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A QUADRATIC SPLINE PROJECTION METHOD FOR COMPUTING STATIONARY DENSITIES OF RANDOM MAPS*

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Abstract. We propose a quadratic spline projection method that computes stationary densities of random maps with position-dependent probabilities. Using a key variation inequality for the corresponding Markov operator, we prove the norm convergence of the numerical scheme for a family of random maps consisting of the Lasota-Yorke class of interval maps. The numerical experimental results show that the new method improves the L^1-norm errors and increases the convergence rate greatly, compared with the previous operator-approximation-based numerical methods for random maps.

Keywords. Projection method; Frobenius-Perron operator; Foias operator; Markov operator; absolutely continuous probability measure; invariant measure; stationary density; random map.

AMS subject classifications. 41A35; 65D07; 65J10.

1. Introduction

Random maps occur in many problems of physical sciences to describe the dynamical system that is governed by several transformations of a phase space. The dynamics of the system is via the iterative process, at each step of which a transformation among them is chosen with its given probability. Such probabilities, whose sum is equal to one naturally, may depend on the location of the iterates. In the ergodic theory of random maps, one is interested in the eventual behavior of the iteration with the probability allocation, that is, the statistical distribution of the iterates for almost all initial points.

Dynamical systems of individual transformations have been extensively studied in both deterministic and statistical senses. The study of the deterministic behavior of the iterates is often related to the concept of chaos that is featured by unpredictability or sensitive dependence on initial conditions, and the investigation of the statistical behavior of the chaotic iterates is usually done with the help of the Frobenius-Perron operator associated with the transformation. A fixed point of the operator, which is a density function, determines the statistical property of the iterates [12]. In the past almost half a century since the publication of the Lasota-Yorke paper [13] on interval maps, various existence results of invariant measures for different classes of one or multi-dimensional transformations have appeared, and since Li's pioneering paper [14] that solved Ulam's conjecture [15] for the Lasota-Yorke class of interval maps, the numerical analysis of Frobenius-Perron operators has led to the construction of higher order computational methods for invariant measures and other related quantities.

On the other hand, the eventual statistical behavior of random maps is determined by the corresponding Markov operator, which can be expressed in terms of the Frobenius-Perron operators associated with the participating transformations respectively. The operator is so named since it is a positive operator and maps density functions to density functions. A density function that is a fixed point of this operator

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is referred to as a stationary density and defines an absolutely continuous probability measure, which determines the statistical distribution of the iterates from the random maps for almost all initial points.

There has been increased research on the existence of such absolutely continuous probability measures of random maps [9]. Designing and analyzing computational methods for random maps have become a recent research topic because of its importance in applied areas. One family of the numerical methods is based on the maximum entropy principle [10], with [3] the first paper for a single deterministic map; see [4] for a survey of maximum entropy methods for the density function recovery in dynamical systems.

Following the idea of Ulam's method [15] and Li's solution to Ulam's conjecture [14], another numerical approach is to use finite dimensional operators to approximate the infinite dimensional Frobenius-Perron operator or more general Markov operator, as a basis of developing efficient numerical schemes. The direct numerical approach for the computation of stationary densities is along Li's direction of employing piecewise polynomials of higher order in the approximation, so that the original fixed point equation is reduced to a finite system of linear algebraic equations. We refer the reader to the monograph [7] for a general theory of numerical analysis of Frobenius-Perron operators and more general Markov operators.

Ulam's original scheme is to approximate the unknown density function by piecewise constant ones. To increase the convergence rate of Ulam's method, Ding and Li [5] proposed a piecewise linear numerical method to compute stationary densities of Frobenius-Perron operators, based on the idea of integral and positivity preserving approximations of integrable functions. The theoretical study [6] of the resulting Markov finite approximations indicates that the piecewise linear method has the convergence rate of $\mathrm{O}(1/n)$ under the BV-norm for the Lasota-Yorke class of piecewise monotonic maps, as compared to the order of $\mathrm{O}(\ln n/n)$ under the weaker L^1-norm for Ulam's method.

