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LINEARIZED STABILITY OF PLANAR RAREFACTION WAVE FOR 3D GAS DYNAMICS IN THERMAL NONEQUILIBRIUM*

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Abstract. For the three-dimensional gas flow in vibrational nonequilibrium, the linearized stability of the planar rarefaction waves is obtained in this paper in terms of the rarefaction wave strength is small enough. The main feature of the problem is that the L^2-norm of the perturbations may grow in time.

Keywords. Thermal nonequilibrium; Weak dissipation.

AMS subject classifications. 35B35; 35B40.

1. Introduction

Gas dynamics in vibrational nonequilibrium in 3D is governed by the following equations in Eulerian coordinates

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ (\rho E)_t + \operatorname{div}(\rho E u + p u) = 0, \\ (\rho q)_t + \operatorname{div}(\rho q u) = \rho \frac{Q-q}{\tau}, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2, u_3)^t$, p, E, q, Q and τ are the gas density, velocity, pressure, specific total energy, specific vibrational energy, local equilibrium value of specific vibrational energy, and local relaxation time, respectively. All these are functions of the space variable $x \in \Omega$ and time variable t. $(1.1)_4$ is used to describe how the non-equilibrium vibrational mode relaxes to its local equilibrium value. $\tau > 0$ is a constant time scale. The symbol $\otimes$ denotes the Kronecker tensor product. In this paper, an infinite long flat nozzle domain $\Omega := \mathbb{R} \times \mathbb{T}^2$ is concerned with $\mathbb{R}$ being a real line and $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ being a two-dimensional unit flat torus. It should be noted that

$$E = e + \frac{1}{2}|u|^2 = e_1 + q + \frac{1}{2}\sum_{i=1}^{3} u_i^2,$$

consists of internal energy e and the kinetic energy $\frac{1}{2}|u|^2 = \frac{1}{2}\sum_{i=1}^{3} u_i^2$. The internal energy e will be divided into two parts: the total of translational energy and rotational energy e_1, and vibrational nonequilibrium energy q. For the simplicity of presentation, the thermal dynamical variables are supposed to satisfy the following relations [34]:

$$\begin{cases} v = \frac{1}{\rho},\ e_1 = \frac{\alpha}{2} p v = \frac{\alpha R}{2} T_1, \\ Q = \frac{\alpha_f}{2} p v = \frac{\alpha_f R}{2} T_1 = w(T_1), \\ s_1 = R(\ln v + \frac{\alpha}{2} \ln e_1),\ s_2 = \frac{\alpha_f R}{2} \ln q, \\ q = \frac{\alpha_f R}{2} T_2 = w(T_2), \\ \chi = Q - q, \end{cases} \tag{1.2}$$

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where α, α_f are positive constants, denoting the degrees of the freedom adjusting instantaneously and taking longer to relax, respectively. $R>0$ is a positive constant and we take R = 1 for convenience. v is called the specific volume for gas. T_1 and T_2 are the temperature for translational and rotational mode, and vibrational mode, respectively. s is the total entropy

$$s=s_1+s_2,$$

where s_1 is the entropy for translational and rotational mode, while s_2 is the non-equilibrium vibrational entropy. For vibrational non-equilibrium mode, the total entropy s could be expressed as

$$s=\frac{\alpha}{2}\ln\Big(\frac{\alpha}{2}pv\Big)+\ln v+\frac{\alpha_f}{2}\ln\Big(\frac{\alpha_f}{2}pv-\chi\Big) \tag{1.3}$$

from (1.2). Because the linearized asymptotic behavior of the planar rarefaction wave to the system (1.1) will be concerned, we consider the following far-field condition on the x_1-direction

$$(\rho,u,s)(x,t)\to(\rho_\pm,u_\pm,s_\pm),\text{ as } x_1\to\pm\infty,\, t>0, \tag{1.4}$$

where $u_\pm=(u_{1\pm},0,0)^t$, $\rho_\pm>0$, $u_{1\pm}$ and $s_+=s_-$ are constant states. And the periodic boundary conditions are imposed on $(x_2,x_3)\in\mathbb{T}^2$ for the solution $(\rho,u,q,s)(x,t)$. The two end states $(\rho_\pm,u_\pm,s_\pm)$ are connected by the rarefaction wave solution to the Riemann problem of the corresponding 1D Euler system:

$$\begin{cases}\rho_t+(\rho u_1)_{x_1}=0,\\ (\rho u_1)_t+(\rho u_1^2+p)_{x_1}=0,\\ (\rho E)_t+(\rho E u_1+pu_1)_{x_1}=0,\end{cases} \tag{1.5}$$

and $(1.5)_3$ is equivalent to

$$s_t+u_1 s_{x_1}=0,$$

with the Riemann initial data

$$(\rho,u_1,s)(x_1,0)=(\rho_0^r,u_{10}^r,s_0^r)(x_1,0)=\begin{cases}(\rho_-,u_{1-},s_-),\ x_1<0,\\ (\rho_+,u_{1+},s_+),\ x_1>0.\end{cases} \tag{1.6}$$

Now we would like to introduce the equilibrium state. If τ is taken to be short enough, then the solution to the vibrational nonequilibrium system (1.1) should well approximate the corresponding equilibrium system: the three-dimensional compressible Euler equations

$$\begin{cases}\rho_t+\mathrm{div}(\rho u)=0,\\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)+\nabla p=0,\\ (\rho E)_t+\mathrm{div}(\rho E u+pu)=0,\end{cases} \tag{1.7}$$

which could be obtained by letting $T_1=T_2$ or Q taking the value of q, where the third equation could be rewritten as

$$s_t+u\cdot\nabla s=0.$$

For further discussions on the relation between equilibrium and nonequilibrium flow, the readers are referred to [32] and references therein. The Riemann initial data for (1.7) is given by

$$(\rho,u,s)(x,0)=(\rho_0^r,u_0^r,s_0^r)(x,0)=\begin{cases}(\rho_-,u_-,s_-),\ x_1<0,\\(\rho_+,u_+,s_+),\ x_1>0.\end{cases}\tag{1.8}$$

It is expected that the large-time behavior of solution for system (1.1)-(1.4) is very closed to Riemann problem of the 3D compressible Euler equations. In this case, the equilibrium pressure p is the function of v and the equilibrium entropy s, defined as

$$p=p(v,s)=Kv^{-\gamma}e^{(\gamma-1)s}\tag{1.9}$$

where

$$\gamma=\frac{\alpha+\alpha_f+2}{\alpha+\alpha_f},\ K=\left(\left(\frac{\alpha}{2}\right)^{-\frac{\alpha}{2}}\left(\frac{\alpha_f}{2}\right)^{-\frac{\alpha_f}{2}}\right)^{(\gamma-1)}.$$

The inviscid compressible Euler system (1.5) or (1.7) is a typical example of the system of hyperbolic conservation laws, which is used to describe an ideal fluid model wherein the dissipative effects have been neglected. Three basic wave patterns to the system of hyperbolic conservation laws are shock waves, rarefaction waves, and contact discontinuity. It is expected that the large-time behavior of the solution to system (1.1)-(1.4) is governed by the solutions of the Riemann problem (1.7)-(1.8), which include planar shock wave, planar rarefaction wave and contact discontinuity in general. Notice that system (1.1) is a hyperbolic system with relaxation, which may produce some dissipative effects. So it is interesting and important for us to study the time-asymptotic stability of three basic planar wave patterns to this system in three-dimensional space. In this paper, we are interested in the nonlinear stability of planar rarefaction wave to the system (1.1) in an infinite long flat nozzle domain $\Omega=\mathbb{R}\times\mathbb{T}^2$.

It is well-known that the essential differences between the 1D Riemann problem (1.5)-(1.6) and the three-dimensional Riemann problem (1.7)-(1.8) exist even when the components u_2 and u_3 are continuous on both sides of $x_1=0$ as in (1.8). See [3,4,6,28] and references therein for details.

As mentioned before, system (1.1) is a hyperbolic system with relaxation, which may induce certain dissipative effects. Deep investigations have been achieved on the global regularity and long-time behavior for some inhomogeneous quasilinear hyperbolic systems, mainly due to the dissipative effects induced by the inhomogeneous terms through the strong coupling with the flux functions. A typical version for such a coupling is the Shizuta-Kawashima condition, [29]. However, systems of gas dynamics in thermal nonequilibrium studied in this paper do not satisfy those conditions of strong coupling, particularly, the Shizuta-Kawashima condition [29], as pointed out in [42, 44]. The right-hand side of $(3.1)_4$ implies that

$$s_t+u\cdot\nabla s=\frac{\chi^2}{\tau pvq}>0.$$

This means that, the dissipation of the relaxation is too weak to have an effect on all variables for system (1.1), the relaxation term does not have, on all the equilibrium characteristic directions, a positive projection on the linear level. Therefore, compared with many dissipative hyperbolic systems, the dissipation of (1.1) induced by the relaxation is extremely weak, too weak to dominate the hyperbolicity. This makes it a challenging task to investigate the global regularity, long-time and small-relaxation behavior of solutions, which is important to understand the physical process from thermal non-equilibrium to equilibrium. In this direction, for (1.1), the pointwise estimates for the Green's functions linearized around a constant state are given in [44] , and the global existence of the smooth irrotational flow was proved in [11]. For the thermal non-equilibrium gas flow in one-dimensional space, the global existence of smooth solutions was proved in [2] for an initial boundary value problem. The global existence and the pointwise behavior of smooth solutions are obtained in [42] for the initial value problem of (1.1), while the pointwise behavior of smooth solutions is obtained too for gas flows with several thermal nonequilibrium modes in [43].

It should be noted that all results mentioned above concerning with system (1.1) concentrate on the global well-posedness for small perturbations around constant equilibrium states. As one of most important physical models of hyperbolic conservation laws with relexation, it is important to understand the nonlinear wave propagations such as shock waves and rarefaction waves, the most important nonlinear waves in gas dynamics [7]. In this respect, the nonlinear asymptotic stability of shock profile, a traveling wave solution and the linearized asymptotic stability of rarefaction waves obtained with convergence rate, for (1.1) in one dimension have been proved in [23] and [24], respectively. Since rarefaction wave is one elementary wave of gas dynamics, the topic concerning the nonlinear stability of rarefaction wave has been one of the fundamental and important issues in fluid mechanics. For instance, in one-dimensional case, the extensive results on the nonlinear asymptotic stability of rarefaction waves for compressible Navier-Stokes equations for large time or small viscosity can be found in [8, 10, 12–14, 19, 25–27, 38, 39] and general viscous conservation laws [21, 31, 36, 37, 41]. The related results for Boltzmann equations can be found in [20] and [39]. For some model problems of hyperbolic system with relaxation, the nonlinear asymptotic stability of rarefaction waves as either time goes to infinity or the relaxation parameter goes to zero has been studied in [9, 18, 22, 33, 40, 45, 46]. It should be noted that, in most of these works, the systems discussed are 2×2 so that the equilibrium equation is scalar, except some for the Broadwell model of the discrete Boltzmann equation which is semilinear. Unlike (1.1), all these systems satisfy the Shizuta-Kawashima condition.

In the present paper, we turn our attention to the asymptotic behavior of the Cauchy problem for the system (1.1) being a perturbation around a planar rarefaction wave in the domain $\mathbb{R}\times\mathbb{T}^2$. While for the hyperbolic conservation laws with relaxation, it is still open due to the higher dimensionality. Note that although there have been rather satisfactory results about the stablity of rarefaction waves for viscous conservation laws in one-dimensional case, the results for Navier-Stokes equation in higher dimensions are few. Recently, Li and Wang [16] proved the stablity of planar rarefaction wave in two-dimensional domain $\mathbb{R}\times\mathbb{T}$. Later, [17] extends the results of [16] to 3D full compressible Navier-Stokes equation with domain $\mathbb{R}\times\mathbb{T}^2$. And [35] investigates the time-asymptotic stability of planar rarefaction wave for the three-dimensional Boltzmann equation.

One may find discussions on the general structure of hyperbolic systems with relaxation in [5] including the dissipative structure and entropy.

It is clear that (1.5) has three distinct eigenvalues

$$\lambda_1(\rho,u_1,s)=u_1-\sqrt{p_\rho(\rho,s)},\ \lambda_2(\rho,u_1,s)=u_1,\ \lambda_3(\rho,u_1,s)=u_1+\sqrt{p_\rho(\rho,s)}$$

with the corresponding right eigenvectors

$$r_1(\rho,u_1,s)=\left(-\rho,\sqrt{p_\rho(\rho,s)},0\right),\quad r_2(\rho,u_1,s)=(p_s,0,-p_\rho),$$
$$r_3(\rho,u_1,s)=\left(\rho,\sqrt{p_\rho(\rho,s)},0\right).$$

Because

$$r_i(\rho,u_1,s)\cdot\nabla\lambda_i(\rho,u_1,s)\neq 0,\ \text{if}\ i=1,3,$$

and

$$r_2(\rho,u_1,s)\cdot\nabla\lambda_2(\rho,u_1,s)=0,$$

there are two families of rarefaction waves for the Euler system (1.5), see [30]. For illustration, we only discuss the 3-rarefaction wave. Indeed the case of 1-rarefaction wave can be treated similarly. There exist two 3-Riemann invariants w_3^1 and w_3^2, respectively, denoted by

$$w_3^1(\rho,u_1,s)=u_1-\int^\rho\frac{\sqrt{p_\rho(z,s)}}{z}dz,\quad w_3^2(\rho,u_1,s)=s.$$

Based on 3-Riemann invariants, if we have $\rho_+>0$, then 3-rarefaction wave can be defined as

$$\begin{aligned}R_3(\rho_+,u_{1+},s_+)=:\big\{(\rho,u_1,s)\big|\lambda_{3x_1}(\rho,u_1,s)>0,&\\ w_3^j(\rho,u_1,s)=w_3^j(\rho_+,u_{1+},s_+),j=1,2.\big\}&\end{aligned}\tag{1.10}$$

The solution to (1.5) and (1.6) can be expressed explicitly by the Riemann solution to the Burgers equation:

$$\begin{cases}w_t+\left(\frac{w^2}{2}\right)_{x_1}=0,\\ w(x_1,0)=w_0^r(x_1)=\begin{cases}w_-,\ x_1<0,\\ w_+,\ x_1>0.\end{cases}\end{cases}\tag{1.11}$$

If $w_+>w_-$, it is obvious that (1.11) admits a centered rarefaction wave like

$$w^r\left(\frac{x_1}{t}\right)=\begin{cases}w_-,\ \frac{x_1}{t}\le w_-,\\ \frac{x_1}{t},\ \ w_-\le\frac{x_1}{t}\le w_+,\\ w_+,\ \frac{x_1}{t}\ge w_+,\end{cases}\tag{1.12}$$

then the 3-rarefaction wave solution $(\rho^r,u_1^r,s^r)\left(\frac{x_1}{t}\right)$ to compressible Euler equations (1.5)-(1.6) could be expressed as

$$\begin{cases}w_\pm=\lambda_3(\rho_\pm,u_{1\pm},s_\pm),\ w^r\left(\frac{x_1}{t}\right)=\lambda_3(\rho^r,u_1^r,s^r)\left(\frac{x_1}{t}\right),\\ w_3^j(\rho^r,u_1^r,s^r)\left(\frac{x_1}{t}\right)=w_3^j(\rho_+,u_{1+},s_+),\ j=1,2.\end{cases}\tag{1.13}$$

So the corresponding density and pressure are given by

$$v^r\left(\frac{x_1}{t}\right)=\frac{1}{\rho^r},\ p^r\left(\frac{x_1}{t}\right)=K\left(v^r\right)^{-\gamma}e^{(\gamma-1)s^r}.$$

And the planar 3-rarefaction wave to 3D compressible Euler Equations (1.7)-(1.8) is defined by

$$(\rho^r, u_1^r, u_2^r, u_3^r, s^r)\left(\frac{x_1}{t}\right) = (\rho^r, u_1^r, 0, 0, s^r)\left(\frac{x_1}{t}\right)$$

with $(\rho^r, u_1^r, s^r)\left(\frac{x_1}{t}\right)$ being the one-dimensional rarefaction wave for (1.5)-(1.6). Because the centered planar rarefaction wave is only Lipschitz continuous but not smooth, a smooth planar 3-rarefaction wave which is time-asymptotically equivalent will be constructed. Inspired by [26], suppose that $w^R(x_1,t)$ is the globally defined smooth solution to the following Cauchy problem for Burgers' equation:

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_{x_1} = 0, \\ w(x_1,0) = w_0(x_1) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \cdot \kappa \int_0^{\epsilon x_1} (1+y^2)^{-2} dy, \end{cases} \tag{1.14}$$

where $\epsilon > 0$ and κ is a constant such that $\kappa \int_0^\infty (1+y^2)^{-2} dy = 1$. Therefore the smooth 3-rarefaction wave $(\rho^R, u_1^R, s^R)(x_1,t)$ for (1.5)-(1.6) could be constructed as

$$\begin{cases} w_\pm = \lambda_3(\rho_\pm, u_{1\pm}, s_\pm),\ w^R(x_1, 1+t) = \lambda_3(\rho^R, u_1^R, s^R), \\ w_3^j(\rho^R, u_1^R, s^R) = w_3^j(\rho_+, u_{1+}, s_+),\ j = 1,2. \end{cases} \tag{1.15}$$

Hence the corresponding smooth density and pressure are given by

$$v^R(x_1,t) = \frac{1}{\rho^R},\ p^R(x_1,t) = K\left(v^R\right)^{-\gamma} e^{(\gamma-1)s^R}. \tag{1.16}$$

By the definition of 3-Riemann invariant, one can see that $s^R = \bar{s} = s_+ = s_-$. The smooth planar 3-rarefaction wave to 3D compressible Euler Equations (1.7)-(1.8) is defined by

$$(\rho^R, u_1^R, u_2^R, u_3^R, s^R)(x_1,t) = (\rho^R, u_1^R, 0, 0, s^R)(x_1,t). \tag{1.17}$$

After a simple calculation, (1.5) could be rewritten as:

$$\begin{cases} p_t^R + u_1^R p_{x_1}^R = -\gamma p^R u_{1x_1}^R, \\ u_{1t}^R + u_1^R u_{1x_1}^R + v^R p_{x_1}^R = 0, \\ u_i^R = 0,\ i = 2,3, \\ \chi^R = 0, \\ s^R = \bar{s}. \end{cases} \tag{1.18}$$

Since $v^R = \frac{1}{\rho^R}$, we also have

$$v_t^R + u_1^R v_{x_1}^R - v^R u_{1x_1}^R = 0. \tag{1.19}$$

As described earlier, the dissipation of (1.1) is extremely weak. At the same time, the planar 3-rarefaction wave is the only solution of the equilibrium which does not solve (3.1), a non-integrable in time error takes place. This prevents us from obtaining the bounds on the L^2 norm of $||(u-u^R, p-p^R, \chi, s-s^R)||$. Therefore, we turn to investigate the linearized stability of the rarefaction wave. Now, for convenience, we give the decomposition to the solution of (3.1) as follows

$$p = p^R + \phi,\ u = u^R + \psi = (u_1^R + \psi_1, u_2^R + \psi_2, u_3^R + \psi_3),\ \chi = \chi,\ s = \bar{s} + \zeta. \tag{1.20}$$

Linearizing (3.1) around (1.18), the linearization system about $(\phi,\psi_1,\psi_2,\psi_3,\chi,\zeta)(x,t)$ is obtained as:

$$\begin{cases}\phi_t+u_1^R\phi_{x_1}+b_2p^R\mathrm{div}u^R+\psi\cdot\nabla p^R+\beta\phi\mathrm{div}u^R+\beta p^R\mathrm{div}\psi=-\frac{2\chi}{\tau\alpha v^R},\\ \psi_t+u_1^R\psi_{x_1}+v^R\nabla\phi+\psi_1\nabla u_1^R-b_3\frac{v^R}{p^R}\phi\nabla p^R+b_4v^R\nabla p^R\zeta+b_4\frac{\chi}{p^R}\nabla p^R=0,\\ \chi_t+u_1^R\chi_{x_1}+b_5u_{1x_1}^R(v^R\phi+p^Rv^R\zeta+\chi)+\frac{\alpha_f}{\alpha}p^Rv^R(u_{1x_1}^R+\mathrm{div}\psi)=-\frac{b_1}{\tau}\chi,\\ \zeta_t+u_1^R\zeta_{x_1}=0,\end{cases}\tag{1.21}$$

where

$$\beta=\frac{\alpha+2}{\alpha},\ b_1=\frac{\alpha+\alpha_f}{\alpha},\ b_2=\frac{2\alpha_f}{\alpha(\alpha+\alpha_f)},$$
$$b_3=\frac{\alpha+\alpha_f}{\alpha+\alpha_f+2}=\frac{1}{\gamma},\ b_4=\frac{2}{\alpha+\alpha_f+2},\ b_5=\frac{2\alpha_f}{\alpha(\alpha+\alpha_f+2)}.$$

The main idea for the linearization procedure is to regard u, p, χ and s as independent variables, and take v as a function of p, χ and s. Then an implicit function F was introduced to linearize v by implicit function's theorem. The details of this derivation of the linearization can be found in Section 3. Let

$$\delta=|u_{1+}-u_{1-}|+|v_+-v_-|.$$

Consider the Cauchy problem of (1.21) with the following initial data

$$(\phi,\psi,\chi,\zeta)(x,0)=(\phi_0,\psi_0,\chi_0,\zeta_0)(x),\ x\in\mathbb{R}\times\mathbb{T}^2,\tag{1.22}$$

then we have the following theorem on the linearized stability of the smooth planar 3-rarefaction wave:

Theorem 1.1. *Let $(u^R,p^R,s^R)(x_1,t)$ be the smooth planar 3-rarefaction wave of the equilibrium system (1.7)-(1.8) given by (1.15)-(1.17) with $\epsilon=\delta^{1/2}$. For the Cauchy problem of the linearized system (1.21)-(1.22) with the initial data satisfying $(\phi_0,\psi_0,\chi_0,\zeta_0)\in H^2(\mathbb{R}\times\mathbb{T}^2)$, it holds that, for all $t>0$,*

$$||(\phi,\chi)||_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le C(1+t)^{-1/16}(\delta\ln(1+t)+1)^{1/2},$$
$$\|div\psi\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\le C(1+t)^{-1/8}(\delta\ln(1+t)+1)^{1/2},$$

and

$$||\psi||_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le C(\delta\ln(1+t)+1)^{1/4},\ ||\zeta(\cdot,t)||_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le C,$$

exist for some positive constant C independent of t, provided the wave strength $\delta=|u_{1+}-u_{1-}|+|v_+-v_-|$ is suitably small.

