COMMUN. MATH. SCI.
Vol. 22, No. 2, pp. 435–460

ON GLOBAL SOLUTIONS TO THE INHOMOGENEOUS, INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH TEMPERATURE-DEPENDENT COEFFICIENTS*

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Abstract. In this paper, we study the initial-boundary value problem for the full inhomogeneous, incompressible Navier-Stokes equations with temperature-dependent viscosity and heat conductivity coefficients. The viscosity coefficient may be degenerate in the sense that it may vanish in the region of absolutely zero temperature. Our main result is to prove the global existence of large weak solutions to such a system. The proof is based on a three-level approximate scheme, the Galerkin method, De Giorgi's method and appropriate compactness arguments.

Keywords. Global existence; Galerkin method; De Giorgi's method.

AMS subject classifications. 35Q35; 35D30; 76D03; 76D05.

1. Introduction and main result

In the present paper, we consider the following three-dimensional inhomogeneous, incompressible Navier-Stokes equations with heat-conducting effects

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \mathrm{div}\mathbb{S}, \\ \partial_t(\rho\theta) + \mathrm{div}(\rho \mathbf{u}\theta) + \mathrm{div}\mathbf{q} = \mathbb{S} : \nabla \mathbf{u}, \\ \mathrm{div}\mathbf{u} = 0. \end{cases} \tag{1.1}$$

This system is supplemented with the initial conditions

$$(\rho, \rho\mathbf{u}, \theta)(0,x) = (\rho_0, \mathbf{m}_0, \theta_0)(x) \quad \text{in}\, \Omega, \tag{1.2}$$

and the boundary conditions

$$\mathbf{u}(t,x) = 0, \quad \nabla\theta(t,x)\cdot \mathbf{n}(x) = 0 \quad \text{on}\, [0,T]\times\partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $C^{2+\nu}$ with $\nu > 0$, $\mathbf{n}(x)$ is the unit outward normal vector to the boundary at $x \in \partial\Omega$. Here, $\rho = \rho(t,x)$ is the density of the fluid, $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field, $\theta = \theta(t,x)$ is the temperature, $P = P(t,x)$ is the pressure, $\mathbb{S}$ denotes the viscous stress tensor given by

$$\mathbb{S} = \mu(\theta)(\nabla\mathbf{u} + \nabla^T\mathbf{u}),$$

where $\nabla^T\mathbf{u}$ is the transposition of $\nabla\mathbf{u}$ and $\mu(\theta) \geq 0$ is the viscosity coefficient which depends on the temperature and may degenerate in the region of absolutely zero temperature, and $\mathbf{q}$ denotes the heat flux of the fluid satisfying Fourier's law

$$\mathbf{q} = -\kappa(\theta)\nabla\theta,$$

with $\kappa(\theta) > 0$ being the heat conductivity coefficient depending on the fluid temperature. Moreover, it is easily seen that our discussion below still holds with more general internal energy $e = c_\nu\theta$, with $c_\nu > 0$ being the specific heat at constant volume.

*Received: October 09, 2022; Accepted (in revised form): July 11, 2023. Communicated by Feimin Huang.

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The global existence of solutions to system (1.1)-(1.3) is an open problem, even for the homogeneous case (i.e. $\rho = \text{const.} > 0$) in the class of the weak solutions introduced by Leray [26]. The main difficulty lies in the dissipative term

$$\mathbb{S} : \nabla \mathbf{u} = \frac{\mu(\theta)}{2} \left| \nabla \mathbf{u} + \nabla^T \mathbf{u} \right|^2.$$

This term is the only source of a priori estimates on $\nabla \mathbf{u}$, and at the same time, only weakly lower semicontinuous with respect to $\nabla \mathbf{u}$. Therefore, the temperature Equation $(1.1)_3$ has to be replaced by the inequalities

$$\partial_t(\rho\theta) + \operatorname{div}(\rho \mathbf{u}\theta) + \operatorname{div}\mathbf{q} \geq \mathbb{S} : \nabla \mathbf{u},$$

and

$$\partial_t \left(\frac{1}{2}\rho|\mathbf{u}|^2 \right) + \operatorname{div}\left(\left(\frac{1}{2}\rho|\mathbf{u}|^2 + P \right) \mathbf{u} \right) - \operatorname{div}(\mathbb{S}\mathbf{u}) + \mathbb{S} : \nabla \mathbf{u} \leq 0.$$

Notably, this difficulty can be tackled if we consider some non-Newtonian fluids (see [8, 31]) or two-dimensional case (see [9]), where better a priori estimates for $\nabla \mathbf{u}$ can be obtained.

Without heat-conducting effects, there has been a lot of literature on the existence of solutions to the inhomogeneous, incompressible Navier-Stokes equations. When the viscosity coefficient is a positive constant, Kazhikhov [1,2,23] proved the global existence of weak solutions and local strong solution in the absence of vacuum. Then, Simon [32] established global weak solutions with finite energy for the case that the initial data may contain vacuum. Later, based on some compatibility condition, Choe-Kim [7] constructed a local strong solution, which was extended to be a global one by Huang-Wang [20] for two-dimensional case and Kim [24] for three-dimensional case if $\|\nabla \mathbf{u}_0\|$ is sufficiently small. Recently, Lü-Shi-Zhong [28] removed the compatibility condition and showed the global existence of strong solutions with large initial data on the whole space $\mathbb{R}^2$. When the viscosity coefficient depends on density, Diperna-Lions [12, 27] constructed global weak solutions. Then, Desjardins [11] proved the global existence of weak solutions with higher regularity for two-dimensional case where μ is a small perturbation of a positive constant in L^∞-norm. Meanwhile, we also mention the existence of global strong solutions in the absence of the vacuum states (see [18, 21, 22, 34, 35]).

When the heat-conducting effects are considered, for the homogeneous case (i.e. $\rho = \text{const.} > 0$), Lions [27] studied (1.1)-(1.3) with constant viscosity and heat conductivity coefficients, which means that system $(1.1)_1$, $(1.1)_2$ and Equation $(1.1)_3$ are uncoupled, therefore, Lions established the existence of global weak solutions by two different approaches. To be specific, the first approach, called decoupled approach is to solve first $(1.1)_1$, $(1.1)_2$, then, given a weak solution $\mathbf{u}$ of $(1.1)_1$, $(1.1)_2$, in particular, for a given L^1-function $\mathbb{S} : \nabla \mathbf{u}$ on the right-hand side of $(1.1)_3$, try to solve $(1.1)_3$. The second approach is to solve $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$ simultaneously, where however, the temperature Equation $(1.1)_3$ is replaced by the total energy equation

$$\partial_t \left(\frac{1}{2}|\mathbf{u}|^2 + \theta \right) + \operatorname{div}\left[\left(\frac{1}{2}|\mathbf{u}|^2 + P + \theta \right) \mathbf{u} \right] + \operatorname{div}\mathbf{q} = \operatorname{div}(\mathbb{S}\mathbf{u}).$$

Then, based on the second approach in [27], Feireisl-Málek [15] established a global weak solution in a periodic domain with the viscosity and heat conductivity coefficients depending on temperature. Later, Bulíček-Feireisl-Málek [5] extended [15] to general three-dimensional domains with Navier's slip boundary condition for the velocity. This result also extended the result of Caffarelli-Kohn-Nirenberg [6], from a purely mechanical context to a complete thermodynamic setting.

For the compressible Navier-Stokes equations with heat-conducting effects, Feireisl first introduced the concept of "variational" solutions, and in [13], he proved the global existence of "variational" solutions in a bounded domain $\Omega\subset\mathbb{R}^N$, $N\geq 2$ with boundary of class $C^{2+\nu}$ for a certain $\nu>0$, where the constant viscosity coefficients are assumed to satisfy

$$\mu>0, \quad \lambda+\frac{2}{N}\mu\geq 0.$$

Then, Feireisl [14] extended the above result to the case that the viscosity coefficients are functions of temperature satisfying

$$\mu(\theta)\geq \underline{\mu}>0, \quad \lambda(\theta)+\frac{2}{N}\mu(\theta)\geq 0,$$

for some positive constant $\underline{\mu}>0$. Recently, Wang-Zuo [33] further strengthened the result [14] and established global weak solutions to the compressible Navier-Stokes equations with degenerate, or even vanishing shear viscosity coefficient.

The existence of global weak solutions to the inhomogeneous, incompressible Navier-Stokes with heat-conducting effects (1.1)-(1.3) was first proposed by Lions in his book [27], Sect. 3.4, where the viscosity and heat conductivity coefficients were assumed to be functions of density and temperature satisfying $\mu,\kappa\in C([0,\infty)\times\mathbb{R})$

$$\inf\{\mu(t,s)||t|\leq R, s\in\mathbb{R}\}>0, \quad \inf\{\kappa(t,s)||t|\leq R, s\in\mathbb{R}\}>0 \text{ for all } R>0.$$

Lions pointed out that it was possible to study such a problem by the methods developed in Chapter 2 and Chapter 3 of his book. However, precise results about this problem were not given therein. Note that the assumption that the viscosity coefficient was bounded below from zero played an essential role in the compactness analysis in [27].

In this paper, we aim to study the global existence of weak solutions to the initial-boundary problem (1.1)-(1.3), with the viscosity and heat conductivity coefficients depending on the temperature. It should be emphasized that the viscosity coefficient is assumed to be degenerate and may vanish in the region of absolutely zero temperature. This assumption is based on the observation that zero viscosity occurs when the temperature is very low in superfluids in [4]. The degeneracy of the viscosity coefficient may lead to the failure of parabolicity of the momentum Equation $(1.1)_2$, which makes it invalid to apply the Galerkin method to the momentum Equation $(1.1)_2$ to establish approximate solutions. Another negative effect of this degeneracy is that we are unable to obtain a priori estimates on $\nabla\mathbf{u}$ from the energy inequality, which makes compactness analysis extremely difficult.

To tackle these two difficulties, we add an artificial viscosity term $\varepsilon\Delta\mathbf{u}$ $(\varepsilon>0)$ in the momentum equation. Such an artificial viscosity term makes it possible to apply the Galerkin method to obtain global approximate solutions and plays an important role in obtaining estimates for the gradient of velocity. At the same time, to eliminate this artificial viscosity term as $\varepsilon\to 0^+$, we need to obtain uniform estimates on $\nabla\mathbf{u}$ independent of $\varepsilon>0$, which is one of the most serious challenges we face.

