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STABILITY AND DECAY RATE OF VISCOUS CONTACT WAVE TO ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS*

XINXIANG BIAN† AND LINGLING XIE‡

Abstract. This paper studies the large-time asymptotic stability and optimal time-decay rate of viscous contact wave to one-dimensional compressible Navier-Stokes equations. We prove that one-dimensional compressible Navier-Stokes equations are asymptotically stable for viscous contact wave with arbitrarily large strength, under large initial perturbations. The time optimal decay rate of viscous contact wave is also obtained under the small initial perturbations. In the proof, the Lagrange transform is used to cancel the convection terms, which are difficult to estimate due to the lower spatial derivatives compared with the diffusion terms.

Keywords. Compressible Navier-Stokes Equations; Stability; Decay Rate; Viscous Contact Wave.

AMS subject classifications. 35Q30; 76N10.

1. Introduction

The two-dimensional compressible isentropic Navier-Stokes equations read as

$$\begin{cases} \rho_t+\mathrm{div}(\rho\boldsymbol{u})=0, \qquad x=(x_1,x_2)\in\mathbb{R}^2, t>0,\\ (\rho\boldsymbol{u})_t+\mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p(\rho)=\mu\Delta\boldsymbol{u}+(\mu+\lambda)\nabla\mathrm{div}\boldsymbol{u}, \end{cases} \tag{1.1}$$

where $\rho=\rho(t,x)>0, \boldsymbol{u}=\boldsymbol{u}(t,x)=(u_1,u_2)(t,x)$ and p represent the fluid density, velocity and pressure, respectively, $x=(x_1,x_2)\in\mathbb{R}^2$ is the spatial variable and $t>0$ is the time variable. The pressure $p=p(\rho)$ is given by the well-known γ-law:

$$p(\rho)=A\rho^\gamma,$$

with the adiabatic constant $\gamma>1$ and the fluid constant $A>0$. If we consider the one-dimensional solution $\rho=\rho(t,x_1)$, $\boldsymbol{u}=(u_1,u_2)(t,x_1)$ to the two-dimensional compressible Navier-Stokes Equations (1.1), then $\rho(t,x)$, $(u_1,u_2)(t,x)$[1] satisfy the following one-dimensional(1D) compressible Navier-Stokes equations

$$\begin{cases} \rho_t+(\rho u_1)_x=0, \qquad x\in\mathbb{R},\ t>0,\\ (\rho u_1)_t+(\rho u_1^2+p(\rho))_x=(2\mu+\lambda)u_{1xx},\\ (\rho u_2)_t+(\rho u_1u_2)_x=\mu u_{2xx}. \end{cases} \tag{1.2}$$

The initial data to (1.2) is prescribed by

$$(\rho,u_1,u_2)(0,x)=(\rho_0,u_{10},u_{20})(x)\to(\bar\rho,\bar u_1,\bar u_{2\pm}), \qquad \text{as } x\to\pm\infty, \tag{1.3}$$

with the far-field constant state $(\bar\rho,\bar u_1,\bar u_{2\pm})$ satisfying $\bar\rho>0$ and $\bar u_{2-}\neq\bar u_{2+}$.

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†School of Information and Mathematics, Yangtze University, Hubei 434000, P.R. China (bianxinxiang17@mails.ucas.ac.cn).

‡Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing, 100190, P.R. China; School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, P.R. China (linglingxie@amss.ac.cn).

[1]For convenience, we write the subscript x_1 as x in the following.

Note that in the above 1D system (1.2), the first two Equations $(1.2)_i (i=1,2)$ are exactly the classical 1D isentropic compressible Navier-Stokes equations decoupled with the third one $(1.2)_3$. However, $(1.2)_3$ depends on the solution to $(1.2)_i (i=1,2)$ and it leads to new solution behaviors. More precisely, the large-time asymptotic behaviors of the solution to 1D compressible Navier-Stokes Equations (1.2)-(1.3) are expected to be determined by the corresponding 1D compressible Euler equations:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, & x \in \mathbb{R},\ t > 0, \\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_x = 0, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = 0, \end{cases} \tag{1.4}$$

with Riemann initial data

$$(\rho, u_1, u_2)(0,x) = \begin{cases} (\bar{\rho}, \bar{u}_1, \bar{u}_{2-}),\ x < 0, \\ (\bar{\rho}, \bar{u}_1, \bar{u}_{2+}),\ x > 0. \end{cases} \tag{1.5}$$

The Euler Equations (1.4) with normal Riemann initial data contains broad classes of basic waves: shock waves, rarefaction waves on genuinely nonlinear characteristic fields, contact discontinuous waves on linearly degenerate characteristic fields by Lax [16] and Riemann [29].

It is observed that the compressible Euler Equations (1.4) are strictly hyperbolic in the non-vacuum region. Due to the third Equation $(1.4)_3$, besides the two genuinely nonlinear characteristic fields, there exists a linearly degenerate characteristic field, which is quite different from the classical isentropic Euler Equations $(1.4)_{1,2}$ with only genuinely nonlinear fields. Therefore, besides the nonlinear shock or rarefaction waves, there is a linearly degenerate wave, i.e., contact discontinuity, to the compressible Euler Equations (1.4).

There is extensive literature on the stability analysis of viscous contact waves. The nonlinear stability of a weak contact discontinuity for the compressible Euler equations with uniform viscosity was proved by Xin [31]. This was generalized by Liu and Xin [21] to the stability of contact discontinuity for a class of general systems of nonlinear conservation laws with uniform viscosity. This result was improved by Xin and Zeng [32]. Huang et al. first [9] showed the asymptotic stability of viscous contact waves for the compressible Navier-Stokes equations with free boundary. When the zero mass condition is added to the initial perturbations, it was studied in [10]. The zero mass condition was removed in [12]. There also exist some interesting results on the stability of composite waves. Zeng [33] showed the large-time asymptotic nonlinear stability of the superposition of viscous shock waves and contact discontinuity for system of viscous conservation laws with artificial viscosity under small initial perturbations. Recently, the stability of the combination of viscous contact waves and rarefaction waves to one-dimensional compressible Navier-Stokes system was studied in [7]. We refer to [4, 8, 18, 19, 22, 30] for viscous shock waves, [20, 24–26] for rarefaction waves, [2, 5, 13, 14, 23] for viscous contact discontinuities, [3, 6, 11, 17] for the composition of viscous contact waves and rarefaction waves, and the references therein.