Remark 1.1. Unlike the estimates for perturbation of planar rarefaction waves for compressible Navier-Stokes equations, cf. [16, 17], where the L^2 norm of the perturbations are bounded in time, the interesting feature for the thermal non-equilibrium system (1.1) is that the L^2 norm of the perturbation (ϕ,ψ,χ) may grow in time at the rate of $(\ln(1+t))^{\frac{1}{2}}$ (Lemma 4.2). Similarly as [24], the smooth planar 3-rarefaction wave of the equilibrium system (1.7)-(1.8) does not solve the non-equilibrium (3.1), the error terms of the form $p^R\mathrm{div}u^R$ and $p^Rv^Ru_{1x_1}^R$ appear in the linearized system $(1.21)_1$ and $(1.21)_3$, respectively. On the linear level, these error terms may induce a growth of the

L^2 norm of the perturbation in time at the rate of $(\ln(1+t))^{\frac{1}{2}}$ due to the slow decay of the L^2 norm of $u^R_{1x_1}$ at the order of $(1+t)^{-1/2}$ whose square is not integrable in $[0,\infty)$ for time, see (4.21) for details. It should be noted that the L^∞ norm could be controlled by one-dimensional Sobolev's inequality ($H^1(\mathbb{R})$ norm) in [24], while Lemma 2.1 tells us that the L^∞ norm should be dominated by $H^2(\Omega)$ in the present paper. Whereas the L^2 norm of the first derivatives of the perturbation $(\nabla\phi,\nabla\psi,\nabla\chi)$ and second derivatives of the perturbation $(\nabla^2\phi,\nabla^2\psi,\nabla^2\chi)$, are found to be uniformly bounded by some postive constant (Lemmas 4.3-4.4). Inspired by [24] and (1.23) (Lemma 4.5), one can show that the L^2 norm of the first derivatives of the perturbation $(\nabla\phi,\nabla\chi)$ and divψ actually deccays in time at the rate of $(1+t)^{-1/8}(\ln(1+t))^{\frac{1}{2}}$ (Lemma 4.6). Hence by means of Lemma 2.1, it has been proved that the perturbation in L^∞ for (ϕ,χ) decays as time goes to infinity on the linear level.

Remark 1.2. In the case of one-dimensional space, it should be pointed out that the results of Theorem 1.1 coincide with those in [24]. Compared with the one-dimensional stablity results in [24], the main difference lies in higher dimensionality. Particularly, we can obtain the similar estimates as

$$\int_0^t\int \left(|\nabla\phi|^2+|\text{div}\psi|^2\right)dxdt' \le C(\delta\ln(1+t)+1). \tag{1.23}$$

During the process to get (1.23), it is not only that more terms will be needed to deal with than [24], but also the terms which include $u_1^R\psi_{x_1}$ should be treated carefully, see (4.126) and (4.128) for details. However, there are essential differences between $|\nabla\psi|$ and divψ if the space dimension is greater than 1. This is because from the first or third equation of (1.21), only divψ could be expressed by other terms due to the positive lower bound of (p^R,v^R). But since u_1^R may take 0 as value, then the structure of (1.21) tells us it is impossible for $\nabla\psi$ to appear alone.

The remaining part of the paper is organized as follows. In Section 2, we give two basic lemmas for later use, including the properties of smooth planar 3-rarefaction waves which can be found in [8, 26]. In Section 3, we give the main idea of the linearization of the original nonlinear problem around the smooth planar rarefaction wave. Based on energy estimates, Section 4 is contributed to proving Theorem 1.1, the linearized stability. Throughout this paper, the L^pnorm on Ω for $f(x,t)$ is given by

$$\|f(\cdot,t)\|_{L^p}\equiv\|f\|_{L^p}\equiv\|f(t)\|_{L^p(\Omega)}=\left(\int_\Omega|f|^p(x,t)dx\right)^{1/p}.$$

And we also use the following notations:

$$||f||\equiv||f(t)||_{L^2(\Omega)},\ ||\cdot||_{L^\infty}\equiv||\cdot||_{L^\infty(\mathbb{R})}.$$

We also use $\int$ to denote $\int_\Omega$ unless otherwise stated. Moreover, C will be used as a generic constant independent of time t.

2. Basic lemmas

In this section, we will list two lemmas which will be used later. Firstly, the properties of smooth planar 3-rarefaction wave have been given and will be frequently used throughout this paper.

Lemma 2.1 ([8, 26]). *The smooth solution (v^R,u^R,p^R,s^R) constructed in (1.15) for $\epsilon=\delta^{1/2}$ has the following properties:*

(1) *For each* $(x_1,t)\in\mathbb{R}\times\mathbb{R}_+$, $u^R_{1x_1}=\frac{2}{\gamma+1}w^R_{x_1}>0$, $v^R_{x_1}=-\frac{1}{\sqrt{\gamma K}}e^{-\frac{\gamma-1}{2}\bar{s}}(v^R)^{\frac{\gamma+1}{2}}u^R_{1x_1}$, $p^R_{x_1}=\sqrt{\gamma K}e^{\frac{\gamma-1}{2}\bar{s}}(v^R)^{-\frac{\gamma+1}{2}}u^R_{1x_1}$;

(2) *For any* p *with* $1\le p\le+\infty$, *there exists a constant* $C_p>0$ *depending only on* p *such that*

$$||(v^R_{x_1},u^R_{x_1},p^R_{x_1})||_{L^p(\mathbb{R})}\le C_p\min\{\delta^{3/2-1/2p},\delta^{1/p}(1+t)^{-1+1/p}\},$$
$$||(v^R_{x_1x_1},u^R_{x_1x_1},p^R_{x_1x_1})||_{L^p(\mathbb{R})}\le C_p\min\{\delta^{2-1/2p},\delta^{1/8-1/8p}(1+t)^{-5/4+1/4p}\};$$

(3) *There exists a positive constant* C *such that*

$$||(v^R_{x_1x_1},u^R_{x_1x_1},p^R_{x_1x_1})||_{L^\infty(\mathbb{R})}\le C\delta^{1/2}||(v^R_{x_1},u^R_{x_1},p^R_{x_1})||_{L^\infty(\mathbb{R})},$$
$$||(v^R_{x_1x_1x_1},u^R_{x_1x_1x_1},p^R_{x_1x_1x_1})||_{L^\infty(\mathbb{R})}\le C\delta^{1/4}||(v^R_{x_1},u^R_{x_1},p^R_{x_1})||^{3/2}_{L^\infty(\mathbb{R})},$$

and

$$|(v^R_t,u^R_t,p^R_t)|\le C|(v^R_{x_1},u^R_{x_1},p^R_{x_1})|;$$

(4) *There exists a positive constant* C *such that*

$$||(v^R-v^r,u^R-u^r,p^R-p^r)||_{L^\infty(\mathbb{R})}\le C\min\{\delta,\delta^{1/4}(1+t)^{-1/3}(\ln(2+t)+|\ln\delta|)\}.$$

REMARK 2.1. It also holds that $\rho^R_{x_1}=\frac{1}{\sqrt{\gamma K}}e^{-\frac{\gamma-1}{2}\bar{s}}(\rho^R)^{-\frac{\gamma-3}{2}}u^R_{1x_1}$.

The following lemma is important for us to study the large-time behavior for the solutions to linearization system.

LEMMA 2.1 ([1,15]). *There exists some positive constant* C *such that for* $f\in H^2(\Omega)$ *with* $\Omega:=\mathbb{R}\times\mathbb{T}^2$, *it holds that*

$$||f||^2_{L^\infty(\Omega)}\le\|f\|\|\nabla f\|+\|\nabla f\|\|\nabla^2 f\|.$$

3. The linearization process

In this section, we derive the linearized system. By means of (1.2), one can see that (1.1) is equivalent to

$$\begin{cases} p_t+u\cdot\nabla p+\beta p\mathrm{div}u=-\frac{2\chi}{\tau\alpha v},\\ u_{it}+u\cdot\nabla u_i+vp_{x_i}=0,\quad 1\le i\le 3,\\ \chi_t+u\cdot\nabla\chi+\frac{\alpha_f}{\alpha}pv\mathrm{div}u=-\frac{b_1\chi}{\frac{\tau}{2}},\\ s_t+u\cdot\nabla s=(\frac{1}{T_2}-\frac{1}{T_1})\frac{\chi}{\tau}=\frac{\chi^2}{\tau pvq}. \end{cases}\tag{3.1}$$

As mentioned in the introduction, taking p,u,χ,s as basic unknowns, then by (1.3), v could be regarded as an implicit function of other three variables (p,χ,s) and $v-v^R$ could be presented by the implicit function theorem. For this purpose, set F as

$$\begin{aligned} F:=&F(v,p,\chi,s)\\ =&s-\left[\frac{\alpha}{2}\ln\left(\frac{\alpha}{2}pv\right)+\ln v+\frac{\alpha_f}{2}\ln\left(\frac{\alpha_f}{2}pv-\chi\right)\right]=0. \end{aligned}\tag{3.2}$$

After a direct computation, one has

$$\begin{cases} F_s=1,\\ F_v=-\frac{\alpha+2}{2}\frac{1}{v}-\frac{(\frac{\alpha_f}{2})^2p}{\frac{\alpha_f}{2}pv-\chi}\neq0,\quad \text{for small }\chi,\\ F_p=-\frac{\alpha}{2p}-\frac{(\frac{\alpha_f}{2})^2v}{\frac{\alpha_f}{2}pv-\chi},\\ F_\chi=\frac{\alpha_f}{2}\frac{1}{\frac{\alpha_f}{2}pv-\chi},\\ v_s=-\frac{F_s}{F_v},\\ v_p=-\frac{F_p}{F_v},\\ v_\chi=-\frac{F_\chi}{F_v}, \end{cases}\tag{3.3}$$

which implies

$$v-v^R=-\frac{\alpha+\alpha_f}{\alpha+\alpha_f+2}\frac{v^R}{p^R}\phi+\frac{2}{\alpha+\alpha_f+2}v^R\zeta+\frac{2}{\alpha+\alpha_f+2}\frac{\chi}{p^R}+o(\phi^2,\chi^2,\zeta^2) \tag{3.4}$$

by the Taylor's formula with higher order $o(\phi^2,\chi^2,\zeta^2)$. One can linearize the system (3.1) around (p^R,u^R,χ^R,s^R), which satisfies the system (1.18). Utilizing the similar argument as [24], then the linearized system (1.21) has been achieved.

4. Proof of Theorem 1.1

Because $\zeta_t+u_1^R\zeta_{x_1}=0$, it is easy to obtain the estimates $\|(\zeta,\nabla\zeta,\nabla^2\zeta)\|$. Hence we would like to get the estimates for ζ firstly.

LEMMA 4.1. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time such that*

$$||(\zeta,\nabla\zeta,\nabla^2\zeta)(t)||^2+\int_0^t\int u_{1x_1}^R|(\zeta,\nabla\zeta,\nabla^2\zeta)|^2(x,t')dxdt'\le C||(\zeta,\nabla\zeta,\nabla^2\zeta)(0)||^2. \tag{4.1}$$

Moreover,

$$||\zeta(\cdot,t)||_{L^\infty}\le C||\zeta(0)||_{H^2(\Omega)}\le C.$$

Proof. It is clear that

$$\frac{\partial}{\partial t}\left(\frac{1}{v^R}\zeta^2\right)=-\left(\frac{1}{v^R}u_1^R\zeta^2\right)_{x_1},$$

by (1.19), which implies

$$||\zeta(t)||^2\le C||\zeta(0)||^2. \tag{4.2}$$

Also, one can show that

$$\frac{\partial}{\partial t}\zeta^2=-\left(u_1^R\zeta^2\right)_{x_1}+u_{1x_1}^R\zeta^2,$$

then

$$\int_0^t\int u_{1x_1}^R\zeta^2dxdt'\le C(||\zeta(t)||^2+||\zeta(0)||^2)\le C||\zeta(0)||^2. \tag{4.3}$$

Similarly,

$$\frac{\partial}{\partial t}\zeta_{x_1}^2+u_{1x_1}^R\zeta_{x_1}^2=-\left(u_1^R\zeta_{x_1}^2\right)_{x_1}$$

gives us

$$||\zeta_{x_1}(t)||^2+\int_0^t\int u_{1x_1}^R\zeta_{x_1}^2dxdt'\le C||\zeta_{x_1}(0)||^2. \tag{4.4}$$

While for ζ_{x_i}, $i=2,3$, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{v^R}\zeta_{x_i}^2\right)=-\left(\frac{1}{v^R}u_1^R\zeta_{x_i}^2\right)_{x_1},$$

which implies that

$$||\zeta_{x_i}(t)||^2 \le C||\zeta_{x_i}(0)||^2. \tag{4.5}$$

Also

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\zeta|^2 + u^R_{1x_1}\zeta^2_{x_1} = -\frac{1}{2}\left(u^R_1|\nabla\zeta|^2\right)_{x_1} + \frac{1}{2}u^R_{1x_1}(\zeta^2_{x_2}+\zeta^2_{x_3})$$

results in

$$||\nabla\zeta(t)||^2 + \int_0^t\int u^R_{1x_1}(\zeta^2_{x_2}+\zeta^2_{x_3})dxdt' \le C||\nabla\zeta(0)||^2 \tag{4.6}$$

by (4.4) and (4.5). Now we only need to show that $||\nabla^2\zeta||^2$ is bounded to finish the proof of this lemma. Since

$$\nabla^2\zeta = \left\{\partial^{l_1}_{x_1}\partial^{l_2}_{x_2}\partial^{l_3}_{x_3}\zeta : l_1,l_2,l_3\in\mathbb{N}_+, l_1+l_2+l_3=2\right\},$$

then we firstly show that $||\zeta_{x_ix_i}||^2$, $i=1,2,3$ is bounded. After a straightforward calculation, the following equation is arrived

$$\frac{1}{2}\frac{\partial}{\partial t}\zeta^2_{x_1x_1} + \frac{3}{2}u^R_{1x_1}\zeta^2_{x_1x_1} = -\frac{1}{2}\left(u^R_1\zeta^2_{x_1x_1} - u^R_{1x_1x_1}\zeta^2_{x_1}\right)_{x_1} + \frac{1}{2}u^R_{1x_1x_1x_1}\zeta^2_{x_1},$$

which implies

$$\begin{aligned}||\zeta_{x_1x_1}(t)||^2 + \int_0^t\int u^R_{1x_1}\zeta^2_{x_1x_1}dxdt' \le& C||\zeta_{x_1x_1}(0)||^2 + C\int_0^t\int u^R_{1x_1x_1x_1}\zeta^2_{x_1}dxdt'\\ \le& ||\zeta_{x_1x_1}(0)||^2 + C\delta\int_0^t\int u^R_{1x_1}\zeta^2_{x_1}dxdt'\\ \le& C||\zeta_{x_1x_1}(0)||^2\end{aligned} \tag{4.7}$$

due to (4.4) and Lemma 2.1. When $i,j=2,3$, on the one hand, one has

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{v^R}\zeta^2_{x_ix_j}\right) = -\frac{1}{2}\left(\frac{u^R_1}{v^R}\zeta^2_{x_ix_j}\right)_{x_1},$$

by (1.19), which implies

$$||\zeta_{x_ix_j}||^2(t) \le C||\zeta_{x_ix_j}||^2(0). \tag{4.8}$$

When $i,j=2,3$, we also have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\zeta^2_{x_ix_j}\right) = -\frac{1}{2}\left(u^R_1\zeta^2_{x_ix_j}\right)_{x_1} + \frac{1}{2}u^R_{1x_1}\zeta^2_{x_ix_j}.$$

So combining the above equation with (4.8), it is easy to see

$$\int_0^t\int u^R_{1x_1}\zeta^2_{x_ix_j}dxdt' \le C||\zeta_{x_ix_j}(0)||^2,\ i,j=2,3. \tag{4.9}$$

While for $||\zeta_{x_1x_j}||^2$, $j=2,3$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\zeta^2_{x_1x_j} + \frac{1}{2}u^R_{1x_1}\zeta^2_{x_1x_j} = -\frac{1}{2}\left(u^R_1\zeta^2_{x_1x_j}\right)_{x_1},$$

which gives

$$||(\zeta_{x_1x_2},\zeta_{x_1x_3})(t)||^2+\int_0^t\int u^R_{1x_1}(\zeta^2_{x_1x_2},\zeta^2_{x_1x_3})dxdt'\le C||(\zeta_{x_1x_2},\zeta_{x_1x_3})(0)||^2. \tag{4.10}$$

Combining (4.2)-(4.4) and (4.6), (4.7)-(4.10), then (4.1) has been proved. Then

$$\begin{aligned}||\zeta(\cdot,t)||^2_{L^\infty}\le&C\left(||\zeta(t)||||\nabla\zeta(t)||+||\nabla\zeta(t)||||\nabla^2\zeta(t)||\right)\\ \le&C\left(||\zeta(0)||||\nabla\zeta(0)||+||\nabla\zeta(0)||||\nabla^2\zeta(0)||\right)\\ \le&C||\zeta(0)||^2_{H^2(\Omega)}\end{aligned}$$

comes from Lemma 2.1 and (4.1). Hence, we have finished the proof of this lemma. □

LEMMA 4.2. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time such that*

$$||(\phi,\psi,\chi)||^2+\int_0^t\int\left[u^R_{1x_1}(\phi^2+\psi_1^2)+\chi^2\right](x,t')dxdt'\le C(\delta\ln(1+t)+1). \tag{4.11}$$

