong solution on }\mathbb{R}^3\times(0,\mathrm{T}]\text{ and (3.1) holds}\}. \tag{4.1}$$

Then $T^*\geqslant T_0>0$. For any $0<\tau<T\le T^*$ with T finite, one deduces from Lemmas 3.1-3.7 and standard embedding that

$$\begin{cases}\rho\in C([0,T];L^{3/2}\cap H^1),\nabla\mu(\rho)\in C([0,T];L^p),\\ u\in C([\tau,T];H_0^1),\nabla d\in C([0,T];L^{3/2}\cap H^1).\end{cases} \tag{4.2}$$

We claim that

$$T^*=\infty.$$

Otherwise, $T^*<\infty$. Proposition 3.1 implies that (3.2) hold at $T=T^*$. Let

$$(\rho^*,u^*,d^*)\triangleq(\rho,u,d)=\lim_{t\to T^*}(\rho,u,d)(x,t) \tag{4.3}$$

and

$$\rho^*\in L^{3/2}\cap H^1,\quad \nabla\mu(\rho^*)\in L^p,\quad u^*\in H_0^1,\quad \nabla d^*\in L^{3/2}\cap H^1.$$

Therefore, one can choose (ρ^*,u^*,d^*) as the initial data and to extend the local strong solution beyond T^*. This contradicts the assumption of T^* in (4.1). Hence, $T^*=\infty$. We finish the proof of Theorem 1.1 since the decay estimate of (1.7), (1.8) and (1.9) follow directly from (3.9), (3.18), (3.38), (3.40) and (3.86), respectively.

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