In the present paper, we are concerned with the nonlinear time-asymptotic stability and optimal time-decay rate of viscous contact wave to the 1D compressible Navier-Stokes Equations (1.2)-(1.3), which is a viscous version of contact discontinuity solution to (1.4)-(1.5). Since $(1.2)_{1,2}$ is decoupled with $(1.2)_3$, we have that $\rho \to \bar{\rho}$, $u_1 \to \bar{u}_1$ as $t \to +\infty$, by Kanel's result [15]. Due to the existence of $(1.2)_3$ and the different far-field states of u_{20}, the asymptotic function $\bar{u}_2(x,t)$ satisfies the diffusion equation

$$(\bar{\rho}\bar{u}_2)_t + (\bar{\rho}\bar{u}_1\bar{u}_2)_x = \mu\bar{u}_{2xx}, \tag{1.6}$$

with

$$\bar{u}_2(0,x) = \begin{cases} \bar{u}_{2-},\ x<0, \\ \bar{u}_{2+},\ x>0. \end{cases}$$

From $(1.2)_3$ and (1.6), we find that the convection terms of the corresponding error equation cannot be controlled due to the lower spatial derivatives compared with the diffusion terms. The main idea in this paper is to use the Lagrange transform to cancel the convection terms in $(1.2)_3$ and (1.6). By this transform, one-dimensional compressible Navier-Stokes Equations (1.2) has the following form:

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p(v)_x = (2\mu+\lambda)(\frac{u_{1x}}{v})_x, \\ u_{2t} = \mu(\frac{u_{2x}}{v})_x, \end{cases} \tag{1.7}$$

where the volume $v = \frac{1}{\rho}$ and pressure $p(v) = v^{-\gamma}$. The initial data (1.3) is converted to

$$(v,u_1,u_2)(0,x) = (v_0,u_{10},u_{20}) \to (\bar{v},\bar{u}_1,\bar{u}_{2\pm}), \qquad as \quad x\to\pm\infty. \tag{1.8}$$

Correspondingly, the large-time behavior of the solution to the 1D compressible Navier-Stokes Equations (1.7) is determined by the viscous wave $(\bar{v},\bar{u}_1,\bar{u}_2)$ with the constant states $(\bar{v} = \frac{1}{\bar{\rho}},\bar{u}_1)$ and $\bar{u}_2 = \bar{u}_2(1+t,x)$ satisfying the different diffusion equation

$$\bar{u}_{2t} = \frac{\mu}{\bar{v}}\bar{u}_{2xx}, \tag{1.9}$$

with

$$\bar{u}_2(0,x) = \begin{cases} \bar{u}_{2-},\ x<0, \\ \bar{u}_{2+},\ x>0. \end{cases} \tag{1.10}$$

The diffusion Equation (1.9)-(1.10) has a self-similar solution $\bar{u}_2(\frac{x}{\sqrt{t}})$ satisfying

$$\frac{\partial^k}{\partial x^k}\bar{u}_2(1+t,x) = O(1)(1+t)^{-\frac{k}{2}} e^{-\frac{C|x|^2}{1+t}}, \quad \forall\ k = 1,2,3,\cdots, \tag{1.11}$$

with the uniform-in-time positive constants $O(1)$ and C.

Let

$$\phi(t,x) := v(t,x) - \bar{v}, \quad \psi_1(t,x) := u_1(t,x) - \bar{u}_1,$$
$$\psi_2(t,x) := u_2(t,x) - \bar{u}_2(1+t,x), \quad \Psi := (\psi_1,\psi_2),$$

By (1.7) and (1.9), we can get:

$$\begin{cases} \phi_t - \psi_{1x} = 0, \\ \psi_{1t} + (p(v)-p(\bar{v}))_x = (2\mu+\lambda)(\frac{\psi_{1x}}{v})_x, \\ \psi_{2t} = \mu(\frac{\psi_{2x}}{v})_x + \mu\big[(\frac{1}{v}-\frac{1}{\bar{v}})\bar{u}_{2x}\big]_x, \end{cases} \tag{1.12}$$

with the initial perturbation

$$\phi_0 = v_0 - \bar{v}, \quad \psi_1 = u_{10} - \bar{u}_1, \quad \psi_2 = u_{20} - \bar{u}_{20},$$

where $\phi_0, \psi_{10}, \psi_{20} \in H^1(\mathbb{R})$. Let

$$X(0,t_1) = \big\{(\phi,\Psi) | (\phi,\psi_1,\psi_2) \in C^0([0,t_1];H^1(\mathbb{R})), \phi_x \in L^2([0,t_1];L^2(\mathbb{R}), \\ (\psi_{1x},\psi_{2x}) \in L^2([0,t_1];H^1(\mathbb{R}))\big\}.$$

The following theorem involves the global existence and the asymptotic stability of (ϕ,Ψ) in $X(0,+\infty)$.

Theorem 1.1 (Stability). *There exists a constant C_1 only depending on the initial perturbations $||(\phi_0,\Psi_0)||_1$, such that the error Equations (1.12) has a unique global smooth solution (ϕ,Ψ) in $X(0,\infty)$ satisfying*

$$\sup_{0 \le t \le \infty} ||(\phi,\Psi)||_1^2 + \int_0^\infty (||\phi_x||^2 + ||\Psi_x||_1^2) d\tau \le C_1, \tag{1.13}$$

and $\lim_{t\to+\infty} ||(\phi,\Psi)(t,.)||_{L^\infty(\mathbb{R})} = 0.$

Remark 1.1. Here viscous contact waves can have arbitrarily large strength and perturbations. The Lagrange transform may not be applied directly to the higher dimension, since the lower spatial derivative still exists and it comes from the diffusion terms even if the compressible Navier-Stokes Equations (1.1) are transformed into the Lagrangian coordinates. Therefore, it is an interesting future work to study the stability for the two-dimensional compressible Navier-Stokes Equations (1.1) with the initial data (1.3).

The optimal time-decay rate is obtained in the following theorem.

Theorem 1.2 (Optimal decay rate). *Assume that E_0 is sufficiently small and F_0 is bounded. Then,*

$$||(\phi,\psi_1)||_\infty \le CE_0(1+t)^{-\frac{1}{2}}, \quad ||\psi_2||_\infty \le CF_0(1+t)^{-\frac{1}{2}},$$

where $E_0 = ||(\phi_0,\psi_{10})||_2 + ||(\phi_0,\psi_{10})||_{L^1}, F_0 = ||\psi_{20}||_2 + ||\psi_{20}||_{L^1}.$

There exists an optimal decay rate $t^{-\frac{1}{2}}$ for the error Equations $(1.12)_{1,2}$ with small perturbations ϕ_0,ψ_{10}. When the initial perturbations are not small, the problem is still open. For our concerned optimal decay rate, the small perturbations are only added to ϕ_0,ψ_{10}.

The rest of the paper is to prove Theorem 1.1 and Theorem 1.2. We present the following notations used in this paper. The notation $H^l(\Omega)$ $(l \ge 0, l \in \mathbb{Z})$ denotes the usual Sobolev space with the norm $||\cdot||_l$, and $L^2(\Omega) := H^0(\Omega)$ with the norm $||\cdot|| := ||\cdot||_0$. We write by C, C_1 a generic positive constant independent of T, but only C_1 depends on ϕ_0, Ψ_0.

2. Proof of Theroem 1.1

In this section, we give the proof of Theroem 1.1. The global existence is based on the local existence and a prior estimate for the solution to the perturbation Equations (1.12) in $X(0,t)$. The asymptotic behavior of (ϕ,Ψ) is proved by the one-dimensional Sobolev's inequality. The local existence is standard and thus omitted. In the following, we will give a prior estimate.