The crucial point in obtaining uniform bounds on $\nabla\mathbf{u}$ independent of $\varepsilon>0$ is to show that the temperature is bounded away from zero by De Giorgi's method [10], which was originally developed by De Giorgi for the regularity of elliptic equations with discontinuous coefficients, then applied by Baer-Vasseur [3] and Mellet-Vasseur [29] to give a positive bound from below for the temperature in compressible Navier-Stokes equations. Those results can be viewed as a priori estimates for the temperature and motivate us to show that the temperature is uniformly bounded away from zero for the approximated solutions of the incompressible Navier-Stokes equations. Furthermore, by the assumptions imposed on $\mu(\theta)$, we can obtain a lower bound for the viscosity coefficient, that is, $\mu(\theta)\geq\underline{\mu}$ for some positive constant $\underline{\mu}$ independent of $\varepsilon>0$. Then the uniform H^1-regularity of the velocity field $\mathbf{u}$ could be derived by the elementary energy inequality.

Another difficulty in proving the global solvability of (1.1)-(1.3) lies in dealing with the temperature concentration, which as in [13, 14] can be resolved by the renormalization of the temperature Equation $(1.1)_3$. To be specific, multiplying $(1.1)_3$ by $h(\theta)$ for some suitable function h, we obtain

$$\partial_t(\rho H(\theta))+\operatorname{div}(\rho\mathbf{u}H(\theta))-\triangle\mathcal{K}_h(\theta)=h(\theta)\mathbb{S}:\nabla\mathbf{u}-h'(\theta)\kappa(\theta)|\nabla\theta|^2,$$

where

$$H(\theta)=\int_0^\theta h(z)dz,\qquad \mathcal{K}_h(\theta)=\int_0^\theta \kappa(z)h(z)dz.$$

The idea of renormalization is inspired by DiPerna-Lions [12] where they replaced the continuity Equation $(1.1)_1$ by

$$\partial_t b(\rho)+\operatorname{div}(b(\rho)\mathbf{u})+(b'(\rho)\rho-b(\rho))\operatorname{div}\mathbf{u}=0,$$

for suitable functions $b=b(\rho)$, and then was used by Feireisl [13] and Lions [27] to overcome the temperature concentration. However, as mentioned above, due to the dissipative term $\mathbb{S}:\nabla\mathbf{u}$, we can only obtain an inequality instead of $(1.1)_3$ by passing to the limit.

The weak solutions to the initial-boundary value problem (1.1)-(1.3) are defined in the following sense:

Definition 1.1. *We call $(\rho,\mathbf{u},\theta,P)$ a weak solution to the initial-boundary value problem (1.1)-(1.3) if*

(i) *the density $\rho\geq 0$ satisfies*

$$\rho\in L^\infty((0,T)\times\Omega)\cap C([0,T];L^p(\Omega)),1\leq p<\infty,$$

the velocity $\mathbf{u}$ belongs to $L^2(0,T;W_0^{1,2}(\Omega))$, and $(\rho,\mathbf{u})$ is a renormalized solution of the continuity Equation $(1.1)_1$ in the sense of distributions, that is,

$$\int_0^T\int_\Omega b(\rho)\partial_t\Phi+b(\rho)\mathbf{u}\cdot\nabla\Phi+(b(\rho)-b'(\rho)\rho)\operatorname{div}\mathbf{u}\Phi dxdt=0,$$

holds for any b satisfying

$$b\in C^1[0,\infty),\quad b'(\rho)=0\,\text{for all}\,\rho\,\text{large enough},$$

and any $\Phi\in C_c^\infty((0,T)\times\Omega)$;

(ii) *the momentum Equation* $(1.1)_2$ *and the incompressibility condition* $(1.1)_4$ *hold in* $\mathcal{D}'((0,T)\times\Omega)$*, that is,*

$$\int_0^T\int_\Omega \rho\mathbf{u}\cdot\partial_t\Phi dxdt+\int_0^T\int_\Omega \rho\mathbf{u}\otimes\mathbf{u}:\nabla\Phi dxdt=\int_0^T\int_\Omega \mathbb{S}:\nabla\Phi dxdt,$$

holds for any $\Phi\in C_c^\infty((0,T)\times\Omega)$ *satisfying* $\mathrm{div}\Phi=0$*, where*

$$\rho\mathbb{S}=\rho\mu(\theta)(\nabla\mathbf{u}+\nabla^T\mathbf{u}),$$

and for any $\eta\in C_c^\infty(\Omega)$*, it holds that for a.a.* $t\in(0,T)$

$$\int_\Omega \mathbf{u}\cdot\nabla\eta dx=0.$$

Moreover, $\rho\mathbf{u}\in C([0,T];L^2_{weak}(\Omega))$ *satisfies the initial condition* (1.2)*;*

(iii) *the temperature* $\theta\geq 0$ *satisfies*

$$\theta\in L^2(0,T;W^{1,2}(\Omega)),\quad \rho\theta\in L^\infty(0,T;L^1(\Omega)),$$

and the temperature inequality holds in the sense of distributions, that is,

$$\begin{aligned}&\int_0^T\int_\Omega \rho\theta\partial_t\varphi dxdt+\int_0^T\int_\Omega (\rho\mathbf{u}\theta\cdot\nabla\varphi+\mathcal{K}(\theta)\triangle\varphi)dxdt\\ &\leq -\int_0^T\int_\Omega \mathbb{S}:\nabla\mathbf{u}\varphi dxdt-\int_\Omega \rho_0\theta_0\varphi(0),\end{aligned}$$

for any $\varphi\in C_c^\infty([0,T]\times\Omega)$ *satisfying*

$$\varphi\geq 0,\ \varphi(T,\cdot)=0,\ \nabla\varphi\cdot\mathbf{n}|_{\partial\Omega}=0,$$

where

$$\mathcal{K}(\theta)=\int_0^\theta \kappa(z)dz.$$

Moreover, $\theta(t,\cdot)\to\theta_0$ *in* $\mathcal{D}'(\Omega)$ *as* $t\to 0^+$*, that is, for any* $\chi\in C_c^\infty(\Omega)$*, it holds*

$$\lim_{t\to 0^+}\int_\Omega \theta(t,x)\chi(x)dx=\int_\Omega \theta_0(x)\chi(x)dx;$$

(iv) *the energy inequality holds, that is, for a.a.* $t\in(0,T)$*,*

$$E[\rho,\mathbf{u},\theta](t)\leq E[\rho,\mathbf{u},\theta](0),$$

where

$$E[\rho,\mathbf{u},\theta](t)=\int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho\theta\right)(t)dx,$$

and

$$E[\rho,\mathbf{u},\theta](0)=\int_\Omega\left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0}+\rho_0\theta_0\right)dx.$$

REMARK 1.1. As mentioned in [15], the reason for introducing the function $\mathcal{K}(\theta) = \int_0^\theta \kappa(z)dz$ with $\nabla\mathcal{K}(\theta) = \kappa(\theta)\nabla\theta = -\mathbf{q}$ is that we are unable to deduce $\kappa(\theta)\nabla\theta$ is locally integrable by a priori estimates that we can obtain. However, for weak formulation, we can write

$$\int_\Omega \Delta\mathcal{K}(\theta)\varphi dxdt = \int_\Omega \mathcal{K}(\theta)\Delta\varphi dxdt,$$

where the right-hand side makes sense for any function $\varphi \in C^2(\Omega)$.

REMARK 1.2. As shown later, the bounds on the velocity fail to ensure the convergence of the term $\mathbb{S}:\nabla\mathbf{u}$ in the sense of distributions. Therefore, as in [13, 14], we replaced the temperature Equation $(1.1)_3$ by the following two inequalities in Definition 1.1

$$\partial_t(\rho\theta) + \operatorname{div}(\rho\mathbf{u}\theta) - \triangle\mathcal{K}(\theta) \geq \mathbb{S}:\nabla\mathbf{u},$$

and

$$E[\rho,\mathbf{u},\theta](t) \leq E[\rho,\mathbf{u},\theta](0).$$

Our main result can be stated as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, with $\nu > 0$. Assume that*

(i) *the heat conductivity coefficient $\kappa(\theta) \in C^1([0,\infty))$ satisfies*

$$\underline{\kappa}(1+\theta^2) \leq \kappa(\theta) \leq \overline{\kappa}(1+\theta^2), \tag{1.4}$$

for constants $\underline{\kappa} > 0$ and $\overline{\kappa} > 0$;

(ii) *the viscosity coefficient $\mu(\theta)$ is globally Lipschitz continuous on $[0,\infty)$ and it is a positive function on $[\bar{\theta},\infty)$, satisfying*

$$\lim_{\theta\to\infty} \mu(\theta) > 0, \quad \textit{and} \quad \mu(\theta) \geq \kappa\theta, \quad \textit{for } 0 \leq \theta \leq \bar{\theta}; \tag{1.5}$$

(iii) *the initial data satisfy*

$$\begin{cases} \rho_0 \in L^\infty(\Omega),\ \rho_0 \geq 0 & \text{on } \Omega, \\ \theta_0 \in L^1(\Omega),\ \theta_0 \geq \underline{\theta} > 0 & \text{on } \Omega, \\ \dfrac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega). \end{cases} \tag{1.6}$$

Then, for any given $T > 0$, the initial-boundary value problem (1.1)-(1.3) *admits a global weak solution $(\rho,\mathbf{u},\theta,P)$ in the sense of Definition 1.1.*

REMARK 1.3. As mentioned in [4], zero viscosity only occurs when the temperature is very low in superfluids. Otherwise, by the second law of thermodynamics, the viscosity of all fluids is positive. Thus, our restriction (1.5) is physical. Moreover, the assumptions imposed on the viscosity coefficient $\mu(\theta)$ in Theorem 1.1 are definitely not optimal, but our goal is to highlight the main ideas of the proof without going into unnecessary technical details.

The proof of Theorem 1.1 will be roughly divided into the following three steps:

Step 1: For fixed ε, $\delta>0$, we solve the approximate system (2.1)-(2.3) by the Galerkin method, to be specific, we first solve the problem in a suitable finite dimensional space X_n, then recover a global solution by passing to the limit as $n\to\infty$;

Step 2: For fixed $\delta>0$, letting $\varepsilon\to 0^+$ to eliminate the artificial viscosity in the momentum Equation $(1.1)_2$. The crucial point is to obtain a below bound for the temperature by De Giorgi's method, which is motivated by the work of Mellet-Vasseur [29]. This implies that the temperature-depending viscosity coefficient is bounded away from zero by the assumptions in Theorem 1.1. Thus, one can obtain uniform bounds for the velocity for passing to the limits. Note that the below bound of temperature is also uniform in $\delta>0$. This means that the uniform bounds are also available for the limits as δ goes to zero;

Step 3: We are able to recover a globally defined weak solution to the initial-boundary value problem (1.1)-(1.3) by letting $\delta\to 0^+$.

