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LIFESPAN ESTIMATES OF SOLUTIONS TO THE WEAKLY COUPLED SYSTEM OF SEMILINEAR WAVE EQUATIONS WITH SPACE DEPENDENT DAMPINGS*

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Abstract. This paper is devoted to investigating the weakly coupled system of semilinear wave equations with space dependent dampings and power nonlinearities $|v|^p$, $|u|^q$, derivative nonlinearities $|v_t|^p$, $|u_t|^q$, mixed nonlinearities $|v|^q$, $|u_t|^p$, combined nonlinearities $|v_t|^{p_1}+|v|^{q_1}$, $|u_t|^{p_2}+|u|^{q_2}$, combined and power nonlinearities $|v_t|^{p_1}+|v|^{q_1}$, $|u|^{q_2}$, combined and derivative nonlinearities $|v_t|^{p_1}+|v|^{q_1}$, $|u_t|^{p_2}$, respectively. Formation of singularities and lifespan estimates of solutions to the problem in the sub-critical and critical cases are illustrated by making use of test function technique. The main innovation is that upper bound lifespan estimates of solutions are associated with the Strauss exponent and Glassey exponent.

Keywords. Weakly coupled system; Semilinear wave equations; Test function technique; Formation of singularities; Lifespan estimates.

AMS subject classifications. 35L70; 58J45.

1. Introduction

Our main goal in this work is to investigate the following weakly coupled system of semilinear wave equations with space dependent dampings

$$\begin{cases} u_{tt}-\Delta u+\dfrac{\mu}{(1+|x|)^{\beta}}u_t=f_1(v,v_t), & x\in\mathbb{R}^n, t>0,\\ v_{tt}-\Delta v+\dfrac{\mu}{(1+|x|)^{\beta}}v_t=f_2(u,u_t), & x\in\mathbb{R}^n, t>0,\\ (u,u_t,v,v_t)(x,0)=\varepsilon(u_0,u_1,v_0,v_1)(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\mu>0$, $\beta>2$, the nonlinear terms $f_1(v,v_t), f_2(u,u_t)$ are illustrated in the forms of power type nonlinearities $f_1(v,v_t)=|v|^p, f_2(u,u_t)=|u|^q$, derivative type nonlinearities $|v_t|^p,|u_t|^q$, mixed type nonlinearities $|v|^q,|u_t|^p$, combined type nonlinearities $|v_t|^{p_1}+|v|^{q_1},|u_t|^{p_2}+|u|^{q_2}$, combined and power type nonlinearities $|v_t|^{p_1}+|v|^{q_1}$, $|u|^{q_2}$, combined and derivative type nonlinearities $|v_t|^{p_1}+|v|^{q_1}$, $|u_t|^{p_2}$, respectively. Indexes in the nonlinear terms satisfy $1<p,p_1,p_2,q,q_1,q_2<\infty$. For brevity, we assume that the constant R satisfies $R>2$. $B_R(0)=\{x\,\big|\,|x|\le R\}$. $u_0,v_0\in H^1(\mathbb{R}^n)$ and $u_1,v_1\in L^2(\mathbb{R}^n)$ are non-negative functions and compactly supported in $B_R(0)\,(R>2)$. In addition, all the initial values u_0,u_1,v_0,v_1 do not vanish identically. $\varepsilon>0$ is a parameter describing the size of initial values. It is well known that a solution u has compact support when the initial values have compact supports. Therefore, we directly assume that the solution has compact support set.

Let us start with a brief review on the Cauchy problem for classical wave equation

$$\begin{cases} u_{tt}-\Delta u=f(u,u_t), & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=\varepsilon f(x),\, u_t(x,0)=\varepsilon g(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.2}$$

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Problem (1.2) with power nonlinearity $f(u,u_t)=|u|^p$ possesses the Strauss exponent $p_S(n)$, which is the positive solution to the quadratic equation

$$r(p,n)=-[(n-1)p^2-(n+1)p-2]=0.$$

We refer readers to the works in [9,20,28,39,42,43] for more details. If $1<p\le p_S(n)$, a solution to problem (1.2) blows up in finite time. There exists a unique global solution when $p>p_S(n)$. Problem (1.2) with derivative nonlinearity $f(u,u_t)=|u_t|^p$ admits the Glassey exponent $p_G(n)=\frac{n+1}{n-1}$. Concerning the Glassey exponent, upper bound lifespan estimate of solution to the problem when $1<p\le p_G(n)$ and global existence result when $p>p_G(n)$ are considered in [16,41]. Ikeda et al. [12] present a simple proof for upper bound lifespan estimates of solutions to semilinear wave equation and related weakly coupled system by introducing a test function, which is related to the hypergeometric function.

The Cauchy problem of linear damped wave equation

$$\begin{cases} u_{tt}-\Delta u+c(x,t)u_t=0, & x\in\mathbb{R}^n, t>0,\\ (u,u_t)(x,0)=\varepsilon(u_0,u_1)(x), & x\in\mathbb{R}^n \end{cases} \tag{1.3}$$

catches a lot of attention (see [24,27,34]), where the coefficient in damping term satisfies $c(x,t)=a_0a(t)b(x)=a_0(1+t)^{-\alpha}(1+|x|)^{-\beta}$, a_0 is a positive constant. We have time dependent damping when $\alpha\in\mathbb{R},\beta=0$. Behavior of solution can be classified in the following cases. If $\alpha\in(-\infty,-1)$, the solution does not decay to zero in general. If $\alpha\in[-1,1)$, the solution behaves like that of heat equation. If $\alpha>1$, the solution behaves like that of wave equation. The damping is scale invariant when $\alpha=1$. Asymptotic behavior of solution depends on the constant a_0. On the other hand, we have space dependent damping when $\alpha=0,\beta\in\mathbb{R}$. The solution behaves like that of heat equation when $\beta\in(-\infty,1)$. In the case $\beta=1$, the damping is scaling invariant weak damping. The damping is scattering when $\beta\in(1,\infty)$. This implies that the solution behaves like that of wave equation without damping.

Recently, the nonlinear wave equation with time or space dependent damping

$$\begin{cases} u_{tt}-\Delta u+c(x,t)u_t=f(u,u_t), & x\in\mathbb{R}^n, t>0,\\ (u,u_t)(x,0)=\varepsilon(u_0,u_1)(x), & x\in\mathbb{R}^n \end{cases} \tag{1.4}$$

attracted more attention (see detailed illustrations in [2,4–8,10,11,13–15,17–19,21–27,29,30,33,38,40]). Blow-up result and lifespan estimate of solution to problem (1.4) with $\alpha=0$, $\beta>2$ and nonlinear terms $f(u,u_t)=|u|^p,|u_t|^p$ are studied in [24] by applying test function method. Taking advantage of the Kato lemma and iteration approach, Lai et al. [21] illustrate upper bound lifespan estimate of solution to semilinear wave equation with scale invariant damping term and mass term. Hamouda et al. [8] establish blow-up result and lifespan estimate of solution to problem (1.4) with scale invariant damping $(\alpha=1,\beta=0)$ and nonlinear term $f(u,u_t)=|u_t|^p+|u|^q$.