PROPOSITION 2.1 (A prior estimate). *Let $(\phi,\Psi)\in X(0,t_0)$, $\forall t_0\in(0,T]$, there exists a constant C_1 only depending on $||(\phi_0,\Psi_0)||_1$ such that*

$$\sup_{0\le t\le t_0}||(\phi,\Psi)||_1^2+\int_0^{t_0}(||\phi_x||^2+||\Psi_x||_1^2)d\tau\le C_1.$$

To get such a prior estimate, we first derive the L^2 energy estimates (cf., Lemma 2.1) and the first-order derivative estimates (cf., Lemma 2.2).

In Lemma 2.1, we give the L^2 energy estimates. In fact, we only need to obtain the energy estimate of ψ_2 by using the fact that ρ is bounded upper and lower, and the estimates of ϕ,ψ_1 in [15].

LEMMA 2.1. *There exists a constant C_1 only depending on $||\phi_0||_1$ and $||\Psi_0||$ such that*

$$\sup_{0\le t\le t_0}(||\phi||_1^2+||\Psi||^2)+\int_0^{t_0}(||\phi_x||^2+||\Psi_x||^2)d\tau\le C_1. \tag{2.1}$$

Proof. By Kanel [15], it holds that

$$\sup_{0\le t\le t_0}(||\phi||_1^2+||\psi_1||^2)+\int_0^{t_0}(||\phi_x||^2+||\psi_{1x}||^2)d\tau\le C(||\phi_0||_1^2+||\psi_{10}||^2), \tag{2.2}$$

and

$$C^{-1}\le v\le C,$$

for some uniform-in-time positive constant C. Thus, we only need to get the L^2 estimate for ψ_2.

We multiply $(1.12)_3$ by ψ_2,

$$(\frac{1}{2}\psi_2^2)_t+\big[\frac{\mu\psi_{2x}}{v}\psi_2+\mu(\frac{1}{v}-\frac{1}{\bar v})\bar u_{2x}\psi_2\big]_x+\mu\frac{\psi_{2x}^2}{v}=-\mu(\frac{1}{\bar v}-\frac{1}{v})\bar u_{2x}\psi_{2x}.$$

By integrating the above equation over $[0,t]\times\mathbb{R}$ and using Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\psi_2^2(t,x)dx+\mu\int_0^t\int_{\mathbb{R}}\frac{\psi_{2x}^2}{v}dxd\tau\\ =&\frac{1}{2}\int_{\mathbb{R}}\psi_{20}^2dx-\mu\int_0^t\int_{\mathbb{R}}(\frac{1}{v}-\frac{1}{\bar v})\bar u_{2x}\psi_{2x}dxd\tau\\ \le&\frac{1}{2}\int_{\mathbb{R}}\psi_{20}^2dx+\frac{\mu}{16}\int_0^t\int_{\mathbb{R}}\frac{\psi_{2x}^2}{v}dxd\tau+C\int_0^t\int_{\mathbb{R}}\bar u_{2x}^2\phi^2dxd\tau,\end{aligned} \tag{2.3}$$

where $\bar u_{2x}\sim e^{-\frac{\alpha|x|^2}{1+t}}/\sqrt{1+t}:=w_\alpha(t,x)$ for some positive constant α.

Inspired by the method in Huang, Li, and Matsumura [7], we can get the estimate for the third term of the right-hand side in (2.3). Let $f(t,x):=\int_{-\infty}^x\big(w_\alpha(t,y)\big)^2dy$, then

$$||f(.,t)||_\infty\le C_\alpha(1+t)^{-\frac{1}{2}},\quad ||f_t(.,t)||_\infty\le C_\alpha(1+t)^{-\frac{3}{2}}.$$

Multiplying $(1.12)_2$ by $(p(v)-p(\bar v))f$,

$$\frac{1}{2}(p(v)-p(\bar v))^2w^2-\big[\frac{1}{2}(p(v)-p(\bar v))^2f-(2\mu+\lambda)\frac{\psi_{1x}}{v}(p(v)-p(\bar v))f\big]_x$$

$$
\begin{aligned}
&=(\psi_1(p(v)-p(\bar v))f)_t-\psi_1(p(v)-p(\bar v))_t f-\psi_1(p(v)-p(\bar v))f_t\\
&\quad+(2\mu+\lambda)\frac{\psi_{1x}}{v}(p(v)-p(\bar v))_x f+(2\mu+\lambda)\frac{\psi_{1x}}{v}(p(v)-p(\bar v))w^2\\
&:=\sum_{i=1}^{5} I_i.
\end{aligned}\tag{2.4}
$$

By Hölder's inequality, it yields that

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} I_1 dxd\tau &= \int_{\mathbb{R}}\psi_1(p(v)-p(\bar v))f dx|_0^t\\
&\le C\big[||(\phi,\psi_1)(t,\cdot)||^2+||(\phi_0,\psi_{10})||^2\big].
\end{aligned}
$$

We integrate by part over I_2 and obtain

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} I_2 dxd\tau &= -\int_0^t\int_{\mathbb{R}} p'(v)\phi_t\psi_1 f dxd\tau\\
&= -\int_0^t\int_{\mathbb{R}} p'(v)\psi_{1x}\psi_1 f dxd\tau\\
&= \frac{1}{2}\int_0^t\int_{\mathbb{R}} p''(v)\phi_x\psi_1^2 f dxd\tau-\frac{1}{2}\int_0^t\int_{\mathbb{R}} |p'(v)|\psi_1^2 w^2 dxd\tau,
\end{aligned}
$$

the first part of which is bounded by

$$
\begin{aligned}
\frac{1}{2}\int_0^t\int_{\mathbb{R}} p''(v)\phi_x\psi_1^2 f dxd\tau &\le C\int_0^t ||f||_\infty||\phi_x||\cdot||\psi_1||_\infty||\psi_1||d\tau\\
&\le C\int_0^t (1+t)^{-\frac{1}{2}}||\phi_x||\cdot||\psi_1||^{\frac{3}{2}}||\psi_{1x}||^{\frac{1}{2}}d\tau\\
&\le \int_0^t ||\phi_x||^2 d\tau + C\int_0^t (1+t)^{-1}||\psi_1||^3||\psi_{1x}||d\tau\\
&\le C\int_0^t ||(\phi_x,\psi_{1x})||^2 d\tau + C\sup_{0\le\tau\le t}||\psi_1(\tau,\cdot)||^6,
\end{aligned}
$$

and the other part is a good term. By Hölder's inequality, we have

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} I_3 dxd\tau &\le C\int_0^t ||\psi_1||\cdot||\phi||\cdot||f_t||_\infty d\tau\\
&\le C\sup_{0\le\tau\le t}||\psi_1||\sup_{0\le\tau\le t}||\phi||\\
&\le C\sup_{0\le\tau\le t}\big[||\phi(\tau,\cdot)||^2+||\psi_1(\tau,\cdot)||^2\big].
\end{aligned}
$$