Now we would like to give some comments on Theorem 1.1 by comparing it with the relevant result [33] for the compressible Navier-Stokes equations. Roughly speaking, there are two major differences between the inhomogeneous, incompressible case and compressible case, that is,

(1) the pressure. The pressure is an independent variable in terms of the density and the velocity for the incompressible case, which can be tackled by choosing suitable divergence-free test functions in Definition 1.1. For the compressible case, the pressure is a function of the density and the main source of integrability of the density;

(2) the continuity equation. For the incompressible case, the continuity equation can be rewritten as the transport form

$$\partial_t\rho+\mathbf{u}\cdot\nabla\rho=0,$$

which can be solved based on Lions' result [27]. However, this method is invalid for the compressible case.

Therefore, the approximate system and compactness arguments established in the present paper are different from that in [33].

The rest of the paper is organized as follows. In Section 2, we construct a suitable approximate system (2.1)-(2.3) and obtain the global solvability by the Galerkin method. In Section 3, we perform the limit $\varepsilon\to 0^+$ to eliminate the artificial viscosity. In Section 4, we let $\delta\to 0^+$ to finish the proof of Theorem 1.1.

2. The construction of approximate solutions

First, we construct the following approximate system

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla P-\operatorname{div}\mathbb{S}-\varepsilon\Delta\mathbf{u}=0,\\ \partial_t((\delta+\rho)\theta)+\operatorname{div}(\rho\mathbf{u}\theta)-\triangle\mathcal{K}(\theta)+\delta\theta^3=(1-\delta)\mathbb{S}:\nabla\mathbf{u},\\ \operatorname{div}\mathbf{u}=0,\end{cases}\tag{2.1}$$

where both ε and δ are positive parameters, supplemented with the initial condition

$$(\rho,\rho\mathbf{u},\theta)(0)=(\rho_{0,\delta},\mathbf{m}_{0,\delta},\theta_{0,\delta})\quad\text{in }\Omega,\tag{2.2}$$

and the boundary condition

$$\mathbf{u}=0,\quad \nabla\theta\cdot\mathbf{n}=0\quad\text{on }\partial\Omega.\tag{2.3}$$

Note that the construction of the above approximate system is motivated but different from [13, 14, 16, 17]. Moreover, the regularized initial data are required to satisfy the following conditions:

$$\begin{cases} \rho_{0,\delta}\in C^{2+\nu}(\bar{\Omega}), \quad 0<\delta\le\rho_{0,\delta}(x)\le\bar{\rho}; \\ \rho_{0,\delta}\to\rho_0 \text{ in } L^2(\Omega), \quad |\{x\in\Omega\,|\,\rho_{0,\delta}(x)<\rho_0(x)\}|\to 0, \text{ as } \delta\to 0; \\ \theta_{0,\delta}\in C^{2+\nu}(\bar{\Omega}), \quad \nabla\theta_{0,\delta}\cdot\mathbf{n}|_{\partial\Omega}=0, \quad 0<\underline{\theta}\le\theta_{0,\delta}; \\ \theta_{0,\delta}\to\theta_0 \text{ in } L^2(\Omega), \text{ as } \delta\to 0; \\ \mathbf{m}_{0,\delta}=\begin{cases}\mathbf{m}_0, & \text{if } \rho_{0,\delta}\ge\rho_0, \\ 0, & \text{if } \rho_{0,\delta}<\rho_0,\end{cases} \end{cases} \tag{2.4}$$

where $\bar{\rho}$ and $\underline{\theta}$ are independent of $\delta>0$. In particular, the regularized initial value of the total energy

$$E_\delta(0)=\int_\Omega\left(\frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}+(\delta+\rho_{0,\delta})\theta_{0,\delta}\right)dx \tag{2.5}$$

is bounded by a constant independent of $\delta>0$.

REMARK 2.1. In Equations $(2.1)_2$ and $(2.1)_3$, the quantities ε and δ are small positive parameters. Roughly speaking, the extra term $\varepsilon\triangle\mathbf{u}$ represents the artificial viscosity which ensures the parabolic property of the momentum Equation $(2.1)_2$. The quantity $\delta\theta^3$ is introduced to improve the integrability of the temperature. The other terms related to the parameter $\delta>0$ are introduced to avoid technicalities in the temperature estimates.

We give our result about the global solvability of the approximate problem (2.1)-(2.3) in the following proposition.

PROPOSITION 2.1. *For any fixed ε, $\delta>0$, under the hypotheses of Theorem 1.1 and the assumptions imposed on the initial data (2.4), the approximate system (2.1)-(2.3) admits a global weak solution $(\rho,\mathbf{u},\theta,P)$ satisfying the following properties:*

(i) *the density $\rho\ge 0$ satisfies*

$$\rho\in L^\infty((0,T)\times\Omega)\cap C([0,T];L^p(\Omega)), 1\le p<\infty,$$

the velocity $\mathbf{u}$ belongs to the space $L^2(0,T;W_0^{1,2}(\Omega))$, and $(\rho,\mathbf{u})$ is a renormalized solution of the continuity Equation $(2.1)_1$ in the sense of distributions;

(ii) *the modified momentum Equation $(2.1)_2$ and the incompressibility condition $(2.1)_4$ hold in $\mathcal{D}'((0,T)\times\Omega)$. Moreover, $\rho\mathbf{u}\in C([0,T];L^2_{weak}(\Omega))$ satisfies the initial condition (2.2);*

(iii) *the temperature $\theta\ge 0$ satisfies*

$$\theta\in L^2(0,T;W^{1,2}(\Omega))\cap L^3((0,T)\times\Omega), \quad \rho\theta\in L^\infty(0,T;L^1(\Omega)),$$

and the renormalized temperature inequality holds in the sense of distributions, that is,

$$\int_0^T\int_\Omega(\delta+\rho)H(\theta)\partial_t\varphi dxdt+\int_0^T\int_\Omega\left(\rho H(\theta)\mathbf{u}\cdot\nabla\varphi+\mathcal{K}_h(\theta)\triangle\varphi-\delta\theta^3h(\theta)\varphi\right)dxdt$$
$$\le\int_0^T\int_\Omega\left((\delta-1)\mathbb{S}:\nabla\mathbf{u}h(\theta)+h'(\theta)\kappa(\theta)|\nabla\theta|^2\right)\varphi dxdt-\int_\Omega(\delta+\rho_{0,\delta})H(\theta_{0,\delta})\varphi(0)dx \tag{2.6}$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$ *satisfying*

$$\varphi \geq 0,\ \varphi(T,\cdot)=0,\ \nabla\varphi\cdot\mathbf{n}|_{\partial\Omega}=0,$$

where $H(\theta)=\int_0^\theta h(z)dz$ *and* $\mathcal{K}_h(\theta)=\int_0^\theta \kappa(z)h(z)dz$*, with the non-increasing* $h\in C^2([0,\infty))$ *satisfying*

$$0<h(0)<\infty,\ \lim_{z\to\infty} h(z)=0, \tag{2.7}$$

and

$$h''(z)h(z)\geq 2(h'(z))^2 \ for\ all\ z\geq 0; \tag{2.8}$$

(iv) *the energy inequality*

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2+(\delta+\rho)\theta\right)(t)dx+\int_0^t\int_\Omega \delta\mathbb{S}:\nabla\mathbf{u}+\varepsilon|\nabla\mathbf{u}|^2+\delta\theta^3dxds\\ &\leq \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_{0,\delta}}+(\delta+\rho_{0,\delta})\theta_{0,\delta}dx\end{aligned} \tag{2.9}$$

holds for a.a. $t\in[0,T]$.

REMARK 2.2. As proved in [13] for the constant viscosity coefficient case, the hypotheses (2.7)-(2.8) are imposed to ensure the convex and weakly lower semi-continuous property of the function

$$(\theta,\nabla\mathbf{u})\mapsto h(\theta)\mathbb{S}:\nabla\mathbf{u}, \tag{2.10}$$

which is still valid for the temperature-depending viscosity coefficient case (cf. [19]).

2.1. Global solvability of the approximate system in a finite dimensional space. Let $\{\eta_n\}$ be a family of divergence-free linearly independent smooth vector functions that vanish on the boundary $\partial\Omega$. Consider a sequence of finite dimensional spaces

$$X_n=\text{span}\{\eta_1,\eta_2,...,\eta_n\},n=1,2,\cdots. \tag{2.11}$$

The global solvability of the approximate problem (2.1)-(2.3) in the finite dimensional space X_n can be achieved by the following four steps:

Step 1: Given $\mathbf{u}=\mathbf{u}_n\in C([0,T];X_n)$, the approximate continuity Equation $(2.1)_1$ can be seen as a transport equation of ρ, which can be solved directly by the characteristics method. We denote the solution by $\rho_n:=\rho[\mathbf{u}_n]$ and give the details in Proposition 2.2.

Step 2: Given $\mathbf{u}=\mathbf{u}_n$ and $\rho=\rho_n$, the approximate temperature Equation $(2.1)_3$ can be seen as a quasi-linear parabolic equation of θ, which can be solved by applying the parabolic theory [25]. See details in Proposition 2.3. Denote the solutions by $\theta_n:=\theta[\mathbf{u}_n]$.

Step 3: Substituting $\rho=\rho_n$ and $\theta=\theta_n$ into the following integral equation

$$\int_\Omega(\rho\mathbf{u}_n)(t)\cdot\eta dx-\int_\Omega \mathbf{m}_{0,\delta}\cdot\eta dx=\int_0^t\int_\Omega(\rho\mathbf{u}_n\otimes\mathbf{u}_n-\mathbb{S}_n-\varepsilon\nabla\mathbf{u}_n):\nabla\eta dxds, \tag{2.12}$$

for any $\eta\in X_n$, with

$$\mathbb{S}_n=\mu(\theta)(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n),$$

we can obtain a local solution $\mathbf{u}_n \in C([0,T_n];X_n)$ with $T_n \leq T$ by the standard fixed-point theorem. See details in Proposition 2.4.

Step 4: By virtue of some uniform (in time) estimates, we can extend T_n to T to obtain a global existence result.

Remark 2.3. Note that the integral Equation (2.12) can be seen as a projection of the momentum Equation $(2.1)_2$ onto the finite dimensional space X_n in the sense of distributions.

Following the steps presented above, for fixed velocity field $\mathbf{u}=\mathbf{u}_n \in C([0,T];X_n)$, we first study the solvability of the approximate continuity Equation $(2.1)_1$.

Proposition 2.2. *Let $\mathbf{u}=\mathbf{u}_n$ be a given vector function belonging to $C([0,T];X_n)$. Assume that the initial data $\rho_{0,\delta}$ satisfy the hypotheses in* (2.4).