Markov finite approximations may not achieve as good error bounds as Galerkin projection methods in general, because of the structure-preserving restriction of the former and the least squares property of the latter for square integrable functions. Although Ulam's method is both a Markov finite approximation method and Galerkin projection method that projects integrable functions onto the subspace of piecewise constant functions, higher order Galerkin projection methods fail to be of Markov finite approximations. Such numerical approaches have been extended to approximate stationary densities of interval maps and random maps. A piecewise quadratic approximation method was proposed and analyzed for interval maps in [16], and a piecewise linear method has been proposed for random maps in [11].

Here we develop a numerical analysis for a quadratic spline projection method applied to computing a stationary density of Markov operators associated with given random maps with position dependent probabilities. For the convergence proof of the method, we use the same approach as in [1], which avoids complicated technical estimates of the upper bound constants for the L^1-norm and the variation norm.

In the next section we give useful properties of random maps and the corresponding Markov operators, and Section 3 is devoted to the construction of the quadratic spline projection method. In Section 4 we prove the convergence theorem for a class of random maps consisting of the Lasota-Yorke interval maps, after establishing the consistency and stability results of the approximation sequence. Numerical experiments of this method will be given in Section 5, and we conclude in Section 6.

2. The Markov operator for random maps

We recall the definition of random maps. Let $S_1,\ldots,S_k$ be k Borel measurable maps from $[0,1]$ into itself, and let $p_1,\ldots,p_k$ be k positive functions whose sum equals 1 identically on $[0,1]$. We call $\tau \equiv \{S_1,\ldots,S_k;p_1,\ldots,p_k\}$ a *random map* with position dependent probabilities. The dynamics of a random map is the iterative process: Given an initial point $x_0 \in [0,1]$, the $(n+1)$-th iterate is given by

$$x_{n+1} = S_{i_n}(x_n),\, n = 0,1,\ldots,$$

where at step n, the map S_{i_n} is chosen with probability $p_{i_n}(x_n)$ at x_n.

The iteration of random maps from the statistical point of view can be studied via its corresponding Markov operator, based on the concept of Frobenius-Perron operators with respect to the individual measurable maps [7, 12]. For any nonsingular Borel measurable map $S:[0,1]\to[0,1]$, which means that the inverse image $S^{-1}(B)$ of a Borel measurable subset B of $[0,1]$ is a Borel set and $m(S^{-1}(B))=0$ whenever $m(B)=0$ with m denoting the Lebesgue measure, the corresponding *Frobenius-Perron operator* $P_S : L^1(0,1)\to L^1(0,1)$ is defined by

$$\int_B P_S f\,dm = \int_{S^{-1}(B)} f\,dm,\ \forall\,\text{Borel measurable sets}\, B\subset[0,1], \tag{2.1}$$

where $L^1(0,1)$ is the space of all Lebesgue integrable functions on $[0,1]$ with the L^1-norm $\|f\| = \int_0^1 |f(x)|\,dx$. This operator is well-defined from the Radon-Nikodym theorem by the non-singularity assumption of S [7,12]. The Frobenius-Perron operator is a *Markov operator*, in other words, P_S maps nonnegative functions to nonnegative functions and preserves their L^1-norm.

Letting $B=[0,x]$ and taking derivative on the both sides of (2.1) with respect to x, we can rewrite the above implicit definition of P_S explicitly as

$$P_S f(x) = \frac{d}{dx}\int_{S^{-1}([0,x])} f(t)dt,\ \forall\, x\in[0,1].$$

A nonnegative function f in $L^1(0,1)$ is called a *density* if $\|f\|=1$. If f^* is a density and fixed point of P_S, then the probability measure μ_{f^*}, defined by $\mu_{f^*}(B) = \int_B f^* dm$ for all Borel measurable sets $B\subset[0,1]$, is absolutely continuous with respect to the Lebesgue measure and is an *invariant measure* for S in the sense that

$$\mu_{f^*}(S^{-1}(B)) = \mu_{f^*}(B) \tag{2.2}$$

for all Borel measurable sets $B\subset[0,1]$. Such a measure gives the statistical distribution of the sequence of iterates $x_{n+1}=S(x_n)$ for almost all initial points x_0 with respect to the invariant measure [7,12].