Proof. After a straightforward calculation, it is easy to see that

$$\begin{cases}\frac{\phi_t}{p^R}=-b_2u^R_{1x_1}-\frac{u_1^R}{p^R}\phi_{x_1}-\frac{p^R_{x_1}}{p^R}\psi_1-\beta\frac{u^R_{1x_1}}{p^R}\phi-\beta\mathrm{div}\psi-\frac{2\chi}{\tau\alpha p^Rv^R},\\ \frac{\psi_t}{v^R}=-\frac{u_1^R}{v^R}\psi_{x_1}-\nabla\phi-\frac{\psi_1}{v^R}\nabla u_1^R+b_3\frac{\phi}{p^R}\nabla p^R-b_4\nabla p^R\zeta-b_4\frac{\chi}{p^Rv^R}\nabla p^R,\\ (\phi-\dfrac{b_6}{v^R}\chi)_t=(\beta-b_2)p^Ru^R_{1x_1}-u_1^R\phi_{x_1}-p^R_{x_1}\psi_1+(b_5b_6-\beta)u^R_{1x_1}\phi+b_7\dfrac{\chi}{\tau v^R}\\ \qquad\qquad+b_6\dfrac{u_1^R\chi_{x_1}}{v^R}+b_5b_6p^Ru^R_{1x_1}\zeta+(b_5+1)b_6\dfrac{u^R_{1x_1}\chi}{v^R}-b_6\dfrac{u_1^Rv^R_{x_1}\chi}{(v^R)^2},\end{cases} \tag{4.12}$$

from (1.21), where

$$b_6=\frac{\alpha+2}{\alpha_f},\ b_7=\frac{\alpha+\alpha_f+2}{\alpha_f}.$$

Multiplying the equations in (4.12) by ϕ, $\beta\psi$ and $b_5(\phi-\frac{b_6}{v^R}\chi)$, respectively, then one can obtain

$$\begin{aligned}&\frac12\frac{\partial}{\partial t}\left(\frac{\phi^2}{p^R}\right)+\frac12\frac{\partial}{\partial t}\left(\frac{\beta|\psi|^2}{v^R}\right)+\frac12\frac{\partial}{\partial t}\left[\frac{b_5}{p^R}\left(\phi-b_6\frac{\chi}{v^R}\right)^2\right]\\ &\quad+\frac{b_8}{p^R}u^R_{1x_1}\phi^2+\frac{\beta}{v^R}u^R_{1x_1}\psi_1^2+\frac{b_5b_6b_7\chi^2}{\tau p^R(v^R)^2}\\ =&-\frac12\left[\beta\frac{u_1^R}{v^R}|\psi|^2+(b_5+1)\frac{u_1^R}{p^R}\phi^2+b_5b_6^2\frac{u_1^R\chi^2}{p^R(v^R)^2}-2b_5b_6\frac{u_1^R\phi\chi}{p^Rv^R}\right]_{x_1}\\ &\quad-\beta\mathrm{div}(\phi\psi)-\beta b_4p^R_{x_1}\psi_1\zeta-\beta b_4\frac{p^R_{x_1}\psi_1\chi}{p^Rv^R}+b_9\frac{u^R_{1x_1}\phi\chi}{p^Rv^R}+b_5^2b_6u^R_{1x_1}\phi\zeta\\ &\quad+b_{10}\frac{u^R_{1x_1}\chi}{v^R}+b_5b_6\frac{p^R_{x_1}\psi_1\chi}{p^Rv^R}-b_5^2b_6^2\frac{u^R_{1x_1}\zeta\chi}{v^R}+b_{11}\frac{u^R_{1x_1}\chi^2}{p^R(v^R)^2}\\ =:&-\frac12\left[\beta\frac{u_1^R}{v^R}|\psi|^2+(b_5+1)\frac{u_1^R}{p^R}\phi^2+b_5b_6^2\frac{u_1^R\chi^2}{p^R(v^R)^2}-2b_5b_6\frac{u_1^R\phi\chi}{p^Rv^R}\right]_{x_1}-\beta\mathrm{div}(\phi\psi)+\sum_{i=1}^8I_i,\end{aligned} \tag{4.13}$$

where

$$b_8 = (b_5+1)(\beta - \frac{\gamma}{2} - \frac{1}{2}) - b_5^2 b_6 = \frac{1}{\alpha} + \frac{\alpha_f(\alpha^2+\alpha\alpha_f+2\alpha+4\alpha_f)}{\alpha^2(\alpha+\alpha_f+2)^2} > 0,$$
$$b_9 = b_5 b_6(b_5 - \gamma + \beta - b_5 b_6),\ b_{10} = b_5 b_6(b_2 - \beta),\ b_{11} = b_5 b_6^2\left[\frac{\gamma+1}{2} - b_5 - 1\right].$$

Thanks to the positivity of b_8, integrating (4.13) over $\Omega\times[0,t]$, then the following inequality is arrived

$$||(\phi,\psi,\chi)||^2 + \int_0^t\int \left[u^R_{1x_1}(\phi^2+\psi_1^2)+\chi^2\right](x,t')dxdt' \le C\sum_{i=1}^8\int_0^t\int |I_i|(x,t')dxdt'. \quad (4.14)$$

Now we begin to deal with the right-hand side of (4.14). By Lemma 2.1, Lemma 4.1 and Young's inequality, we have

$$\begin{aligned} C\int_0^t\int |I_1|(x,t')dxdt' \le& C\int_0^t\int |p^R_{x_1}\psi_1\zeta|(x,t')dxdt' \\ \le& C\int_0^t\int |u^R_{1x_1}\psi_1\zeta|(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\psi_1^2(x,t')dxdt' + C\int_0^t\int u^R_{1x_1}\zeta^2(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\psi_1^2(x,t')dxdt' + C, \end{aligned} \quad (4.15)$$

and by a similar way to have the following inequality

$$\begin{aligned} C\int_0^t\int |I_4|(x,t')dxdt' \le& C\int_0^t\int |u^R_{1x_1}\phi\zeta|(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\phi^2(x,t')dxdt' + C\int_0^t\int u^R_{1x_1}\zeta^2(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\phi^2(x,t')dxdt' + C. \end{aligned} \quad (4.16)$$

It is easy to get

$$\begin{aligned} C\int_0^t\int |I_2|(x,t')dxdt' \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\psi_1^2(x,t')dxdt' + C\int_0^t\int u^R_{1x_1}\chi^2(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\psi_1^2(x,t')dxdt' + C\delta\int_0^t\int \chi^2(x,t')dxdt' \end{aligned} \quad (4.17)$$

due to Lemma 2.1, Lemma 4.1 and Young's inequality. One can use the same way as (4.17) to have

$$C\int_0^t\int |I_3|(x,t')dxdt' \le \frac{1}{8}\int_0^t\int u^R_{1x_1}\phi^2(x,t')dxdt' + C\delta\int_0^t\int \chi^2(x,t')dxdt', \quad (4.18)$$

$$\begin{aligned} C\int_0^t\int |I_6|(x,t')dxdt' \le& C\int_0^t\int |u^R_{1x_1}\psi_1\chi|(x,t')dxdt' \\ \le& \frac{1}{8}\int_0^t\int u^R_{1x_1}\psi_1^2(x,t')dxdt' + C\delta\int_0^t\int \chi^2(x,t')dxdt' \end{aligned} \quad (4.19)$$

and

$$
\begin{aligned}
C\int_0^t\int |I_7|(x,t')dxdt' \le & C\int_0^t\int |u^R_{1x_1}\chi\zeta|(x,t')dxdt'\\
\le & \int_0^t\int u^R_{1x_1}\chi^2(x,t')dxdt'+C\int_0^t\int u^R_{1x_1}\zeta^2(x,t')dxdt'\\
\le & C\delta\int_0^t\int \chi^2(x,t')dxdt'+C.
\end{aligned}
\tag{4.20}
$$

Finally, we only need to estimate $C\int_0^t\int|(I_5,I_8)|(x,t')dxdt'$ through Lemma 2.1, that is

$$
\begin{aligned}
C\int_0^t\int |I_5|(x,t')dxdt' \le & C\int_0^t\int |u^R_{1x_1}\chi|(x,t')dxdt'\\
\le & C\int_0^t\int |u^R_{1x_1}|^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int \chi^2(x,t')dxdt'\\
\le & C\delta\ln(1+t)+\frac{1}{8}\int_0^t\int \chi^2(x,t')dxdt'
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
C\int_0^t\int |I_8|(x,t')dxdt' \le & C\int_0^t\int |u^R_{1x_1}\chi^2|(x,t')dxdt'\\
\le & C\delta\int_0^t\int \chi^2(x,t')dxdt'.
\end{aligned}
\tag{4.22}
$$

Therefore, putting (4.14)-(4.22) together, the desired estimate is obtained if δ is small enough. □

LEMMA 4.3. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time t such that*

$$
||(\nabla\phi,\nabla\chi)||^2+\int_0^t\int \left(u^R_{1x_1}|\nabla\phi|^2+|\nabla\chi|^2\right)(x,t')dxdt'\le C,
$$

and

$$
||\nabla\psi||^2+\int_0^t\int u^R_{1x_1}\left(|\psi_{x_1}|^2+\psi^2_{1x_2}+\psi^2_{1x_3}\right)(x,t')dxdt'\le C.
$$

Proof. Since the smooth planar rarefaction is only a function of x_1, we would like to firstly consider x_i, $i=2,3$ for simplicity.

$$
\begin{cases}
\frac{\phi_{tx_i}}{p^R}=-\frac{u_1^R}{p^R}\phi_{x_1x_i}-\frac{p^R_{x_1}}{p^R}\psi_{1x_i}-\beta\frac{u^R_{1x_1}}{p^R}\phi_{x_i}-\beta(\mathrm{div}\psi)_{x_i}-\frac{2\chi_{x_i}}{\tau\alpha p^Rv^R},\\
\frac{\psi_{tx_i}}{v^R}=-\frac{u_1^R}{v^R}\psi_{x_1x_i}-(\nabla\phi)_{x_i}-\frac{\psi_{x_i}}{v^R}\nabla u_1^R+b_3\frac{\phi_{x_i}}{p^R}\nabla p^R-b_4\nabla p^R\zeta_{x_i}-b_4\frac{\chi_{x_i}}{p^Rv^R}\nabla p^R,\\
\phi_{x_it}-\frac{b_6}{v^R}\chi_{x_it}=-u_1^R\phi_{x_1x_i}-p^R_{x_1}\psi_{1x_i}+(b_5b_6-\beta)u^R_{1x_1}\phi_{x_i}+b_7\frac{\chi_{x_i}}{\tau v^R}\\
\qquad\qquad +b_6\frac{u_1^R\chi_{x_1x_i}}{v^R}+b_5b_6p^Ru^R_{1x_1}\zeta_{x_i}+b_5b_6\frac{u^R_{1x_1}\chi_{x_i}}{v^R}.
\end{cases}
\tag{4.23}
$$

Multiplying the equations in (4.23) by ϕ_{x_i}, $\beta\psi_{x_i}$ and $b_5(\phi_{x_i}-\frac{b_6}{v^R}\chi_{x_i})$, respectively, then a calculation gives us

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\phi_{x_i}^2}{p^R}\right)+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\beta|\psi_{x_i}|^2}{v^R}\right)+\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{b_5}{p^R}\left(\phi_{x_i}-b_6\frac{\chi_{x_i}}{v^R}\right)^2\right]\\
&\quad+\frac{b_8}{p^R}u_{1x_1}^R\phi_{x_i}^2+\frac{\beta}{v^R}u_{1x_1}^R|\psi_{1x_i}|^2+\frac{b_5b_6b_7\chi_{x_i}^2}{\tau p^R(v^R)^2}\\
=&-\frac{1}{2}\left[\beta\frac{u_1^R}{v^R}|\psi_{x_i}|^2+(b_5+1)\frac{u_1^R}{p^R}\phi_{x_i}^2+b_5b_6^2\frac{u_1^R\chi_{x_i}^2}{p^R(v^R)^2}-2b_5b_6\frac{u_1^R\phi_{x_i}\chi_{x_i}}{p^Rv^R}\right]_{x_1}\\
&\quad-\beta\mathrm{div}(\phi_{x_i}\psi_{x_i})-\beta b_4p_{x_1}^R\psi_{1x_i}\zeta_{x_i}-\beta b_4\frac{p_{x_1}^R\psi_{1x_i}\chi_{x_i}}{p^Rv^R}+b_9\frac{u_{1x_1}^R\phi_{x_i}\chi_{x_i}}{p^Rv^R}\\
&\quad+b_5^2b_6u_{1x_1}^R\phi_{x_i}\zeta_{x_i}+b_5b_6\frac{p_{x_1}^R\psi_{1x_i}\chi_{x_i}}{p^Rv^R}-b_5^2b_6^2\frac{u_{1x_1}^R\zeta_{x_i}\chi_{x_i}}{v^R}+b_{11}\frac{u_{1x_1}^R\chi_{x_i}^2}{p^R(v^R)^2}\\
=:&-\frac{1}{2}\left[\beta\frac{u_1^R}{v^R}|\psi_{x_i}|^2+(b_5+1)\frac{u_1^R}{p^R}\phi_{x_i}^2+b_5b_6^2\frac{u_1^R\chi_{x_i}^2}{p^R(v^R)^2}-2b_5b_6\frac{u_1^R\phi_{x_i}\chi_{x_i}}{p^Rv^R}\right]_{x_1}\\
&\quad-\beta\mathrm{div}(\phi_{x_i}\psi_{x_i})+\sum_{k=1}^{7}J_k,\ i=2,3.
\end{aligned}\tag{4.24}$$

Here we have observed that (4.24) doesn't have the terms like I_5 in (4.13), so integrating (4.24) over $(x,t')\in\Omega\times[0,t]$, as the way we estimate (4.13), then for $i=2,3$, the following inequality is arrived

$$\begin{aligned}
&||(\phi_{x_i},\psi_{x_i},\chi_{x_i})||^2+\int_0^t\int\left(u_{1x_1}^R(\phi_{x_i}^2+\psi_{1x_i}^2)+\chi_{x_i}^2\right)dxdt'\\
\le& C\sum_{k=1}^{7}\int_0^t\int|J_k|(x,t')dxdt'\le C
\end{aligned}\tag{4.25}$$

due to Lemma 2.1, Lemmas 4.1-4.2, Young's inequality and the smallness of δ. Now $(\phi_{x_1},\psi_{x_1},\chi_{x_1})$ will be studied and we have

$$\begin{cases}
\phi_{tx_1}=-b_2(p^R\mathrm{div}u^R)_{x_1}-(u_1^R\phi_{x_1})_{x_1}-(\psi_1p_{x_1}^R)_{x_1}-\beta(u_{1x_1}^R\phi)_{x_1}-\beta(p^R\mathrm{div}\psi)_{x_1}-\frac{2}{\tau\alpha}\left(\frac{\chi}{v^R}\right)_{x_1},\\
\frac{p^R}{v^R}\psi_{tx_1}=\frac{p^R}{v^R}[-u_1^R\psi_{x_1}-v^R\nabla\phi-\psi_1\nabla u_1^R+b_3\frac{v^R}{p^R}\phi\nabla p^R-b_4v^R\nabla p^R\zeta-b_4\frac{\chi}{p^R}\nabla p^R]_{x_1},\\
(v_{x_1}^R\phi+v^R\phi_{x_1}-b_6\chi_{x_1})_t=[\gamma p^Rv^Ru_{1x_1}^R-u_1^Rv^R\phi_{x_1}-p_{x_1}^Rv^R\psi_1-b_5u_{1x_1}^Rv^R\phi]_{x_1}+\dfrac{b_7\chi_{x_1}}{\tau}\\
\qquad+[b_6u_1^R\chi_{x_1}+b_5b_6u_{1x_1}^Rp^Rv^R\zeta+b_5b_6u_{1x_1}^R\chi-u_1^Rv_{x_1}^R\phi]_{x_1}.
\end{cases}\tag{4.26}$$

Testing the equations of (4.26) by $\frac{1}{b_5}\phi_{x_1}$, $b_{12}\psi_{x_1}$ and $\frac{1}{(v^R)^2}(v_{x_1}^R\phi+v^R\phi_{x_1}-b_6\chi_x)$, respectively, where

$$b_{12}=\frac{(\alpha+2)(\alpha+\alpha_f+2)}{2\alpha_f},$$

then a straightforward calculation shows

$$\begin{aligned}
&\frac{1}{2b_5}\frac{\partial}{\partial t}\phi_{x_1}^2+\frac{b_{12}}{2}\frac{\partial}{\partial t}\left(\frac{p^R}{v^R}|\psi_{x_1}|^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{(v_{x_1}^R\phi+v^R\phi_{x_1}-b_6\chi_{x_1})^2}{(v^R)^2}\right]\\
&\quad+b_{13}u_{1x_1}^R\phi_{x_1}^2+\frac{b_{12}}{2}\frac{p^R}{v^R}u_{1x_1}^R\left[(\gamma+2)|\psi_{x_1}|^2+\psi_{1x_1}^2\right]+\frac{b_6b_7}{\tau(v^R)^2}\chi_{x_1}^2
\end{aligned}$$

$$
\begin{aligned}
=&-W_{x_1}-b_{12}\mathrm{div}(p^R\psi_{x_1}\phi_{x_1})+b_{14}p^R_{x_1}\phi_{x_1}\psi_{1x_1}+\gamma\frac{p^Ru^R_{1x_1}v^R_{x_1}\phi_{x_1}}{v^R}\\
&-[(b_5+b_{12})u^R_{1x_1x_1}+(b_5+3)\frac{u^R_{1x_1}v^R_{x_1}}{v^R}]\phi\phi_{x_1}\\
&-b_{14}\left[p^R_{x_1x_1}\psi_1\phi_{x_1}-\left(\frac{p^R_{x_1}v^R_{x_1}}{v^R}-\frac{(p^R_{x_1})^2}{p^R}+p^R_{x_1x_1}\right)\phi\psi_{1x_1}\right]\\
&-\frac{p^R_{x_1}v^R_{x_1}}{v^R}\psi_1\phi_{x_1}-b_{12}p^R_{x_1}(\psi_{2x_2}+\psi_{3x_3})\phi_{x_1}-\frac{b_{12}p^R_{x_1}v^R_{x_1}}{v^R}(\phi_{x_2}\psi_{2x_1}+\phi_{x_3}\psi_{3x_1})\\
&-\frac{b_7}{\tau}\left(\frac{v^R_{x_1}}{(v^R)^2}\right)_{x_1}\phi\chi-\frac{b_{12}p^R}{v^R}u^R_{1x_1x_1}\psi_{1x_1}\psi_1-\frac{b_4b_{12}p^R}{v^R}(\nabla p^Rv^R\zeta)_{x_1}\cdot\psi_{x_1}\\
&-\frac{b_4b_{12}p^R}{v^R}\left(\frac{\nabla p^R\chi}{p^R}\right)_{x_1}\cdot\psi_{x_1}+2b_6\left(u^R_{1x_1}-\frac{u^R_1v^R_{x_1}}{v^R}\right)\frac{v^R_{x_1}\chi_{x_1}\phi}{(v^R)^2}\\
&-\frac{3b_6^2u^R_{1x_1}}{2}\frac{\chi^2_{x_1}}{(v^R)^2}+\left(\frac{u^R_1v^R_{x_1}}{v^R}-u^R_{1x_1}\right)\frac{(v^R_{x_1})^2\phi^2}{(v^R)^2}+\frac{3b_6u^R_{1x_1}\phi_{x_1}\chi_{x_1}}{v^R}\\
&+(\gamma p^Rv^Ru^R_{1x_1}-p^R_{x_1}v^R\psi_1-b_5u^R_{1x_1}v^R\phi-u^R_1v^R_{x_1}\phi)_{x_1}\frac{v^R_{x_1}\phi-b_6\chi_{x_1}}{(v^R)^2}\\
&+b_6\frac{(u^R_1\chi_{x_1})_{x_1}v^R_{x_1}\phi}{(v^R)^2}+b_5b_6\frac{(u^R_{1x_1}p^Rv^R\zeta+u^R_{1x_1}\chi)_{x_1}}{(v^R)^2}(v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})\\
=&-W_{x_1}-b_{12}\mathrm{div}(p^R\psi_{x_1}\phi_{x_1})+b_{14}p^R_{x_1}\phi_{x_1}\psi_{1x_1}+\sum_{i=1}^{17}K_i,
\end{aligned}\tag{4.27}
$$

where

$$
b_{13}=b_5+b_{12}+\frac{3}{2}+\frac{1}{2b_5},\ b_{14}=b_3b_{12}=1+\frac{1}{b_5}=\frac{(\alpha+\alpha_f)(\alpha+2)}{2\alpha_f},
$$

$$
W=\frac{b_{14}}{2}u^R_1\phi^2_{x_1}+\frac{b_{12}p^R}{2v^R}u^R_1|\psi_{x_1}|^2+\frac{u^R_1}{v^R}\left(\frac{b_6^2}{2}\chi^2_{x_1}+v^R_{x_1}\phi\phi_{x_1}-b_6\phi_{x_1}\chi_{x_1}\right)-\frac{b_7v^R_{x_1}}{\tau(v^R)^2}\phi\chi
$$

and we have used the following facts

$$
p^R_{x_1}=-\gamma\frac{p^R}{v^R}v^R_{x_1},\tag{4.28}
$$

$$
b_{14}p^R_{x_1}\psi_{1x_1}\phi_{x_1}=-\left(\left(\frac{1}{b_5}-b_3b_{12}+1\right)p^R_{x_1}+\frac{b_{12}p^R}{v^R}v^R_{x_1}\right)\psi_{1x_1}\phi_{x_1}.\tag{4.29}
$$

Now we only have the basic energy estimates in Lemma 4.2 and the estimates for ζ in Lemma 4.1, so the term $b_{14}p^R_{x_1}\psi_{1x_1}\phi_{x_1}$ should be treated carefully. It is expected to be canceled by the right-hand side of (4.27). Observing that

$$
(p^R_{x_1})^2=\frac{\gamma p^R}{v^R}(u^R_{1x_1})^2,\tag{4.30}
$$

we have

$$
b_{14}p^R_{x_1}\psi_{1x_1}\phi_{x_1}\le\frac{b_{12}\gamma p^R}{2v^R}u^R_{1x_1}\psi^2_{1x_1}+\frac{b_{14}}{2}u^R_{1x_1}\phi^2_{x_1},\tag{4.31}
$$

which implies

$$
\frac{1}{2b_5}\frac{\partial}{\partial t}\int\phi^2_{x_1}dx+\frac{b_{12}}{2}\frac{\partial}{\partial t}\int\left(\frac{p^R}{v^R}|\psi_{x_1}|^2\right)dx+\frac{1}{2}\frac{\partial}{\partial t}\int\frac{(v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})^2}{(v^R)^2}dx
$$

$$+\int\left[u^R_{1x_1}(\phi^2_{x_1}+|\psi_{x_1}|^2)+\chi^2_{x_1}\right](x,t')dxdt'\le C\sum_{i=1}^{17}\left|\int K_i dx\right|. \tag{4.32}$$