The weakly coupled system of semilinear wave equations with time dependent dampings

$$\begin{cases} u_{tt}-\Delta u+\frac{\mu}{(1+t)^\alpha}u_t=f_1(v,v_t), & x\in\mathbb{R}^n, t>0,\\ v_{tt}-\Delta v+\frac{\mu}{(1+t)^\alpha}v_t=f_2(u,u_t), & x\in\mathbb{R}^n, t>0,\\ (u,u_t,v,v_t)(x,0)=(\varepsilon u_0,\varepsilon u_1,\varepsilon v_0,\varepsilon v_1)(x) & x\in\mathbb{R}^n \end{cases} \tag{1.5}$$

has been studied by many scholars (see [1,3,31,32,35–37]). Palmieri et al. [35] consider problem (1.5) with scattering damping ($\alpha>1$) and $f_1(v,v_t)=|v|^p$, $f_2(u,u_t)=|u|^q$. Upper bound lifespan estimates of solutions to the Cauchy problem in the sub-critical and critical cases are derived by exploiting test function technique and iteration approach. Taking advantage of iteration technique, Palmieri et al. [36] investigate problem (1.5) with scattering damping term and $f_1(v,v_t)=|v_t|^p$, $f_2(u,u_t)=|u_t|^q$ in the sub-critical and critical cases. Palmieri et al. [37] derive blow-up results and lifespan estimates of solutions to problem (1.5) with $f_1(v,v_t)=|v|^q$, $f_2(u,u_t)=|u_t|^p$ in the sub-critical and critical cases by employing iteration technique and slicing method. Ming et al. [31] establish upper bound lifespan estimates of solutions to problem (1.5) with combined nonlinearities $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}+|u|^{q_2}$ in the sub-critical and critical cases by making use of iteration method.

Enlightened by the works in [12,24,31,35–37], our interest is to establish blow-up results and lifespan estimates of solutions to problem (1.1) in the sub-critical and critical cases. Owing to the similarity of structure of the equations, we expect that lifespan estimates of solutions to problem (1.1) are the same as those of wave equation. It is worth pointing out that problem (1.5) with power nonlinearities $|v|^p$, $|u|^q$, derivative nonlinearities $|v_t|^p$, $|u_t|^q$ and mixed nonlinearities $|v|^q$, $|u_t|^p$ in the case $\mu=0$ is considered in [12]. Upper bound lifespan estimates of solutions are verified by applying test function method, which is connected with the hypergeometric function. Making use of iterative technique, Palmieri et al. [35–37] consider problem (1.5) with power nonlinearities $|v|^p$, $|u|^q$, derivative nonlinearities $|v_t|^p$, $|u_t|^q$ and mixed nonlinearities $|v|^q$, $|u_t|^p$, respectively. In this work, the innovation is that upper bound lifespan estimates of solutions to problem (1.1) are derived by applying test function approach, which is based on the test function in [24]. In addition, this test function is different from the function in [12]. It is worth mentioning that blow-up results and lifespan estimates of solutions to problem (1.4) with power nonlinearity $|u|^p$, derivative nonlinearity $|u_t|^p$ in the case $\alpha=0,\beta>2$ are verified in Lai et al. [24], which is a special case of problem (1.1) with power nonlinearities $|v|^p$, $|u|^q$ and derivative nonlinearities $|v_t|^p$, $|u_t|^q$ when $p=q$. In addition, we establish upper bound lifespan estimates of solutions to problem (1.1) with mixed nonlinearities $|v|^q,|u_t|^p$, combined nonlinearities $|v_t|^{p_1}+|v|^{q_1},|u_t|^{p_2}+|u|^{q_2}$, combined and power nonlinearities $|v_t|^{p_1}+|v|^{q_1},|u|^{q_2}$, combined and derivative nonlinearities $|v_t|^{p_1}+|v|^{q_1},|u_t|^{p_2}$. To our knowledge, the results in Theorems 1.3-1.6 are new.

Throughout the paper, we denote

$$F_{SS}(n,p,q)=(p+2+\frac{1}{q})(pq-1)^{-1}-\frac{n-1}{2},$$

$$F_{GG}(n,p,q)=\frac{p+1}{pq-1}-\frac{n-1}{2},$$

$$F_{SG,1}(n,p,q)=(\frac{1}{p}+1+q)(pq-1)^{-1}-\frac{n-1}{2},$$

$$F_{SG,2}(n,p,q)=(2+\frac{1}{q})(pq-1)^{-1}-\frac{n-1}{2},$$

$$\Gamma_{CC}(n,p_1,p_2,q_1,q_2)=\max\{\frac{q_1q_2+2q_2+1}{p_2(q_1q_2-1)}-\frac{n-1}{2},\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}\},$$

$$\Gamma_{CS}(n,p_1,q_1,q_2)=\max\{\frac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\frac{n-1}{2},\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}\},$$

$$\Gamma_{CG}(n,p_1,q_1,p_2)=\max\{\frac{2q_1+1}{p_1(q_1p_2-1)}-\frac{n-1}{2},\frac{q_1p_2+p_2+1}{p_2(q_1p_2-1)}-\frac{n-1}{2}\},$$

$$\Gamma_S(n,q_1)=\frac{r(n,q_1)}{2q_1(q_1-1)},$$

$$\Gamma_G(n,p_1)=\frac{1}{p_1-1}-\frac{n-1}{2},$$

$$\Gamma_{Comb}(n,p_1,q_1)=\frac{q_1+1}{p_1(q_1-1)}-\frac{n-1}{2}.$$

Exponent p' stands for the conjugate exponent of p, which satisfies $\frac{1}{p}+\frac{1}{p'}=1$.

Definition of weak solutions and the main results in this paper are illustrated as follows.

DEFINITION 1.1. *Let $(u_0,u_1),(v_0,v_1)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Assume that*

$$u\in C([0,T),H^1(\mathbb{R}^n))\cap C^1([0,T),L^2(\mathbb{R}^n)),$$
$$v\in C([0,T),H^1(\mathbb{R}^n))\cap C^1([0,T),L^2(\mathbb{R}^n)).$$

$u_t\in L^p_{loc}([0,T)\times\mathbb{R}^n)$ and $v\in L^q_{loc}([0,T)\times\mathbb{R}^n)$ when the nonlinear terms are $f_1(v,v_t)=|v|^q$, $f_2(u,u_t)=|u_t|^p$ in problem (1.1). $u_t\in L^{p_2}_{loc}([0,T)\times\mathbb{R}^n)$, $u\in L^{q_2}_{loc}([0,T)\times\mathbb{R}^n)$, $v_t\in L^{p_1}_{loc}([0,T)\times\mathbb{R}^n)$ and $v\in L^{q_1}_{loc}([0,T)\times\mathbb{R}^n)$ when the nonlinear terms are $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}+|u|^{q_2}$ in problem (1.1). $u\in L^{q_2}_{loc}([0,T)\times\mathbb{R}^n)$, $v_t\in L^{p_1}_{loc}([0,T)\times\mathbb{R}^n)$ and $v\in L^{q_1}_{loc}([0,T)\times\mathbb{R}^n)$ when the nonlinear terms are $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u|^{q_2}$ in problem (1.1). $u_t\in L^{p_2}_{loc}([0,T)\times\mathbb{R}^n)$, $v_t\in L^{p_1}_{loc}([0,T)\times\mathbb{R}^n)$ and $v\in L^{q_1}_{loc}([0,T)\times\mathbb{R}^n)$ when the nonlinear terms are $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}$ in problem (1.1). It holds that