Similarly, I_4 and I_5 can be estimated as

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} I_4 dxd\tau &= (2\mu+\lambda)\int_0^t\int_{\mathbb{R}}\frac{\psi_{1x}}{v}p'(v)\phi_x f dxd\tau\\
&\le C\int_0^t ||\psi_{1x}||\cdot||\phi_x||\cdot||f||_\infty d\tau\\
&\le \int_0^t ||\psi_{1x}||^2 d\tau + C\int_0^t ||\phi_x||^2 d\tau.
\end{aligned}
$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}} I_5 dxd\tau \leq & C\int_0^t ||\psi_{1x}||\cdot||(p(v)-p(\bar{v}))w||\cdot||w||_\infty d\tau\\ \leq & \frac{1}{16}\int_0^t ||(p(v)-p(\bar{v}))w||^2 d\tau + C\int_0^t ||\psi_{1x}||^2 d\tau.\end{aligned}$$

Combining these estimates, (2.2) and (2.4) imply that

$$\int_0^t\int_{\mathbb{R}}(\phi^2+\psi_1^2)w^2 dxd\tau \leq C_1. \tag{2.5}$$

We combine (2.5) with (2.3) and then get the following estimates

$$\sup_{0\leq\tau\leq t} ||\psi_2||^2 + \int_0^t ||\psi_{2x}||^2 d\tau \leq C_1, \tag{2.6}$$

for some uniform-in-time positive constant C_1 only depending on $||\phi_0||_1$ and $||\Psi_0||$. Due to (2.2) and (2.6), Lemma 2.1 is proved. □

In the following, we will give the first-order derivative estimates. In fact, it suffices to derive the first-order derivative estimate of ψ_2.

LEMMA 2.2. *There exists a positive constant C_1 only depending on $||(\phi_0,\Psi_0)||_1$ such that*

$$\sup_{0\leq t\leq t_0} ||(\phi,\Psi)||_1^2 + \int_0^{t_0}(||\phi_x||^2+||\Psi_x||_1^2)d\tau \leq C_1.$$

Proof. By Kanel's result [15], we have

$$\sup_{0\leq t\leq t_0} ||(\phi,\psi_1)||_1^2 + \int_0^{t_0}(||\phi_x||^2+||\psi_{1x}||_1^2)dt \leq C_1, \tag{2.7}$$

and

$$C^{-1}\leq\rho\leq C,$$

for some uniform-in-time positive constant C_1 that only depends on $||(\phi_0,\psi_{10})||_1$. We only need to get the L^2 estimate for ψ_{2x}. Multiplying $(1.12)_3$ by $-\psi_{2xx}$,

$$(\frac{1}{2}\psi_{2x}^2)_t + (\psi_{2x}\psi_{2t})_x + \mu\frac{\psi_{2xx}^2}{v} = -\mu\psi_{2x}\psi_{2xx}(\frac{1}{v})_x + \mu\big[(\frac{1}{v}-\frac{1}{\bar{v}})\bar{u}_{2x}\big]_x\psi_{2xx}.$$

By Hölder's inequality and Young's inequality, it holds that

$$\begin{aligned}-\int_0^t\int_{\mathbb{R}}\mu\psi_{2x}\psi_{2xx}(\frac{1}{v})_x dxd\tau \leq & C\mu\int_0^t ||\psi_{2x}||_\infty \left\|\frac{\psi_{2xx}}{v^{\frac{1}{2}}}\right\|\cdot||\phi_x||d\tau\\ \leq & C\mu\int_0^t ||\psi_{2x}||^{\frac{1}{2}}\cdot\left\|\frac{\psi_{2xx}}{v^{\frac{1}{2}}}\right\|^{\frac{3}{2}}\cdot||\phi_x||d\tau\\ \leq & \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac{1}{2}}}\right\|^2 d\tau + C\sup_{0\leq\tau\leq t}||\phi_x||^4\int_0^t ||\psi_{2x}||^2 d\tau\\ \leq & \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac{1}{2}}}\right\|^2 d\tau + C,\end{aligned}$$

where the last inequality uses Lemma 2.1. Then we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\mu(\frac{1}{v}-\frac{1}{\bar v})\bar u_{2xx}\psi_{2xx}dxd\tau &\le \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|^2d\tau + C\int_0^t||\bar u_{2xx}\phi||^2d\tau\\ &\le \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|^2d\tau + C\int_0^t(1+t)^{-\frac32}d\tau\\ &\le \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|^2d\tau + C,\end{aligned}$$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\mu(\frac{1}{v})_x\bar u_{2x}\psi_{2xx}dxd\tau &\le C\mu\int_0^t||\bar u_{2x}||_\infty\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|\cdot||\phi_x||d\tau\\ &\le \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|^2d\tau + C\int_0^t||\phi_x||^2d\tau\\ &\le \frac{\mu}{16}\int_0^t\left\|\frac{\psi_{2xx}}{v^{\frac12}}\right\|^2d\tau + C.\end{aligned}$$

Thus,

$$\sup_{0\le\tau\le t}||\psi_{2x}||^2+\int_0^t||\psi_{2xx}||^2d\tau\le C_1, \tag{2.8}$$

for some uniform-in-time positive constant C_1 only depending on $||(\phi_0,\Psi_0)||_1$. The relations (2.7) and (2.8) imply that Lemma 2.2 is true. □

Combining Lemma 2.1 and Lemma 2.2, we can directly get Proposition 2.1.

Proof. **(Proof of Theorem 1.1.)** With the standard and omitted local existence, the first part of Theorem 1.1 is obtained. Next we turn to the remaining part of the Theorem 1.1. We claim that

$$\int_0^\infty\left\{||\phi_x,\Psi_x||^2+\left|\frac{d}{d\tau}||\phi_x,\Psi_x||\right|\right\}d\tau<+\infty.$$

Since the estimates (1.13) are established, in fact, it suffices that we check $\int_0^\infty\left|\frac{d}{d\tau}||\phi_x,\Psi_x||\right|d\tau<+\infty$. By Cauchy's inequality and the Equation $(1.12)_1$, we obtain

$$\begin{aligned}\int_0^\infty\left|\frac{d}{d\tau}||\phi_x||^2\right|d\tau &= 2\int_0^\infty\left|\int_{\mathbb{R}}\phi_x\phi_{\tau x}dx\right|d\tau\\ &= 2\int_0^\infty\left|\int_{\mathbb{R}}\phi_x\psi_{1xx}dx\right|d\tau\\ &\le \int_0^\infty||\phi_x||^2d\tau+\int_0^\infty||\psi_{1xx}||^2d\tau\\ &< +\infty.\end{aligned}$$