Now we will estimate K_i, $1\le i\le 17$ in turn by invoking Lemma 2.1, Lemmas 4.1-4.2, (4.25) and Young's inequality. For K_1, one has

$$\begin{aligned}C\int|K_1|dx\le& C\int|u^R_{1x_1}v^R_{x_1}\phi_{x_1}|dx\\ \le& C||u^R_{1x_1}||^{3/2}_{L^\infty}\int|u^R_{1x_1}|dx+C||u^R_{1x_1}||^{1/2}_{L^\infty}\int u^R_{1x_1}\phi^2_{x_1}dx\\ \le& C\delta(1+t)^{-3/2}+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1}dx.\end{aligned} \tag{4.33}$$

Since $|K_2|\le C(|u^R_{1x_1x_1}\phi\phi_{x_1}|+|u^R_{1x_1}v^R_{x_1}\phi\phi_{x_1}|)$, then

$$C\int|K_2|dx\le C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1}dx \tag{4.34}$$

comes from the following two inequalities

$$\begin{aligned}&C\int|u^R_{1x_1x_1}\phi\phi_{x_1}|dx\le C||u^R_{1x_1x_1}||_{L^\infty}\int\phi^2dx+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{1x_1}dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{1x_1}dx,\end{aligned} \tag{4.35}$$

$$\begin{aligned}&C\int|u^R_{1x_1}v^R_{x_1}\phi\phi_{x_1}|dx\le C||u^R_{1x_1}v^R_{x_1}||_{L^\infty}\int\phi^2dx+C||v^R_{x_1}||_{L^\infty}\int u^R_{1x_1}\phi^2_{x_1}dx\\ \le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int u^R_{1x_1}\phi^2_{1x_1}dx.\end{aligned} \tag{4.36}$$

A calculation gives us

$$|K_3|\le C(|p^R_{x_1x_1}\phi_{x_1}\psi_1|+|p^R_{x_1x_1}\phi\psi_{1x_1}|+|P^R_{x_1}v^R_{x_1}\phi\psi_{1x_1}|+|(p^R_{x_1})^2\phi\psi_{1x_1}|),$$

and

$$\begin{aligned}&C\int|p^R_{x_1x_1}\phi_{x_1}\psi_1|dx\le C||p^R_{x_1x_1}||_{L^\infty}\int\psi_1^2dx+C\int|p^R_{1x_1x_1}|\phi^2_{x_1}dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1}dx,\end{aligned} \tag{4.37}$$

$$\begin{aligned}&C\int|p^R_{x_1x_1}\phi\psi_{1x_1}|dx\le C||p^R_{x_1x_1}||_{L^\infty}\int\phi^2dx+C\int|p^R_{1x_1x_1}|\psi^2_{1x_1}dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1}dx,\end{aligned} \tag{4.38}$$

$$\begin{aligned}&C\int|p^R_{x_1}(p^R_{x_1},v^R_{x_1})\phi\psi_{1x_1}|dx\\ \le& C||p^R_{x_1}(p^R_{x_1},v^R_{x_1})||_{L^\infty}\int\phi^2dx+C\int|p^R_{x_1}(p^R_{x_1},v^R_{x_1})|\psi^2_{1x_1}dx\\ \le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int u^R_{1x_1}\psi^2_{1x_1}dx,\end{aligned} \tag{4.39}$$

which implies that

$$C\int|K_3|\,dx\le C(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}(\phi^2_{x_1}+\psi^2_{1x_1})dx. \tag{4.40}$$

For K_4, one has

$$\begin{aligned}C\int|K_4|\,dx\le& C\|p^R_{x_1}v^R_{x_1}\|_{L^\infty}\int\psi_1^2dx+C\int|p^R_{x_1}v^R_{x_1}|\phi^2_{x_1}dx\\ \le& C(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta\int u^R_{1x_1}\phi^2_{x_1}dx.\end{aligned} \tag{4.41}$$

Since

$$\begin{aligned}-\frac{K_5}{b_{12}}=&(p^R_{x_1}\psi_2\phi_{x_1})_{x_2}-(p^R_{x_1}\psi_2\phi_{x_2})_{x_1}+p^R_{x_1x_1}\psi_2\phi_{x_2}+p^R_{x_1}\psi_{2x_1}\phi_{x_2}\\ &+(p^R_{x_1}\psi_3\phi_{x_1})_{x_3}-(p^R_{x_1}\psi_3\phi_{x_3})_{x_1}+p^R_{x_1x_1}\psi_3\phi_{x_3}+p^R_{x_1}\psi_{3x_1}\phi_{x_3},\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}C\int|p^R_{x_1x_1}\psi_2\phi_{x_2}|dx\le& C\|p^R_{x_1x_1}\|_{L^\infty}\int(\psi_2^2+\phi^2_{x_2})dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1),\end{aligned} \tag{4.43}$$

$$\begin{aligned}C\int|p^R_{x_1x_1}\psi_3\phi_{x_3}|dx\le& C\|p^R_{x_1x_1}\|_{L^\infty}\int(\psi_3^2+\phi^2_{x_3})dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1),\end{aligned} \tag{4.44}$$

then

$$\begin{aligned}C\left|\int K_5dx\right|\le& C\int\left(|p^R_{x_1}\psi_{2x_1}\phi_{x_2}|+|p^R_{x_1}\psi_{3x_1}\phi_{x_3}|\right)dx\\ &+C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1).\end{aligned} \tag{4.45}$$

It is obvious that

$$\begin{aligned}C\int|p^R_{x_1}v^R_{x_1}\phi_{x_2}\psi_{2x_1}|dx\le& C\|p^R_{x_1}v^R_{x_1}\|_{L^\infty}\int\phi^2_{x_2}dx+\int|p^R_{x_1}v^R_{x_1}\psi^2_{2x_1}|dx\\ \le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int u^R_{x_1}\psi^2_{2x_1},\end{aligned} \tag{4.46}$$

$$\begin{aligned}C\int|p^R_{x_1}v^R_{x_1}\phi_{x_3}\psi_{3x_1}|dx\le& C\|p^R_{x_1}v^R_{x_1}\|_{L^\infty}\int\phi^2_{x_3}dx+\int|p^R_{x_1}v^R_{x_1}\psi^2_{3x_1}|dx\\ \le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int u^R_{x_1}\psi^2_{3x_1},\end{aligned} \tag{4.47}$$

which tells us

$$C\int|K_6|\,dx\le C(1+t)^{-2}+C\delta\int u^R_{1x_1}(\psi^2_{2x_1}+\psi^2_{3x_1})dx. \tag{4.48}$$

For K_7, one can obtain

$$|K_7|\le C(|v^R_{x_1x_1}\phi\chi|+|(v^R_{x_1})^2\phi\chi|),$$

which shows

$$C\int|K_7|dx\le C(1+t)^{-5/4}(\delta\ln(1+t)+1),\tag{4.49}$$

due to

$$\begin{aligned}C\int|v^R_{x_1x_1}\phi\chi|dx\le& C||v^R_{x_1x_1}||_{L^\infty}\int(\phi^2+\chi^2)dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1),\end{aligned}\tag{4.50}$$

$$\begin{aligned}C\int|(v^R_{x_1})^2\phi\chi|dx\le& C||(v^R_{x_1})^2||_{L^\infty}\int(\phi^2+\chi^2)dx\\ \le& C(1+t)^{-2}(\delta\ln(1+t)+1).\end{aligned}\tag{4.51}$$

For K_8, the following estimate is given

$$\begin{aligned}C\int|K_8|dx\le& C\int|u^R_{1x_1x_1}\psi_{1x_1}\psi_1|dx\\ \le& C||u^R_{1x_1x_1}||_{L^\infty}\int\psi_1^2dx+C\int|u^R_{1x_1x_1}\psi^2_{1x_1}|\\ \le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1}dx.\end{aligned}\tag{4.52}$$

For K_9, we have

$$\begin{aligned}K_9=&-\frac{b_4b_{12}p^R\psi_{1x_1}}{v^R}[p^R_{x_1x_1}v^R\zeta+p^R_{x_1}v^R_{x_1}\zeta+p^R_{x_1}v^R\zeta_{x_1}]\\ \le& C(|p^R_{x_1x_1}\psi_{1x_1}\zeta|+|p^R_{x_1}v^R_{x_1}\psi_{1x_1}\zeta|+|p^R_{x_1}\psi_{1x_1}\zeta_{x_1}|),\end{aligned}\tag{4.53}$$

$$\begin{aligned}&C\int|p^R_{x_1x_1}\psi_{1x_1}\zeta|dx\le C||p^R_{x_1x_1}||_{L^\infty}\int\zeta^2+C\int|p^R_{x_1x_1}\psi^2_{1x_1}|dx\\ \le& C\delta^{1/8}(1+t)^{-5/4}+C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1}dx,\end{aligned}\tag{4.54}$$

$$\begin{aligned}&C\int|p^R_{x_1}v^R_{x_1}\psi_{1x_1}\zeta|dx\le C||p^R_{x_1}v^R_{x_1}||_{L^\infty}\int\zeta^2dx+C||p^R_{x_1}||_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1}dx\\ \le& C(1+t)^{-2}+C\delta\int u^R_{1x_1}\psi^2_{1x_1}dx,\end{aligned}\tag{4.55}$$

hence combining (4.53)-(4.55) together to obtain

$$C\int|K_9|dx\le C\left[\delta^{1/8}(1+t)^{-5/4}+\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1}dx\right]+C\int|p^R_{x_1}\psi_{1x_1}\zeta_{x_1}|dx.\tag{4.56}$$

While for K_{10}, because

$$\begin{aligned}K_{10}=&\frac{b_4b_{12}\psi_{1x_1}}{v^R}\left[\frac{(p^R_{x_1})^2}{p^R}\chi-p^R_{x_1x_1}\chi-p^R_{x_1}\chi_{x_1}\right]\\ \le& C(|p^R_{x_1x_1}\psi_{1x_1}\chi|+|(p^R_{x_1})^2\psi_{1x_1}\chi|+|p^R_{x_1}\psi_{1x_1}\chi_{x_1}|),\end{aligned}$$

and

$$C\int|p^R_{x_1x_1}\psi_{1x_1}\chi|dx\le C||p^R_{x_1x_1}||_{L^\infty}\int\chi^2+C\int|p^R_{x_1x_1}\psi^2_{1x_1}|dx$$

$$\leq C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1}dx \quad (4.57)$$

$$C\int|(p^R_{x_1})^2\psi_{1x_1}\chi|dx \leq C||(p^R_{x_1})^2||_{L^\infty}\int\chi^2dx+C||p^R_{x_1}||_{L^\infty}\int p^R_{x_1}\psi^2_{1x_1}dx$$
$$\leq C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int u^R_{1x_1}\psi^2_{1x_1}dx, \quad (4.58)$$

$$C\int|p^R_{x_1}\psi_{1x_1}\chi_{x_1}|dx \leq C||p^R_{x_1}||^{1/2}_{L^\infty}\int|p^R_{x_1}\psi^2_{1x_1}|dx+C||p^R_{x_1}||^{1/2}_{L^\infty}\int\chi^2_{x_1}dx$$
$$\leq C\delta^{1/2}\int(u^R_{1x_1}\psi^2_{1x_1}+\chi^2_{x_1})dx, \quad (4.59)$$

hold, then

$$C\int|K_{10}|dx \leq C(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int(u^R_{1x_1}\psi^2_{1x_1}+\chi^2_{x_1})dx. \quad (4.60)$$

Observe that

$$|K_{11}|=\left|2b_6\left(u^R_{1x_1}-\frac{v^R_{x_1}}{v^R}\right)\frac{v^R_{x_1}\chi_{x_1}\phi}{(v^R)^2}\right|\leq C[|u^R_{1x_1}v^R_{x_1}\chi_{x_1}\phi|+|(v^R_{x_1})^2\chi_{x_1}\phi|],$$

so the following inequality forms

$$C\int|K_{11}|dx \leq C||(u^R_{1x_1},v^R_{x_1})v^R_{x_1}||_{L^\infty}\int(\phi^2+\chi^2_{x_1})dx$$
$$\leq C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int\chi^2_{x_1}dx. \quad (4.61)$$

It is easy to see that

$$C\int|K_{12}|dx \leq C\int u^R_{1x_1}\chi^2_{x_1}dx \leq C\delta\int\chi^2_{x_1}dx. \quad (4.62)$$

And K_{13} could be controlled by

$$|K_{13}|=\left|\left(\frac{u^R_1v^R_{x_1}}{v^R}-u^R_{1x_1}\right)\frac{(v^R_{x_1})^2\phi^2}{(v^R)^2}\right|\leq C(v^R_{x_1})^2\phi^2,$$

which shows

$$C\int|K_{13}|dx \leq C\int(v^R_{x_1})^2\phi^2dx \leq C(1+t)^{-2}(\delta\ln(1+t)+1). \quad (4.63)$$

For K_{14}, we have

$$C\int|K_{14}|dx \leq C\int|u^R_{1x_1}\phi_{x_1}\chi_{x_1}|dx$$
$$\leq C||u^R_{x_1}||^{1/2}_{L^\infty}\int u^R_{1x_1}\phi^2_{x_1}dx+C||u^R_{x_1}||^{1/2}_{L^\infty}\int\chi^2_{x_1}dx$$
$$\leq C\delta^{1/2}\int(u^R_{1x_1}\phi^2_{x_1}+\chi^2_{x_1})dx. \quad (4.64)$$

After a straightforward computation, the following estimate for K_{15} is arrived

$$
\begin{aligned}
|K_{15}|=&\left|(\gamma p^R v^R u^R_{1x_1}-p^R_{x_1}v^R\psi_1-b_5u^R_{1x_1}v^R\phi-u^R_1v^R_{x_1}\phi)_{x_1}\frac{v^R_{x_1}\phi-b_6\chi_{x_1}}{(v^R)^2}\right|\\
\le& C\big(|p^R_{x_1}u^R_{1x_1}v^R_{x_1}\phi|+|u^R_{1x_1}(v^R_{x_1})^2\phi|+|p^R_{x_1}u^R_{1x_1}\chi_{x_1}|+|u^R_{1x_1}v^R_{x_1}\chi_{x_1}|\\
&+|u^R_{1x_1x_1}v^R_{x_1}\phi|+|u^R_{1x_1x_1}\chi_{x_1}|+|p^R_{x_1}(v^R_{x_1})^2\phi\psi_1|+|u^R_{1x_1}(v^R_{x_1})^2\phi^2|\\
&+|p^R_{x_1x_1}v^R_{x_1}\phi\psi_1|+|u^R_{1x_1x_1}v^R_{x_1}\phi^2|+|v^R_{1x_1x_1}v^R_{x_1}\phi^2|+|p^R_{x_1x_1}\psi_1\chi_{x_1}|\\
&+|u^R_{1x_1x_1}\phi\chi_{x_1}|+|v^R_{x_1x_1}\phi\chi_{x_1}|+|p^R_{x_1}v^R_{x_1}\psi_1\chi_{x_1}|+|u^R_{1x_1}v^R_{x_1}\phi\chi_{x_1}|\\
&+|p^R_{x_1}v^R_{x_1}\phi\psi_{1x_1}|+|u^R_{1x_1}v^R_{x_1}\phi_{x_1}\phi|+|(v^R_{x_1})^2\phi_{x_1}\phi|+|p^R_{x_1}\psi_{1x_1}\chi_{x_1}|\\
&+|v^R_{x_1}\phi_{x_1}\chi_{x_1}|+|u^R_{1x_1}\phi_{x_1}\chi_{x_1}|\big).
\end{aligned}\tag{4.65}
$$

Now we would like to control the term $\int|K_{15}|(x,t)dx$. The following inequalities have been obtained by Lemma 2.1, Lemmas 4.1-4.2, Young's inequality and the smallness of δ:

$$
\begin{aligned}
&C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}v^R_{x_1}\phi|dx\\
\le& C||u^R_{1x_1}||^2_{L^\infty}\int\phi^2dx+C\int|(p^R_{x_1},v^R_{x_1})v^R_{x_1}|^2dx\\
\le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C||(p^R_{x_1},v^R_{x_1})^2v^R_{x_1}||_{L^\infty}\int|v^R_{x_1}|dx\\
\le& C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta(1+t)^{-3},
\end{aligned}\tag{4.66}
$$

$$
\begin{aligned}
&C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}\chi_{x_1}|dx\le C||(p^R_{x_1},v^R_{x_1})||^2_{L^\infty}\int|u^R_{1x_1}|dx+C\int u^R_{1x_1}\chi^2_{x_1}dx\\
\le& C\delta(1+t)^{-2}+C\delta\int\chi^2_{x_1}dx,
\end{aligned}\tag{4.67}
$$

$$
\begin{aligned}
&C\int|u^R_{1x_1x_1}v^R_{x_1}\phi|dx\le C||u^R_{1x_1x_1}||_{L^\infty}\int\phi^2dx+C\int|u^R_{1x_1x_1}(v^R_{x_1})^2|dx\\
\le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{9/8}(1+t)^{-9/4}\\
\le& C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1),
\end{aligned}\tag{4.68}
$$

$$
\begin{aligned}
&C\int|u^R_{1x_1x_1}\chi_{x_1}|dx\le C||u^R_{1x_1x_1}||^{1/2}_{L^\infty}\int|u^R_{1x_1x_1}|dx+C||u^R_{1x_1x_1}||^{1/2}_{L^\infty}\int\chi^2_{x_1}dx\\
\le& C\delta^{1/16}(1+t)^{-13/8}+C\delta\int\chi^2_{x_1}dx,
\end{aligned}\tag{4.69}
$$

$$
\begin{aligned}
&C\int|p^R_{x_1}(v^R_{x_1})^2\phi\psi_1|dx\le C||p^R_{x_1}(v^R_{x_1})^2||_{L^\infty}\int(\phi^2+\psi_1^2)dx\\
\le& C(1+t)^{-3}(\delta\ln(1+t)+1).
\end{aligned}\tag{4.70}
$$

By the same way as (4.70), one has

$$
C\int|u^R_{1x_1}(v^R_{x_1})^2\phi^2|dx\le C(1+t)^{-3}(\delta\ln(1+t)+1),\tag{4.71}
$$

$$C\int|p^R_{x_1x_1}v^R_{x_1}\phi\psi_1|dx\leq C\|p^R_{x_1x_1}v^R_{x_1}\|_{L^\infty}\int(\phi^2+\psi_1^2)dx$$
$$\leq C\delta^{1/8}(1+t)^{-9/4}(\delta\ln(1+t)+1). \tag{4.72}$$

Similarly, $C\int|(u^R_{1x_1x_1},v^R_{x_1x_1})v^R_{x_1}\phi^2|dx$ could be estimated as (4.72) and the same result is obtained.

$$C\int|p^R_{x_1x_1}\psi_1\chi_{x_1}|dx\leq C\|p^R_{x_1x_1}\|_{L^\infty}\int(\psi_1^2+\chi_{x_1}^2)dx$$
$$\leq C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta\int\chi_{x_1}^2dx \tag{4.73}$$

and $C\int|(u^R_{1x_1x_1},v^R_{x_1x_1})\phi\chi_{x_1}|dx$ follow the idea of (4.73) to obtain the same results. Moreover, the following inequalities hold

$$C\int|p^R_{x_1}v^R_{x_1}\psi_1\chi_{x_1}|dx\leq C\|p^R_{x_1}v^R_{x_1}\|_{L^\infty}\int(\psi_1^2+\chi_{x_1}^2)dx$$
$$\leq C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int\chi_{x_1}^2dx, \tag{4.74}$$

$$C\int|u^R_{1x_1}v^R_{x_1}\phi\chi_{x_1}|dx\leq C\|u^R_{1x_1}v^R_{x_1}\|_{L^\infty}\int(\phi^2+\chi_{x_1}^2)dx$$
$$\leq C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int\chi_{x_1}^2dx, \tag{4.75}$$

$$C\int|(v^R_{x_1})^2\phi\phi_{x_1}|dx\leq C\|v^R_{x_1}\|^2_{L^\infty}\int(\phi^2+\phi_{x_1}^2)dx$$
$$\leq C(1+t)^{-2}(\delta\ln(1+t)+1)+C\delta\int\phi_{x_1}^2dx, \tag{4.76}$$

$$C\int|(v^R_{x_1},u^R_{1x_1})\phi_{x_1}\chi_{x_1}|dx\leq C\|(v^R_{x_1},u^R_{1x_1})\|^{1/2}_{L^\infty}\int\left(|(v^R_{x_1},u^R_{1x_1})\phi^2_{x_1}|+\chi^2_{x_1}\right)dx$$
$$\leq C\delta^{1/2}\int(u^R_{1x_1}\phi^2_{x_1}+\chi^2_{x_1})dx. \tag{4.77}$$

Hence, putting (4.36), (4.39), (4.59) and (4.65)-(4.77) together, one can obtain

$$C\int|K_{15}|dx\leq C(1+t)^{-5/4}(\delta\ln(1+t)+1)+\delta^{1/2}\int[u^R_{1x_1}(\phi^2_{x_1}+\psi^2_{1x_1})+\chi^2_{x_1}]dx. \tag{4.78}$$