Then there exists a mapping $\rho_n = \rho[\mathbf{u}_n]$:

$$\rho_n : C([0,T];X_n) \to C([0,T];C^2(\bar{\Omega}))$$

having the following properties:

- *the initial value problem $(2.1)_1$, (2.2) possesses a unique classical solution ρ_n;*
- $0<\delta\leq \rho[\mathbf{u}_n] \leq \overline{\rho}$ *for all $t \in [0,T]$;*
- *continuity of the mapping:*

$$\|\rho_{n_1}-\rho_{n_2}\|_{C([0,T];C^2(\bar{\Omega}))} \leq CT \|\mathbf{u}_{n_1}-\mathbf{u}_{n_2}\|_{C([0,T];X_n)}. \tag{2.13}$$

Proof. Taking $\mathbf{u}=\mathbf{u}_n$, we can rewrite the continuity Equation $(2.1)_1$ as the following transport equation

$$\partial_t \rho + \mathbf{u}_n \cdot \nabla \rho = 0.$$

By characteristics method, we have

$$\rho(t,x) = \rho_{0,\delta}(x-\mathbf{u}_n t), \tag{2.14}$$

which, combined with the assumptions imposed on the initial data $\rho_{0,\delta}$ in (2.4), yields the properties in Proposition 2.2. □

For the temperature Equation $(2.1)_3$, similarly as in [13, 16], we have the following proposition.

Proposition 2.3. *Let $\mathbf{u}=\mathbf{u}_n$ be a given vector function belonging to $C([0,T];X_n)$ and $\rho = \rho_n$ be the unique solution in Proposition 2.2. Suppose that the initial data $\theta_{0,\delta}$ satisfy the hypotheses in* (2.4).

Then there exists a mapping $\theta_n = \theta[\mathbf{u}_n]$ having the following properties:

- *the initial-boundary value problem $(2.1)_3$, (2.2) and (2.3) admits a unique strong solution $\theta_n = \theta[\mathbf{u}_n]$;*
- *the solution θ_n has the following regularity properties:*

$$\nabla \theta_n \in L^2((0,T)\times\Omega), \quad \partial_t \theta_n \in L^2((0,T)\times\Omega);$$

- *continuity of the mapping:*

$$\|\theta_{n_1}-\theta_{n_2}\|_{L^2(0,T;H^1(\Omega))} \leq C\sqrt{T}\|\mathbf{u}_{n_1}-\mathbf{u}_{n_2}\|_{C([0,T];X_n)}. \tag{2.15}$$

Based on Proposition 2.2 and Proposition 2.3, we can follow the same idea as in [13]-[16] to obtain the local existence as follows.

PROPOSITION 2.4. *For fixed $\varepsilon,\delta>0$, assume that the initial data satisfy (2.4) and X_n is defined by (2.11). Denote $\mathbf{u}_n(0):=\mathbf{u}_{0,\delta,n}$ and suppose $\rho_{0,\delta}\mathbf{u}_{0,\delta,n}=\mathbf{m}_{0,\delta}$ for any n.*

Then the approximate problem (2.1)-(2.3) admits a local solution $(\rho_n,\mathbf{u}_n,\theta_n)$ on a short time interval $[0,T_n]$ with $T_n\le T$ satisfying Proposition 2.2 and Proposition 2.3.

Now, in order to show $T_n=T$ for any n, it is enough to obtain uniform (in time) bounds on the norm $\|\mathbf{u}_n(t)\|_{X_n}$ for $t\in[0,T_n]$ independent of T_n, which is often obtained by energy estimates.

First, by (2.12), we obtain that the velocity $\mathbf{u}_n$ is continuously differentiable, which implies that the following integral identity holds on $(0,T_n)$ for any $\eta\in X_n$

$$\int_\Omega \partial_t(\rho_n\mathbf{u}_n)\cdot\eta dx=\int_\Omega(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n-\mathbb{S}_n-\varepsilon\nabla\mathbf{u}_n):\nabla\eta dx. \tag{2.16}$$

Taking $\eta=\mathbf{u}_n$ in (2.16), we obtain

$$\frac{d}{dt}\int_\Omega\frac{1}{2}\rho_n|\mathbf{u}_n|^2dx+\int_\Omega\frac{1}{2}\mu(\theta_n)\left(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n\right)^2dx+\varepsilon\int_\Omega|\nabla\mathbf{u}_n|^2dx=0. \tag{2.17}$$

Integrating (2.17) over $(0,\tau)$ for any $\tau\in[0,T_n]$, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega(\rho_n|\mathbf{u}_n|^2)(\tau)dx+\int_0^\tau\int_\Omega\frac{1}{2}\mu(\theta_n)\left(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n\right)^2dxds+\varepsilon\int_0^\tau\int_\Omega|\nabla\mathbf{u}_n|^2dxds\\ =&\frac{1}{2}\int_\Omega\mathbf{m}_{0,\delta}\cdot\mathbf{u}_n(0)dx,\end{aligned}$$

where the term on the right-hand side can be controlled by

$$\begin{aligned}&\int_\Omega\mathbf{m}_{0,\delta}\cdot\mathbf{u}_n(0)dx=\int_\Omega\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,n}dx\\ \le&\frac{1}{2}\int_\Omega\left(\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}+\rho_{0,\delta}|\mathbf{u}_{0,\delta,n}|^2\right)dx=\frac{1}{2}\int_\Omega\left(\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}+\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,n}\right)dx.\end{aligned}$$

Thus, we deduce that for any $\tau\in[0,T_n]$

$$\begin{aligned}&\frac{1}{2}\int_\Omega(\rho_n|\mathbf{u}_n|^2)(\tau)dx+\int_0^\tau\int_\Omega\frac{1}{2}\mu(\theta_n)\left(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n\right)^2dxds+\varepsilon\int_0^\tau\int_\Omega|\nabla\mathbf{u}_n|^2dxds\\ \le&\frac{1}{2}\int_\Omega\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}dx.\end{aligned} \tag{2.18}$$

This implies

$$\|\sqrt{\rho_n}\mathbf{u}_n\|_{L^\infty(0,T_n;L^2(\Omega))}\le C,$$

where C is independent of n and T_n.

Since ρ_n is bounded from below by a positive constant, we deduce

$$\|\mathbf{u}_n\|_{L^\infty(0,T_n;L^2(\Omega))}\le C. \tag{2.19}$$

By virtue of the fact that all norms are equivalent on X_n, we have

$$\|\mathbf{u}_n\|_{L^\infty(0,T_n;X_n)}\le C, \tag{2.20}$$

with C independent of n and T_n, which allows to extend the local existence result on T_n to the global existence on T by repeating the fixed-point argument above after finite steps.

2.2. Passing to the limit for $n\to\infty$. The goal of this subsection is to pass limit to the approximate solutions $(\rho_n,\mathbf{u}_n,\theta_n)$ to recover a weak solution to the approximate system (2.1)-(2.3) as $n\to\infty$, for any fixed ε, $\delta>0$. For convenience, in the rest of this subsection, we denote C a generic positive constant which is independent of n. In fact, we have the following uniform bounds.

PROPOSITION 2.5. *For fixed ε, $\delta>0$, under the hypotheses of Proposition 2.1, we have*

$$\|\rho_n\|_{L^\infty((0,T)\times\Omega)}\le C, \tag{2.21a}$$
$$\|\mathbf{u}_n\|_{L^2(0,T;W_0^{1,2}(\Omega))}\le C, \tag{2.21b}$$
$$\|\theta_n\|_{L^2(0,T;W^{1,2}(\Omega))}\le C, \tag{2.21c}$$
$$\|\theta_n\|_{L^3((0,T)\times\Omega)}\le C, \tag{2.21d}$$
$$\|\sqrt{\rho}_n\mathbf{u}_n\|_{L^\infty(0,T;L^2(\Omega))}\le C, \tag{2.21e}$$
$$\|\rho_n\theta_n\|_{L^\infty(0,T;L^1(\Omega))}\le C. \tag{2.21f}$$

Proof. First, replacing T_n by T in the energy inequality (2.18) and thanks to Poincaré's inequality, we have (2.21b) and (2.21e). Moreover, as proved in [27], by the divergence-free property of $\mathbf{u}_n$, we have for all $0\le\alpha\le\beta<\infty$

$$meas\{x\in\Omega\,|\,\alpha\le\rho_n(t,x)\le\beta\}\ \text{is independent of}\ t\ge 0,$$

which, combined with $0<\delta\le\rho_{0,\delta}(x)\le\bar\rho$ implies (2.21a).

Then, integrating the Equation $(2.1)_3$ over $(0,\tau)\times\Omega$ for any $\tau\in[0,T]$ and adding to (2.18), we obtain the following total energy inequality

$$\begin{aligned}&\textstyle\int_\Omega\left(\frac12\rho_n|\mathbf{u}_n|^2+(\delta+\rho_n)\theta_n\right)(\tau)dx+\varepsilon\int_0^\tau\int_\Omega|\nabla\mathbf{u}_n|^2dxds+\delta\int_0^\tau\int_\Omega\left(\theta_n^3+\mathbb{S}_n:\nabla\mathbf{u}_n\right)dxds\\ &\le\int_\Omega\left(\frac12\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}+(\delta+\rho_{0,\delta})\theta_{0,\delta}\right)dx,\end{aligned}\tag{2.22}$$

which implies (2.21d) and (2.21f).

Next, multiplying $(2.1)_3$ by $h(\theta_n)$, we have

$$\begin{aligned}&\partial_t((\delta+\rho_n)H(\theta_n))+\operatorname{div}(\rho_n\mathbf{u}_nH(\theta_n))-\triangle\mathcal{K}_h(\theta_n)+\kappa(\theta_n)h'(\theta_n)|\nabla\theta_n|^2+\delta\theta_n^3h(\theta_n)\\ &=(1-\delta)\mathbb{S}_n:\nabla\mathbf{u}_nh(\theta_n),\end{aligned}\tag{2.23}$$

where $H(\theta)=\int_0^\theta h(z)dz$ and $\mathcal{K}_h(\theta)=\int_0^\theta\kappa(z)h(z)dz$, with h meeting (2.7) and (2.8). Integrating (2.23) over $(0,\tau)\times\Omega$ for any $\tau\in[0,T]$, one obtains

$$\begin{aligned}&\int_\Omega((\delta+\rho_n)H(\theta_n))(\tau)dx+\int_0^\tau\int_\Omega\left(\kappa(\theta_n)h'(\theta_n)|\nabla\theta_n|^2+\delta\theta_n^3h(\theta_n)\right)dxds\\ &=(1-\delta)\int_0^\tau\int_\Omega\mathbb{S}_n:\nabla\mathbf{u}_nh(\theta_n)dxds+\int_\Omega(\delta+\rho_{0,\delta})H(\theta_{0,\delta})dx.\end{aligned}\tag{2.24}$$

Choosing $h(\theta_n)=\frac{1}{1+\theta_n}$ in (2.24), we obtain

$$\begin{aligned}\int_0^\tau\int_\Omega\frac{\kappa(\theta_n)}{(1+\theta_n)^2}|\nabla\theta_n|^2dxds=&\int_\Omega((\delta+\rho_n)H(\theta_n))(\tau)dx-\int_\Omega(\delta+\rho_{0,\delta})H(\theta_{0,\delta})dx\\ &+\delta\int_0^\tau\int_\Omega\frac{\theta_n^3}{1+\theta_n}dxds-(1-\delta)\int_0^\tau\int_\Omega\frac{\mathbb{S}_n:\nabla\mathbf{u}_n}{1+\theta_n}dxds,\end{aligned}\tag{2.25}$$

where by the growth restriction imposed on $\kappa(\theta)$ (1.4), the term on the left-hand side can be controlled by

$$0<C\int_0^\tau\int_\Omega|\nabla\theta_n|^2dxds\leq\int_0^\tau\int_\Omega\frac{\kappa(\theta_n)}{(1+\theta_n)^2}|\nabla\theta_n|^2dxds,$$

hence (2.21c) holds. □

2.2.1. Strong convergence of the approximate densities. In this subsection, we recall a strong convergence result for the densities ρ_n.