Similarly, we use $(1.12)_2$ to prove that

$$\begin{aligned}&\int_0^\infty\left|\frac{d}{d\tau}||\psi_{1x}||^2\right|d\tau = 2\int_0^\infty\left|\int_{\mathbb{R}}\psi_{1x}\psi_{1\tau x}dx\right|d\tau\\ &= 2\int_0^\infty\left|\int_{\mathbb{R}}((\psi_{1x}\psi_{1\tau})_x-\psi_{1xx}\psi_{1\tau})dx\right|d\tau\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{\infty}\left|\int_{\mathbb{R}}\psi_{1xx}((2\mu+\lambda)(\frac{\psi_{1x}}{v})_x-(p(v)-p(\bar{v}))_x)dx\right|d\tau\\
&\leq C\int_0^{\infty}||\psi_{1xx}||^2d\tau+C\int_0^{\infty}||\psi_{1x}||_{\infty}||\psi_{1xx}||\cdot||\phi_x||d\tau+C\int_0^{\infty}||\psi_{1xx}||\cdot||\phi_x||d\tau\\
&\leq C\int_0^{\infty}||\psi_{1xx}||^2d\tau+C\int_0^{\infty}||\psi_{1x}||^{\frac{1}{2}}\cdot||\psi_{1xx}||^{\frac{3}{2}}\cdot||\phi_x||d\tau+C\int_0^{\infty}||\phi_x||^2d\tau\\
&\leq C\int_0^{\infty}||\psi_{1xx}||^2d\tau+C\int_0^{\infty}||\phi_x||^2d\tau+C\sup_{0\leq\tau\leq\infty}||\phi_x||^4\int_0^{\infty}||\psi_{1x}||^2d\tau\\
&<+\infty.
\end{aligned}
$$

Finally, it holds from $(1.12)_3$ that

$$
\begin{aligned}
&\int_0^{\infty}\left|\frac{d}{d\tau}||\psi_{2x}||^2\right|d\tau=2\int_0^{\infty}\left|\int_{\mathbb{R}}\psi_{2x}\psi_{2\tau x}dx\right|d\tau\\
&=2\int_0^{\infty}\left|\int_{\mathbb{R}}((\psi_{2x}\psi_{2\tau})_x-\psi_{2xx}\psi_{2\tau})dx\right|d\tau\\
&=2\int_0^{\infty}\left|\int_{\mathbb{R}}\psi_{2xx}(\mu(\frac{\bar{u}_{2x}+\psi_{2x}}{v})_x-\mu\frac{\bar{u}_{2xx}}{\bar{v}})dx\right|d\tau\\
&\leq C\int_0^{\infty}||\psi_{2xx}||^2d\tau+C\int_0^{\infty}||\bar{u}_{2xx}||^2d\tau+C\int_0^{\infty}||\bar{u}_{2x}||_{\infty}||\psi_{2xx}||\cdot||\phi_x||d\tau\\
&\quad+C\int_0^{\infty}||\psi_{2x}||_{\infty}||\psi_{2xx}||\cdot||\phi_x||d\tau\\
&\leq C\int_0^{\infty}||\psi_{2xx}||^2d\tau+C\int_0^{\infty}(1+\tau)^{-\frac{3}{2}}d\tau+C\int_0^{\infty}||\phi_x||^2d\tau\\
&\quad+C\int_0^{\infty}||\psi_{2x}||^{\frac{1}{2}}\cdot||\psi_{2xx}||^{\frac{3}{2}}\cdot||\phi_x||d\tau\\
&\leq C\int_0^{\infty}||\psi_{2xx}||^2d\tau+C\int_0^{\infty}||\phi_x||^2d\tau+C\sup_{0\leq\tau\leq\infty}||\phi_x||^4\int_0^{\infty}||\psi_{2x}||^2d\tau+C\\
&<+\infty.
\end{aligned}
$$

Therefore, $\int_0^{\infty}\left|\frac{d}{d\tau}||\phi_x,\Psi_x||\right|d\tau<+\infty$ is checked. By the one-dimensional Sobolev's inequality, it implies that

$$
\lim_{t\to+\infty}||(\phi,\Psi)(t,.)||_{L^{\infty}(\mathbb{R})}=0.
$$

□

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The proof is based on the linear spectral theory discussed in Matsumura [27, 28]. First we introduce the pointwise difference between the fundamental solution to the linearlized hyperbolic-parabolic equations and to the linearlized parabolic equations in Zeng [34], which is crucial in our analysis of decay rate to viscous contact wave. The solution to the linearlized parabolic equations is equivalent to heat kernel in the usual Sobolev's norm. Combining local linearization with suitably small initial data E_0 defined in (3.6), we obtain Lemma 3.2 and Lemma 3.3.

For the convenience of readers, we review the results obtained in Zeng [34]. The system $(1.12)_{1,2}$ around the constant state $(\bar{v},\bar{u}_1)$ is linearized as

$$\begin{cases} \phi_t - \psi_{1x} = 0, \quad t>0, \quad x\in\mathbb{R}, \\ \psi_{1t} + p'\phi_x = \mu_1\psi_{1xx}, \\ \Phi(0,x) = \Phi_0(x), \end{cases} \tag{3.1}$$

with $p' = p'(\bar{v}) < 0$, $\mu_1 = \frac{2\mu+\lambda}{\bar{v}} > 0$ and $\Phi = (\phi,\psi_1)$, $\Phi_0 = (\phi_0,\psi_{10})$. We denote the Fourier transform of $v(x)$ by $v^\wedge(\xi)$. Then the linearized system (3.1) is transformed into

$$\begin{cases} \phi_t^\wedge - i\xi\psi_1^\wedge = 0, \\ \psi_{1t}^\wedge + p' i\xi\phi^\wedge = \mu_1(i\xi)^2\psi_1^\wedge, \\ \Phi(0,\xi)^\wedge = \Phi_0^\wedge(\xi). \end{cases}$$

The solution is expressed as

$$\Phi^\wedge(t,\xi) = [e^{\lambda_-(\xi)t}P_-(\xi) + e^{\lambda_+(\xi)t}P_+(\xi)]\Phi_0^\wedge(\xi),$$

where

$$\lambda_\pm(\xi) = -\frac{\mu_1}{2}\xi^2 \mp i\xi\sqrt{-p' - (\frac{\mu_1\xi}{2})^2},$$
$$P_\pm(\xi) = \begin{bmatrix} \frac{1}{2} \pm \frac{\frac{i\mu_1\xi}{2}}{2\sqrt{-p'-(\frac{\mu_1\xi}{2})^2}} & \mp\frac{1}{2\sqrt{-p'-(\frac{\mu_1\xi}{2})^2}} \\ \pm\frac{p'}{2\sqrt{-p'-(\frac{\mu_1\xi}{2})^2}} & \frac{1}{2} \mp \frac{\frac{i\mu_1\xi}{2}}{2\sqrt{-p'-(\frac{\mu_1\xi}{2})^2}} \end{bmatrix}.$$

By taking the inverse Fourier transform, it holds that

$$\begin{aligned} \Phi(t,x) &= F^{-1}\{[e^{\lambda_-(\xi)t}P_-(\xi) + e^{\lambda_+(\xi)t}P_+(\xi)]\Phi_0^\wedge(\xi)\} \\ &= \frac{1}{\sqrt{2\pi}}G(t,x)*\Phi_0(x) + e^{\frac{p'}{\mu_1}t}A\Phi_0(x), \end{aligned}$$

where

$$G(t,x) = F^{-1}\{e^{\lambda_-(\xi)t}P_-(\xi) + e^{\lambda_+(\xi)t}P_+(\xi) - e^{\frac{p'}{\mu_1}t}A\}, \tag{3.2}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and "$*$" denotes the convolution with respect to x.