Now we proceed to estimate K_{16}, since

$$K_{16}=b_6\frac{(u_1^R\chi_{x_1})_{x_1}v^R_{x_1}\phi}{(v^R)^2}$$
$$=b_6\left[\left(\frac{u_1^Rv^R_{x_1}\phi\chi_{x_1}}{(v^R)^2}\right)_{x_1}-\frac{u_1^Rv^R_{x_1x_1}\phi\chi_{x_1}}{(v^R)^2}-\frac{u_1^Rv^R_{x_1}\phi_{x_1}\chi_{x_1}}{(v^R)^2}+\frac{2u_1^R(v^R_{x_1})^2\phi\chi_{x_1}}{(v^R)^3}\right] \tag{4.79}$$

and due to (4.61), (4.73) and (4.77), one can check that

$$C\left|\int K_{16}dx\right|\leq C\int|(v^R_{x_1})^2\phi\chi_{x_1}|+|v^R_{x_1x_1}\phi\chi_{x_1}|+|v^R_{x_1}\phi_{x_1}\chi_{x_1}|dx$$

$$\leq C(1+t)^{-5/4}(\delta\ln(1+t)+1)+\delta^{1/2}\int(u^R_{1x_1}\phi^2_{x_1}+\chi^2_{x_1})dx. \tag{4.80}$$

While for K_{17}, it is clear that

$$C\int|K_{17}|dx\leq C\int\Big(\delta^{1/2}(u^R_{1x_1}\phi^2_{x_1}+\chi^2_{x_1})+|u^R_{1x_1}\phi_{x_1}\zeta_{x_1}|\Big)dx \\ +C(1+t)^{-5/4}(\delta\ln(1+t)+1), \tag{4.81}$$

where (4.62), (4.64), (4.75) and the following estimates have been used,

$$\begin{aligned}|K_{17}|=&\left|b_5b_6\frac{(u^R_{1x_1}p^Rv^R\zeta+u^R_{1x_1}\chi)_{x_1}}{(v^R)^2}(v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})\right|\\ \leq& C(|u^R_{1x_1x_1}v^R_{x_1}\phi\zeta|+|u^R_{1x_1x_1}\phi_{x_1}\zeta|+|u^R_{1x_1x_1}\zeta\chi_{x_1}|+|p^R_{x_1}u^R_{1x_1}v^R_{x_1}\phi\zeta|\\ &+|u^R_{1x_1}(v^R_{x_1})^2\phi\zeta|+|p^R_{x_1}u^R_{1x_1}\phi_{x_1}\zeta|+|u^R_{1x_1}v^R_{x_1}\phi_{x_1}\zeta|+|p^R_{x_1}u^R_{1x_1}\zeta\chi_{x_1}|\\ &+|u^R_{1x_1}v^R_{x_1}\zeta\chi_{x_1}|+|u^R_{1x_1}v^R_{x_1}\phi\zeta_{x_1}|+|u^R_{1x_1}\phi_{x_1}\zeta_{x_1}|+|u^R_{1x_1}\zeta_{x_1}\chi_{x_1}|\\ &+|u^R_{1x_1x_1}v^R_{x_1}\phi\chi|+|u^R_{1x_1x_1}\phi_{x_1}\chi|+|u^R_{1x_1x_1}\chi_{x_1}\chi|+|u^R_{1x_1}v^R_{x_1}\phi\chi_{x_1}|\\ &+|u^R_{1x_1}\phi_{x_1}\chi_{x_1}|+|u^R_{1x_1}\chi^2_{x_1}|),\end{aligned} \tag{4.82}$$

and

$$\begin{aligned}&C\int|u^R_{1x_1x_1}v^R_{x_1}\phi\zeta|dx\leq C\|u^R_{1x_1x_1}\|^2_{L^\infty}\int|\phi^2_{x_1}|dx+C\|v^R_{x_1}\|^2_{L^\infty}\int\zeta^2dx\\ \leq& C\delta^{1/4}(1+t)^{-5/2}(\delta\ln(1+t)+1)+C(1+t)^{-2},\end{aligned} \tag{4.83}$$

$$\begin{aligned}&C\int|u^R_{1x_1x_1}\phi_{x_1}\zeta|dx\leq C\int|u^R_{1x_1x_1}\phi^2_{x_1}|dx+C\|u^R_{1x_1x_1}\|_{L^\infty}\int\zeta^2dx\\ \leq& C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1}dx+C\delta^{1/8}(1+t)^{-5/4},\end{aligned} \tag{4.84}$$

$$\begin{aligned}&C\int|u^R_{1x_1x_1}\zeta\chi_{x_1}|dx\leq +C\|u^R_{1x_1x_1}\|_{L^\infty}\int(\zeta^2+\chi^2_{x_1})dx\\ \leq& C\delta\int\chi^2_{x_1}dx+C\delta^{1/8}(1+t)^{-5/4},\end{aligned} \tag{4.85}$$

$$\begin{aligned}&C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}v^R_{x_1}\phi\zeta|dx\leq C\|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}v^R_{x_1}\|_{L^\infty}\int(\phi^2+\zeta^2)dx\\ \leq& C(1+t)^{-3}(\delta\ln(1+t)+1),\end{aligned} \tag{4.86}$$

$$\begin{aligned}&C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}\phi_{x_1}\zeta|dx\leq C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}|(\phi^2_{x_1}+\zeta^2)dx\\ \leq& C\delta\int u^R_{1x_1}\phi^2_{x_1}dx+C(1+t)^{-2},\end{aligned} \tag{4.87}$$

$$\begin{aligned}&C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}\zeta\chi_{x_1}|dx\leq C\int|(p^R_{x_1},v^R_{x_1})u^R_{1x_1}|(\chi^2_{x_1}+\zeta^2)dx\\ \leq& C\delta\int\chi^2_{x_1}dx+C(1+t)^{-2},\end{aligned} \tag{4.88}$$

$$\begin{aligned}&C\int|u^R_{1x_1}v^R_{x_1}\phi\zeta_{x_1}|dx\le C\|u^R_{1x_1}v^R_{x_1}\|_{L^\infty}\int(\phi^2+\zeta^2_{x_1})dx\\&\le C(1+t)^{-2}(\delta\ln(1+t)+1)\end{aligned}\tag{4.89}$$

$$\begin{aligned}&C\int|u^R_{1x_1}\zeta_{x_1}\chi_{x_1}|dx\le C\|u^R_{1x_1}\|^{3/2}_{L^\infty}\int\zeta^2_{x_1}dx+C\|u^R_{1x_1}\|^{1/2}_{L^\infty}\int\chi^2_{x_1}dx\\&\le C(1+t)^{-3/2}+C\delta^{1/2}\int\chi^2_{x_1}dx,\end{aligned}\tag{4.90}$$

$$\begin{aligned}&C\int|u^R_{1x_1x_1}v^R_{x_1}\phi\chi|dx\le C\|u^R_{1x_1x_1}v^R_{x_1}\|_{L^\infty}\int(\phi^2+\chi^2)dx\\&\le C\delta^{1/8}(1+t)^{-9/4}(\delta\ln(1+t)+1),\end{aligned}\tag{4.91}$$

$$\begin{aligned}&C\int|u^R_{1x_1x_1}\phi_{x_1}\chi|dx\le C\|u^R_{1x_1x_1}\|_{L^\infty}\int\chi^2dx+C\int|u^R_{1x_1x_1}\phi^2_{x_1}|\\&\le C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1},\end{aligned}\tag{4.92}$$

$$\begin{aligned}&C\int|u^R_{1x_1x_1}\chi_{x_1}\chi|dx\le C\|u^R_{1x_1x_1}\|_{L^\infty}\int\chi^2dx+C\int|u^R_{1x_1x_1}\chi^2_{x_1}|\\&\le C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1)+C\delta^{1/2}\int u^R_{1x_1}\chi^2_{x_1}.\end{aligned}\tag{4.93}$$

On the one hand, putting the estimates for $C\left|\int K_i dx\right|$, $1\le i\le 17$ together and integrating the inequality (4.32) over $t'\in[0,t]$, then we have

$$\begin{aligned}&\int[\phi^2_{x_1}+\psi^2_{x_1}+(v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})^2]dx+\int_0^t\int\left[u^R_{1x_1}(\phi^2_{x_1}+|\psi_{x_1}|^2)+\chi^2_{x_1}\right]dxdt'\\&\le C\int_0^t\int(|p^R_{x_1}\psi_{2x_1}\phi_{x_2}|+|p^R_{x_1}\psi_{3x_1}\phi_{x_3}|+|p^R_{x_1}\psi_{1x_1}\zeta_{x_1}|+|u^R_{1x_1}\phi_{x_1}\zeta_{x_1}|)dxdt'\\&\quad+C\int_0^t(1+t')^{-5/4}(\delta\ln(1+t)+1)dt'+C\\&\le C\int_0^t\int(|p^R_{x_1}\psi_{2x_1}\phi_{x_2}|+|p^R_{x_1}\psi_{3x_1}\phi_{x_3}|+|p^R_{x_1}\psi_{1x_1}\zeta_{x_1}|+|u^R_{1x_1}\phi_{x_1}\zeta_{x_1}|)dxdt'+C.\end{aligned}\tag{4.94}$$

On the other hand, based on the results of Lemma 4.1 and (4.25), the following inequalities are established:

$$\begin{aligned}C\int_0^t\int|p^R_{x_1}\psi_{2x_1}\phi_{x_2}|(x,t')dxdt'&\le\frac18\int_0^t\int u^R_{1x_1}\psi^2_{2x_1}dxdt'+C\int_0^t\int u^R_{1x_1}\phi^2_{x_2}dxdt'\\&\le\frac18\int_0^t\int u^R_{1x_1}\psi^2_{2x_1}dxdt'+C,\end{aligned}\tag{4.95}$$

$$\begin{aligned}C\int_0^t\int|p^R_{x_1}\psi_{3x_1}\phi_{x_3}|(x,t')dxdt'&\le\frac18\int_0^t\int u^R_{1x_1}\psi^2_{3x_1}dxdt'+C\int_0^t\int u^R_{1x_1}\phi^2_{x_3}dxdt'\\&\le\frac18\int_0^t\int u^R_{1x_1}\psi^2_{3x_1}dxdt'+C,\end{aligned}\tag{4.96}$$

$$C\int_0^t\int|p^R_{x_1}\psi_{1x_1}\zeta_{x_1}|(x,t')dxdt'\le\frac18\int_0^t\int u^R_{1x_1}\psi^2_{1x_1}dxdt'+C\int_0^t\int u^R_{1x_1}\zeta^2_{x_1}dxdt'$$

$$\leq \frac{1}{8}\int_0^t\int u_{1x_1}^R\psi_{1x_1}^2 dxdt' + C, \tag{4.97}$$

$$\begin{aligned} C\int_0^t\int |u_{1x_1}^R\phi_{x_1}\zeta_{x_1}|(x,t')dxdt' &\leq \frac{1}{8}\int_0^t\int u_{1x_1}^R\phi_{x_1}^2 dxdt' + C\int_0^t\int u_{1x_1}^R\zeta_{x_1}^2 dxdt' \\ &\leq \frac{1}{8}\int_0^t\int u_{1x_1}^R\phi_{x_1}^2 dxdt' + C. \end{aligned} \tag{4.98}$$

Hence, summing up (4.94)-(4.98), then

$$||(\phi_{x_1},\psi_{x_1})||^2 + \int_0^t\int \left[u_{1x_1}^R(\phi_{x_1}^2 + |\psi_{x_1}|^2) + \chi_{x_1}^2\right] dxdt' \leq C \tag{4.99}$$

and

$$\int (v_{x_1}^R\phi + v^R\phi_{x_1} - b_6\chi_{x_1})^2 dx \leq C \tag{4.100}$$

hold. Moreover, noting that

$$\int (v_{x_1}^R\phi)^2 dx \leq C(1+t)^{-2}(\delta\ln(1+t)+1) \leq C \tag{4.101}$$

forms, then

$$\int (v_{x_1}^R\phi + v^R\phi_{x_1})^2 dx \leq C, \tag{4.102}$$

which implies that

$$\int \chi_{x_1}^2(t) dx \leq C \tag{4.103}$$

due to (4.100). Combining (4.25), (4.99) with (4.103), the desired results are obtained. □

Next, we would like to give the second-order estimates.

Lemma 4.4. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time t such that*

$$||(\nabla^2\phi,\nabla^2\chi)||^2 + \int_0^t\int \left(u_{1x_1}^R|\nabla^2\phi|^2 + |\nabla^2\chi|^2\right)(x,t')dxdt' \leq C,$$

and

$$||\nabla^2\psi||^2 + \int_0^t\int u_{1x_1}^R\left(|\psi_{x_1x_1}|^2 + |\psi_{1x_ix_j}|^2 + |\psi_{x_1x_j}|^2\right)(x,t')dxdt' \leq C,$$

where $i,j=2,3$.

Proof. Because we have

$$\begin{cases} \frac{\phi_{tx_ix_j}}{p^R} = -\frac{u_1^R}{p^R}\phi_{x_1x_ix_j} - \frac{p_{x_1}^R}{p^R}\psi_{1x_ix_j} - \beta\frac{u_{1x_1}^R}{p^R}\phi_{x_ix_j} - \beta\mathrm{div}\psi_{x_ix_j} - \frac{2\chi_{x_ix_j}}{\tau\alpha p^R v^R}, \\ \frac{\psi_{tx_ix_j}}{v^R} = -\frac{u_1^R}{v^R}\psi_{x_1x_ix_j} - \nabla\phi_{x_ix_j} - \frac{\psi_{1x_ix_j}}{v^R}\nabla u_1^R + b_3\frac{\phi_{x_ix_j}}{p^R}\nabla p^R - b_4\nabla p^R\zeta_{x_ix_j} - b_4\frac{\chi_{x_ix_j}}{p^R v^R}\nabla p^R, \\ \phi_{x_ix_jt} - \frac{b_6}{v^R}\chi_{x_ix_jt} = -u_1^R\phi_{x_1x_ix_j} - p_{x_1}^R\psi_{1x_ix_j} + (b_5b_6-\beta)u_{1x_1}^R\phi_{x_ix_j} + b_7\frac{\chi_{x_ix_j}}{\tau v^R} \\ \qquad + b_6\frac{u_1^R\chi_{x_1x_ix_j}}{v^R} + b_5b_6p^R u_{1x_1}^R\zeta_{x_ix_j} + b_5b_6\frac{u_{1x_1}^R\chi_{x_ix_j}}{v^R}, \end{cases}$$

where $i,j=2,3$. Multiplying the above equations by $\phi_{x_ix_j}$, $\beta\psi_{x_ix_j}$ and $\frac{b_5}{p^R}(\phi_{x_ix_j}-\frac{b_6}{v^R}\chi_{x_ix_j})$, respectively, then a calculation gives us

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\phi^2_{x_ix_j}}{p^R}\right)+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\beta|\psi_{x_ix_j}|^2}{v^R}\right)+\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{b_5}{p^R}\left(\phi_{x_ix_j}-b_6\frac{\chi_{x_ix_j}}{v^R}\right)^2\right]\\
&\quad+\frac{b_8}{p^R}u^R_{1x_1}\phi^2_{x_ix_j}+\frac{\beta}{v^R}u^R_{1x_1}|\psi_{1x_ix_j}|^2+\frac{b_5b_6b_7\chi^2_{x_ix_j}}{\tau p^R(v^R)^2}\\
&=-\frac{1}{2}\left[\beta\frac{u^R_1}{v^R}|\psi_{x_ix_j}|^2+(b_5+1)\frac{u^R_1}{p^R}\phi^2_{x_ix_j}+b_5b_6^2\frac{u^R_1\chi^2_{x_ix_j}}{p^R(v^R)^2}-2b_5b_6\frac{u^R_1\phi_{x_ix_j}\chi_{x_ix_j}}{p^Rv^R}\right]_{x_1}\\
&\quad-\beta\mathrm{div}(\phi_{x_ix_j}\psi_{x_ix_j})-\beta b_4p^R_{x_1}\psi_{1x_ix_j}\zeta_{x_ix_j}-\beta b_4\frac{p^R_{x_1}\psi_{1x_ix_j}\chi_{x_ix_j}}{p^Rv^R}+b_9\frac{u^R_{1x_1}\phi_{x_ix_j}\chi_{x_ix_j}}{p^Rv^R}\\
&\quad+b_5^2b_6u^R_{1x_1}\phi_{x_ix_j}\zeta_{x_ix_j}+b_5b_6\frac{p^R_{x_1}\psi_{1x_ix_j}\chi_{x_ix_j}}{p^Rv^R}-b_5^2b_6^2\frac{u^R_{1x_1}\zeta_{x_ix_j}\chi_{x_ix_j}}{v^R}+b_{11}\frac{u^R_{1x_1}\chi^2_{x_ix_j}}{p^R(v^R)^2}\\
&=:-\frac{1}{2}\left[\beta\frac{u^R_1}{v^R}|\psi_{x_ix_j}|^2+(b_5+1)\frac{u^R_1}{p^R}\phi^2_{x_ix_j}+b_5b_6^2\frac{u^R_1\chi^2_{x_ix_j}}{p^R(v^R)^2}-2b_5b_6\frac{u^R_1\phi_{x_ix_i}\chi_{x_ix_i}}{p^Rv^R}\right]_{x_1}\\
&\quad-\beta\mathrm{div}(\phi_{x_ix_j}\psi_{x_ix_j})+\sum_{k=1}^{7}L_k,\ i,j=2,3.
\end{aligned}$$

So one can follow the same idea to deal with (4.24), by Lemma 2.1, Lemmas 4.1-4.2, Young's inequality and the smallness of δ, the following inequality could be arrived

$$||(\phi_{x_ix_j},\psi_{x_ix_j},\chi_{x_ix_j})||^2+\int_0^t\int\left[u^R_{1x_1}(\phi^2_{x_ix_j}+\psi^2_{1x_ix_j})+\chi^2_{x_ix_j}\right](x,t')dxdt'\le C,\qquad(4.104)$$

for $i,j=2,3$. By means of (4.26), it is apparent to have the following system

$$\begin{cases}
\phi_{tx_1x_i}=-(u^R_1\phi_{x_1}+\psi_1p^R_{x_1})_{x_1x_i}-\beta(u^R_{1x_1}\phi+p^R\mathrm{div}\psi)_{x_1x_i}-\frac{2}{\tau\alpha}\left(\frac{\chi}{v^R}\right)_{x_1x_i},\\
\frac{p^R}{v^R}\psi_{tx_1x_i}=\frac{p^R}{v^R}\left[-u^R_1\psi_{x_1}-v^R\nabla\phi-\psi_1\nabla u^R_1+b_3\frac{v^R}{p^R}\phi\nabla p^R-b_4v^R\nabla p^R\zeta-b_4\frac{\chi}{p^R}\nabla p^R\right]_{x_1x_i},\\
(v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})_{tx_i}=[\gamma p^Rv^Ru^R_{1x_1}-u^R_1v^R\phi_{x_1}-p^R_{x_1}v^R\psi_1-b_5u^R_{1x_1}v^R\phi]_{x_1x_i}+\frac{b_7\chi_{x_1x_i}}{\tau}\\
\qquad\qquad+[b_6u^R_1\chi_{x_1}+b_5b_6u^R_{1x_1}p^Rv^R\zeta+b_5b_6u^R_{1x_1}\chi-u^R_1v^R_{x_1}\phi]_{x_1x_i},
\end{cases}$$

for $i=2,3$. Then a straightforward calculation gives

$$\begin{aligned}
&\frac{1}{2b_5}\frac{\partial}{\partial t}\phi^2_{x_1x_i}+\frac{b_{12}}{2}\frac{\partial}{\partial t}\left(\frac{p^R}{v^R}|\psi_{x_1x_i}|^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{(v^R_{x_1}\phi_{x_i}+v^R\phi_{x_1x_i}-b_6\chi_{x_1x_i})^2}{(v^R)^2}\right]\\
&\quad+b_{13}u^R_{1x_1}\phi^2_{x_1x_i}+\frac{b_{12}}{2}\frac{p^R}{v^R}u^R_{1x_1}\left[(\gamma+2)|\psi_{x_1x_i}|^2+\psi^2_{1x_1x_i}\right]+\frac{b_6b_7}{\tau(v^R)^2}\chi^2_{x_1x_i}\\
&=-\tilde{W}_{x_1}-b_{12}\mathrm{div}(p^R\psi_{x_1x_i}\phi_{x_1x_i})+b_{14}p^R_{x_1}\phi_{x_1x_i}\psi_{1x_1x_i}\\
&\quad-[(b_5+b_{12})u^R_{1x_1x_1}+(b_5+3)\frac{u^R_{1x_1}v^R_{x_1}}{v^R}]\phi_{x_i}\phi_{x_1x_i}-\frac{b_7}{\tau}\left(\frac{v^R_{x_1}}{(v^R)^2}\right)_{x_1}\phi_{x_i}\chi_{x_i}\\
&\quad-b_{14}\left[p^R_{x_1x_1}\psi_{1x_i}\phi_{x_1x_i}-\left(\frac{p^R_{x_1}v^R_{x_1}}{v^R}-\frac{(p^R_{x_1})^2}{p^R}+p^R_{x_1x_1}\right)\phi_{x_i}\psi_{1x_1x_i}\right]-\frac{p^R_{x_1}v^R_{x_1}}{v^R}\psi_{1x_i}\phi_{x_1x_i}
\end{aligned}$$