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}u_1(x)\phi(x,0)dx\\&\quad+\varepsilon\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u_0(x)\phi(x,0)dx+\int_0^T\int_{\mathbb{R}^n}f_1(v,v_t)(x,s)\phi(x,s)dxds\\&=-\int_0^T\int_{\mathbb{R}^n}u_t(x,s)\phi_t(x,s)dxds+\int_0^T\int_{\mathbb{R}^n}\nabla u(x,s)\nabla\phi(x,s)dxds\\&\quad-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u(x,s)\phi_t(x,s)dxds\end{aligned}\tag{1.6}$$

and

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}v_1(x)\phi(x,0)dx\\&\quad+\varepsilon\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}v_0(x)\phi(x,0)dx+\int_0^T\int_{\mathbb{R}^n}f_2(u,u_t)(x,s)\phi(x,s)dxds\\&=-\int_0^T\int_{\mathbb{R}^n}v_t(x,s)\phi_t(x,s)dxds+\int_0^T\int_{\mathbb{R}^n}\nabla v(x,s)\nabla\phi(x,s)dxds\\&\quad-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}v(x,s)\phi_t(x,s)dxds,\end{aligned}\tag{1.7}$$

where $\phi \in C_0^\infty([0,T)\times\mathbb{R}^n)$, $t\in(0,T)$. *Then,* (u,v) *are called weak solutions of problem* (1.1).

THEOREM 1.1. *Let* (u,v) *be weak solutions to problem* (1.1) *with* $f_1(v,v_t)=|v|^p$, $f_2(u,u_t)=|u|^q$. *If supp*$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, *then there exists a positive constant* $\varepsilon_0=\varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,R,\mu,\beta)$ *such that for all* $0<\varepsilon\le\varepsilon_0$, *the solutions* (u,v) *blow up in finite time. The upper bound lifespan estimates of solutions* $T(\varepsilon)$ *satisfy*

$$T(\varepsilon)\le\begin{cases} C\varepsilon^{-\Gamma_{SS}^{-1}(n,p,q)}, & \Gamma_{SS}(n,p,q)>0,\\ \exp(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}), & \Gamma_{SS}(n,p,q)=0,\ p\neq q,\\ \exp(C\varepsilon^{-p(p-1)}), & \Gamma_{SS}(n,p,q)=0,\ p=q,\end{cases} \tag{1.8}$$

where $\Gamma_{SS}(n,p,q)=\max\{F_{SS}(n,p,q),F_{SS}(n,q,p)\}\ge 0$, $C>0$ *is independent of* ε.

THEOREM 1.2. *Let* (u,v) *be weak solutions to problem* (1.1) *with* $f_1(v,v_t)=|v_t|^p$, $f_2(u,u_t)=|u_t|^q$. *If supp*$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, *then the solutions* (u,v) *blow up in finite time. The upper bound lifespan estimates satisfy*

$$T(\varepsilon)\le\begin{cases} C\varepsilon^{-\Gamma_{GG}^{-1}(n,p,q)}, & \Gamma_{GG}(n,p,q)>0,\\ \exp(C\varepsilon^{-(pq-1)}), & \Gamma_{GG}(n,p,q)=0,\ p\neq q,\\ \exp(C\varepsilon^{-(p-1)}), & \Gamma_{GG}(n,p,q)=0,\ p=q,\end{cases} \tag{1.9}$$

where $\Gamma_{GG}(n,p,q)=\max\{F_{GG}(n,p,q),F_{GG}(n,q,p)\}\ge 0$, $C>0$ *is independent of* ε.

REMARK 1.1. Similar to the derivation in [24] with some modifications, we can derive the lifespan estimates of solutions in (1.8) and (1.9). We omit the detailed proofs of Theorems 1.1 and 1.2 for simplicity.

THEOREM 1.3. *Let* (u,v) *be weak solutions to problem* (1.1) *with* $f_1(v,v_t)=|v|^q$, $f_2(u,u_t)=|u_t|^p$. *If supp*$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, *then the solutions* (u,v) *blow up in finite time. The upper bound lifespan estimates satisfy*

$$T(\varepsilon)\le\begin{cases} C\varepsilon^{-\Gamma_{SG}^{-1}(n,p,q)}, & \Gamma_{SG}(n,p,q)>0,\\ \exp(C\varepsilon^{-p(pq-1)}), & F_{SG,1}(n,p,q)=0>F_{SG,2}(n,p,q),\\ \exp(C\varepsilon^{-q(pq-1)}), & F_{SG,1}(n,p,q)<0=F_{SG,2}(n,p,q),\\ \exp(C\varepsilon^{-(pq-1)}), & F_{SG,1}(n,p,q)=0=F_{SG,2}(n,p,q),\end{cases} \tag{1.10}$$

where $\Gamma_{SG}(n,p,q)=\max\{F_{SG,1}(n,p,q),F_{SG,2}(n,p,q)\}\ge 0$, $C>0$ *is independent of* ε.

THEOREM 1.4. *Let* (u,v) *be weak solutions to problem* (1.1) *with* $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}+|u|^{q_2}$. *If supp*$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, *then the* *solutions (u,v) blow up in finite time. The upper bound lifespan estimates satisfy*

$$T(\varepsilon)\le\begin{cases}C\varepsilon^{-\Gamma_{CC}(n,p_1,p_2,q_1,q_2)^{-1}}, \Gamma_{CC}(n,p_1,p_2,q_1,q_2)>0,\\ \exp(C\varepsilon^{-p_2(q_1p_2-1)}),\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)=0>\dfrac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\dfrac{n-1}{2}, \quad p_2=q_2,\\ \exp(C\varepsilon^{-p_1(p_1q_2-1)}),\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)=0>\dfrac{q_1q_2+2q_2+1}{p_2(q_1q_2-1)}-\dfrac{n-1}{2}, \quad p_1=q_1,\\ \exp(C\varepsilon^{-(p_1-1)}),\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0, p_1=p_2, q_1=q_2, p_1=p_G(n), \quad q_1>1+\dfrac{4}{n-1},\\ C\varepsilon^{-\Gamma_G(n,p_1)^{-1}},\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0, p_1=p_2, q_1=q_2, p_1<p_G(n), \quad q_1>2p_1-1,\\ C\varepsilon^{-\Gamma_{Comb}(n,p_1,q_1)^{-1}},\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0, p_1=p_2, q_1=q_2,\\ \quad \Gamma_{Comb}(n,p_1,q_1)>0, p_1\le q_1\le 2p_1-1,\\ C\varepsilon^{-\Gamma_S(n,q_1)^{-1}},\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0, p_1=p_2, q_1=q_2, q_1<p_S(n), \quad p_1>q_1,\\ \exp(C\varepsilon^{-q_1(q_1-1)}),\\ \quad \Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0, p_1=p_2, q_1=q_2, q_1=p_S(n), \quad p_1>q_1,\end{cases} \tag{1.11}$$

where $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0$. The second and third lifespan estimates in (1.11) are derived in the case $p_1\ne p_2$ or $q_1\ne q_2$ when $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)=0$. $\max\{\Gamma_S(n,q_1),\Gamma_G(n,p_1)\}\ge 0$ or $\Gamma_{Comb}(n,p_1,q_1)>0$, $C>0$ is independent of ε.