In order to give pointwise estimate for $G(t,x)$, we need to consider the linearlized parabolic system

$$\begin{cases} \tilde{\phi}_t - \tilde{\psi}_{1x} = \frac{\mu_1}{2}\tilde{\psi}_{1xx}, \\ \tilde{\psi}_{1t} + p'\tilde{\phi}_x = \frac{\mu_1}{2}\tilde{\psi}_{1xx}, \\ \tilde{\Phi}(0,x) = \tilde{\Phi}_0(x). \end{cases} \tag{3.3}$$

We take the Fourier transform to (3.3) and it yields that

$$\begin{cases} \tilde{\phi}_t^{\wedge} - i\xi\tilde{\psi}^{\wedge} = \frac{\mu_1}{2}(i\xi)^2\tilde{\psi}^{\wedge}, \\ \tilde{\psi}_{1t}^{\wedge} + p' i\xi\tilde{\phi}^{\wedge} = \frac{\mu_1}{2}(i\xi)^2\tilde{\psi}^{\wedge}, \\ \tilde{\Phi}(0,\xi)^{\wedge} = \tilde{\Phi}_0^{\wedge}(\xi). \end{cases}$$

The solution is expressed as

$$\tilde{\Phi}^{\wedge}(t,\xi) = [e^{\tilde{\lambda}_-(\xi)t}\tilde{P}_- + e^{\tilde{\lambda}_+(\xi)t}\tilde{P}_+]\tilde{\Phi}_0^{\wedge}(\xi),$$

where

$$\tilde{\lambda}_{\pm}(\xi) = -\frac{\mu_1}{2}\xi^2 \mp i\xi\sqrt{-p'},$$
$$\tilde{P}_{\pm} = \begin{bmatrix} \frac{1}{2} & \mp\frac{1}{2\sqrt{-p'}} \\ \mp\frac{\sqrt{-p'}}{2} & \frac{1}{2} \end{bmatrix}.$$

By taking the inverse Fourier transform, we have that

$$\tilde{\Phi}(t,x) = \frac{1}{\sqrt{2\pi}}\tilde{G}(t,x) * \tilde{\Phi}_0(x),$$

where

$$\begin{aligned} \tilde{G}(t,x) &= F^{-1}\{e^{\tilde{\lambda}_-(\xi)t}\tilde{P}_- + e^{\tilde{\lambda}_+(\xi)t}\tilde{P}_+\} \\ &= \frac{1}{\sqrt{\mu_1 t}}e^{-\frac{(x+t\sqrt{-p'})^2}{2\mu_1 t}}\tilde{P}_- + \frac{1}{\sqrt{\mu_1 t}}e^{-\frac{(x-t\sqrt{-p'})^2}{2\mu_1 t}}\tilde{P}_+. \end{aligned} \tag{3.4}$$

The following lemma describes the pointwise difference between $G(t,x)$ and $\tilde{G}(t,x)$.

LEMMA 3.1 (Lemma 2.8, [34]). *Let $G(t,x)$ and $\tilde{G}(t,x)$ be defined by (3.2) and (3.4). Then*

$$|G(t,x) - \tilde{G}(t,x)| \le C(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}(e^{-\frac{(x+t\sqrt{-p'})^2}{Ct}} + e^{-\frac{(x-t\sqrt{-p'})^2}{Ct}}).$$

Due to (3.4) and Lemma 3.1, we obtain the estimates

$$\begin{aligned} &||\partial_x^l G(t,.)|| \le C(1+t)^{-\frac{1+2l}{4}}, \quad 0 \le l \le 3, l \in \mathbb{Z}, \\ &||\partial_x^2 G(t,.)||_{L^1} \le C(1+t)^{-1}. \end{aligned} \tag{3.5}$$

Set

$$E_0 = ||(\phi_0,\psi_{10})||_2 + ||(\phi_0,\psi_{10})||_{L^1},\ F_0 = ||\psi_{20}||_2 + ||\psi_{20}||_{L^1}. \tag{3.6}$$

Now, we could exploit (3.5) and $(1.12)_{1,2}$ to derive the decay rate to (ϕ,ψ_1).

LEMMA 3.2. *Assume that E_0 is suitably small. Then*

$$||(\phi,\psi_1)|| \le CE_0(1+t)^{-\frac{1}{4}},$$
$$||(\phi_x,\psi_{1x})||_1 \le CE_0(1+t)^{-\frac{3}{4}}.$$

Proof. We consider the initial value problem to the system (1.12)$_{1,2}$

$$\begin{cases} \phi_t - \psi_{1x} = 0, \\ \psi_{1t} + p'(\bar{v})\phi_x - \frac{2\mu+\lambda}{\bar{v}}\psi_{1xx} = f_1(t,x), \\ \Phi(x,0) = \Phi_0(x), \end{cases} \tag{3.7}$$

where

$$f_1(t,x) = [(-p(v) + p(\bar{v}) + p'(\bar{v})\phi) + (2\mu+\lambda)\psi_{1x}(\frac{1}{v} - \frac{1}{\bar{v}})]_x := g_{1x}(t,x).$$

We utilize $G(t,x)$ defined by (3.2) to obtain the integral equation

$$\Phi(t,x) = \frac{1}{\sqrt{2\pi}} G(t,x) * \Phi_0(x) + e^{\frac{p'}{\mu_1}t} A\Phi_0(x) + \frac{1}{\sqrt{2\pi}} \int_0^t G(t-s,x) * (0, f_1(s,x))^\top ds. \tag{3.8}$$

Set

$$M_1(t) = \sup_{0\le\tau\le t} [(1+\tau)^{\frac{1}{4}} ||\Phi|| + (1+\tau)^{\frac{3}{4}} ||\Phi_x||_1].$$

We consider L^2 estimate for $\Phi(t,x)$. By Minkowshi's inequality and Young's inequality for the convolution of two functions (cf., [1]), the first two terms on the right-hand side of (3.8) are bounded by

$$\left\| \frac{1}{\sqrt{2\pi}} G(t,x) * \Phi_0(x) + e^{\frac{p'}{\mu_1}t} A\Phi_0(x) \right\| \le C||G(t,.)|| \cdot ||\Phi_0||_{L^1} + e^{\frac{p'}{\mu_1}t} ||\Phi_0|| \le CE_0(1+t)^{-\frac{1}{4}}.$$