LEMMA 2.1 ([12,27]). *Assume that the sequence $(\rho_n,\mathbf{u}_n)$ solves Equations $(2.1)_1$ and $(2.1)_4$ in the sense of distributions, and satisfies the estimates (2.21a) and (2.21b).*
Then we have

$$\rho_n\to\rho\,\text{in}\,C([0,T];L^p(\Omega)) \tag{2.26}$$

for any $1\leq p<\infty$. Moreover, ρ satisfies

$$meas\{x\in\Omega\,|\,\alpha\leq\rho(t,x)\leq\beta\}\,\text{is independent of}\,t\geq0,$$

for all $0\leq\alpha\leq\beta<\infty$.

2.2.2. Limit in the approximate continuity equation. By (2.21b), we assume

$$\mathbf{u}_n\rightharpoonup\mathbf{u}\,\text{weakly in}\,L^2(0,T;W_0^{1,2}(\Omega)), \tag{2.27}$$

at least for a suitable subsequence.

Therefore, in order to prove that $\{\rho,\mathbf{u}\}$ solves the continuity Equation $(2.1)_1$ in the sense of distributions, it suffices to show

$$\rho_n\mathbf{u}_n\to\rho\mathbf{u}\,\text{in}\,\mathcal{D}'((0,T)\times\Omega),$$

which can be achieved by (2.26) and (2.27).

2.2.3. Strong convergence of the approximate temperatures. Before proving the strong convergence of the approximate temperature θ_n, we need the following variant of the Aubin-Lions lemma, which plays an essential role in our proof.

LEMMA 2.2 ([13]). *Let $\{\mathbf{v}_n\}_{n=1}^\infty$ be a sequence of functions such that*

$$\mathbf{v}_n\,\text{is bounded in}\,L^2(0,T;L^q(\Omega))\cap L^\infty(0,T;L^1(\Omega)),\quad\text{with}\,q>6/5,$$

furthermore, assume that

$$\partial_t\mathbf{v}_n\geq l_n\,\text{in}\,\mathcal{D}'((0,T)\times\Omega),$$

where

$$l_n\,\text{is bounded in}\,L^1(0,T;W^{-m,r}(\Omega))$$

for a certain $m\geq1$, $r>1$.
Then $\{\mathbf{v}_n\}_{n=1}^\infty$ contains a subsequence such that

$$\mathbf{v}_n\to\mathbf{v}\,\text{in}\,L^2(0,T;H^{-1}(\Omega)).$$

Now we apply Lemma 2.2 to $(2.1)_3$ with $\mathbf{v}_n = (\delta+\rho_n)\theta_n$. By estimates (2.21a), (2.21c) and (2.21f), we have

$$(\delta+\rho_n)\theta_n \text{ is bounded in } L^2((0,T)\times\Omega)\cap L^\infty(0,T;L^1(\Omega)),$$

and

$$\partial_t((\delta+\rho_n)\theta_n) = l_n \text{ in } \mathcal{D}'((0,T)\times\Omega),$$

where

$$l_n = -\mathrm{div}(\rho_n\mathbf{u}_n\theta_n) + \triangle\mathcal{K}(\theta_n) - \delta\theta_n^3 + (1-\delta)\mathbb{S}_n : \nabla\mathbf{u}_n.$$

Owing to estimates (2.21a)-(2.21d) and the compact imbedding of $L^1(\Omega)$ into $W^{-1,s}(\Omega)$, with $s\in(1,\frac{4}{3})$, we have

$$l_n \text{ is bounded in } L^1(0,T;W^{-3,r}(\Omega)),\ r>1.$$

Therefore, we obtain

$$(\delta+\rho_n)\theta_n \to (\delta+\rho)\theta \text{ in } L^2(0,T;H^{-1}(\Omega)).$$

Thanks to (2.21c), this yields

$$(\delta+\rho_n)\theta_n^2 \to (\delta+\rho)\theta^2 \text{ in } \mathcal{D}'((0,T)\times\Omega),$$

which, combined with (2.26), implies

$$\begin{aligned}
&\int_0^T\int_\Omega (\delta+\rho)\theta_n^2 dxdt \\
&= \int_0^T\int_\Omega [(\delta+\rho)-(\delta+\rho_n)]\theta_n^2 dxdt + \int_0^T\int_\Omega (\delta+\rho_n)\theta_n^2 dxdt \\
&\to \int_0^T\int_\Omega (\delta+\rho)\theta^2 dxdt.
\end{aligned}$$

Thus, we have

$$\theta_n \to \theta \text{ in } L^2((0,T)\times\Omega). \tag{2.28}$$

2.2.4. Limit in the approximate momentum equation. In order to take the limit $n\to\infty$ in the approximate momentum Equation (2.12), it suffices to deal with the nonlinear terms.

First, similarly as in the previous subsection, applying Lemma 2.2 to (2.12) with $\mathbf{v}_n = \rho_n\mathbf{u}_n$, we have

$$\rho_n\mathbf{u}_n \to \rho\mathbf{u} \text{ in } L^2([0,T];H^{-1}(\Omega)),$$

which, combined with (2.27), gives

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u} \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

Then, for the nonlinear term $\mu(\theta_n)(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n)$, by (2.28) and the Lipschitz continuity of $\mu(\theta)$, we have

$$\mu(\theta_n) \to \mu(\theta) \text{ in } L^2((0,T)\times\Omega),$$

which, with help of (2.27) yields

$$\mu(\theta_n)(\nabla\mathbf{u}_n+\nabla^T\mathbf{u}_n) \to \mu(\theta)(\nabla\mathbf{u}+\nabla^T\mathbf{u}) \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

2.2.5. Limit in the renormalized temperature equation. First, by (2.28) and the properties of $h(\theta)$ in Proposition 2.1, we have

$$H(\theta_n) \to H(\theta) \text{ in } L^2((0,T)\times\Omega), \tag{2.29}$$

which, combined with (2.26), yields

$$(\delta+\rho_n)H(\theta_n) \to (\delta+\rho)H(\theta) \text{ in } L^2(0,T;L^m(\Omega)),$$

for any $1\leq m<2$. Moreover, by (2.26), (2.27) and (2.29), we have

$$\rho_n H(\theta_n)\mathbf{u}_n \rightharpoonup \rho H(\theta)\mathbf{u} \text{ weakly in } L^1((0,T)\times\Omega). \tag{2.30}$$

Then, thanks to Proposition 2.1 in [13], we are able to deal with the terms $\mathcal{K}_h(\theta_n)$ and $\delta\theta_n^3 h(\theta_n)$. To be specific, by the property

$$\lim_{z\to\infty} h(z)=0,$$

we have

$$\lim_{z\to\infty}\frac{\mathcal{K}_h(z)}{\mathcal{K}(z)}=0.$$

By (2.21d) and the growth restriction imposed on $\kappa(\theta)$ (1.4), we have

$$\sup_{n\geq 1}\int_\Omega \mathcal{K}(\theta_n)dy<\infty,$$

thus, Proposition 2.1 in [13] yields

$$\mathcal{K}_h(\theta_n) \rightharpoonup \mathcal{K}_h(\theta) \text{ weakly in } L^1((0,T)\times\Omega). \tag{2.31}$$

Similarly, for the term $\delta\theta_n^3 h(\theta_n)$, owing to

$$\lim_{z\to\infty}\frac{z^3h(z)}{z^3}=0,$$

and

$$\sup_{n\geq 1}\int_\Omega \theta_n^3 dy<\infty,$$

we have

$$\delta\theta_n^3 h(\theta_n) \rightharpoonup \delta\theta^3 h(\theta) \text{ weakly in } L^1((0,T)\times\Omega). \tag{2.32}$$

Next, owing to $0<\underline{\kappa}\leq\kappa(\theta)$ and the non-increasing property of h, we have

$$-\int_0^T\int_\Omega \kappa(\theta)h'(\theta)|\nabla\theta|^2\varphi dxdt \leq -\liminf_{n\to\infty}\int_0^T\int_\Omega \kappa(\theta_n)h'(\theta_n)|\nabla\theta_n|^2\varphi dxdt, \tag{2.33}$$

for any non-negative function $\varphi\in C_c^\infty((0,T)\times\Omega)$.

Finally, it is crucial to have the following lemma to deal with the term $\mathbb{S}_n:\nabla\mathbf{u}_n h(\theta_n)$.

Lemma 2.3 ([19]). *Let $g(\theta)$ be a bounded, continuous and non-negative function from $[0,\infty)$ to $\mathbb{R}$. Suppose that θ_n and $\mathbf{u}_n$ are two sequences of functions defined on Ω satisfying*

$$\theta_n \to \theta \, a.e. \, \text{in} \, \Omega,$$

and

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \, \text{weakly in} \, W^{1,2}(\Omega).$$

Then

$$\int_\Omega g(\theta)h(\theta)|\nabla \mathbf{u}|^2 dx \le \liminf_{n\to\infty} \int_\Omega g(\theta_n)h(\theta_n)|\nabla \mathbf{u}_n|^2 dx,$$

where the function $h(\theta)$ satisfies (2.7) and (2.8). In particular,

$$\int_\Omega \mathbb{S}:\nabla \mathbf{u} h(\theta)\varphi dx \le \liminf_{n\to\infty} \int_\Omega \mathbb{S}_n : \nabla \mathbf{u}_n h(\theta_n)\varphi dx, \tag{2.34}$$

for any non-negative function $\varphi \in C_c^\infty((0,T)\times\Omega)$.

With this lemma at hand, combining (2.29)-(2.34), we can obtain (2.6) by letting $n\to\infty$ in (2.23). We are also able to deduce the energy inequality (2.9) in Proposition 2.1 by letting $n\to\infty$ in (2.22).