THEOREM 1.5. *Let (u,v) be weak solutions to problem (1.1) with $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u|^{q_2}$. If supp$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, then the solutions (u,v) blow up in finite time. The upper bound lifespan estimates satisfy*

$$T(\varepsilon)\le\begin{cases}C\varepsilon^{-\Gamma_{CS}(n,p_1,q_1,q_2)^{-1}}, \quad \Gamma_{CS}(n,p_1,q_1,q_2)>0,\\ \exp(C\varepsilon^{-p_1(p_1q_2-1)}),\\ \quad \Gamma_{CS}(n,p_1,q_1,q_2)=0>\dfrac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\dfrac{n-1}{2}, p_1=q_1,\\ \exp(C\varepsilon^{-q_2(q_1q_2-1)}),\\ \quad \Gamma_{CS}(n,p_1,q_1,q_2)=0>\dfrac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\dfrac{n-1}{2}\\ \quad \text{or } \Gamma_{CS}(n,p_1,q_1,q_2)=0, p_1=q_1=q_2,\end{cases} \tag{1.12}$$

where $\Gamma_{CS}(n,p_1,q_1,q_2)\ge 0$, $C>0$ is independent of ε.

THEOREM 1.6. *Let (u,v) be weak solutions to problem (1.1) with $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}$. If supp$(u,v)\subset\{(x,t)\in\mathbb{R}^n\times[0,T)\big||x|\le t+R\}$, then the* *solutions (u,v) blow up in finite time. The upper bound lifespan estimates satisfy*

$$T(\varepsilon)\le\begin{cases} C\varepsilon^{-\Gamma_{CG}(n,p_1,q_1,p_2)^{-1}}, & \Gamma_{CG}(n,p_1,q_1,p_2)>0,\\ \exp(C\varepsilon^{-p_1(p_1p_2-1)}), & \\ \quad \Gamma_{CG}(n,p_1,q_1,p_2)=0>\dfrac{q_1p_2+p_2+1}{p_2(q_1p_2-1)}-\dfrac{n-1}{2}, p_1=q_1, & \\ \exp(C\varepsilon^{-p_2(q_1p_2-1)}), & \\ \quad \Gamma_{CG}(n,p_1,q_1,p_2)=0>\dfrac{2q_1+1}{p_1(q_1p_2-1)}-\dfrac{n-1}{2}, & \\ \exp(C\varepsilon^{-\min\{p_1(p_1p_2-1),p_2(q_1p_2-1)\}}), & \\ \quad \dfrac{q_1p_2+p_2+1}{p_2(q_1p_2-1)}-\dfrac{n-1}{2}=0, \dfrac{2q_1+1}{p_1(q_1p_2-1)}-\dfrac{n-1}{2}=0, p_1=q_1, & \end{cases} \tag{1.13}$$

where $\Gamma_{CG}(n,p_1,q_1,p_2)\ge 0$, $C>0$ is independent of ε.

REMARK 1.2. We recognize that Lai et al. [24] investigate the single equation with power nonlinearity $|u|^p$, derivative nonlinearity $|u_t|^p$, which are special cases of problem (1.1) with power nonlinearities $|v|^p, |u|^q$ and derivative nonlinearities $|v_t|^p, |u_t|^q$ when $p=q$, respectively. Based on the proof of lifespan estimate for solution to single wave equation with space dependent damping term $\frac{\mu}{(1+|x|)^\beta}u_t\,(\beta>1)$ in [19], we believe that the results in our paper still hold when $1<\beta\le 2$ by making use of the test function in [19], especially for the case when $p=q$.

REMARK 1.3. It is worth noticing that lifespan estimates of solutions to problem (1.1) with $f_1(v,v_t)=|v|^q$, $f_2(u,u_t)=|u_t|^p$ are established in Theorem 1.3. Problem (1.1) with $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}+|u|^{q_2}$ when $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)\ge 0$, $p_1=p_2$ and $q_1=q_2$ is equivalent to problem (1.4) with $f(u,u_t)=|u_t|^{p_1}+|u|^{q_1}$ when $\alpha=0$. We derive the lifespan estimates of solutions in Theorem 1.4. Lifespan estimates of solutions to problem (1.1) with nonlinearities $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u|^{q_2}$ and $f_1(v,v_t)=|v_t|^{p_1}+|v|^{q_1}$, $f_2(u,u_t)=|u_t|^{p_2}$ are illustrated in Theorems 1.5 and 1.6, respectively. Our main new contribution of this paper is to derive lifespan estimates of solutions to the problem (1.1) with different nonlinear terms by utilizing different test functions.

2. Proof of Theorem 1.3

2.1. Several lemmas.

Before going further, we collect four related lemmas.

LEMMA 2.1. *[24] Let $\beta>0$. It holds that*

$$\int_0^{t+R}(1+r)^\alpha e^{-\beta(t-r)}dr\le C(t+R)^\alpha,$$

where $\alpha\in\mathbb{R}$, C is a positive constant.

Based on the key observation in [24], we will choose $\Phi(x,t)$ (see Lemma 2.2 below) as the test function.

LEMMA 2.2 ([24]). *Assume that $\Delta\phi_1=\phi_1$ and $\Phi=\Phi(x,t)=e^{-t}\phi_1(x)$. It holds that*

$$\partial_t^2\Phi-\Delta\Phi-\frac{\mu}{(1+|x|)^\beta}\partial_t\Phi=0,\qquad \Delta\Phi=\Phi,$$

where $\phi_1(x)$ satisfies

$$\phi_1(x)=\begin{cases} e^x+e^{-x}, & n=1,\\ \displaystyle\int_{S^{n-1}} e^{x\cdot w}dS_w, & n\geq 2.\end{cases}$$

Moreover, $0<\phi_1(x)\leq C(1+|x|)^{-\frac{n-1}{2}}e^{|x|}$, *$C$ is a positive constant.*

LEMMA 2.3 ([24]). *Let $b_a(x,t)=\int_0^1 e^{-\eta t}\psi_\eta(x)\eta^{a-1}d\eta$ with $a>0$. It holds that*

$$\partial_t^2 b_a-\Delta b_a-\frac{\mu}{(1+|x|)^\beta}\partial_t b_a=0, \tag{2.1}$$

where $\psi_\eta(x)\sim\varphi_\eta(x)=\int_{\mathbb{S}^{n-1}}e^{\eta x\cdot w}dw(\sim|\eta x|^{\frac{1-n}{2}}e^{|\eta x|})$ for large $|\eta x|$, $\Delta\psi_\eta=\eta^2\psi_\eta$. Moreover, $b_a(x,t)$ satisfies

$$\frac{\partial}{\partial t}b_a(x,t)=-b_{a+1}(x,t),\quad \frac{\partial^2}{\partial t^2}b_a(x,t)=b_{a+2}(x,t),\quad \Delta b_a(x,t)=b_{a+2}(x,t)$$

and

$$b_a(x,t)\sim\begin{cases}(t+R+|x|)^{-a}, & 0<a<\dfrac{n-1}{2},\\ (t+R+|x|)^{-\frac{n-1}{2}}(t+R-|x|)^{\frac{n-1}{2}-a}, & a>\dfrac{n-1}{2}.\end{cases}$$

LEMMA 2.4 ([12]). *Let $2<t_0<T$ and $0\leq\phi\in C^1([t_0,T))$. Assume that*

$$\begin{cases}\delta\leq K_1 t\phi'(t), & t\in(t_0,T),\\ \phi(t)^{p_1}\leq K_2 t(\log t)^{p_2-1}\phi'(t), & t\in(t_0,T),\end{cases}$$

where $\delta,K_1,K_2>0$, $p_1,p_2>1$. If $p_2<p_1+1$, then there exist two positive constants δ_0 and K_3 such that

$$T\leq\exp(K_3\delta^{-\frac{p_1-1}{p_1-p_2+1}})$$

for all $\delta\in(0,\delta_0)$, where K_3 is independent of δ.