$$-b_{12}p^R_{x_1}(\psi_{2x_2x_i}+\psi_{3x_3x_i})\phi_{x_1x_i}-\frac{b_{12}p^R_{x_1}v^R_{x_1}}{v^R}(\phi_{x_2x_i}\psi_{2x_1x_i}+\phi_{x_3x_i}\psi_{3x_1x_i})$$
$$-\frac{b_{12}p^R}{v^R}u^R_{1x_1x_1}\psi_{1x_1x_i}\psi_{1x_i}-\frac{b_4b_{12}p^R}{v^R}(\nabla p^Rv^R\zeta)_{x_1}\psi_{x_1x_i}-\frac{b_4b_{12}p^R}{v^R}\left(\frac{\nabla p^R\chi}{p^R}\right)_{x_1x_i}\psi_{x_1x_i}$$
$$+2b_6\left(u^R_{1x_1}-\frac{u^R_1v^R_{x_1}}{v^R}\right)\frac{v^R_{x_1}\chi_{x_1x_i}\phi_{x_i}}{(v^R)^2}-\frac{3b_6^2u^R_{1x_1}}{2}\frac{\chi^2_{x_1x_i}}{(v^R)^2}+\left(\frac{u^R_1v^R_{x_1}}{v^R}-u^R_{1x_1}\right)\frac{(v^R_{x_1})^2\phi^2_{x_i}}{(v^R)^2}$$
$$+\frac{3b_6u^R_{1x_1}\phi_{x_1x_i}\chi_{x_1x_i}}{v^R}-(p^R_{x_1}v^R\psi_1+b_5u^R_{1x_1}v^R\phi+u^R_1v^R_{x_1}\phi)_{x_1x_i}\frac{v^R_{x_1}\phi_{x_i}-b_6\chi_{x_1x_i}}{(v^R)^2}$$
$$+b_6\frac{(u^R_1\chi_{x_1x_i})_{x_1}v^R_{x_1}\phi_{x_i}}{(v^R)^2}+b_5b_6\frac{(u^R_{1x_1}p^Rv^R\zeta+u^R_{1x_1}\chi)_{x_1x_i}}{(v^R)^2}(v^R_{x_1}\phi_{x_i}+v^R\phi_{x_1x_i}-b_6\chi_{x_1x_i}),$$

for $i=2,3$, where we have used (4.28) and let

$$\tilde{W}=:\frac{b_{14}u^R_1\phi^2_{x_1x_i}}{2}+\frac{b_{12}p^Ru^R_1|\psi_{x_1x_i}|^2}{2v^R}-\frac{b_7v^R_{x_1}\phi_{x_i}\chi_{x_i}}{\tau(v^R)^2}$$
$$+\frac{u^R_1}{v^R}\left(\frac{b_6^2}{2}\chi^2_{x_1x_i}+v^R_{x_1}\phi_{x_i}\phi_{x_1x_i}-b_6\phi_{x_1x_i}\chi_{x_1x_i}\right).$$

By a similar way as (4.27), one can get

$$||(\phi_{x_1x_i},\psi_{x_1x_i},\chi_{x_1x_i})||^2+\int_0^t\int\left[u^R_{1x_1}(\phi^2_{x_1x_i}+|\psi_{x_1x_i}|^2)+\chi^2_{x_1x_i}\right]dxdt'\le C, \tag{4.105}$$

for $i=2,3$. One can also obtain

$$\begin{cases}\phi_{tx_1x_1}=-b_2(p^R\mathrm{div}u^R)_{x_1x_1}-(u^R_1\phi_{x_1}+\psi_1p^R_{x_1})_{x_1x_1}-\beta(u^R_{1x_1}\phi+p^R\mathrm{div}\psi)_{x_1x_1}-\frac{2}{\tau\alpha}\left(\frac{\chi}{v^R}\right)_{x_1x_1},\\ \frac{p^R}{v^R}\psi_{tx_1x_1}=\frac{p^R}{v^R}\left[-u^R_1\psi_{x_1}-v^R\nabla\phi-\psi_1\nabla u^R_1+b_3\frac{v^R}{p^R}\phi\nabla p^R-b_4v^R\nabla p^R\zeta-b_4\frac{\chi}{p^R}\nabla p^R\right]_{x_1x_1},\\ (v^R_{x_1}\phi+v^R\phi_{x_1}-b_6\chi_{x_1})_{tx_1}=[\gamma p^Rv^Ru^R_{1x_1}-u^R_1v^R\phi_{x_1}-p^R_{x_1}v^R\psi_1-b_5u^R_{1x_1}v^R\phi]_{x_1x_1}+\frac{b_7\chi_{x_1x_1}}{\tau}\\ \qquad+[b_6u^R_1\chi_{x_1}+b_5b_6u^R_{1x_1}p^Rv^R\zeta+b_5b_6u^R_{1x_1}\chi-u^R_1v^R_{x_1}\phi]_{x_1x_1}.\end{cases}$$

Multiplying the above equations by $\frac{1}{b_5}\phi_{x_1x_1}$, $b_{12}\frac{p^R}{v^R}\psi_{x_1x_1}$ and $\frac{1}{(v^R)^2}(v^R\phi-b_6\chi)_{x_1x_1}$, respectively, then a calculation tells us that

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{|\phi_{x_1x_1}|^2}{b_5}+\frac{b_{12}p^R}{v^R}|\psi_{x_1x_1}|^2+\frac{(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}+v^R\phi_{x_1x_1}-b_6\chi_{x_1x_1})^2}{(v^R)^2}\right]$$
$$+b_{15}u^R_{1x_1}\phi^2_{x_1x_1}+\frac{p^R}{v^R}u^R_{1x_1}\left[b_{16}|\psi_{x_1x_1}|^2+b_{12}\psi^2_{1x_1x_1}\right]+\left(b_6u^R_{1x_1}+\frac{b_7}{\tau}\right)\frac{b_6\chi^2_{x_1x_1}}{(v^R)^2}$$
$$=\bar{W}_{x_1}-b_{12}\mathrm{div}(p^R\psi_{x_1x_1}\phi_{x_1x_1})+\gamma\left(2(p^Ru^R_{x_1})_{x_1}v^R_{x_1}+p^Ru^R_{x_1}v^R_{x_1x_1}\right)\frac{\phi_{x_1x_1}}{v^R}$$
$$-b_{17}u^R_{1x_1x_1}\phi_{x_1}\phi_{x_1x_1}-\left[b_{18}u^R_{1x_1x_1}+b_{19}\frac{u^R_{1x_1}v^R_{x_1x_1}}{v^R}+2b_5\frac{u^R_{1x_1x_1}v^R_{x_1}}{v^R}\right]\phi\phi_{x_1x_1}$$
$$+2\frac{\phi_{x_1}\phi_{x_1x_1}}{v^R}\left[u^R_1v^R_{x_1x_1}-b_{19}u^R_{1x_1}v^R_{x_1}\right]+b_{12}(\gamma-2)\frac{p^R}{v^R}v^R_{x_1}\psi_{1x_1x_1}\phi_{x_1x_1}$$
$$-b_{14}\phi_{x_1x_1}\left(2p^R_{x_1x_1}\psi_{1x_1}+p^R_{x_1x_1x_1}\psi_1\right)-2b_{12}p^R_{x_1}\phi_{x_1x_1}(\psi_{2x_1x_2}+\psi_{3x_1x_3})$$
$$-\frac{2b_{12}p^Rv^R_{x_1}}{v^R}(\phi_{x_1x_2}\psi_{2x_1x_1}+\phi_{x_1x_3}\psi_{3x_1x_1})+b_3b_{12}\frac{p^R\psi_{1x_1x_1}}{v^R}\left[\left(\frac{v^R}{p^R}p^R_{x_1}\right)_{x_1}\phi\right]_{x_1}$$

$$
\begin{aligned}
&-b_{12}\frac{p^R}{v^R}v^R_{x_1x_1}\nabla\phi\psi_{x_1x_1}+b_3b_{12}\phi_{x_1}\psi_{1x_1x_1}\left[\frac{p^R_{x_1}}{v^R}v^R_{x_1}+p^R_{x_1x_1}-\frac{(p^R_{x_1})^2}{p^R}\right]\\
&-\frac{\phi_{x_1x_1}}{v^R}\left[p^R_{x_1}v^R_{x_1x_1}\psi_1+2v^R_{x_1}(p^R_{x_1}\psi_1)_{x_1}\right]-b_{12}p^R_{x_1x_1}\mathrm{div}\psi\phi_{x_1x_1}\\
&+\frac{b_7}{\tau(v^R)^2}\left[\left(v^R_{x_1x_1}\chi+2v^R_{x_1}\chi_{x_1}-2\frac{(v^R_{x_1})^2\chi}{v^R}\right)\phi_{x_1x_1}+\chi_{x_1x_1}\left(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}\right)\right]\\
&-\frac{b_{12}p^R}{v^R}\left[\psi_{1x_1x_1}\left(u^R_{1x_1x_1x_1}\psi_1-2u^R_{1x_1x_1}\psi_{1x_1}\right)-u^R_{1x_1x_1}\psi_{x_1x_1}\cdot\psi_{x_1}\right]\\
&-\frac{b_4b_{12}p^R\psi_{x_1x_1}}{v^R}\left(v^R\zeta\nabla p^R+\frac{\chi\nabla p^R}{p^R}\right)_{x_1x_1}+\frac{u^R_1v^R_{x_1}}{v^R}\left[(v^R_{x_1x_1}\phi)^2+\frac{b_6^2\chi^2_{x_1x_1}}{(v^R)^2}\right]\\
&+\frac{2u^R_1(v^R_{x_1})^2v^R_{x_1x_1}}{(v^R)^3}\phi\phi_{x_1}+\frac{u^R_1v^R_{x_1}\phi_{x_1x_1}}{(v^R)^2}\left(2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1}\right)\\
&-\frac{u^R_{1x_1}}{(v^R)^2}\left[(v^R_{x_1x_1}\phi)^2+8(v^R_{x_1}\phi_{x_1})^2+6v^R_{x_1}v^R_{x_1x_1}\phi\phi_{x_1}\right]\\
&+\frac{b_6u^R_{1x_1}\chi_{x_1x_1}}{v^R}\left(5\phi_{x_1x_1}+2\frac{v^R_{x_1x_1}\phi}{v^R}\right)+\frac{6b_6u^R_{1x_1}v^R_{x_1}}{(v^R)^2}\phi_{x_1}\chi_{x_1x_1}\\
&+\left(\frac{u^R_{1x_1}}{v^R}+\frac{u^R_1v^R_{x_1}}{v^R}\right)\left(v^R_{x_1x_1x_1}\phi\phi_{x_1}+3v^R_{x_1x_1}\phi^2_{x_1}\right)+\frac{u^R_1v^R_{x_1}}{(v^R)^2}v^R_{x_1x_1x_1}\phi\\
&+\frac{b_6u^R_{1x_1}\chi_{x_1}}{(v^R)^2}\left[v^R_{x_1}(2\frac{v^R_{x_1x_1}}{v^R}\phi+\frac{4v^R_{x_1}\phi}{v^R}-2\phi_{x_1x_1})-v^R_{x_1x_1x_1}\phi-3v^R_{x_1x_1}\phi_{x_1}\right]\\
&+\frac{b_6u^R_{1x_1x_1}}{v^R}\phi_{x_1x_1}\chi_{x_1}-\frac{b_6\chi_{x_1x_1}}{(v^R)^2}\left[u^R_1v^R_{x_1x_1x_1}\phi+\phi_{x_1}\left(2u^R_1v^R_{x_1x_1}-v^Ru^R_{1x_1x_1}\right)\right]\\
&-\frac{b_6^2\chi_{x_1x_1}}{(v^R)^2}(u^R_1\chi_{x_1})_{x_1x_1}-\frac{v^R_{x_1}\phi_{x_1x_1}}{v^R}\left(u^R_{1x_1x_1}\phi+2u^R_{1x_1}\phi_{x_1}\right)\\
&+\frac{(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1})}{(v^R)^2}\left[b_5u^R_{1x_1}v^R\phi+\gamma p^Ru^R_{1x_1}v^R-p^R_{x_1}v^R\psi_1\right]_{x_1x_1}\\
&+\frac{b_5b_6(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1})}{(v^R)^2}\left[p^Ru^R_{1x_1}v^R\zeta+u^R_{1x_1}\chi\right]_{x_1x_1}\\
&-\frac{v^R_{x_1x_1}\phi_{x_1x_1}}{v^R}\left(u^R_{1x_1}\phi+2u^R_1\phi_{x_1}\right)-\frac{(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1})}{(v^R)^2}(u^R_1v^R_{x_1}\phi)_{x_1x_1}
\end{aligned}
\tag{4.106}
$$

where we have used (4.28) and set

$$
\begin{aligned}
&b_{15}=\frac{3b_{14}}{2}+b_{12}+b_5+1>0,\ b_{16}=\frac{b_{12}}{2}(\gamma+4),\\
&b_{17}=2b_{12}+b_{14}+2b_5,\ b_{18}=b_5+b_{12},\ b_{19}=b_5+2,\\
\bar{W}=:&-\frac{b_{14}u^R_1\phi^2_{x_1x_1}}{2}-\frac{b_{12}p^Ru^R_1|\psi_{x_1x_1}|^2}{2v^R}-\frac{b_6u^R_1\chi_{x_1x_1}\phi_{x_1x_1}}{v^R}\\
&-\frac{v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}}{(v^R)^2}\left(u^R_1(v^R\phi_{x_1}+b_6\chi_{x_1})\right)_{x_1}.
\end{aligned}
$$

Now we will deal with some terms as follows. Utilizing Lemma 2.1, Lemma 4.3 and

Cauchy's inequality, one has

$$\begin{aligned}&\int \gamma\left[2(p^R u^R_{1x_1})_{x_1} v^R_{x_1} + p^R u^R_{1x_1} v^R_{x_1x_1}\right]\frac{\phi_{x_1x_1}}{v^R}dx\\ \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\left(\|u^R_{1x_1}\|^2_{L^\infty}\|u^R_{1x_1}\|^3_{L^3} + \|u^R_{1x_1}\|_{L^\infty}\|u^R_{1x_1x_1}\|^2\right)\\ \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\delta^{1/8}(1+t)^{-9/4}, \end{aligned}\tag{4.107}$$

$$\begin{aligned}-\int b_{17}u^R_{1x_1x_1}\phi_{x_1}\phi_{x_1x_1}dx \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\|u^R_{1x_1x_1}\|_{L^\infty}\|\phi_{x_1}\|^2\\ \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4}, \end{aligned}\tag{4.108}$$

$$\begin{aligned}&\int\left[-b_{18}u^R_{1x_1x_1x_1} + b_{19}\frac{u^R_{1x_1}v^R_{x_1x_1}}{v^R} + 2b_5\frac{u^R_{1x_1x_1}v^R_{x_1}}{v^R}\right]\phi\phi_{x_1x_1}dx\\ \leq&\int\left(|u^R_{1x_1x_1x_1}|(\phi^2+\phi^2_{x_1x_1})\right)dx + \frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\|u^R_{1x_1}\|_{L^\infty}\|u^R_{1x_1x_1}\|^2_{L^\infty}\int\phi^2dx\\ \leq&(C\delta+\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4}[ln(1+t)+1], \end{aligned}\tag{4.109}$$

and

$$\begin{aligned}-\int 2b_{19}\frac{u^R_{1x_1}v^R_{x_1}}{v^R}\phi_{x_1}\phi_{x_1x_1}dx \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\|u^R_{1x_1}\|^3_{L^\infty}\|\phi_{x_1}\|^2\\ \leq&\frac{b_5}{32}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4}. \end{aligned}\tag{4.110}$$

It is clear that

$$\begin{aligned}\int b_{12}(\gamma-2)\frac{p^R}{v^R}v^R_{x_1}\psi_{1x_1x_1}\phi_{x_1x_1}dx \leq&\int|b_{12}p^R_{x_1}\psi_{1x_1x_1}\phi_{x_1x_1}|dx\\ \leq&\frac{b_{12}}{2}\int\left(\frac{p^R}{v^R}u^R_{1x_1}\psi^2_{1x_1x_1} + u^R_{1x_1}\phi^2_{x_1x_1}\right)dx \end{aligned}\tag{4.111}$$

where we have used Cauchy's inequality and the facts: $|\gamma-2|\leq\gamma$, and (4.28)-(4.30). In the same way, it gives that

$$\begin{aligned}&-\int 2b_{12}\frac{p^R}{v^R}v^R_{x_1}\left(\psi_{2x_1x_1}\phi_{x_1x_1} + \phi_{x_1x_3}\psi_{3x_1x_1}\right)dx\\ \leq&\frac{2}{\gamma}\int|b_{12}p^R_{x_1}\psi_{2x_1x_1}\phi_{x_1x_1}|dx + C\int|v^R_{x_1}\phi_{x_1x_3}\psi_{3x_1x_1}|\\ \leq&2b_{12}\int\frac{p^R}{v^R}u^R_{1x_1}\psi^2_{2x_1x_1}dx + \frac{b_{12}}{2}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\int|v^R_{x_1}\phi_{x_1x_3}\psi_{3x_1x_1}|. \end{aligned}\tag{4.112}$$

Obivously, one can check

$$\begin{aligned}-b_{12}\int\frac{p^R}{v^R}v^R_{x_1x_1}\nabla\phi\psi_{x_1x_1}dx \leq& C\delta^{1/2}\int u^R_{1x_1}|\psi_{x_1x_1}|^2dx + \|v^R_{x_1x_1}\|_{L^\infty}\|\nabla\phi\|^2\\ \leq& C\delta^{1/2}\int u^R_{1x_1}|\psi_{x_1x_1}|^2dx + C\delta^{1/8}(1+t)^{-5/4}, \end{aligned}\tag{4.113}$$

and

$$-b_{12}\int p^R_{x_1x_1}\mathrm{div}\psi\phi_{x_1}dx\le C\|p^R_{x_1x_1}\|^2_{L^\infty}(\|\nabla\psi\|^2+\|\nabla\phi\|^2)\le C\delta^{1/8}(1+t)^{-5/4}. \quad (4.114)$$

A direct computation shows us that

$$\begin{aligned}
&\int\frac{b_7}{\tau(v^R)^2}\left[\phi_{x_1x_1}\left(v^R_{x_1x_1}\chi+2v^R_{x_1}\chi_{x_1}-2\frac{(v^R_{x_1})^2\chi}{v^R}\right)+\chi_{x_1x_1}\left(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}\right)\right]dx\\
\le&C\int\left(|v^R_{x_1x_1}\phi_{x_1x_1}\chi|+|v^R_{x_1}\phi_{x_1x_1}\chi_{x_1}|+|(v^R_{x_1})^2\phi_{x_1x_1}\chi|\right)dx\\
&+C\int\left(|v^R_{x_1x_1}\phi\chi_{x_1x_1}|+|v^R_{x_1}\phi_{x_1}\chi_{x_1x_1}|\right)dx\\
\le&C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1x_1}dx+\|v^R_{x_1x_1}\|^2_{L^\infty}\|\chi\|^2+C\int|v^R_{x_1}\phi_{x_1x_1}\chi_{x_1}|dx+\|v^R_{x_1}\|^2_{L^\infty}\|\chi\|^2\\
&+\|v^R_{x_1x_1}\|_{L^\infty}\|\phi\|^2+\|v^R_{x_1}\|^2_{L^\infty}\|\phi_{x_1}\|^2+(C\delta+\frac{1}{32})\int\chi^2_{x_1x_1}dx\\
\le&C\delta^{1/2}\int u^R_{1x_1}\phi^2_{x_1x_1}dx+\left(C\delta+\frac{1}{32}\right)\int\chi^2_{x_1x_1}dx+C\int|v^R_{x_1}\phi_{x_1x_1}\chi_{x_1}|dx\\
&+C\delta^{1/8}(1+t)^{-5/4}(\delta\ln(1+t)+1).
\end{aligned}$$