3. Limit passage for ε tends to zero

In this section, we use $(\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\epsilon)$ to denote the weak solutions constructed in Proposition 2.1. The main task is to pass limits to $(\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon)$ as $\varepsilon\to 0^+$. Note that, for any fixed $\varepsilon>0$, $\sqrt{\varepsilon}\nabla \mathbf{u}_n$ is bounded in $L^2((0,T)\times\Omega)$, which is crucial to show the compactness of weak solutions as n goes to infinity. However, this estimate is not uniform on ε. This will lead to the loss of compactness of weak solutions. Our alternative way is to show that the temperature-depending viscosity coefficient is bounded below from zero, which can provide the uniform bound of $\nabla\mathbf{u}_\varepsilon$ in $L^2((0,T)\times\Omega)$. With such a bound, we are able to obtain the exact same compactness as in Section 2. Thus, to pass to the limits as $\varepsilon\to 0^+$, we mainly need to prove the temperature is bounded below from zero. Therefore, this section will be devoted to obtain a positive bound from below for the temperature, which is uniform in terms of $\varepsilon>0$ and $\delta>0$.

3.1. A positive bound from below for the temperature. We first give our result about the positive bound for the temperature θ_ε.

Proposition 3.1. *Let $(\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon)$ be a weak solution to the approximate system (2.1)-(2.3) in the sense of Proposition 2.1. Assume that the initial temperature satisfies the assumptions in (2.4), that is,*

$$\theta_\varepsilon(0)=\theta_{0,\delta}\ge \underline{\theta}>0.$$

Then there exists a constant $\widetilde{\theta}>0$ such that

$$\theta_\varepsilon(t,x)\ge\widetilde{\theta}>0 \tag{3.1}$$

for all $t\in[0,T]$ and almost all $x\in\Omega$.

Remark 3.1. We emphasize here that the constant $\widetilde{\theta}$ does not depend on the parameters $\varepsilon>0$ and $\delta>0$, which is essential in the limit passage later.

To prove Proposition 3.1, we need the following important lemmas.

LEMMA 3.1 ([30]). *Let U_k be a sequence satisfying*

(i) $0 \le U_0 \le C$;

(ii) *for some constants* $A \ge 1$, $1 < \beta_1 < \beta_2$ *and* $C > 0$,

$$0 \le U_k \le C \frac{A^k}{K}(U_{k-1}^{\beta_1} + U_{k-1}^{\beta_2}).$$

Then there exists some K_0 such that for every $K > K_0$, the sequence U_k converges to 0 *when k goes to infinity.*

LEMMA 3.2 ([13]). *Let ρ be a non-negative function such that*

$$0 < M_1 \le \int_\Omega \varrho dx, \int_\Omega \varrho^\gamma dx \le M_2, \text{ with } \gamma > \frac{6}{5}.$$

Then there exists a positive constant C depending only on M_1, M_2 such that

$$\|\mathbf{v}\|_{H^1(\Omega)} \le C\left(\|\nabla \mathbf{v}\|_{L^2(\Omega)} + \int_\Omega \varrho |\mathbf{v}| dx\right).$$

Our proof is in the spirit of the work of Mellet-Vasseur [29], where they first used De Giorgi's method to give a positive bound from below for the temperature.

Proof. **(Proof of Proposition 3.1.)** Taking

$$H(\theta) = -\int_0^\theta h(z) dz$$

with $h(z) = \frac{1}{z+\omega} 1_{\{z+\omega \le C\}}$ for some constant $\omega > 0$, we have

$$H(\theta_\varepsilon) = \begin{cases} -\ln(\theta_\varepsilon + \omega) + \ln \omega, & \text{if } \theta_\varepsilon + \omega \le C, \\ -\ln C + \ln \omega, & \text{if } \theta_\varepsilon + \omega > C. \end{cases}$$

Thanks to (2.6), the weak solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ satisfies the following temperature inequality in the sense of distributions

$$\begin{aligned} &\partial_t((\delta + \rho_\varepsilon) H(\theta_\varepsilon)) + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon H(\theta_\varepsilon)) - \triangle \mathcal{K}_h(\theta_\varepsilon) - h'(\theta_\varepsilon)\kappa(\theta_\varepsilon)|\nabla \theta_\varepsilon|^2 \\ &\le \delta \theta_\varepsilon^3 h(\theta_\varepsilon) - (1-\delta)\mathbb{S}_\varepsilon : \nabla \mathbf{u}_\varepsilon h(\theta_\varepsilon), \end{aligned} \tag{3.2}$$

with $H(\theta) = -\int_0^\theta h(z)dz$ and $\mathcal{K}_h(\theta) = -\int_0^\theta \kappa(z) h(z) dz$.

Then, letting $\phi(\theta_\varepsilon) = H(\theta_\varepsilon) + \ln C - \ln \omega = \left[\ln\left(\frac{C}{\theta_\varepsilon + \omega}\right)\right]_+$ and integrating (3.2) over $(s,t) \times \Omega$ for any $0 \le s \le t \le T$, we deduce

$$\begin{aligned} &\int_\Omega ((\delta + \rho_\varepsilon)\phi(\theta_\varepsilon))(t) dx - 2(1-\delta)\int_s^t \int_\Omega \mu(\theta_\varepsilon)|D(\mathbf{u}_\varepsilon)|^2 \phi'(\theta_\varepsilon) dx d\tau \\ &\quad + \int_s^t \int_\Omega \phi''(\theta_\varepsilon)\kappa(\theta_\varepsilon)|\nabla \theta_\varepsilon|^2 dx d\tau \\ &\le \int_\Omega ((\delta + \rho_\varepsilon)\phi(\theta_\varepsilon))(s) dx - \delta \int_s^t \int_\Omega \theta_\varepsilon^3 \phi'(\theta_\varepsilon) dx d\tau, \end{aligned} \tag{3.3}$$

where $D(\mathbf{u})=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right)$. Now, introducing a sequence of real numbers

$$C_k=e^{-M[1-2^{-k}]} \quad \text{for all positive integers } k, \tag{3.4}$$

where M is a positive number to be chosen later. We define $\phi_{k,\omega}$ as

$$\phi_{k,\omega}(\theta_\varepsilon)=\left[\ln\left(\frac{C_k}{\theta_\varepsilon+\omega}\right)\right]_+, \tag{3.5}$$

then

$$\phi'_{k,\omega}(\theta_\varepsilon)=-\frac{1}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}, \tag{3.6}$$

$$\phi''_{k,\omega}(\theta_\varepsilon)\ge\frac{1}{(\theta_\varepsilon+\omega)^2}1_{\{\theta_\varepsilon+\omega\le C_k\}}. \tag{3.7}$$

Next define $U_{k,\omega}$ by

$$\begin{aligned} U_{k,\omega}:=&\sup_{T_k\le t\le T}\left(\int_\Omega(\delta+\rho_\varepsilon)\phi_{k,\omega}(\theta_\varepsilon)dx\right)+2(1-\delta)\int_{T_k}^T\int_\Omega\frac{\mu(\theta_\varepsilon)}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}|D(\mathbf{u}_\varepsilon)|^2dxdt\\ &+\int_{T_k}^T\int_\Omega\frac{\kappa(\theta_\varepsilon)}{(\theta_\varepsilon+\omega)^2}1_{\{\theta_\varepsilon+\omega\le C_k\}}|\nabla\theta_\varepsilon|^2dxdt, \end{aligned} \tag{3.8}$$

where $\{T_k\}$ is a sequence of non-negative numbers. Note that $U_{k,\omega}$ depends on ε, δ and ω, that is, $U_{k,\omega}=U_{k,\varepsilon,\delta,\omega}$, and for convenience, we still write it as $U_{k,\omega}$.

Assuming $T_k=0$ for all $k\in\mathbb{N}$, from (3.3) and (3.8), we claim that

$$U_{k,\omega}\le\int_\Omega(\delta+\rho_{0,\delta})\phi_{k,\omega}(\theta_{0,\delta})dx+\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxdt. \tag{3.9}$$

In fact, taking $0\le T_{k-1}\le s\le T_k\le t\le T$ in (3.3) and using (3.6)-(3.7), one gets

$$\begin{aligned} &\int_\Omega((\delta+\rho_\varepsilon)\phi_{k,\omega}(\theta_\varepsilon))(t)dx+2(1-\delta)\int_{T_k}^t\int_\Omega\frac{\mu(\theta_\varepsilon)}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}|D(\mathbf{u}_\varepsilon)|^2dxd\tau\\ &\quad+\int_{T_k}^t\int_\Omega\frac{\kappa(\theta_\varepsilon)}{(\theta_\varepsilon+\omega)^2}1_{\{\theta_\varepsilon+\omega\le C_k\}}|\nabla\theta_\varepsilon|^2dxd\tau\\ &\le\int_\Omega((\delta+\rho_\varepsilon)\phi_{k,\omega}(\theta_\varepsilon))(s)dx+\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxd\tau. \end{aligned} \tag{3.10}$$

Taking the supremum over $t\in[T_k,T]$ on both sides of (3.10), one deduces that

$$U_{k,\omega}\le\int_\Omega((\delta+\rho_\varepsilon)\phi_{k,\omega}(\theta_\varepsilon))(s)dx+\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxdt. \tag{3.11}$$

If $T_k=0$ for all $k\in N$, then $s=0$ in (3.11), thus we get (3.9).

Next, we prove that the second term on the right-hand side of (3.9) can be controlled by $U_{k-1,\omega}^\gamma$ for some $\gamma>1$. More precisely, we claim that

$$\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxdt\le C\frac{2^{k\alpha}}{M^\alpha}U_{k-1,\omega}^\gamma, \tag{3.12}$$

for some $\gamma>1$, where the constant C is independent of $\varepsilon,\delta>0$.