2.2. Proof of Theorem 1.3.

Proof. Let $\eta(t)\in C^\infty([0,\infty))$ satisfy

$$\eta(t)=\begin{cases}1, & t\leq\dfrac{1}{2},\\ \textit{decreasing}, & \dfrac12<t<1,\\ 0, & t\geq 1\end{cases}$$

and $|\eta'(t)|\leq C$, $|\eta''(t)|\leq C$. We set $\eta_T(t)=\eta(\frac{t}{T})$ for $t\in(1,T)$. We define

$$\theta(t)=\begin{cases}0, & t<\dfrac12,\\ \eta(t), & t\geq\dfrac12,\end{cases}\qquad \theta_M(t)=\theta(\frac{t}{M}),$$

where $M\in[1,T)$.

Letting $\psi(x,t)=-\eta_M^{2p'}(t)\Phi(x,t)=-\eta_M^{2p'}(t)e^{-t}\phi_1(x)$, replacing $\phi(x,s)$ with $\Psi(x,s)=\partial_t\psi(x,s)$ in (1.6), integrating by parts, sending $t\to T$ and employing Lemma 2.2 yield

$$\begin{aligned}&\varepsilon C(u_0,u_1)+\int_0^T\int_{\mathbb{R}^n}|v(x,s)|^q\partial_t\psi(x,s)dxds\\ =&\int_0^T\int_{\mathbb{R}^n}u_t(\partial_t^2\eta_M^{2p'}\Phi+2\partial_t\eta_M^{2p'}\partial_t\Phi-\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2p'}\Phi)dxds.\end{aligned}\tag{2.2}$$

Similar to the derivation in (3.16) in [24] and utilizing Lemma 2.1, we acquire

$$(\varepsilon C(u_0,u_1))^pT^{n-\frac{n-1}{2}p}\le\int_0^T\int_{\mathbb{R}^n}\theta_M^{2p'}|u_t|^pdxds.\tag{2.3}$$

Analogously, we have

$$(\varepsilon C(v_0,v_1))^qT^{n-\frac{n-1}{2}q}\le\int_0^T\int_{\mathbb{R}^n}\theta_M^{2p'}|v|^qdxds.\tag{2.4}$$

Choosing the test function $\phi(x,s)=\eta_T^{2p'}$ in (1.6) gives rise to

$$\begin{aligned}&C(u_0,u_1)\varepsilon+\int_0^T\int_{\mathbb{R}^n}|v(x,s)|^q\eta_T^{2p'}dxds\\ =&\int_0^T\int_{\mathbb{R}^n}u_t(x,s)(-\partial_t\eta_T^{2p'}+\frac{\mu}{(1+|x|)^\beta}\eta_T^{2p'})dxds\\ =&I_1+I_2.\end{aligned}\tag{2.5}$$

Estimates of the terms I_1-I_2 in (2.5) are similar to (3.4) in [24], we achieve

$$\int_0^T\int_{\mathbb{R}^n}|v|^q\eta_T^{2p'}dxds\le CT^{\frac{-1+n(p-1)}{p}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2p'}dxds)^{\frac{1}{p}}.\tag{2.6}$$

In a similar way, we conclude

$$\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2p'}dxds\le CT^{\frac{-2+(n-1)(q-1)}{q}}(\int_0^T\int_{\mathbb{R}^n}|v|^q\eta_T^{2p'}dxds)^{\frac{1}{q}}.\tag{2.7}$$

Applying (2.6) and (2.7) yields

$$\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2p'}dxds\le CT^{n-\frac{pq+p+1}{pq-1}}\tag{2.8}$$

and

$$\int_0^T\int_{\mathbb{R}^n}|v|^q\eta_T^{2p'}dxds\le CT^{n-\frac{2q+1}{pq-1}}.\tag{2.9}$$

Combining (2.3) and (2.8), we come to the estimate

$$T(\varepsilon)\le C\varepsilon^{-F_{SG,1}^{-1}(n,p,q)}.$$

Similarly, utilizing (2.4) and (2.9) leads to

$$T(\varepsilon) \leq C\varepsilon^{-F^{-1}_{SG,2}(n,p,q)}.$$

Therefore, we derive the first lifespan estimate in (1.10).

In the case $F_{SG,1}(n,p,q)=0>F_{SG,2}(n,p,q)$, applying (2.3), (2.7) and Lemma 2.3, we acquire

$$\int_0^T \int_{\mathbb{R}^n} |v(x,s)|^q b_a \theta_M^{2p'} dxds \geq C\varepsilon^{pq}, \tag{2.10}$$

where $a=\frac{n+1}{2}-\frac{1}{p}$. Replacing ϕ with $\partial_t\psi=\eta_M^{2p'}b_a$ in (1.6) and using (2.1) yield

$$\begin{aligned}
&\varepsilon C(u_0,u_1)+\int_0^T \int_{\mathbb{R}^n} |v(x,s)|^q \eta_M^{2p'} b_a dxds \\
=&\int_0^T \int_{\mathbb{R}^n} u_t(x,s)(-\partial_t^2\eta_M^{2p'}b_a-2\partial_t\eta_M^{2p'}\partial_t b_a+\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2p'}b_a)dxds \\
&+\int_0^T \int_{\mathbb{R}^n} -u_t(x,s)\eta_M^{2p'}\left(\partial_t^2 b_a-\Delta b_a-\frac{\mu}{(1+|x|)^\beta}\partial_t b_a\right)dxds \\
=&\int_0^T \int_{\mathbb{R}^n} u_t(x,s)(-\partial_t^2\eta_M^{2p'}b_a-2\partial_t\eta_M^{2p'}\partial_t b_a+\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2p'}b_a)dxds \\
=&I_3+I_4+I_5.
\end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned}
|I_3|\leq& CM^{-2}(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}}(\int_{\frac{T}{2}}^T \int_{\{|x|\leq s+R\}} b_a^{\frac{p}{p-1}} dxds)^{\frac{p-1}{p}} \\
\leq& CM^{-1+(\frac{-n-1}{2}+\frac{n(p-1)}{p})}(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
|I_4|\leq& CM^{-1}(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}}(\int_{\frac{T}{2}}^T \int_{\{|x|\leq s+R\}} b_{a+1}^{\frac{p}{p-1}} dxds)^{\frac{p-1}{p}} \\
\leq& CM^{-1+(\frac{-n-1}{2}+\frac{n(p-1)}{p})}(\log M)^{\frac{p-1}{p}} \times (\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}}
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
|I_5|\leq& CM^{-1}(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}} \\
&\times(\int_{\frac{M}{2}}^M \int_0^{s+R} (s+2+r)^{-(\frac{n+1}{2}-\frac{1}{p})p'} \frac{(1+r)^{n-1-p'}}{(1+r)^{p'(\beta-1)}} dxds)^{\frac{p-1}{p}} \\
\leq& CM^{-1+(\frac{-n-1}{2}+\frac{n(p-1)}{p})}(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxds)^{\frac{1}{p}}.
\end{aligned} \tag{2.14}$$