The last term can be divided into two terms, one of which can be majored by

$$\begin{aligned} &\left\| \frac{1}{\sqrt{2\pi}} \int_0^{\frac{t}{2}} G(t-s,x) * (0, f_1(s,x))^\top ds \right\| \\ &= \left\| \frac{1}{\sqrt{2\pi}} \int_0^{\frac{t}{2}} G_x(t-s,x) * (0, g_1(s,x))^\top ds \right\| \\ &\le C \int_0^{\frac{t}{2}} ||G_x(t-s,.)|| \cdot ||g_1(s,.)||_{L^1} ds \\ &\le C \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{3}{4}} (||\phi||^2 + ||\psi_{1x}|| \cdot ||\phi||) ds \\ &\le CM_1(t)^2 \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{3}{4}} (1+s)^{-\frac{1}{2}} dt \\ &\le CM_1(t)^2 (1+t)^{-\frac{1}{4}}, \end{aligned}$$

and the other term is estimated by

$$\begin{aligned} &\left\| \frac{1}{\sqrt{2\pi}} \int_{\frac{t}{2}}^t G(t-s,x) * (0, f_1(s,x))^\top ds \right\| \\ &\le C \int_{\frac{t}{2}}^t ||G(t-s,.)|| \cdot ||f_1(s,.)||_{L^1} ds \end{aligned}$$

$$\leq C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{1}{4}}||\Phi||_1\cdot||\Phi_x||_1ds$$
$$\leq CM_1(t)^2\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{1}{4}}(1+s)^{-1}ds$$
$$\leq CM_1(t)^2(1+t)^{-\frac{1}{4}},$$

where we use the definition of $M_1(t)$. Thus,

$$||\Phi||\leq CE_0(1+t)^{-\frac{1}{4}}+CM_1(t)^2(1+t)^{-\frac{1}{4}}. \tag{3.9}$$

We differentiate the integral Equation (3.8) with respect to x,

$$\Phi_x(t,x)=\frac{1}{\sqrt{2\pi}}G_x(t,x)*\Phi_0(x)+e^{\frac{p'}{\mu_1}t}A\Phi_{0x}+\frac{1}{\sqrt{2\pi}}\int_0^t G_x(t-s,x)*(0,f_1(s,x))^\top ds. \tag{3.10}$$

Similarly,

$$\left\|\frac{1}{\sqrt{2\pi}}G_x(t,x)*\Phi_0(x)+e^{\frac{p'}{\mu_1}t}A\Phi_{0x}\right\|\leq C||G_x(t,.)||\cdot||\Phi_0||_{L^1}+e^{\frac{p'}{\mu_1}t}||\Phi_{0x}||\leq CE_0(1+t)^{-\frac{3}{4}}.$$

The last term is controlled by

$$\begin{aligned}\left\|\frac{1}{\sqrt{2\pi}}\int_0^{\frac{t}{2}}G_x(t-s,x)*(0,f_1(s,x))^\top ds\right\|&=\left\|\frac{1}{\sqrt{2\pi}}\int_0^{\frac{t}{2}}G_{xx}(t-s,x)*(0,g_1(s,x))^\top ds\right\|\\&\leq C\int_0^{\frac{t}{2}}||G_{xx}(t-s,.)||\cdot||g_1(s,.)||_{L^1}ds\\&\leq C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{5}{4}}(||\phi||^2+||\psi_{1x}||\cdot||\phi||)ds\\&\leq CM_1(t)^2\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{1}{2}}ds\\&\leq CM_1(t)^2(1+t)^{-\frac{3}{4}},\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{1}{\sqrt{2\pi}}\int_{\frac{t}{2}}^{t}G_x(t-s,x)*(0,f_1(s,x))^\top ds\right\|&\leq C\int_{\frac{t}{2}}^{t}||G_x(t-s,.)||\cdot||f_1(s,.)||_{L^1}ds\\&\leq C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{3}{4}}||\Phi||_1\cdot||\Phi_x||_1ds\\&\leq CM_1(t)^2\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{3}{4}}(1+s)^{-1}ds\\&\leq CM_1(t)^2(1+t)^{-\frac{3}{4}}.\end{aligned}$$

Thus,

$$||\Phi_x||\leq CE_0(1+t)^{-\frac{3}{4}}+CM_1(t)^2(1+t)^{-\frac{3}{4}}. \tag{3.11}$$

Finally, we differentiate the integral Equation (3.10) with respect to x

$$\Phi_{xx}(t,x)=\tfrac{1}{\sqrt{2\pi}}G_{xx}(t,x)*\Phi_0(x)+e^{\frac{p'}{\mu_1}t}A\Phi_{0xx}+\tfrac{1}{\sqrt{2\pi}}\int_0^t G_{xx}(t-s,x)*(0,f_1(s,x))^\top ds, \tag{3.12}$$

to get the L^2 estimate of Φ_{xx}. Similarly,

$$\left\|\tfrac{1}{\sqrt{2\pi}}G_{xx}(t,x)*\Phi_0+e^{\frac{p'}{\mu_1}t}A\Phi_{0xx}\right\|\le C||G_{xx}(t,.)||\cdot||\Phi_0||_{L^1}+e^{\frac{p'}{\mu_1}t}||\Phi_{0xx}||\le CE_0(1+t)^{-\frac{5}{4}}.$$

The last term is estimated by

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{2\pi}}\int_0^{\frac{t}{2}}G_{xx}(t-s,x)*(0,f_1(s,x))^\top ds\right\|=\left\|\frac{1}{\sqrt{2\pi}}\int_0^{\frac{t}{2}}G_{xxx}(t-s,x)*(0,g_1(s,x))^\top ds\right\|\\
&\le C\int_0^{\frac{t}{2}}||G_{xxx}(t-s,.)||\cdot||g_1(s,.)||_{L^1}ds\\
&\le C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{7}{4}}(||\phi||^2+||\psi_{1x}||\cdot||\phi||)ds\\
&\le CM_1(t)^2\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{1}{2}}ds\\
&\le CM_1(t)^2(1+t)^{-\frac{5}{4}},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{2\pi}}\int_{\frac{t}{2}}^{t}G_{xx}(t-s,x)*(0,f_1(s,x))^\top ds\right\|\le C\int_{\frac{t}{2}}^{t}||G_{xx}(t-s,.)||_{L^1}\cdot||f_1(s,.)||ds\\
&\le C\int_{\frac{t}{2}}^{t}||G_{xx}(t-s,.)||_{L^1}\cdot(||\phi||_\infty\cdot||\phi_x||+||\psi_{1x}||\cdot||\phi_x||_\infty+||\psi_{1xx}||\cdot||\phi||_\infty)ds\\
&\le C\int_{\frac{t}{2}}^{t}(1+t-s)^{-1}(||\phi||^{\frac{1}{2}}\cdot||\phi_x||^{\frac{3}{2}}+||\psi_{1x}||\cdot||\phi_x||^{\frac{1}{2}}||\phi_{xx}||^{\frac{1}{2}}+||\phi||^{\frac{1}{2}}||\phi_x||^{\frac{1}{2}}||\psi_{1xx}||)ds\\
&\le CM_1(t)^2\int_{\frac{t}{2}}^{t}(1+t-s)^{-1}(1+s)^{-\frac{5}{4}}ds\\
&\le CM_1(t)^2(1+t)^{-\frac{5}{4}}ln(1+t).
\end{aligned}$$

Thus,

$$||\Phi_{xx}||\le CE_0(1+t)^{-\frac{5}{4}}+CM_1(t)^2(1+t)^{-\frac{5}{4}}ln(1+t). \tag{3.13}$$

Combining these estimates (3.9), (3.11) and (3.13), we obtain the inequality $M_1(t)\le CE_0+CM_1(t)^2$, from which follows the desired estimate $M_1(t)\le CE_0$ if E_0 is sufficiently small. Therefore, this lemma is proved. □

Finally, we exploit $(1.12)_3$ to obtain the decay estimate of ψ_2.