Indeed, if $\theta_\varepsilon+\omega\le C_k$, we have

$$\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}\le 1, \text{ for any } \omega>0, \tag{3.13}$$

by taking M large enough such that C_k is small enough, and

$$\phi_{k-1,\omega}(\theta_\varepsilon)=\left[\ln\left(\frac{C_{k-1}}{\theta_\varepsilon+\omega}\right)\right]_+\ge \ln\frac{C_{k-1}}{C_k},$$

which implies

$$1_{\{\theta_\varepsilon+\omega\le C_k\}}\le\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}\phi_{k-1,\omega}(\theta_\varepsilon)^\alpha, \text{ for any } \alpha>0. \tag{3.14}$$

Taking (3.13) and (3.14) into account, we deduce

$$\begin{aligned}
&\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxdt\\
\le&\delta^{1-\beta}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}\int_{T_{k-1}}^T\int_\Omega(\delta+\rho_\varepsilon)^\beta\phi_{k-1,\omega}(\theta_\varepsilon)^\alpha dxdt\\
\le&C\delta^{1-\beta}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}T^{1/p'}|\Omega|^{1/q'}\|(\delta+\rho_\varepsilon)^\beta\phi_{k-1,\omega}(\theta_\varepsilon)^\alpha\|_{L^p(T_{k-1},T;L^q(\Omega))},
\end{aligned}\tag{3.15}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. For the last term in (3.15), we have

$$\begin{aligned}
&\|(\delta+\rho_\varepsilon)^\beta\phi_{k-1,\omega}(\theta_\varepsilon)^\alpha\|_{L^p(T_{k-1},T;L^q(\Omega))}\\
=&\|((\delta+\rho_\varepsilon)\phi_{k-1,\omega}(\theta_\varepsilon))^{\beta/\alpha}\phi_{k-1,\omega}(\theta_\varepsilon)^{1-\beta/\alpha}\|^\alpha_{L^{p\alpha}(T_{k-1},T;L^{q\alpha}(\Omega))}\\
\le&\|((\delta+\rho_\varepsilon)\phi_{k-1,\omega}(\theta_\varepsilon))^{\beta/\alpha}\|^\alpha_{L^\infty(T_{k-1},T;L^{\alpha/\beta}(\Omega))}\\
&\qquad\|\phi_{k-1,\omega}(\theta_\varepsilon)^{1-\beta/\alpha}\|^\alpha_{L^{\frac{2}{1-\beta/\alpha}}(T_{k-1},T;L^{\frac{6}{1-\beta/\alpha}}(\Omega))}\\
=&\|(\delta+\rho_\varepsilon)\phi_{k-1,\omega}(\theta_\varepsilon)\|^\beta_{L^\infty(T_{k-1},T;L^1(\Omega))}\|\phi_{k-1,\omega}(\theta_\varepsilon)\|^{\alpha-\beta}_{L^2(T_{k-1},T;L^6(\Omega))}\\
\le&\|(\delta+\rho_\varepsilon)\phi_{k-1,\omega}(\theta_\varepsilon)\|^\beta_{L^\infty(T_{k-1},T;L^1(\Omega))}\\
&\qquad\left(\|\rho_\varepsilon\phi_{k-1,\omega}(\theta_\varepsilon)\|_{L^\infty(T_{k-1},T;L^1(\Omega))}+\|\nabla\phi_{k-1,\omega}(\theta_\varepsilon)\|_{L^2((T_{k-1},T)\times\Omega)}\right)^{\alpha-\beta}\\
\le&CU^\beta_{k-1,\omega}\left(U_{k-1,\omega}+U^{1/2}_{k-1,\omega}\right)^{\alpha-\beta}\\
\le&C\left(U^\alpha_{k-1,\omega}+U^{\frac{\alpha+\beta}{2}}_{k-1,\omega}\right),
\end{aligned}\tag{3.16}$$

where we used Lemma 3.2 in the third inequality from below, the growth restriction imposed on $\kappa(\theta)$ (1.4) in the second inequality from below, and the coefficients p, q, α and β satisfy

$$\frac{1}{p\alpha}=\frac{1-\beta/\alpha}{2},\quad \frac{1}{q\alpha}=\frac{\beta}{\alpha}+\frac{1-\beta/\alpha}{6}.$$

Substituting (3.16) into (3.15), we have

$$\delta\int_{T_{k-1}}^T\int_\Omega\frac{\theta_\varepsilon^3}{\theta_\varepsilon+\omega}1_{\{\theta_\varepsilon+\omega\le C_k\}}dxdt\le C\delta^{1-\beta}\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}\left(U^\alpha_{k-1,\omega}+U^{\frac{\alpha+\beta}{2}}_{k-1,\omega}\right). \tag{3.17}$$

Then, by (3.4), we have

$$\left[\ln\frac{C_{k-1}}{C_k}\right]^{-\alpha}=\frac{2^{k\alpha}}{M^\alpha}. \tag{3.18}$$

Meanwhile, we can choose $\beta<1$, $\alpha>1$ such that

$$\gamma:=\min\left(\frac{\alpha+\beta}{2},\alpha\right)>1, \tag{3.19}$$

and

$$\delta^{1-\beta}\leq 1. \tag{3.20}$$

Combining (3.17)-(3.20) together, we obtain (3.12).

We are now ready to complete the proof of Proposition 3.1. By assumption $\theta_{0,\delta}\geq \underline{\theta}>0$, choosing M large enough such that $e^{-M/2}<\underline{\theta}$, we have for any $\omega>0$

$$\phi_{k,\omega}(\theta_{0,\delta})=\left[\ln\left(\frac{e^{-M[1-2^{-k}]}}{\theta_{0,\delta}+\omega}\right)\right]_+=0. \tag{3.21}$$

Substituting (3.12) and (3.21) into (3.9), we obtain

$$U_{k,\omega}\leq C\frac{2^{k\alpha}}{M^\alpha}U^\gamma_{k-1,\omega}\ \text{with}\ \gamma>1. \tag{3.22}$$

Thanks to Lemma 3.1, for M large enough (independently on ε, δ and ω), we have

$$\lim_{k\to\infty}U_{k,\omega}=0, \tag{3.23}$$

which, combined with the definition of $U_{k,\omega}$ (3.8) yields

$$\int_0^T\int_\Omega\kappa(\theta_\varepsilon)\left|\nabla\left[\ln\frac{e^{-M}}{\theta_\varepsilon+\omega}\right]_+\right|^2dxdt=0, \tag{3.24}$$

and

$$\int_\Omega(\delta+\rho_\varepsilon)\left[\ln\frac{e^{-M}}{\theta_\varepsilon+\omega}\right]_+dx=0. \tag{3.25}$$

By (1.4) and (3.24), we obtain

$$\left[\ln\frac{e^{-M}}{\theta_\varepsilon+\omega}\right]_+\ \text{is constant in}\ \Omega\ \text{for all}\ t\in[0,T], \tag{3.26}$$

and with help of (3.25), this implies

$$\left[\ln\frac{e^{-M}}{\theta_\varepsilon+\omega}\right]_+=0.$$

This yields

$$\theta_\varepsilon+\omega\geq e^{-M}$$

for any $\omega>0$, which completes our proof. □

3.2. Estimates independent of $\varepsilon>0$. By virtue of Proposition 3.1 and the assumptions imposed on $\mu(\theta)$ in Theorem 1.1, we can obtain a positive constant $\underline{\mu}$ independent of $\varepsilon,\delta>0$ such that

$$\mu(\theta_\varepsilon)\geq\underline{\mu}>0. \tag{3.27}$$

This, combined with the energy inequality (2.9), yields

$$\nabla\mathbf{u}_\varepsilon \text{ is bounded in } L^2((0,T)\times\Omega) \tag{3.28}$$

by a positive constant independent of $\varepsilon>0$. Thanks to Poincaré's inequality, we have

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega))}\leq C, \tag{3.29}$$

where C is independent of $\varepsilon>0$. Thus, we are able to get the following uniform bounds for $(\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon)$ as in Section 2.

PROPOSITION 3.2. *For fixed $\delta>0$, under the hypotheses of Theorem 1.1 and Proposition 2.1, we have*

$$\|\rho_\varepsilon\|_{L^\infty((0,T)\times\Omega)}\leq C, \tag{3.30a}$$
$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega))}\leq C, \tag{3.30b}$$
$$\|\theta_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))}\leq C, \tag{3.30c}$$
$$\|\theta_\varepsilon\|_{L^3((0,T)\times\Omega)}\leq C, \tag{3.30d}$$
$$\|\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leq C, \tag{3.30e}$$
$$\|\rho_\varepsilon\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\leq C, \tag{3.30f}$$

where all constants C are independent of $\varepsilon>0$.

Thanks to (3.30a)-(3.30f), we are able to derive the same compactness structure for $(\rho_\varepsilon,\mathbf{u}_\varepsilon,\theta_\varepsilon)$ as $(\rho_n,\mathbf{u}_n,\theta_n)$. Thus, we can show the following proposition by passing to the limits as $\varepsilon\to0^+$.

PROPOSITION 3.3. *For fixed $\delta>0$, under the hypotheses of Theorem 1.1 and Proposition 2.1, the initial-boundary value problem (1.1)-(1.3) with the parameter $\delta>0$ admits an approximate solution $(\rho,\mathbf{u},\theta,P)$, which is also the limit of the weak solution to (2.1)-(2.3) when $\varepsilon\to0^+$, satisfying*

(i) *the density $\rho\geq0$ satisfies*

$$\rho\in L^\infty((0,T)\times\Omega)\cap C([0,T];L^p(\Omega)),\quad 1\leq p<\infty,$$

the velocity $\mathbf{u}$ belongs to the space $L^2(0,T;W_0^{1,2}(\Omega))$, and $(\rho,\mathbf{u})$ is a renormalized solution of the continuity Equation $(1.1)_1$ in the sense of distributions;

(ii) *the momentum Equation $(1.1)_2$ and the incompressibility condition $(1.1)_4$ hold in $\mathcal{D}'((0,T)\times\Omega)$. Moreover, $\rho\mathbf{u}\in C([0,T];L^2_{weak}(\Omega))$ satisfies the initial condition (2.2);*

(iii) *the temperature $\theta\geq0$ satisfies*

$$\theta\in L^2(0,T;W^{1,2}(\Omega))\cap L^3((0,T)\times\Omega),\quad \rho\theta\in L^\infty(0,T;L^1(\Omega)),$$

and the renormalized temperature inequality holds in the sense of distributions, that is,

$$\int_0^T\int_\Omega(\delta+\rho)H(\theta)\partial_t\varphi dxdt+\int_0^T\int_\Omega\left(\rho H(\theta)\mathbf{u}\cdot\nabla\varphi+\mathcal{K}_h(\theta)\triangle\varphi-\delta\theta^3h(\theta)\varphi\right)dxdt$$

$$\leq \int_0^T \int_\Omega \left((\delta-1)\mathbb{S}:\nabla \mathbf{u} h(\theta) + h'(\theta)\kappa(\theta)|\nabla\theta|^2\right)\varphi dxdt - \int_\Omega (\delta+\rho_{0,\delta})H(\theta_{0,\delta})\varphi(0)dx, \quad (3.31)$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$ *satisfying*

$$\varphi \geq 0,\ \varphi(T,\cdot)=0,\ \nabla\varphi\cdot\mathbf{n}|_{\partial\Omega}=0,$$

where $H(\theta)=\int_0^\theta h(z)dz$ *and* $\mathcal{K}_h(\theta)=\int_0^\theta \kappa(z)h(z)dz$, *with non-increasing* $h\in C^2([0,\infty))$ *satisfying* (2.7) *and* (2.8)*;*

(iv) *the energy inequality holds, that is, for a.a.* $t\in(0,T)$*,*

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + (\delta+\rho)\theta\right)(t)dx + \delta\int_0^t\int_\Omega \mathbb{S}:\nabla\mathbf{u} + \theta^3 dxds$$
$$\leq \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}} + (\delta+\rho_{0,\delta})\theta_{0,\delta}dx. \quad (3.32)$$

4. Limit passage for δ tends to zero

The final step is to recover a weak solution to the initial-boundary value problem (1.1)-(1.3) by passing to the limit as $\delta\to 0^+$. In this section, we denote by $(\rho_\delta,\mathbf{u}_\delta,\theta_\delta)$ the weak solutions constructed in Proposition 3.3. Note that the positive below bound of the temperature in Proposition 3.1 does not depend on $\delta>0$, so we have

$$\mu(\theta_\delta)\geq \underline{\mu}>0 \quad (4.1)$$

for some positive constant $\underline{\mu}$ independent of $\delta>0$ in this whole section.