Combining (2.11) with (2.12)-(2.14), we arrive at

$$\int_0^T \int_{\mathbb{R}^n} |v(x,s)|^q \eta_M^{2p'} b_a dxds$$

$$\begin{aligned}&\le CM^{-1+(\frac{-n-1}{2}+\frac{n(p-1)}{p})}(\log M)^{\frac{p-1}{p}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxds)^{\frac{1}{p}}\\&\le CM^{-1+(\frac{-n+1}{2}+\frac{n(p-1)}{p})}(\log M)^{\frac{p-1}{p}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxds)^{\frac{1}{p}}.\end{aligned}\tag{2.15}$$

Plugging (2.7) into (2.15), applying condition $F_{SG,1}(n,p,q)=0$ and Lemma 2.3 yield

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|v(x,s)|^q\eta_M^{2p'}b_a dxds)^{pq}\\&\le C(\log M)^{q(p-1)}\int_0^T\int_{\mathbb{R}^n}|v(x,s)|^q\eta_M^{2p'}b_a dxds.\end{aligned}\tag{2.16}$$

We set

$$Y[w](M)=\int_1^M(\int_0^T\int_{\mathbb{R}^n}w(x,s)\theta_\sigma^{2p'}(s)dxds)\sigma^{-1}d\sigma.\tag{2.17}$$

Consequently, we conclude

$$\frac{dY[w](M)}{dM}=M^{-1}\int_0^T\int_{\mathbb{R}^n}w(x,s)\theta_M^{2p'}(s)dxds\tag{2.18}$$

and

$$Y[w](M)\le C\log 2\int_0^T\int_{\mathbb{R}^n}w(x,s)\eta_M^{2p'}(s)dxds.\tag{2.19}$$

Utilizing (2.10), (2.16), (2.17)-(2.19) with $w(x,s)=|v(x,s)|^q b_a(x,s)$, Lemma 2.4 with $\delta=\varepsilon^{pq}$, $p_1=pq$ and $p_2=q(p-1)+1$, we obtain the second lifespan estimate in (1.10).

In the case $F_{SG,2}(n,p,q)=0>F_{SG,1}(n,p,q)$, recalling (2.4), (2.6) and Lemma 2.3, we deduce

$$\int_0^T\int_{\mathbb{R}^n}|u_t|^p b_a\theta_M^{2p'}dxds\ge C\varepsilon^{pq},\tag{2.20}$$

where $a=\frac{n-1}{2}-\frac{1}{q}$. Replacing ϕ with $\psi=\eta_M^{2p'}b_a$ in (1.7) yields

$$\begin{aligned}&\varepsilon C(v_0,v_1)+\int_0^T\int_{\mathbb{R}^n}|u_t(x,s)|^p\eta_M^{2p'}b_a dxds\\&=\int_0^T\int_{\mathbb{R}^n}v(\partial_t^2\eta_M^{2p'}b_a+2\partial_t\eta_M^{2p'}\partial_t b_a-\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2p'}b_a)dxds\\&=I_6+I_7+I_8.\end{aligned}\tag{2.21}$$

Estimates of the terms I_6-I_8 in (2.21) is similar to (2.12)-(2.14). Applying (2.6), (2.21) and Lemma 2.3 leads to

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|u_t(x,s)|^p\eta_M^{2p'}b_a dxds)^{pq}\\&\le(\log M)^{p(q-1)}\int_0^T\int_{\mathbb{R}^n}|u_t(x,s)|^p b_a\eta_M^{2p'}dxds.\end{aligned}\tag{2.22}$$

Utilizing (2.17)-(2.19) with $w(x,s)=|u_t(x,s)|^p b_a(x,s)$, (2.20), (2.22) and Lemma 2.4, we arrive at the third lifespan estimate in (1.10).

In the case $F_{SG,2}(n,p,q)=0=F_{SG,1}(n,p,q)$, it is deduced from (2.3) that

$$\begin{aligned}(\varepsilon C(u_0,u_1))^p &\le T^{-n+\frac{n-1}{2}p}\int_0^T\int_{\mathbb{R}^n}\theta_M^{2p'}|u_t|^p dxds\\ &\le C\int_0^T\int_{\mathbb{R}^n}|u_t(x,s)|^p b_a\theta_M^{2p'}dxds. \qquad (2.23)\end{aligned}$$

Combining (2.22) with (2.23), we derive the fourth lifespan estimate in (1.10). This finishes the proof of Theorem 1.3. □

3. Proof of Theorem 1.4

Proof. Choosing the test function $\phi=\eta_M^{2q_2'}$ in (1.6) yields

$$\begin{aligned}&(\varepsilon C(u_0,u_1))+\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_M^{2q_2'}dxds\\ =&\int_0^T\int_{\mathbb{R}^n}u(x,s)(\partial_t^2\eta_M^{2q_2'}-\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2q_2'})dxds. \qquad (3.1)\end{aligned}$$

Similar to the derivation in (2.7), we achieve

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_M^{2q_2'}dxds)^{q_2}\\ \le& CT^{-2+(n-1)(q_2-1)}\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}\eta_T^{2q_2'}dxds. \qquad (3.2)\end{aligned}$$

Analogously, we have

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|u_t|^{p_2}+|u|^{q_2})\eta_M^{2q_2'}dxds)^{q_1}\\ \le& CT^{-2+(n-1)(q_1-1)}\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}\eta_T^{2q_2'}dxds. \qquad (3.3)\end{aligned}$$

From (3.2) and (3.3), we acquire

$$\int_0^T\int_{\mathbb{R}^n}(|u_t|^{p_2}+|u|^{q_2})\eta_M^{2q_2'}dxds\le CT^{n-1-\frac{2(q_2+1)}{q_1q_2-1}}. \qquad (3.4)$$

Letting $\psi=-\eta_M^{2q_2'}\Phi=-\eta_M^{2q_2'}e^{-t}\phi_1(x)$ and replacing ϕ with $\Psi=\partial_t\psi$ in (1.6) give rise to

$$\begin{aligned}&\varepsilon C(u_0,u_1)+\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\partial_t\psi(x,s)dxds\\ =&\int_0^T\int_{\mathbb{R}^n}u_t(\partial_t^2\eta_M^{2q_2'}\Phi+2\partial_t\eta_M^{2q_2'}\partial_t\Phi-\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2q_2'}\Phi)dxds. \qquad (3.5)\end{aligned}$$

Similar to the derivation in (2.3), it holds that

$$(\varepsilon C(u_0,u_1))^{p_2}T^{n-\frac{n-1}{2}p_2}\le\int_0^T\int_{\mathbb{R}^n}\theta_M^{2q_2'}|u_t|^{p_2}dxds. \qquad (3.6)$$

Analogously, we obtain

$$(\varepsilon C(v_0,v_1))^{p_1}T^{n-\frac{n-1}{2}p_1}\leq\int_0^T\int_{\mathbb{R}^n}\theta_M^{2q_2'}|v_t|^{p_1}dxds. \tag{3.7}$$

Employing (3.4) and (3.6), we conclude the lifespan estimate

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{q_1q_2+2q_2+1}{p_2(q_1q_2-1)}-\frac{n-1}{2})^{-1}}.$$

Analogously,, we acquire

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2})^{-1}}.$$

Hence, we derive the first lifespan estimate in (1.11).

In the case $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)=\frac{q_1q_2+2q_2+1}{p_2(q_1q_2-1)}-\frac{n-1}{2}=0>\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}$ and $p_2=q_2$, utilizing (3.3) and (3.6) yields

$$\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})b_a\eta_T^{2q_2'}dxds\geq\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_T^{2q_2'}dxds\geq\varepsilon^{q_1p_2}, \tag{3.8}$$

where $a=\frac{n-1}{2}-\frac{1}{p_2}$.