We can extend the concept of Frobenius-Perron operators with respect to one map to a Markov operator associated with a random map. For a random map $\tau = \{S_1,\ldots,S_k;p_1,\ldots,p_k\}$, the corresponding operator, which is called the *Foias operator* in [12], $P_\tau : L^1(0,1)\to L^1(0,1)$ is defined as

$$P_\tau f(x) = \sum_{i=1}^{k} P_{S_i}(p_i f)(x),\ \forall\, f\in L^1(0,1). \tag{2.3}$$

The following shows that the Foias operator is a Markov operator.

LEMMA 2.1. *The Foias operator P_τ is a Markov operator.*

Proof. Let $f \in L^1(0,1)$ and $f \geq 0$. Since $p_i \geq 0$, we have $p_i f \geq 0$ for every i. So $P_{S_i}(p_i f) \geq 0$ since P_{S_i} is a positive operator. Thus, $P_\tau f = \sum_{i=1}^k P_{S_i}(p_i f) \geq 0$. Then, from the expression (2.3) of P_τ, since P_{S_i} preserves integrals and $\sum_{i=1}^k p_i = 1$,

$$\|P_\tau f\| = \int_0^1 P_\tau f dm = \int_0^1 \sum_{i=1}^k P_{S_i}(p_i f) dm = \sum_{i=1}^k \int_0^1 P_{S_i}(p_i f) dm$$
$$= \sum_{i=1}^k \int_0^1 p_i f dm = \int_0^1 \left(\sum_{i=1}^k p_i\right) f dm = \int_0^1 f dm = \|f\|.$$

$\square$

Since P_τ is a Markov operator, $\|P_\tau f\| \leq \|f\|$ for $f \in L^1(0,1)$ [12]. It follows that

$$\|P_\tau\| = 1. \tag{2.4}$$

Another important property of the Markov operator is that if f is a fixed point of P_τ, then its positive and negative parts f^+ and f^-, defined as

$$f^+(x) \equiv \max\{f(x), 0\}, \ \ f^-(x) \equiv \max\{-f(x), 0\}, \ x \in [0,1],$$

are also fixed points of P_τ. This property will be used for the convergence analysis of our numerical method.

A fixed point f^* of P_τ that is also a density is called a *stationary density* of P_τ, and it defines the absolutely continuous probability measure $\mu_{f^*}(B) = \int_B f^* dm$, which is the invariant measure of τ in the sense that

$$\sum_{i=1}^k \int_{S_i^{-1}(B)} p_i(x) d\mu_{f^*}(x) = \mu_{f^*}(B) \tag{2.5}$$

for every Borel subset B of $[0,1]$. As a special case, when $k = 1$ so that the random map consists of just one single map, the invariance (2.5) is reduced to (2.2).

In particular, if each probability function p_i is a constant, then

$$P_\tau f(x) = \sum_{i=1}^k p_i P_{S_i} f(x),$$

in other words, the corresponding Markov operator is just a convex combination of the Frobenius-Perron operators $P_{S_1}, \ldots, P_{S_k}$ with the coefficients $p_1, \ldots, p_k$.

The next section introduces a quadratic spline projection method to approximate stationary densities for random maps.