4.1. Estimates independent of $\delta>0$.

Observe that estimates for ρ_δ are similar as in the previous sections, and estimates for $\mathbf{u}_\delta$ can be deduced after some calculations, thus our main task in this section is to deal with terms related to θ_δ. For convenience, in the rest of this section, we denote C a generic positive constant independent of $\delta>0$.

First, by (2.5) and the energy inequality (3.32), we have the following estimates

$$\|\sqrt{\rho_\delta}\mathbf{u}_\delta\|_{L^\infty(0,T;L^2(\Omega))}\leq C, \quad (4.2a)$$
$$\|(\delta+\rho_\delta)\theta_\delta\|_{L^\infty(0,T;L^1(\Omega))}\leq C, \quad (4.2b)$$
$$\delta\int_0^T\int_\Omega \mathbb{S}_\delta:\nabla\mathbf{u}_\delta dxdt\leq C, \quad (4.2c)$$
$$\delta\int_0^T\int_\Omega \theta_\delta^3 dxdt\leq C. \quad (4.2d)$$

Then, taking $\varphi(t,x)=\psi(t)$ satisfying $0\leq\psi\leq 1$, $\psi\in C_c^\infty(0,T)$ and $h(\theta)=\frac{1}{(1+\theta)^l}$ with $0<l<1$ in (3.31), we have

$$\int_0^T\int_\Omega \left(\frac{1-\delta}{(1+\theta_\delta)^l}\mathbb{S}_\delta:\nabla\mathbf{u}_\delta + l\frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^{l+1}}|\nabla\theta_\delta|^2\right)\psi dxdt$$
$$\leq \delta\int_0^T\int_\Omega \frac{\theta_\delta^3}{(1+\theta_\delta)^l}\psi dxdt - \int_0^T\int_\Omega (\delta+\rho_\delta)H(\theta_\delta)\partial_t\psi dxdt, \quad (4.3)$$

with $H(\theta)=\int_0^\theta \frac{1}{(1+z)^l}dz$, which, combined with estimates (4.2b)-(4.2d) implies

$$\int_0^T\int_\Omega \left(\frac{\mathbb{S}_\delta:\nabla\mathbf{u}_\delta}{(1+\theta_\delta)^l} + l\frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^{l+1}}|\nabla\theta_\delta|^2\right)\psi dxdt\leq C \quad (4.4)$$

for some constant C independent of $\delta>0$. Letting $l\to 0$ in (4.4), we obtain

$$\int_0^T\int_\Omega \mathbb{S}_\delta:\nabla \mathbf{u}_\delta dxdt\le C. \tag{4.5}$$

This, with help of (4.1) and Poincaré's inequality, yields

$$\|\mathbf{u}_\delta\|_{L^2(0,T;W_0^{1,2}(\Omega))}\le C. \tag{4.6}$$

In addition, for fixed $0<l<1$, by virtue of (4.4) and the growth restriction imposed on $\kappa(\theta)$ (1.4), we obtain

$$\|\nabla\theta_\delta^{\frac{3-l}{2}}\|_{L^2((0,T)\times\Omega)}\le C(l), \tag{4.7}$$

together with

$$\|\nabla\theta_\delta\|_{L^2((0,T)\times\Omega)}\le C(l), \tag{4.8}$$

with the constant $C(l)$ depending on $l\in(0,1)$. Thanks to Lemma 3.2, estimates (4.2b) and (4.8) yield

$$\|\theta_\delta\|_{L^2(0,T;W^{1,2}(\Omega))}\le C(l). \tag{4.9}$$

Bootstraping (4.7) and (4.9), we deduce

$$\|\theta_\delta^{\frac{3-l}{2}}\|_{L^2(0,T;W^{1,2}(\Omega))}\le C(l). \tag{4.10}$$

Combining (4.2b) with (4.10) and thanks to the interpolation inequality, we deduce for a certain $p>1$ and a small positive number ω

$$\theta_\delta^3 \,\text{is bounded in}\, L^p(\{\rho_\delta(t,x)\ge\omega>0\}) \tag{4.11}$$

by a positive constant independent of $\delta>0$.

Putting all estimates independent of $\delta>0$ together, we have the following result.

Proposition 4.1. *Under the hypotheses of Theorem 1.1 and Proposition 2.1, we have*

$$\|\rho_\delta\|_{L^\infty((0,T)\times\Omega)}\le C, \tag{4.12a}$$
$$\|\mathbf{u}_\delta\|_{L^2(0,T;W_0^{1,2}(\Omega))}\le C, \tag{4.12b}$$
$$\|\theta_\delta^{\frac{3-l}{2}}\|_{L^2(0,T;W^{1,2}(\Omega))}\le C(l),\,\text{with}\, l\in(0,1], \tag{4.12c}$$
$$\|\theta_\delta^3\|_{L^p(\{\rho_\delta(t,x)\ge\omega>0\})}\le C,\,\text{for some}\, p>1. \tag{4.12d}$$

By virtue of (4.12a) and (4.12b), we can obtain the same compactness result for $(\rho_\delta,\mathbf{u}_\delta)$ as $(\rho_n,\mathbf{u}_n)$. Thus, in the rest of this section, we focus our attention on the compactness of θ_δ.

4.2. Strong convergence of the approximate temperatures. Similarly as the process in obtaining the strong convergence of θ_n, the main difficulty is to prove

$$(\delta+\rho_\delta)\theta_\delta\to\rho\overline{\theta}\,\text{in}\, L^2(0,T;H^{-1}(\Omega)),\,\text{as}\,\delta\to 0. \tag{4.13}$$

In order to obtain (4.13), by virtue of Lemma 2.2 and estimates (4.12a)-(4.12d), it suffices to prove that

$$\theta_\delta\,\text{is bounded in}\, L^3(\{\rho_\delta(x,t)<\omega\}) \tag{4.14}$$

by a positive constant independent of $\delta > 0$, with ω being a sufficiently small positive number.

As in [13,14], for each $t \in (0,T)$, one can solve the Neumann problem

$$\begin{cases} \triangle \eta_\delta(t) = B(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega B(\rho_\delta(t))dx \quad \text{in } \Omega, \\ \nabla \eta_\delta \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega \eta_\delta(t) dx = 0, \end{cases}$$

with $B \in C^\infty(\mathbb{R})$ non-increasing and satisfying

$$B(z) = \begin{cases} 0, & \text{if } z \le \omega, \\ -1, & \text{if } z \ge 2\omega. \end{cases}$$

Now the estimate (4.14) can be achieved by taking

$$\varphi(t,x) = \psi(t)(\eta(t,x) - \underline{\eta}), \quad \text{with } \underline{\eta} = \inf_{t \in [0,T], x \in \Omega} \eta(t,x),$$

where $\psi \in C_c^\infty(0,T)$

$$0 \le \psi \le 1, \ \psi \text{ is non-decreasing on } (0,a] \text{ and non-increasing on } [a,T),$$

as a test function in (3.31). Since this process is similar to that in [13,14], we omit the details here.

Combining (4.12c) and (4.13), we have

$$\theta_\delta \to \overline{\theta} \text{ in } L^2(\{\rho > 0\}). \tag{4.15}$$

Taking (4.15) into account, we can perform the limit $\delta \to 0^+$ as in Section 2. For convenience, we only give the details of the limit passage in the renormalized temperature inequality.

4.3. Limit in the renormalized temperature inequality. First, for fixed h, passing to the limit $\delta \to 0^+$ in the same way as in Section 2 for the renormalized temperature inequality (3.31), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \left(\rho H(\bar\theta)\partial_t \varphi + \rho \mathbf{u} H(\bar\theta) \cdot \nabla \varphi + \overline{\mathcal{K}_h(\theta)} \triangle \varphi \right) dxdt \\ &\le -\int_0^T \int_\Omega h(\bar\theta) \mathbb{S} : \nabla \mathbf{u} \varphi dxdt - \int_\Omega \rho_0 H(\theta_0) \varphi(0) dx, \end{aligned} \tag{4.16}$$

where

$$\rho \overline{\mathcal{K}_h(\theta)} = \rho \mathcal{K}_h(\bar\theta),$$

and

$$\rho \mathbb{S} = \rho \mu(\bar\theta)(\nabla \mathbf{u} + \nabla^T \mathbf{u}).$$

Next, taking

$$h(\theta) = \frac{1}{(1+\theta)^l}, \ 0 < l < 1$$

in (4.16), letting $l \to 0$ and using the monotone convergence theorem, we have

$$\int_0^T \int_\Omega \left(\rho\bar{\theta}\partial_t\varphi + \rho\mathbf{u}\bar{\theta}\cdot\nabla\varphi + \overline{\mathcal{K}(\theta)}\triangle\varphi\right) dxdt \le -\int_0^T \int_\Omega \mathbb{S}:\nabla\mathbf{u}\varphi dxdt - \int_\Omega \rho_0\theta_0\varphi(0)dx,$$

where

$$\rho\overline{\mathcal{K}(\theta)} = \rho\mathcal{K}(\bar{\theta}),$$

and

$$\rho\mathbb{S} = \rho\mu(\bar{\theta})(\nabla\mathbf{u} + \nabla^T\mathbf{u}).$$

Finally, denote

$$\theta = \mathcal{K}^{-1}\left(\overline{\mathcal{K}(\theta)}\right).$$

We observe the new function θ satisfies

$$\rho\bar{\theta} = \rho\theta \; a.e. \text{ in } (0,T)\times\Omega.$$

Therefore, we have

$$\int_0^T \int_\Omega (\rho\theta\partial_t\varphi + \rho\mathbf{u}\theta\cdot\nabla\varphi + \mathcal{K}(\theta)\triangle\varphi)\, dxdt \le -\int_0^T \int_\Omega \mathbb{S}:\nabla\mathbf{u}\varphi dxdt - \int_\Omega \rho_0\theta_0\varphi(0)dx,$$

with

$$\rho\mathbb{S} = \rho\mu(\theta)(\nabla\mathbf{u} + \nabla^T\mathbf{u}),$$

which is exactly the temperature inequality in Definition 1.1.

Acknowledgments. The author would like to thank Professor Yi Wang and Professor Cheng Yu for the valuable comments and the helpful suggestions on the manuscript. This work was supported by the National Natural Science Foundation of China (Grant No. 12101154), and by the Natural Science Foundation of Heilongjiang Province (Grant No. LH2022A005).

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