Similar to the derivation in (2.15), applying (3.3) and Lemma 2.3 yields

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})b_a\eta_M^{2q_2'}dxds)^{q_1p_2}\\&\leq(\log M)^{q_1(p_2-1)}\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})b_a\eta_M^{2q_2'}dxds.\end{aligned} \tag{3.9}$$

Making use of (2.17)-(2.19) with $w(x,s)=(|v_t|^{p_1}+|v|^{q_1})b_a(x,s)$, (3.8), (3.9) and Lemma 2.4, we arrive at

$$T(\varepsilon)\leq\exp(C\varepsilon^{-p_2(q_1p_2-1)}).$$

Therefore, we obtain the second lifespan estimate in (1.11).

In the case $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)=\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}=0>\frac{q_1q_2+2q_2+1}{p_2(q_1q_2-1)}-\frac{n-1}{2}$ and $p_1=q_1$, employing (3.2) and (3.7) gives rise to

$$\int_0^T\int_{\mathbb{R}^n}(|u_t|^{p_2}+|u|^{q_2})b_a\eta_T^{2q_2'}dxds\geq\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}b_a\eta_T^{2q_2'}dxds\geq\varepsilon^{p_1q_2}, \tag{3.10}$$

where $a=\frac{n-1}{2}-\frac{1}{p_1}$.

Similarly, thanks to (3.2) and Lemma 2.3, we have

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|u_t|^{p_2}+|u|^{q_2})b_a\eta_M^{2q_2'}dxds)^{p_1q_2}\\&\leq(\log M)^{q_2(p_1-1)}\int_0^T\int_{\mathbb{R}^n}(|u_t|^{p_2}+|u|^{q_2})b_a\eta_M^{2q_2'}dxds.\end{aligned} \tag{3.11}$$

Combining (2.17)-(2.19) with $w(x,s)=(|u_t|^{p_2}+|u|^{q_2})b_a(x,s)$, (3.10), (3.11) with Lemma 2.4, it holds that

$$T(\varepsilon)\leq\exp(C\varepsilon^{-p_1(p_1q_2-1)}). \tag{3.12}$$

As a result, we prove the third lifespan estimate in (1.11).

For the case $\Gamma_{CC}(n,p_1,p_2,q_1,q_2)\geq 0$, $p_1=p_2$ and $q_1=q_2$, we consider the single semilinear wave equation with nonlinear term $f(u,u_t)=|u_t|^{p_1}+|u|^{q_1}$. Similar to the derivation in (3.6), we obtain

$$(\varepsilon C(u_0,u_1))^{p_1}T^{n-\frac{n-1}{2}p_1}\leq \int_0^T\int_{\mathbb{R}^n}\theta_M^{2q_1'}|u_t|^{p_1}dxds. \tag{3.13}$$

Choosing the test function $\phi=\eta_M^{2q_1'}$ in (1.6) leads to

$$\begin{aligned}&C(u_0,u_1)\varepsilon+\int_0^T\int_{\mathbb{R}^n}(|u_t(x,s)|^{p_1}+|u(x,s)|^{q_1})\eta_M^{2q_1'}dxds\\&=\int_0^T\int_{\mathbb{R}^n}u(x,s)(\partial_t^2\eta_M^{2q_1'}-\frac{\mu}{(1+|x|)^\beta}\partial_t\eta_M^{2q_1'})dxds.\end{aligned} \tag{3.14}$$

Similar to the derivation in (3.4) in [24], we derive

$$C(u_0,u_1)\varepsilon+\int_0^T\int_{\mathbb{R}^n}(|u_t(x,s)|^{p_1}+|u(x,s)|^{q_1})\eta_T^{2q_1'}dxds\leq CT^{n-\frac{q_1+1}{q_1-1}}. \tag{3.15}$$

Plugging (3.13) into (3.15) yields

$$T(\varepsilon)\leq C\varepsilon^{-\frac{1}{\frac{q_1+1}{p_1(q_1-1)}-\frac{n-1}{2}}}.$$

Thus, we obtain the sixth lifespan estimate in (1.11).

It is worth noticing that if

$$\Gamma_{Comb}(n,p_1,q_1)^{-1}\leq\Gamma_S(n,q_1)^{-1}, \tag{3.16}$$

we acquire $p_1\leq q_1$. If $q_1\leq 2p_1-1$, then we have

$$\Gamma_{Comb}(n,p_1,q_1)^{-1}\leq\Gamma_G(n,p_1)^{-1}. \tag{3.17}$$

We recognize that the sixth lifespan estimate in (1.11) is better than the fifth and seventh lifespan estimates in (1.11) when $p_1\leq q_1\leq 2p_1-1$. On the other hand, we derive the seventh lifespan estimate in (1.11) for $q_1<p_S(n)$, $p_1>q_1$ and the eighth lifespan estimate in (1.11) for $q_1=p_S(n)$, $p_1>q_1$. Moreover, we obtain the fifth lifespan estimate in (1.11) when $p_1<p_G(n)$, $q_1>2p_1-1$ and the fourth lifespan estimate in (1.11) when $p_1=p_G(n)=\frac{n+1}{n-1}$, $q_1>2p_1-1$, which is equivalent to $p_1=p_G(n)$, $q_1>1+\frac{4}{n-1}$. Precise illustrations of the fourth to eighth lifespan estimates in (1.11) in blow-up region are presented in [29]. The proof of Theorem 1.4 is finished. □

4. Proof of Theorem 1.5

Proof. Similar to the derivation in (3.2) and (3.6), we acquire

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_T^{2q_2'}dxds)^{q_2}\\&\leq CT^{-2+(n-1)(q_2-1)}\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}\eta_T^{2q_2'}dxds\end{aligned} \tag{4.1}$$

and

$$(C(u_0,u_1))^{q_2}\varepsilon^{q_2}T^{n-\frac{n-1}{2}q_2}\leq\int_0^T\int_{\mathbb{R}^n}\eta_M^{2q_2'}|u|^{q_2}dxds. \tag{4.2}$$

Similar to the derivation in (3.3) and (3.7), we derive

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}\eta_T^{2q_2'}dxds)^{q_1}\\ \leq & CT^{-2+(n-1)(q_1-1)}\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}\eta_T^{2q_2'}dxds\end{aligned} \tag{4.3}$$

and

$$(\varepsilon C(v_0,v_1))^{p_1}T^{n-\frac{n-1}{2}p_1}\leq\int_0^T\int_{\mathbb{R}^n}\theta_M^{2q_2'}|v_t|^{p_1}dxds. \tag{4.4}$$

Combining (4.1) and (4.3), it holds that

$$\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_T^{2q_2'}dxds\leq CT^{n-1-\frac{2(q_1+1)}{q_1q_2-1}}. \tag{4.5}$$

In a similar way, we acquire

$$\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}\eta_T^{2q_2'}dxds\leq CT^{n-1-\frac{2(q_2+1)}{q_1q_2-1}}. \tag{4.6}$$

Inserting (4.6) into (4.2) yields

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\frac{n-1}{2})^{-1}}.$$

Applying (4.4) and (4.5) leads to

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2})^{-1}}.$$

Hence, we conclude the first lifespan estimate in (1.12).