LEMMA 3.3. *Assume that E_0 is suitably small and F_0 is bounded. Then,*

$$||\partial_x^l\psi_2||\le CF_0(1+t)^{-\frac{1+2l}{4}},\quad 0\le l\le 2, l\in\mathbb{Z}.$$

Proof. We rewrite (1.12$_3$) as

$$\begin{cases} \psi_{2t}-\frac{\mu}{\bar v}\psi_{2xx}=\mu\big[u_{2x}(\frac{1}{v}-\frac{1}{\bar v})\big]_x:=f_2(x,t)=g_{2x}(x,t),\\ \psi_{20}=u_{20}-\bar u_2(1,x). \end{cases}$$

The solution can be expressed as

$$\begin{aligned}\psi_2(t,x)&=H(t,x)*\psi_{20}+\mu\int_0^t H(t-s,x)*\big[u_{2x}(\frac{1}{v}-\frac{1}{\bar v})\big]_x ds,\\ &=H(t,x)*\psi_{20}+\int_0^{\frac{t}{2}}H_x(t-s,x)*g_2(s,x)ds+\int_{\frac{t}{2}}^t H(t-s,x)*f_2(s,x)ds,\end{aligned}$$

where $H(t,x)$ is the fundamental solution to heat equation $\psi_{2t}=\frac{\mu}{\bar v}\psi_{2xx}$ satisfying

$$||\partial_x^l H(t,x)||\le C(1+t)^{-\frac{1+2l}{4}},\quad 0\le l\le 2, l\in\mathbb{Z}.$$

Set

$$M_2(t)=\sup_{0\le\tau\le t}[(1+\tau)^{\frac14}||\psi_2||+(1+\tau)^{\frac34}||\psi_{2x}||+(1+\tau)^{\frac54}||\psi_{2xx}||].$$

Thus, by Hölder's inequality and the definition of $M_2(t)$, we get

$$\begin{aligned}||f_2||_{L^1}&\le C(||\bar u_{2x}||+||\psi_{2x}||)\cdot||\phi_x||+C(||\bar u_{2xx}||+||\psi_{2xx}||)||\phi||\\ &\le CE_0(1+t)^{-1}+CE_0M_2(t)(1+t)^{-\frac32},\\ ||g_2||_{L^1}&\le C(||\bar u_{2x}||+||\psi_{2x}||)||\phi||\le CE_0(1+t)^{-\frac12}+CE_0M_2(t)(1+t)^{-1}.\end{aligned}$$

By Lemma 3.2, we can get the following estimate for $||\psi_2||$

$$\begin{aligned}||\psi_2||&\le||H(t)||\cdot||\psi_{20}||_{L^1}+\int_0^{\frac t2}||H_x(t-s)||\cdot||g_2(s)||_{L^1}ds+\int_{\frac t2}^t||H(t-s)||\cdot||f_2(s)||ds\\ &\le CF_0(1+t)^{-\frac14}+CE_0\int_0^{\frac t2}(1+t-s)^{-\frac34}(1+s)^{-\frac12}ds\\ &\quad+CE_0M_2(t)\int_0^{\frac t2}(1+t-s)^{-\frac34}(1+s)^{-1}ds+CE_0\int_{\frac t2}^t(1+t-s)^{-\frac14}(1+s)^{-1}ds\\ &\quad+CE_0M_2(t)\int_{\frac t2}^t(1+t-s)^{-\frac14}(1+s)^{-\frac32}ds\\ &\le CF_0(1+t)^{-\frac14}+CE_0(1+t)^{-\frac14}+CE_0M_2(t)(1+t)^{-\frac34}ln(1+t).\end{aligned}\tag{3.14}$$

$||\psi_{2x}||$ can be bounded by

$$\begin{aligned}||\psi_{2x}||&\le||H_x(t-s)||\cdot||\psi_{20}||_{L^1}+\int_0^{\frac t2}||H_{xx}(t-s)||\cdot||g_2||_{L^1}ds+\int_{\frac t2}^t||H_x(t-s)||\cdot||f_2||_{L^1}ds\\ &\le CF_0(1+t)^{-\frac34}+CE_0\int_0^{\frac t2}(1+t-s)^{-\frac54}(1+s)^{-\frac12}ds\\ &\quad+CE_0M_2(t)\int_0^{\frac t2}(1+t-s)^{-\frac54}(1+s)^{-1}ds+CE_0\int_{\frac t2}^t(1+t-s)^{-\frac34}(1+s)^{-1}ds\end{aligned}$$

$$+CE_0M_2(t)\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{2}}ds$$
$$\leq CF_0(1+t)^{-\frac{3}{4}}+CE_0(1+t)^{-\frac{3}{4}}+CE_0M_2(t)(1+t)^{-\frac{5}{4}}ln(1+t). \tag{3.15}$$

Finally, $||\psi_{2xx}||$ is majored by

$$||\psi_{2xx}||\leq ||H_{xx}(t-s)||\cdot||\psi_{20}||_{L^1}+\int_0^{\frac{t}{2}}||H_{xxx}(t-s)||\cdot||g_2(s)||_{L^1}ds$$
$$+\int_0^{\frac{t}{2}}||H_{xx}(t-s)||\cdot||f_2||_{L^1}ds$$
$$\leq CF_0(1+t)^{-\frac{5}{4}}+CE_0\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{1}{2}}ds$$
$$+CE_0M_2(t)\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{7}{4}}(1+s)^{-1}ds+CE_0\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{5}{4}}(1+s)^{-1}ds$$
$$+CE_0M_2(t)\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{2}}ds$$
$$\leq CF_0(1+t)^{-\frac{5}{4}}+CE_0(1+t)^{-\frac{5}{4}}+CE_0M_2(t)(1+t)^{-\frac{7}{4}}ln(1+t). \tag{3.16}$$

Therefore, combining estimates (3.14), (3.15) and (3.16), we obtain the inequality

$$M_2(t)\leq CF_0+CE_0+CE_0M_2(t)(1+t)^{-\frac{1}{2}}ln(1+t),$$

from which follows the desired estimate $M(t)\leq CF_0$ if E_0 is sufficiently small. Therefore, this lemma is proved. □

Proof. **(Proof of Theorem 1.2.)** From one-dimensional Sobolev's inequality and Lemma 3.2, we can get

$$||(\phi,\psi_1)||_\infty\leq CE_0(1+t)^{-\frac{1}{2}},$$

if E_0 is sufficiently small. By Lemma 3.3, we have that

$$||\psi_2||_\infty\leq CF_0(1+t)^{-\frac{1}{2}},$$

Thus, we finish the proof. □

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