For the following term, it is easy to see that

$$\begin{aligned}
&-b_4b_{12}\frac{p^R}{v^R}\psi_{x_1x_1}\left(v^R\zeta\nabla p^R+\frac{\chi\nabla p^R}{p^R}\right)_{x_1x_1}\\
\le&C\{|v^R_{x_1x_1}p^R_{x_1}\zeta\psi_{1x_1x_1}|+|v^R_{x_1}p^R_{x_1}\zeta_{x_1}\psi_{1x_1x_1}|+|p^R_{x_1}\zeta_{x_1x_1}\psi_{1x_1x_1}|\\
&+|p^R_{x_1x_1x_1}\zeta\psi_{1x_1x_1}|+|p^R_{x_1}\chi_{x_1x_1}\psi_{1x_1x_1}|+|(p^R_{x_1})^2\chi_{x_1}\psi_{1x_1x_1}|\\
&+|(p^R_{x_1})^3\chi\psi_{1x_1x_1}|+|p^R_{x_1x_1}p^R_{x_1}\chi\psi_{1x_1x_1}|+|p^R_{x_1x_1x_1}\chi\psi_{1x_1x_1}|\\
&+|v^R_{x_1}p^R_{x_1x_1}\zeta\psi_{1x_1x_1}|+|p^R_{x_1x_1}\zeta_{x_1}\psi_{1x_1x_1}|+|p^R_{x_1x_1}\chi_{x_1}\psi_{1x_1x_1}|\} \quad (4.115)
\end{aligned}$$

and we also have

$$\begin{aligned}
&\int|(v^R_{x_1x_1}p^R_{x_1},p^R_{x_1x_1}v^R_{x_1})\zeta\psi_{1x_1x_1}|dx\\
\le&C\int|u^R_{1x_1}(v^R_{x_1x_1},p^R_{x_1x_1})\zeta\psi_{1x_1x_1}|dx\\
\le&C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx+\|(v^R_{x_1x_1},p^R_{x_1x_1})\|_{L^\infty}\|u^R_{1x_1}\|^2_{L^\infty}\|\zeta\|^2\\
\le&C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx+C\delta^{1/8}(1+t)^{-5/4},\\
&\int|v^R_{x_1}p^R_{x_1}\zeta_{x_1}\psi_{1x_1x_1}|dx\\
\le&C\|v^R_{x_1}\|^{1/2}_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx+\|v^R_{x_1}\|^{1/2}_{L^\infty}\|p^R_{x_1}\|^2_{L^\infty}\|\zeta_{x_1}\|^2\\
\le&C\delta^{3/4}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx+C\delta^{1/4}(1+t)^{-2},
\end{aligned}$$

$$\begin{aligned}
\int |p^R_{x_1x_1x_1}\zeta\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1x_1x_1}\|_{L^\infty}\|\zeta\|^2 + C\int |p^R_{x_1x_1x_1}|\psi^2_{1x_1x_1}dx \\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4},\\
\int |p^R_{x_1}\chi_{x_1x_1}\psi_{1x_1x_1}|dx \le & \|p^R_{x_1}\|^{1/2}_{L^\infty}\int \chi^2_{x_1x_1}dx + C\|p^R_{x_1}\|^{1/2}_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta^{1/2}\int \chi^2_{x_1x_1}dx + C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx,\\
\int |\left(p^R_{x_1}\right)^2\chi_{x_1}\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1}\|^2_{L^\infty}\|\chi_{x_1}\|^2 + C\|p^R_{x_1}\|_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4},\\
\int |(p^R_{x_1})^3\chi\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1}\|^3_{L^\infty}\|\chi\|^2 + C\|p^R_{x_1}\|^2_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1],\\
\int |p^R_{x_1x_1}p^R_{x_1}\chi\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1}p^R_{x_1x_1}\|_{L^\infty}\|\chi\|^2 + C\|p^R_{x_1x_1}\|_{L^\infty}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1],\\
\int |p^R_{x_1x_1x_1}\chi\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1x_1x_1}\|_{L^\infty}\|\chi\|^2 + C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/4}(1+t)^{-3/2}[\ln(1+t)+1],\\
\int |p^R_{x_1x_1}(\zeta_{x_1},\chi_{x_1})\psi_{1x_1x_1}|dx \le & C\|p^R_{x_1x_1}\|_{L^\infty}\|(\zeta_{x_1},\chi_{x_1})\|^2 + C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx\\
\le & C\delta\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/8}(1+t)^{-5/4},
\end{aligned}$$

by using Cauchy's inequality. Then the above inequalities imply that

$$\begin{aligned}
&-\int b_4b_{12}\frac{p^R}{v^R}\psi_{x_1x_1}\left(v^R\zeta\nabla p^R + \frac{\chi\nabla p^R}{p^R}\right)_{x_1x_1}dx\\
\le & C\delta^{1/2}\int u^R_{1x_1}\psi^2_{1x_1x_1}dx + C\delta^{1/2}\int \frac{\chi^2_{x_1x_1}}{(v^R)^2}dx + C\int |p^R_{x_1}\zeta_{x_1x_1}\psi_{1x_1x_1}|dx\\
&+ C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]. \qquad (4.116)
\end{aligned}$$

Meanwhile, one can check that

$$\begin{aligned}
2\int \frac{u^R_1 v^R_{x_1x_1}(v^R_{x_1})^2}{(v^R)^3}\phi\phi_{x_1}dx \le & C\|v^R_{x_1}\|^2_{L^\infty}\|v^R_{x_1x_1}\|_{L^\infty}\left(\|\phi\|^2 + \|\phi_{x_1}\|^2\right)\\
\le & C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1], \qquad (4.117)\\
-b_6\int \frac{u^R_1 v^R_{x_1}}{(v^R)^2}\phi_{x_1x_1}\chi_{x_1x_1}dx \le & C\|v^R_{x_1}\|^{1/2}_{L^\infty}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\|v^R_{x_1}\|^{1/2}_{L^\infty}\int \chi^2_{x_1x_1}dx\\
\le & C\delta^{3/4}\int u^R_{1x_1}\phi^2_{x_1x_1}dx + C\delta^{3/4}\int \chi^2_{x_1x_1}dx \qquad (4.118)
\end{aligned}$$

$$
\begin{aligned}
&\int\left(\frac{u^R_{1x_1}}{v^R}+\frac{u^R_1v^R_{x_1}}{(v^R)^2}\right)\left(v^R_{x_1x_1x_1}\phi\phi_{x_1}+3v^R_{x_1x_1}\phi^2_{x_1}\right)dx\\
\le& C||u^R_{1x_1}||_{L^\infty}||v^R_{x_1x_1x_1}||_{L^\infty}(\|\phi\|^2+\|\phi_{x_1}\|^2)+C||u^R_{1x_1}||_{L^\infty}||v^R_{x_1x_1}||_{L^\infty}\|\phi_{x_1}\|^2\\
\le& C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]. \qquad (4.119)
\end{aligned}
$$

Integration by parts and Cauchy's inequality give us

$$
\begin{aligned}
-\int\frac{b_6^2\chi_{x_1x_1}}{(v^R)^2}\left(u^R_1\chi_{x_1}\right)_{x_1x_1}dx\le& C\int\left(|u^R_{1x_1x_1}\chi_{x_1}\chi_{x_1x_1}|+|u^R_{1x_1}\chi^2_{x_1x_1}|+|v^R_{x_1}\chi^2_{x_1x_1}|\right)dx\\
\le& C\delta\int\chi^2_{1x_1x_1}dx+C\int|u^R_{1x_1x_1}\chi^2_{x_1}|dx\\
\le& C\delta\int\chi^2_{1x_1x_1}dx+C\delta^{1/8}(1+t)^{-5/4}, \qquad (4.120)
\end{aligned}
$$

where we have used the following equation

$$
\frac{u^R_1\chi_{x_1x_1}}{(v^R)^2}\chi_{x_1x_1x_1}=\frac{1}{2}\left(\frac{u^R_1\chi^2_{x_1x_1}}{(v^R)^2}\right)_{x_1}-\frac{1}{2}\left(\frac{u^R_1}{(v^R)^2}\right)_{x_1}\chi^2_{x_1x_1}.
$$

By means of Cauchy's inequality, Lemma 2.1 and

$$
\begin{aligned}
&|(p^R_{x_1x_1},v^R_{x_1x_1})|\le C\left(|u^R_{1x_1}|^2+|u^R_{1x_1x_1}|\right),\\
&|(p^R_{x_1x_1x_1},v^R_{x_1x_1x_1})|\le C\left(|u^R_{1x_1}|^3+|u^R_{1x_1}u^R_{1x_1x_1}|+|u^R_{1x_1x_1x_1}|\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int\frac{(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1})}{(v^R)^2}\left[b_5u^R_{1x_1}v^R\phi+\gamma p^Ru^R_{1x_1}v^R-p^R_{x_1}v^R\psi_1\right]_{x_1x_1}dx\\
\le&\int\left(|u^R_{1x_1x_1x_1}v^R_{x_1x_1}\phi|+|u^R_{1x_1x_1x_1}v^R_{x_1}\phi_{x_1}|+|u^R_{1x_1x_1x_1}\chi_{x_1x_1}|\right)dx\\
&+C\delta\int u^R_{1x_1}\left(\psi^2_{1x_1x_1}+\phi^2_{x_1x_1}\right)dx+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]\\
\le& C||u^R_{1x_1x_1x_1}||_{L^\infty}\left(||v^R_{x_1x_1}||^2+||v^R_{x_1}||^2\right)+||u^R_{1x_1x_1x_1}||_{L^\infty}\left(||\phi||^2+||\phi_{x_1}||^2\right)\\
&+C||u^R_{1x_1x_1x_1}||^2+\frac{b_6b_7}{32\tau}\int\frac{\chi^2_{x_1x_1}}{(v^R)^2}dx+C\delta\int u^R_{1x_1}\left(\psi^2_{1x_1x_1}+\phi^2_{x_1x_1}\right)dx\\
&+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]\\
\le&\frac{b_6b_7}{32\tau}\int\frac{\chi^2_{x_1x_1}}{(v^R)^2}dx+C\delta\int u^R_{1x_1}\left(\psi^2_{1x_1x_1}+\phi^2_{x_1x_1}\right)dx\\
&+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1] \qquad (4.121)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\frac{b_5b_6(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}-b_6\chi_{x_1x_1})}{(v^R)^2}\left[p^Ru^R_{1x_1}v^R\zeta+u^R_{1x_1}\chi\right]_{x_1x_1}dx\\
\le& C\int\left(|u^R_{1x_1}\zeta_{x_1x_1}\phi_{x_1x_1}|+|u^R_{1x_1}\zeta_{x_1x_1}\chi_{x_1x_1}|+|u^R_{1x_1x_1x_1}\zeta\chi_{x_1x_1}|+|u^R_{1x_1x_1x_1}\chi\chi_{x_1x_1}|\right)dx\\
&+C\delta^{1/4}\int\left(u^R_{1x_1}\phi^2_{x_1x_1}+\chi^2_{x_1x_1}\right)dx+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int |u^R_{1x_1}\zeta_{x_1x_1}\phi_{x_1x_1}|dx+\frac{b_6b_7}{32\tau}\int\frac{\chi^2_{x_1x_1}}{(v^R)^2}dx+||u^R_{1x_1}||^2_{L^\infty}||\zeta_{x_1x_1}||^2\\
&\quad+||u^R_{1x_1x_1x_1}||_{L^\infty}\left(||\zeta||^2+||\chi||^2+||\chi_{x_1x_1}||^2\right)\\
&\quad+C\delta^{1/4}\int\left(u^R_{1x_1}\phi^2_{x_1x_1}+\chi^2_{x_1x_1}\right)dx+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]\\
&\le C\delta^{1/4}\int u^R_{1x_1}\phi^2_{x_1x_1}dx+\frac{b_6b_7}{16\tau}\int\frac{\chi^2_{x_1x_1}}{(v^R)^2}dx+C\int|u^R_{1x_1}\zeta_{x_1x_1}\phi_{x_1x_1}|dx\\
&\quad+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]. \qquad (4.122)
\end{aligned}
$$

While the other terms could be easily controlled by

$$
\int\left(\frac{b_5}{32}u^R_{1x_1}\phi^2_{x_1x_1}+\frac{b_6b_7}{16\tau}\frac{\chi^2_{x_1x_1}}{(v^R)^2}+C\delta^{1/4}u^R_{1x_1}|\psi_{x_1x_1}|^2\right)dx+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]
$$

due to Cauchy's inequality, Lemma 2.1 and Lemmas 4.1-4.3. Hence, there is a positive constant c_0 independent of t such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{b_5}\|\phi_{x_1x_1}\|^2+b_{12}\left\|\sqrt{\frac{p^R}{v^R}}\psi_{x_1x_1}\right\|^2+\left\|\frac{v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}+v^R\phi_{x_1x_1}-b_6\chi_{x_1x_1}}{v^R}\right\|^2\right)\\
&\quad+c_0\int\left(u^R_{1x_1}\phi^2_{x_1x_1}+\frac{p^Ru^R_{1x_1}}{v^R}|\psi_{x_1x_1}|^2+\chi^2_{x_1x_1}\right)\\
&\le -2b_{12}\int p^R_{x_1}(\psi_{2x_1x_2}+\psi_{3x_1x_3})\phi_{x_1x_1}dx+C\int\left(|v^R_{x_1}\phi_{x_1x_3}\psi_{3x_1x_1}|+|v^R_{x_1}\phi_{x_1x_1}\chi_{x_1}|\right)dx\\
&\quad+C\int\left(|p^R_{x_1}\zeta_{x_1x_1}\psi_{1x_1x_1}|+|u^R_{x_1}\zeta_{x_1x_1}\phi_{x_1x_1}|\right)dx+C\delta^{1/8}(1+t)^{-5/4}[\ln(1+t)+1]. \qquad (4.123)
\end{aligned}
$$

Integrating the above inequality (4.123) with respect to $t'\in[0,t]$, it shows us that

$$
\begin{aligned}
&\|\left(\phi_{x_1x_1},\psi_{x_1x_1},(v_{x_1x_1\phi}+2v^R_{x_1}\phi_{x_1}+v^R\phi_{x_1x_1}-b_6\chi_{x_1x_1})\right)(t)\|^2\\
&\quad+\int_0^t\int\left(u^R_{1x_1}\phi^2_{x_1x_1}+u^R_{1x_1}|\psi_{x_1x_1}|^2+\chi^2_{x_1x_1}\right)(x,t')dxdt'\\
&\le\left\|\left(\phi_{x_1x_1},\psi_{x_1x_1},(v^R_{x_1x_1}\phi+2v^R_{x_1}\phi_{x_1}+v^R\phi_{x_1x_1}-b_6\chi_{x_1x_1})\right)(0)\right\|^2\\
&\quad+C\int_0^t\int\left[u^R_{1x_1}\left(\phi^2_{x_1x_3}+\psi^2_{2x_1x_2}+\psi^2_{3x_1x_3}+\zeta^2_{x_1x_1}\right)+\chi^2_{x_1}\right](x,t')dxdt'\\
&\quad+\frac{1}{32}\int_0^t\int\left[(1+C\delta)u^R_{1x_1}\phi^2_{x_1x_1}+u^R_{1x_1}|\psi_{x_1x_1}|^2\right](x,t')dxdt'+C\le C, \qquad (4.124)
\end{aligned}
$$

by using Cauchy's inequality, Lemma 4.1 and Lemma 4.3. It should be noted that

$$
\int(v^R_{x_1x_1}\phi)^2dx\le||v^R_{x_1x_1}||^2_{L^\infty}||\phi||^2\le C\delta^{1/4}(1+t)^{-5/2}[\ln(1+t)+1]\le C,
$$

$$
\int(v^R_{x_1}\phi_{x_1})^2dx\le||v^R_{x_1}||^2_{L^\infty}||\phi_{x_1}||^2\le C(1+t)^{-2}\le C
$$

and

$$
\int(v^R\phi_{x_1x_1})^2dx\le C,
$$

thus we have

$$\int \chi_{x_1x_1}^2 dx \le C. \tag{4.125}$$

□

Now we would like to estimate $\int_0^t\int(|\nabla\phi|^2,|\mathrm{div}\psi|^2)(x,t')dxdt'$.

LEMMA 4.5. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time t such that*

$$\int_0^t\int(|\nabla\phi|^2,|div\psi|^2)(x,t')dxdt' \le C(1+t)^{-1/16}(\delta\ln(1+t)+1).$$

Proof. Multiplying the third equation in (1.21) by $\frac{\alpha}{\alpha_f}\mathrm{div}\psi$, we obtain

$$\begin{aligned}|\mathrm{div}\psi|^2 =& -\frac{\alpha}{\alpha_f}\left(\frac{1}{p^Rv^R}\mathrm{div}\psi\chi\right)_t + \frac{\alpha}{\alpha_f}\left(\frac{1}{p^Rv^R}\right)_t \mathrm{div}\psi\chi\\ &+\frac{\alpha}{\alpha_f}\left(\mathrm{div}\left(\frac{1}{p^Rv^R}\psi_t\chi\right)-\nabla\left(\frac{\chi}{p^Rv^R}\right)(\psi_t+u_1^R\psi_{x_1})+\nabla\left(\frac{\chi}{p^Rv^R}\right)u_1^R\psi_{x_1}\right)\\ &-\frac{\alpha}{\alpha_f}\frac{u_1^R\mathrm{div}\psi\chi_{x_1}}{p^Rv^R}+\frac{b_4u_{1x_1}^R\mathrm{div}\psi}{p^Rv^R}(v^R\phi+p^Rv^R\zeta+\chi)\\ &-u_{1x_1}^R\mathrm{div}\psi-\frac{\alpha+\alpha_f}{\tau\alpha_fp^Rv^R}\mathrm{div}\psi\chi.\end{aligned}$$

Integrating the above inequality over $\Omega\times[0,t]$, then we have the following inequalities

$$\begin{aligned}\frac{\alpha}{\alpha_f}\int_0^t\int\left(\frac{1}{p^Rv^R}\right)_t\mathrm{div}\psi\chi(x,t')dxdt' \le& C\int_0^t\int|u_{1x_1}^R\mathrm{div}\psi\chi|(x,t')dxdt'\\ \le& C\int_0^t\int u_{1x_1}^R|\mathrm{div}\psi|^2(x,t')dxdt'+C\int_0^t\int\chi^2(x,t')dxdt'\\ \le& C\delta\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C\delta\ln(1+t)+C,\end{aligned}$$

$$\begin{aligned}C\int_0^t\int\mathrm{div}\psi\chi_{x_1}(x,t')dxdt' \le& \frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+\int_0^t\int\chi_{x_1}^2(x,t')dxdt'\\ \le& \frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C,\end{aligned}$$

$$\begin{aligned}C\int_0^t\int u_{1x_1}^R(\phi,\zeta,\chi)\mathrm{div}\psi(x,t')dxdt' \le& C\int_0^t\int u_{1x_1}^R|(\phi,\zeta,\chi)|^2(x,t')dxdt'+\frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2dxdt'\\ \le& \frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C\delta\ln(1+t)+C,\end{aligned}$$

$$\begin{aligned}\int_0^t\int u_{1x_1}^R\mathrm{div}\psi(x,t')dxdt' \le& C\int_0^t\int(u_{1x_1}^R)^2(x,t')dxdt'+\frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2dxdt'\\ \le& \frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C\delta\ln(1+t)\end{aligned}$$

and

$$\begin{aligned}C\int_0^t\int\mathrm{div}\psi\chi(x,t')dxdt' \le& C\int_0^t\int\chi^2(x,t')dxdt'+\frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'\\ \le& \frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C\delta\ln(1+t)+C\end{aligned}$$

by terms of Lemma 2.1, Lemmas 4.2-4.3 and Cauchy's inequality. Because

$$\begin{aligned}|\psi_t+u_1^R\psi_{x_1}|\le& C\left(|\nabla\phi|+|u_{1x_1}^R\psi_1|+|p_{x_1}^R\phi|+|p_{x_1}^R\zeta|+|p_{x_1}^R\chi|\right)\\ \le& C\left(|\nabla\phi|+|u_{1x_1}^R||(\psi_1,\phi,\zeta,\chi)|\right),\end{aligned}$$

and

$$\left|\nabla\left(\frac{\chi}{p^Rv^R}\right)\right|\le C\left(|u_{1x_1}^R||\chi|+|\nabla\chi|\right),$$

it is clear that

$$\begin{aligned}&\int_0^t\int\nabla\left(\frac{\chi}{p^Rv^R}\right)(\psi_t+u_1^R\psi_{x_1})(x,t')dxdt'\\ \le& C\int_0^t\int\left(|u_{1x_1}^R||\chi||\nabla\phi|+|u_{1x_1}^R||(\chi,\nabla\chi)|(\psi_1,\phi,\zeta,\chi)|+|\nabla\chi||\nabla\phi|\right)(x,t')dxdt'\\ \le& C\int_0^t\int|\nabla\phi\cdot\nabla\chi|(x,t')dxdt'+C\left(\delta\ln(1+t)+1\right)\end{aligned}$$

where we have used Lemmas 4.2-4.3, Cauchy's inequality, the boundedness of $|u_{1x_1}^R|$ and the smallness of δ. We can't have the estimate for $\int_0^t\int|\psi_{x_1}|^2(x,t')dxdt'$ since u_1^R may take value near 0, the term $\int_0^t\int\frac{\nabla\chi}{p^Rv^R}u_1^R\psi_{x_1}(x,t')dxdt'$ should be treated carefully.