In the case $\Gamma_{CS}(n,p_1,q_1,q_2)=\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}=0>\frac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\frac{n-1}{2}$ and $p_1=q_1$, employing (4.1) and (4.4) yields

$$\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}b_a\eta_T^{2q_2'}dxds\geq\varepsilon^{p_1q_2} \tag{4.7}$$

with $a=\frac{n-1}{2}-\frac{1}{p_1}$.

Similar to the derivation in (3.11), taking into account (4.1), we have

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}b_a\eta_M^{2q_2'}dxds)^{p_1q_2}\\ \leq&(\log M)^{q_2(p_1-1)}\int_0^T\int_{\mathbb{R}^n}|u|^{q_2}b_a\eta_M^{2q_2'}dxds.\end{aligned} \tag{4.8}$$

Applying (2.17)-(2.19) with $w(x,s)=|u|^{q_2}b_a(x,s)$, (4.7), (4.8) and Lemma 2.4 gives rise to

$$T(\varepsilon)\leq\exp(C\varepsilon^{-p_1(p_1q_2-1)}).$$

Hence, we verify the second lifespan estimate in (1.12).

In the case $\Gamma_{CS}(n,p_1,q_1,q_2)=\frac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\frac{n-1}{2}=0>\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}$, combining (4.2) with (4.3), we acquire

$$\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2q_2'}dxds\geq C\varepsilon^{q_1q_2}, \tag{4.9}$$

where $a=\frac{n-1}{2}-\frac{1}{q_2}$.

Similar to the derivation in (2.22), employing (4.3) yields

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2q_2'}dxds)^{q_1q_2}\\ &\leq(\log M)^{q_1(q_2-1)}\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2q_2'}dxds.\end{aligned} \tag{4.10}$$

Utilizing (2.17)-(2.19) with $w(x,s)=|v|^{q_1}b_a(x,s)$, (4.9), (4.10) and Lemma 2.4 yields

$$T(\varepsilon)\leq\exp(C\varepsilon^{-q_2(q_1q_2-1)}).$$

In the case $\Gamma_{CS}(n,p_1,q_1,q_2)=\frac{q_1q_2+2q_1+1}{p_1(q_1q_2-1)}-\frac{n-1}{2}=0=\frac{q_1q_2+2q_2+1}{q_2(q_1q_2-1)}-\frac{n-1}{2}$ and $p_1=q_1$, taking advantage of (4.7)-(4.10), we deduce the third lifespan estimate in (1.12). The proof of Theorem 1.5 is finished. ∎

5. Proof of Theorem 1.6

Proof. Similar to the derivation in (2.3) and (2.6), it holds that

$$(\varepsilon C(u_0,u_1))^{p_2}T^{n-\frac{n-1}{2}p_2}\leq\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}\theta_M^{2p_2'}dxds \tag{5.1}$$

and

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_T^{2p_2'}dxds)^{p_2}\\ &\leq CT^{-1+n(p_2-1)}\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}\theta_M^{2p_2'}dxds.\end{aligned} \tag{5.2}$$

Similar to the derivation in (4.3) and (4.4), we derive

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}\eta_T^{2p_2'}dxds)^{q_1}\\ &\leq CT^{-2+(n-1)(q_1-1)}\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}\eta_T^{2p_2'}dxds\end{aligned} \tag{5.3}$$

and

$$(\varepsilon C(v_0,v_1))^{p_1}T^{n-\frac{n-1}{2}p_1}\leq\int_0^T\int_{\mathbb{R}^n}\theta_M^{2p_2'}|v_t|^{p_1}dxds. \tag{5.4}$$

Making use of (5.2) and (5.3) gives rise to

$$\int_0^T\int_{\mathbb{R}^n}(|v_t|^{p_1}+|v|^{q_1})\eta_T^{2p_2'}dxds\leq CT^{n-\frac{2q_1+1}{p_2q_1-1}} \tag{5.5}$$

and

$$\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}\eta_T^{2p_2'}dxds\le CT^{n-\frac{q_1p_2+p_2+1}{q_1p_2-1}}. \tag{5.6}$$

Utilizing (5.4) and (5.5) leads to

$$T(\varepsilon)\le C\varepsilon^{-(\frac{2q_1+1}{p_1(q_1p_2-1)}-\frac{n-1}{2})^{-1}}.$$

It is deduced from (5.1) and (5.6) that

$$T(\varepsilon)\le C\varepsilon^{-(\frac{q_1p_2+p_2+1}{p_2(q_1p_2-1)}-\frac{n-1}{2})^{-1}}.$$

As a consequence, we arrive at the first lifespan estimate in (1.13).

In the case $\Gamma_{CG}(n,p_1,q_1,p_2)=\frac{2q_1+1}{p_1(p_2q_1-1)}-\frac{n-1}{2}=0>\frac{p_2q_1+p_2+1}{p_2(p_2q_1-1)}-\frac{n-1}{2}$ and $p_1=q_1$, according to (5.2) and (5.4), we conclude

$$\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}b_a\eta_T^{2p_2'}dxds\ge\varepsilon^{p_1p_2} \tag{5.7}$$

with $a=\frac{n-1}{2}-\frac{1}{p_1}$.

Similar to the derivation in (4.8), taking into account (5.2) yields

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}b_a\eta_M^{2p_2'}dxds)^{p_1p_2}\\&\le(\log M)^{p_2(p_1-1)}\int_0^T\int_{\mathbb{R}^n}|u_t|^{p_2}b_a\eta_M^{2p_2'}dxds.\end{aligned} \tag{5.8}$$

Thanks to (2.17)-(2.19) with $w(x,s)=|u_t|^{p_2}b_a(x,s)$, (5.7), (5.8) and Lemma 2.4, we achieve

$$T(\varepsilon)\le\exp(C\varepsilon^{-p_1(p_1p_2-1)}).$$

Hence, we verify the second lifespan estimate in (1.13).

In the case $\Gamma_{CG}(n,p_1,q_1,p_2)=\frac{p_2q_1+p_2+1}{p_2(p_2q_1-1)}-\frac{n-1}{2}=0>\frac{2q_1+1}{p_1(p_2q_1-1)}-\frac{n-1}{2}$, An application of (5.1) and (5.3) shows

$$\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2p_2'}dxds\ge C\varepsilon^{q_1p_2} \tag{5.9}$$

with $a=\frac{n+1}{2}-\frac{1}{p_2}$.

Similar to the derivation in (2.16), utilizing (5.3), we arrive at

$$\begin{aligned}&(\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2p_2'}dxds)^{q_1p_2}\\&\le(\log M)^{q_1(p_2-1)}\int_0^T\int_{\mathbb{R}^n}|v|^{q_1}b_a\eta_M^{2p_2'}dxds.\end{aligned} \tag{5.10}$$

Exploiting (2.17)-(2.19) with $w(x,s)=|v|^{q_1}b_a(x,s)$, (5.9), (5.10) and Lemma 2.4 leads to

$$T(\varepsilon)\le\exp(C\varepsilon^{-p_2(q_1p_2-1)}).$$

Thus, we conclude the third lifespan estimate in (1.13).

In the case $\Gamma_{CG}(n,p_1,q_1,p_2)=\frac{2q_1+1}{p_1(p_2q_1-1)}-\frac{n-1}{2}=0=\frac{p_2q_1+p_2+1}{p_2(p_2q_1-1)}-\frac{n-1}{2}$ and $p_1=q_1$, taking advantage of (5.7)-(5.10) leads to the fourth lifespan estimate in (1.13). The proof of Theorem 1.6 is finished. □

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