3. Quadratic spline projections for stationary densities

In this section, we develop the quadratic spline projection method for approximating a stationary density of a random map. We divide the interval $[0,1]$ into n equal subintervals $I_i = [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ with step $h = x_{i+1} - x_i = \frac{1}{n}$. Let $S_n^2[0,1]$ be the corresponding $(n+2)$-dimensional space of continuously differentiable piecewise quadratic functions on $[0,1]$. Then its canonical basis is made of the quadratic basic splines

$$\phi_i(x) \ = \ q\left(\frac{x - x_i}{h}\right), \ i = -2, -1, \ldots, n-1, \text{for } x \in [0,1],$$

where

$$q(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x < 1, \\ \frac{3}{4} - \left(x - \frac{3}{2}\right)^2, & 1 \le x < 2, \\ \frac{1}{2}(x-3)^2, & 2 \le x \le 3, \\ 0, & x \notin [0,3]. \end{cases} \tag{3.1}$$

We shall use the following important fact of the spline functions:

$$\sum_{i=-2}^{n-1} \phi_i(x) = 1, \forall x \in [0,1]. \tag{3.2}$$

For any $f \in L^1(0,1)$, we define $Q_n f \in S_n^2[0,1]$ to be such that

$$\langle Q_n f, \phi_i \rangle = \langle f, \phi_i \rangle, \forall i = -2, -1, \dots, n-1, \tag{3.3}$$

where $\langle g, \phi_i \rangle = \int_0^1 g(x)\phi_i(x)dx$ for $g \in L^1(0,1)$. We write $Q_n f = \sum_{j=-2}^{n-1} c_j \phi_j$. Then the coefficients $c_{-2}, c_{-1}, \dots, c_{n-1}$ can be determined uniquely by the $n+2$ linear equations

$$\sum_{j=-2}^{n-1} \langle \phi_i, \phi_j \rangle c_j = \langle f, \phi_i \rangle, \ \forall i = -2, -1, \dots, n-1. \tag{3.4}$$

Let $B_n = [\langle \phi_i, \phi_j \rangle]$ for $-2 \le i, j \le n-1$, $c = (c_{-2}, c_{-1}, \dots, c_{n-1})^T$, and $b = (b_{-2}, b_{-1}, \dots, b_{n-1})^T$ with each $b_i = \langle f, \phi_i \rangle$. Using (3.1), we rewrite the system (3.4) as

$$\hat{B}_n c = \hat{b}, \tag{3.5}$$

where

$$\hat{B}_n = \frac{120}{h} B_n \text{ and } \hat{b} = \frac{120}{h} b. \tag{3.6}$$

A simple computation gives

$$\hat{B}_n = \begin{bmatrix} 6 & 13 & 1 & & & & \\ 13 & 60 & 26 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & 26 & 60 & 13 \\ & & & & 1 & 13 & 6 \end{bmatrix} \in \mathbb{R}^{(n+2)\times(n+2)}.$$

Note that $\hat{B}_n$ is a symmetric band-matrix with band-width two.

If Q_n is restricted to the subspace $L^2(0,1)$ of $L^1(0,1)$, then $Q_n : L^2(0,1) \to S_n^2[0,1]$ is the (orthogonal) projection from $L^2(0,1)$ onto $S_n^2[0,1]$. So $Q_n g$ is the least squares approximation of g under the L^2-norm:

$$\|g - Q_n g\|_2 = \min \left\{ \left\| g - \sum_{j=-2}^{n-1} a_j \phi_j \right\|_2 : \forall a_j \in \mathbb{R} \right\},$$

where $g \in L^2(0,1)$ means that $\|g\|_2^2 \equiv \int_0^1 |g(x)|^2 dx$ is finite.