$$\begin{aligned}&\int_0^t\int\nabla\left(\frac{\chi}{p^Rv^R}\right)u_1^R\psi_{x_1}(x,t')dxdt'\\ \le&\int_0^t\int\frac{\nabla\chi}{p^Rv^R}u_1^R\psi_{x_1}(x,t')dxdt'+C\int_0^t\int|u_{1x_1}^R||\chi||\phi_{x_1}|(x,t')dxdt'\\ \le&-\int_0^t\int\left[\left(\frac{u_1^R}{p^Rv^R}\right)_{x_1}\nabla\chi+\frac{u_1^R}{p^Rv^R}\nabla\chi_{x_1}\right]\cdot\psi dxdt'+C\int_0^t\int|u_{1x_1}^R||\chi||\phi_{x_1}|dxdt'\\ \le&\int_0^t\int\mathrm{div}\left(\frac{u_1^R\psi}{p^Rv^R}\right)\cdot\chi_{x_1}(x,t')dxdt'+C\int_0^t\int(|u_{1x_1}^R||\nabla\chi||\psi|+|u_{1x_1}^R||\chi||\phi_{x_1}|)dxdt'\\ \le& C\int_0^t\int(|u_{1x_1}^R||\chi_{x_1}||\psi_1|+|u_{1x_1}^R||\nabla\chi||\psi|+|u_{1x_1}^R||\chi||\phi_{x_1}|+|\mathrm{div}\psi||\chi_{x_1}|)dxdt'\\ \le&\frac{1}{36}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C(\delta\ln(1+t)+1).\end{aligned}\tag{4.126}$$

Summing up the above inequalities, it shows

$$\begin{aligned}\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'\le& C(\|(\chi,\mathrm{div}\psi)(t)\|^2+\|(\chi,\mathrm{div}\psi)(0)\|^2)\\ &+C\int_0^t\int|\nabla\phi\cdot\nabla\chi|(x,t')dxdt'+C\delta\ln(1+t)\\ \le& C\int_0^t\int|\nabla\phi\cdot\nabla\chi|(x,t')dxdt'+C\left(\delta\ln(1+t)+1\right).\end{aligned}$$

By the similar argument, we test the second equation in (1.21) by $\nabla\phi$ to derive

$$\begin{aligned}|\nabla\phi|^2=&-\left(\frac{\nabla\phi\cdot\psi}{v^R}\right)_t+\mathrm{div}\left(\frac{\phi_t\psi}{v^R}\right)-\frac{\phi_t\mathrm{div}\psi}{v^R}-\nabla\left(\frac{1}{v^R}\right)\cdot\psi\phi_t+\left(\frac{1}{v^R}\right)_t\nabla\phi\cdot\psi\\ &-\frac{u_1^R}{v^R}\psi_{x_1}\cdot\nabla\phi-\left(\frac{\psi_1\nabla u_1^R}{v^R}-b_3\frac{\nabla p^R\phi}{p^R}+b_4\nabla p^R\zeta+b_4\frac{\chi\nabla p^R}{p^Rv^R}\right)\cdot\nabla\phi.\end{aligned}$$

Integrating above equality over $\Omega\times[0,t]$ to get

$$
\begin{aligned}
&\int_0^t\int|\nabla\phi|^2(x,t')dxdt'\\
\le&C\left(||(\nabla\phi,\psi)(t)||^2+||(\nabla\phi,\psi)(0)||^2\right)-\int_0^t\int\frac{u_1^R\psi_{x_1}\cdot\nabla\phi}{v^R}(x,t')dxdt'\\
&+C\int_0^t\int\left(|\phi_t||\mathrm{div}\psi|+|u^R_{1x_1}||(\psi,\psi_1,\phi,\zeta,\chi)||\nabla\phi|\right)dxdt'\\
\le&\int_0^t\int\left(-\frac{u_1^R\psi_{x_1}\cdot\nabla\phi}{v^R}+|\phi_t||\mathrm{div}\psi|\right)(x,t')dxdt'+C\delta(\ln(1+t)+1),
\end{aligned}\tag{4.127}
$$

due to Lemma 2.1, Lemma 4.2 and Cauchy's inequality. Similarly as (4.126), we have

$$
\begin{aligned}
&-\int_0^t\int\frac{u_1^R\psi_{x_1}\cdot\nabla\phi}{v^R}(x,t')dxdt'\\
=&\int_0^t\int\left[\nabla\left(\frac{u_1^R}{v^R}\right)\psi_{x_1}\phi+\left(\frac{u_1^R}{v^R}\right)\mathrm{div}\psi_{x_1}\phi\right](x,t')dxdt'\\
\le&C\int_0^t\int|u^R_{1x_1}\psi_{x_1}\phi|(x,t')dxdt-\int_0^t\int\left[\left(\frac{u_1^R}{v^R}\right)_{x_1}\phi+\frac{u_1^R}{v^R}\phi_{x_1}\right]\mathrm{div}\psi(x,t')dxdt'\\
\le&C\int_0^t\int\left(|u^R_{1x_1}\psi_{x_1}\phi|+|u^R_{1x_1}\mathrm{div}\psi\phi|+|\mathrm{div}\psi\phi_{x_1}|\right)(x,t')dxdt'\\
\le&\frac{1}{36}\int_0^t\int|\nabla\phi|^2dxdt'+C\int_0^t\int\left(|u^R_{1x_1}|(|\psi_{x_1}|^2+|\phi|^2+|\mathrm{div}\psi|^2)+|\mathrm{div}\psi|^2\right)dxdt'\\
\le&\frac{1}{36}\int_0^t\int|\nabla\phi|^2(x,t')dxdt'+C\int_0^t\int|\mathrm{div}\psi|^2dxdt'+C(\delta\ln(1+t)+1),
\end{aligned}\tag{4.128}
$$

holds by the boundedness of $|u^R_{1x_1}|$, Lemmas 4.2-4.3 and Cauchy's inequality. Since

$$
\begin{aligned}
C|\phi_t||\mathrm{div}\psi|\le&C\left(|\phi_{x_1}|+|u^R_{1x_1}|+|p^R_{x_1}\psi_1|+|u^R_{1x_1}\phi|+|\mathrm{div}\psi|+|\chi|\right)|\mathrm{div}\psi|\\
\le&\frac{1}{36}|\phi_{x_1}|^2+C\left(|\mathrm{div}\psi|^2+|u^R_{1x_1}|^2+|u^R_{1x_1}||(\psi_1,\phi)|^2+|\chi|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\left|-\nabla\left(\frac{1}{v^R}\right)\cdot\psi\phi_t\right|\le&C\left(|\phi_{x_1}|+|u^R_{1x_1}|+|p^R_{x_1}\psi_1|+|u^R_{1x_1}\phi|+|\mathrm{div}\psi|+|\chi|\right)|v^R_{x_1}\psi_1|\\
\le&\frac{1}{36}|\phi_{x_1}|^2+C\left(|\mathrm{div}\psi|^2+|u^R_{1x_1}|^2|(\psi_1,\phi)|+|v^R_{x_1}|^2|\psi_1|^2+|\chi|^2\right)\\
\le&\frac{1}{36}|\phi_{x_1}|^2+C\left(|\mathrm{div}\psi|^2+|u^R_{1x_1}|^2+|u^R_{1x_1}|^2|(\psi_1,\phi)|^2+|\chi|^2\right)
\end{aligned}
$$

then

$$
\begin{aligned}
&C\int_0^t\int|\phi_t||\mathrm{div}\psi|+\left|-\nabla\left(\frac{1}{v^R}\right)\cdot\psi\phi_t\right|(x,t')dxdt'\\
\le&\frac{1}{36}\int_0^t\int|\phi_{x_1}|^2(x,t')dxdt'+C\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'\\
&+C(\delta\ln(1+t)+1)
\end{aligned}\tag{4.129}
$$

exists because of Lemma 2.1, Lemma 4.2 and Cauchy's inequality. Thanks to

$$\begin{aligned} &C\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'\le C\int_0^t\int|\nabla\phi\cdot\nabla\chi|(x,t')dxdt'+C(\delta\ln(1+t)+1)\\ &\le\frac{1}{8}\int_0^t\int|\nabla\phi|^2(x,t')dxdt'+C\int_0^t\int|\nabla\chi|^2(x,t')dxdt'+C(\delta\ln(1+t)+1)\\ &\le\frac{1}{8}\int_0^t\int|\nabla\phi|^2(x,t')dxdt'+C(\delta\ln(1+t)+1), \end{aligned} \tag{4.130}$$

then it is obvious that

$$\begin{aligned} \int_0^t\int|\nabla\phi|^2(x,t')dxdt'\le& C\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'+C(\delta\ln(1+t)+1)\\ \le& C(\delta\ln(1+t)+1), \end{aligned} \tag{4.131}$$

where we also have used the estimates (4.127)-(4.129) and Cauchy's inequality. At the same time, we can see

$$\int_0^t\int|\mathrm{div}\psi|^2(x,t')dxdt'\le C(\delta\ln(1+t)+1) \tag{4.132}$$

holds by (4.130)- (4.131). □

Inspired by Lemma 4.5 and [24], we show $||(\nabla\phi,\mathrm{div}\psi,\nabla\chi)(t)||$ actually decays in time.

LEMMA 4.6. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time t such that*

$$||(\nabla\phi,div\psi,\nabla\chi)(t)||^2dxdt'\le C(1+t)^{-1/4}(\delta\ln(1+t)+1).$$

Proof. Because we have

$$\begin{cases} \nabla\phi_t=-\nabla\left(b_2p^R\mathrm{div}u^R-u_1^R\phi_{x_1}-\psi_1p^R_{x_1}-\beta u^R_{1x_1}\phi-\beta p^R\mathrm{div}\psi-\frac{2}{\tau\alpha}\frac{\chi}{v^R}\right),\\ \frac{p^R}{v^R}\mathrm{div}\psi_t=\frac{p^R}{v^R}\mathrm{div}[-u_1^R\psi_{x_1}-v^R\nabla\phi-\psi_1\nabla u_1^R+b_3\frac{v^R}{p^R}\phi\nabla p^R-b_4v^R\nabla p^R\zeta-b_4\frac{\chi}{p^R}\nabla p^R],\\ \nabla(v^R\phi-b_6\chi)_t=\nabla[\gamma p^Rv^Ru^R_{1x_1}-u_1^Rv^R\phi_{x_1}-p^R_{x_1}v^R\psi_1-b_5u^R_{1x_1}v^R\phi]+\dfrac{b_7\nabla\chi}{\tau}\\ \qquad+\nabla[b_6u_1^R\chi_{x_1}+b_5b_6u^R_{1x_1}p^Rv^R\zeta+b_5b_6u^R_{1x_1}\chi-u_1^Rv^R_{x_1}\phi]. \end{cases} \tag{4.133}$$

Testing the equations of (4.133) by $\frac{1}{b_5}\nabla\phi$, $b_{12}\frac{p^R}{v^R}\mathrm{div}\psi$ and $\frac{1}{(v^R)^2}\nabla(v^R\phi-b_6\chi)$, respectively, then a straightforward calculation gives

$$\begin{aligned} &\frac{1}{2b_5}\frac{\partial}{\partial t}|\nabla\phi|^2+\frac{b_{12}}{2}\frac{\partial}{\partial t}\left(\frac{p^R}{v^R}|\mathrm{div}\psi|^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{(v^R)^2}|\nabla(v^R\phi-b_6\chi)|^2\right]\\ &\quad+b_{14}u^R_{1x_1}\phi^2_{x_1}+(b_{13}-b_{14})u^R_{1x_1}|\nabla\phi|^2+\frac{b_{12}\gamma}{2}\frac{p^R}{v^R}u^R_{1x_1}|\mathrm{div}\psi|^2+\frac{b_6b_7}{\tau(v^R)^2}|\nabla\chi|^2\\ &=-\left[\frac{b_{14}}{2}u_1^R|\nabla\phi|^2+\frac{b_{12}p^R}{2v^R}u_1^R|\mathrm{div}\psi|^2+\frac{u_1^R}{v^R}\left(\frac{b_6^2}{2}|\nabla\chi|^2-b_6\nabla\phi\cdot\nabla\chi\right)\right]_{x_1}\\ &\quad-\mathrm{div}\left(b_{12}p^R\mathrm{div}\psi\nabla\phi+\frac{u_1^R\nabla v^R}{v^R}\phi\phi_{x_1}+\frac{b_7\nabla v^R}{\tau(v^R)^2}\phi\chi\right)+b_{14}p^R_{x_1}\nabla\phi\cdot\nabla\psi_1 \end{aligned}$$

$$
\begin{aligned}
&+\gamma\frac{p^R u^R_{1x_1} v^R_{x_1}\phi_{x_1}}{v^R} - [(b_5+b_{12})u^R_{1x_1x_1} + (b_5+3)\frac{u^R_{1x_1}v^R_{x_1}}{v^R}]\phi\phi_{x_1} \\
&-\left(b_{14}p^R_{x_1x_1} + \frac{p^R_{x_1}v^R_{x_1}}{v^R}\right)\psi_1\phi_{x_1} + b_{14}\left(\frac{p^R_{x_1}v^R_{x_1}}{v^R} - \frac{(p^R_{x_1})^2}{p^R} + p^R_{x_1x_1}\right)\phi\mathrm{div}\psi \\
&-\frac{b_7}{\tau}\left(\frac{v^R_{x_1}}{(v^R)^2}\right)_{x_1}\phi\chi - \frac{b_6 u^R_1 v^R_{x_1}}{(v^R)^2}(\phi_{x_2}\chi_{x_2} + \phi_{x_3}\chi_{x_3}) + \frac{b_6\nabla(u^R_1\chi_{x_1})\cdot\nabla v^R\phi}{(v^R)^2} \\
&-\frac{b_{12}p^R}{v^R}\mathrm{div}\psi\mathrm{div}\left(\nabla u^R_1\psi_1 + b_4 v^R\zeta\nabla p^R + b_4\frac{\chi\nabla p^R}{p^R}\right) \\
&+2b_6\left(u^R_{1x_1} - \frac{u^R_1 v^R_{x_1}}{v^R}\right)\frac{v^R_{x_1}\chi_{x_1}\phi}{(v^R)^2} - \frac{b_6^2 u^R_{1x_1}}{2(v^R)^2}(2\chi^2_{x_1} + |\nabla\chi|^2) \\
&+\left(\frac{u^R_1 v^R_{x_1}}{v^R} - u^R_{1x_1}\right)\frac{(v^R_{x_1})^2\phi^2}{(v^R)^2} + \frac{b_6 u^R_{1x_1}}{v^R}(2\phi_{x_1}\chi_{x_1} + \nabla\phi\cdot\nabla\chi) \\
&+\nabla(\gamma p^R v^R u^R_{1x_1} - p^R_{x_1}v^R\psi_1 - b_5 u^R_{1x_1}v^R\phi - u^R_1 v^R_{x_1}\phi)\frac{\nabla v^R\phi - b_6\nabla\chi}{(v^R)^2} \\
&+b_5 b_6\frac{\nabla(u^R_{1x_1}p^R v^R\zeta + u^R_{1x_1}\chi)}{(v^R)^2}\cdot(\nabla v^R\phi + v^R\nabla\phi - b_6\nabla\chi),
\end{aligned}
\tag{4.134}
$$

where

$$
b_{13} - b_{14} = b_5 + b_{12} - \frac{1}{2b_5} - \frac{1}{2} > 0.
$$

Integrating the equation (4.134) over space variable $x\in\Omega$ and employing the similar method as Lemma 4.3, applying Lemma 2.1, Lemmas 4.1-4.3, Young's inequality and the smallness of δ, one can show that

$$
\begin{aligned}
\frac{dE(t)}{dt} \le& C\int\left(|p^R_{x_1}\nabla\phi\cdot\nabla\psi_1| + |p^R_{x_1}\mathrm{div}\psi\zeta_{x_1}| + |u^R_{1x_1}\mathrm{div}\psi\psi_{1x_1}| + |u^R_{1x_1}\nabla\zeta\cdot\nabla\phi|\right)(x,t)dx \\
&+C(1+t)^{-5/4}(\delta\ln(1+t)+1) \\
=:& G(t) + C(1+t)^{-5/4}(\delta\ln(1+t)+1),
\end{aligned}
\tag{4.135}
$$

where $E(t)$ is introduced as

$$
E(t) = \int\left(\frac{|\nabla\phi|^2}{2b_5} + \frac{b_{12}p^R}{2v^R}|\mathrm{div}\psi|^2 + \frac{1}{2(v^R)^2}|\nabla(v^R\phi - b_6\chi)|^2\right)(x,t)dx.
$$

Therefore we have

$$
\begin{aligned}
&\int_0^t ((1+t')E)_{t'}\,dt' = \int_0^t ((1+t')E_{t'} + E)(t')dt' \\
\le& C\int_0^t (1+t')^{-1/4}(\delta\ln(1+t')+1)dt' + C\int_0^t (1+t')G(t')dt' + \int_0^t E(t')dt'
\end{aligned}
\tag{4.136}
$$

due to (4.135). Utilizing Lemma 2.1, Lemma 4.1, Lemma 4.3, Lemma 4.6 and Cauchy's inequality, we have

$$
\int_0^t\int (1+t')\left(|p^R_{x_1}\nabla\phi\cdot\nabla\psi_1| + |u^R_{1x_1}\nabla\zeta\cdot\nabla\phi|\right)(x,t')dxdt'
$$

$$\leq C\int_0^t(1+t')\int\|u^R_{1x_1}\|^{1/2}_{L^\infty}|\nabla\phi|^2(x,t')dxdt'+C\int_0^t(1+t')\int\|u^R_{1x_1}\|^{3/2}_{L^\infty}|(\nabla\psi_1,\nabla\zeta)|^2dxdt'$$
$$\leq C(1+t)^{1/2}\int_0^t\int|\nabla\phi|^2(x,t')dxdt'+C\int_0^t(1+t')^{-1/2}\int|(\nabla\psi_1,\nabla\zeta)|^2(x,t')dxdt'$$
$$\leq C\delta(1+t)^{1/2}\ln(1+t)+C(1+t)^{1/2}$$

and

$$\int_0^t\int(1+t')\left(|p^R_{x_1}\mathrm{div}\psi\zeta_{x_1}|+|u^R_{1x_1}\mathrm{div}\psi\psi_{1x_1}|\right)(x,t')dxdt'$$
$$\leq C\int_0^t(1+t')\int\|u^R_{1x_1}\|^{1/2}_{L^\infty}|\mathrm{div}\psi|^2dxdt'+\int_0^t(1+t')\int\|u^R_{1x_1}\|^{3/2}_{L^\infty}|(\zeta_{x_1},\psi_{1x_1})|^2(x,t')dxdt'$$
$$\leq C(1+t)^{1/2}\int_0^t\int|\nabla\phi|^2(x,t')dxdt'+C\int_0^t(1+t')^{-1/2}\int|(\zeta_{x_1},\psi_{1x_1})|^2(x,t')dxdt'$$
$$\leq C\delta(1+t)^{1/2}\ln(1+t)+C(1+t)^{1/2},$$

then the second term on the right side of (4.136) could be controlled as

$$\int_0^t(1+t')G(t')dt'\leq C(1+t)^{1/2}(\delta\ln(1+t)+1). \tag{4.137}$$

Moreover, we can estimate $\int_0^t E(t')dt'$ as

$$\begin{aligned}\int_0^t Edt'=&\int_0^t\int\left(\frac{|\nabla\phi|^2}{2b_5}+\frac{b_{12}p^R}{v^R}|\mathrm{div}\psi|^2+\frac{1}{2(v^R)^2}|\nabla(v^R\phi-b_6\chi)|^2\right)(x,t')dxdt'\\ \leq& C\int_0^t\int\left(|\nabla\phi|^2+|\mathrm{div}\psi|^2+(v^R_{x_1}\phi)^2+|\nabla\chi|^2\right)(x,t')dxdt'\\ \leq& C(\delta\ln(1+t)+1)\end{aligned} \tag{4.138}$$

from (4.102), Lemmas 4.2-4.3. Based on the above inequalities (4.136)-(4.138), the following estimate

$$E(t)\leq C\delta(1+t)^{-1/4}\ln(1+t)+C(1+t)^{-1/4}$$

holds out, which implies

$$||(\nabla\phi,\mathrm{div}\psi,\nabla\chi)(t)||^2\leq C(1+t)^{-1/4}(\delta\ln(1+t)+1).$$

Therefore, we have finished the proof of Lemma 4.6. ◻

Proof. **(Proof of Theorem 1.1.)** Based on the results of Lemmas 4.2-4.4 and Lemma 4.6, applying Lemma 2.1, then it implies

$$\begin{aligned}||(\phi,\chi)||^2_{L^\infty(\Omega)}\leq& C\left(||(\phi,\chi)(t)||||(\nabla\phi,\nabla\chi)(t)||+||(\nabla\phi,\nabla\chi)(t)||||(\nabla^2\phi,\nabla^2\chi)(t)||\right)\\ \leq& C(1+t)^{-1/8}(\delta\ln(1+t)+1)\end{aligned}$$

and

$$||\psi||^2_{L^\infty(\Omega)}\leq C\left(||\psi(t)||||\nabla\psi(t)||+||\nabla\psi(t)||||\nabla^2\psi(t)||\right)$$

$$\leq C(\delta \ln(1+t)+1)^{1/2}.$$

□

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