Note that a stationary density f^* fulfills the following equation

$$P_\tau f - f = 0, \ \ f \in L^1(0,1),$$

which will be modified to be valid only on the finite dimensional subspace $S_n^2[0,1]$ via the Galerkin projection principle. Thus, the above equation becomes

$$\langle P_\tau f_n - f_n, \phi_i \rangle = 0, \ i = -2, -1, \ldots, n-1; \ f_n \in S_n^2[0,1]. \tag{3.7}$$

Let $f_n = \sum_{j=-2}^{n-1} d_j \phi_j$. Then the above system can be written as

$$\sum_{j=-2}^{n-1} \langle P_\tau \phi_j - \phi_j, \phi_i \rangle d_j = 0, \, i = -2, -1, \ldots, n-1,$$

or in the matrix-vector form,

$$(A_n - B_n)d = 0, \tag{3.8}$$

where $A_n = [a_{ij}]$ with $a_{ij} = \langle P_\tau \phi_j, \phi_i \rangle$ for $-2 \le i, j \le n-1$, and $d = (d_{-2}, d_{-1}, \ldots, d_{n-1})^T$.

We need to prove that there exists a nonzero solution $f_n = \sum_{j=-2}^{n-1} d_j \phi_j$ of (3.7) in $S_n^2[0,1]$, and we do so by proving that (3.8) has a nonzero vector d as a solution.

LEMMA 3.1. *There is a nonzero function $f_n \in S_n^2[0,1]$ that solves the equation (3.7).*

Proof. Since $\sum_{i=-2}^{n-1} \phi_i = 1$ and P_τ is a Markov operator,

$$\begin{aligned}\sum_{i=-2}^{n-1} (a_{ij} - b_{ij}) &= \sum_{i=-2}^{n-1} \langle \phi_i, P_\tau \phi_j - \phi_j \rangle = \left\langle \sum_{i=-2}^{n-1} \phi_i, P_\tau \phi_j - \phi_j \right\rangle = \langle 1, P_\tau \phi_j - \phi_j \rangle \\ &= \langle 1, P_\tau \phi_j \rangle - \langle 1, \phi_j \rangle = \langle 1, \phi_j \rangle - \langle 1, \phi_j \rangle = 0, \ \forall j = -2, -1, \ldots, n-1.\end{aligned}$$

It follows that $A_n - B_n$ is singular. So there is a nonzero vector $d = (d_{-2}, d_{-1}, \ldots, d_{n-1})^T$ that satisfies (3.8), and hence a nonzero function $f_n = \sum_{j=-2}^{n-1} d_j \phi_j$ that solves (3.7). □

From the definition of Q_n (see (3.3)), since $f_n \in S_n^2[0,1]$ solves (3.7),

$$\langle Q_n P_\tau f_n, \phi_i \rangle = \langle P_\tau f_n, \phi_i \rangle = \langle f_n, \phi_i \rangle, \ \ i = -2, -1, \ldots, n-1.$$

Hence, for each n, we have

$$Q_n P_\tau f_n = f_n. \tag{3.9}$$

Note that the entries a_{ij} can be computed by

$$\begin{aligned}\langle P_\tau \phi_j, \phi_i \rangle &= \int_0^1 P_\tau \phi_j(x) \phi_i(x) \ dx = \int_0^1 \sum_{l=1}^{k} P_{S_l}(p_l(x) \phi_j(x)) \phi_i(x) \ dx \\ &= \sum_{l=1}^{k} \int_0^1 P_{S_l}(p_l(x) \phi_j(x)) \phi_i(x) \ dx = \sum_{l=1}^{k} \int_0^1 p_l(x) \phi_j(x) \phi_i(S_l(x)) \ dx\end{aligned}$$

since $\int_0^1 P_{S_l} f(x) g(x) \ dx = \int_0^1 f(x) g(S_l(x)) dx$ [12] for bounded measurable functions g.

The quadratic spline projection algorithm for random maps can be summarized as follows: For a chosen positive integer n, we find a nonzero vector d satisfying $(A_n - B_n)d = 0$ and let $f_n = \sum_{j=-2}^{n-1} d_j \phi_j$. Finally, after normalizing f_n so that $\|f_n\| = 1$, the quadratic spline function f_n is taken as an approximation to f^*. The convergence of $\{f_n\}$ to f^* when $S_1, \dots, S_k$ belong to the Lasota-Yorke class of piecewise C^2 and stretching maps will be proven in the following section.

4. Convergence analysis

To prove the norm convergence of the sequence $\{f_n\}$ to the stationary density f^* as $n \to \infty$, we need to study the sequence of projection operators $\{Q_n\}$. We first establish that the matrix 1-norms of the inverses of $\hat{B}_n$ (see (3.6)) are uniformly bounded. Here the vector 1-norm is $\|v\| \equiv \sum_{i=1}^{r} |v_i|$ for any $v \in \mathbf{R}^r$, and the matrix 1-norm is the operator norm induced by the vector 1-norm.

We used MATLAB to get the following results (to the accuracy shown) for $\|\hat{B}_n^{-1}\|$:

$$\|\hat{B}_{16}^{-1}\| = 0.534554,$$
$$\|\hat{B}_{32}^{-1}\| = 0.534552,$$
$$\|\hat{B}_{64}^{-1}\| = 0.534552,$$
$$\|\hat{B}_{128}^{-1}\| = 0.534552.$$

Hence,

$$\|\hat{B}_n^{-1}\| \le 0.54 \text{ for all } n \ge 16. \tag{4.1}$$

Before showing the consistency of the sequence $\{Q_n\}$ (by consistency we mean Lemma 4.1 part (ii)), we observe that $\|\phi_{-2}\| = \|\phi_{n-1}\| = h/6$, $\|\phi_{-1}\| = \|\phi_{n-2}\| = 5h/6$, and $\|\phi_i\| = h$ with $i = 0, 1, \dots, n-3$. Now we have the following lemma.

LEMMA 4.1. *The operator sequence $\{Q_n\}$ satisfies the following properties:*

(i) $\|Q_n\| \le 65$ *for all* $n \ge 16$.

(ii) $\lim_{n\to\infty} \|Q_n f - f\| = 0$ *for any* $f \in L^1(0,1)$.

Proof. (i) First, using (3.2), we have

$$\|b\| = \sum_{i=-2}^{n-1} |\langle f, \phi_i \rangle| \le \sum_{i=-2}^{n-1} \langle |f|, \phi_i \rangle = \left\langle |f|, \sum_{i=-2}^{n-1} \phi_i \right\rangle = \langle |f|, 1 \rangle = \|f\|. \tag{4.2}$$

Since by (3.6), $B_n = (h/120)\hat{B}_n$. So (4.1) gives

$$\|B_n^{-1}\| = \frac{120}{h} \|\hat{B}_n^{-1}\| \le \frac{120}{h} \cdot 0.54 \le \frac{65}{h}. \tag{4.3}$$

It follows from the fact $\|\phi_i\| \le h$ for all i, (3.5), (3.6), (4.2) and (4.3) that

$$\begin{aligned} \|Q_n f\| &= \left\| \sum_{i=-2}^{n-1} c_i \phi_i \right\| \le \sum_{i=-2}^{n-1} |c_i| \|\phi_i\| \le h \sum_{i=-2}^{n-1} |c_i| \\ &= h\|c\| \le h \|\hat{B}_n^{-1}\| \|\hat{b}\| \le h \frac{h}{120} \|B_n^{-1}\| \frac{120}{h} \|b\| \\ &= h \|B_n^{-1}\| \|b\| \le h \frac{65}{h} \|f\| = 65 \|f\|. \end{aligned}$$

(ii) Let $f\in L^1(0,1)$ and $\epsilon>0$ be given. Since $C^3[0,1]$ is dense in $L^1(0,1)$, there exists $g\in C^3[0,1]$ such that

$$\|g-f\|\leq \frac{\epsilon}{132}.$$

Let $\{0=z_0,z_1,z_2,\ldots,z_n,z_{n+1}=1\}$ be such that $z_i=\frac{x_{i-1}+x_i}{2}$ for $i=1,\ldots,n$. Let $s_n\in S_n^2[0,1]$ be defined by

$$s_n(z_i)=g(z_i),\ i=0,1,\ldots,n.$$

Then by [8]

$$\max_{x\in[0,1]}|g(x)-s_n(x)|\leq \frac{1}{24n^3}\max_{x\in[0,1]}\left|g^{'''}(x)\right|.$$

Note that, since s_n,g, and $Q_ng\in L^2(0,1)$, using Hölder's inequality and the fact that Q_ng is the least squares approximation to g from $S_n^2[0,1]$ we get,

$$\begin{aligned}\|Q_ng-g\|&\leq\|Q_ng-g\|_2\leq\|s_n-g\|_2\\&\leq\max_{x\in[0,1]}|g(x)-s_n(x)|\leq\frac{1}{24n^3}\max_{x\in[0,1]}|g^{'''}(x)|.\end{aligned}$$

So if n is big enough, then $\|Q_ng-g\|<\frac{\epsilon}{2}$. Thus, if n is big enough, we have

$$\begin{aligned}\|Q_nf-f\|&\leq\|Q_nf-Q_ng\|+\|Q_ng-g\|+\|g-f\|\\&\leq\|Q_n\|\,\|f-g\|+\|Q_ng-g\|+\|g-f\|\\&=(\|Q_n\|+1)\,\|f-g\|+\|Q_ng-g\|\\&\leq 66\,\|f-g\|+\|Q_ng-g\|\\&<\frac{66\epsilon}{132}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

□

The following lemma was proven in Lemma 4.4 of [16].

LEMMA 4.2. *For any $f\in L^1(0,1)$,*

$$\bigvee_0^1 Q_nf\leq\sum_{i=-1}^{n-1}|c_i-c_{i-1}|,$$

where $Q_nf=\sum_{j=-2}^{n-1}c_j\phi_j$.

Our next goal is to find another upper bound of $\bigvee_0^1 Q_nf$ by finding an upper bound of $\sum_{i=-1}^{n-1}|c_i-c_{i-1}|$ in terms of $\bigvee_0^1 f$. It was shown in [16] that $\tilde{B}_n\tilde{c}=\tilde{b}$, where

$$\tilde{B}_n=\begin{bmatrix}17&22&1&&&&\\18&65&26&1&&&\\1&26&66&26&1&&\\&\ddots&\ddots&\ddots&\ddots&\ddots&\\&&1&26&66&26&1\\&&&1&26&65&18\\&&&&1&22&17\end{bmatrix}\in\mathbb{R}^{(n+1)\times(n+1)},$$

$$\tilde{b} = \begin{bmatrix} \hat{b}_{-1} - 5\hat{b}_{-2} \\ \hat{b}_0 - \hat{b}_{-1} - \hat{b}_{-2} \\ \hat{b}_1 - \hat{b}_0 \\ \vdots \\ \hat{b}_{n-3} - \hat{b}_{n-4} \\ \hat{b}_{n-1} + \hat{b}_{n-2} - \hat{b}_{n-3} \\ 5\hat{b}_{n-1} - \hat{b}_{n-2} \end{bmatrix}$$

and $\tilde{c} = (c_{-1} - c_{-2}, c_0 - c_{-1}, \ldots, c_{n-1} - c_{n-2})^T \in \mathbb{R}^{n+1}$. Thus, $\tilde{c} = \tilde{B}_n^{-1}\tilde{b}$ implies that

$$\sum_{i=-1}^{n-1} |c_i - c_{i-1}| = \|\tilde{c}\| \le \|\tilde{B}_n^{-1}\| \|\tilde{b}\|.$$

MATLAB again gives the following results (to the accuracy shown):

$$\begin{aligned} \|\tilde{B}_{16}^{-1}\| &= 0.159835, \\ \|\tilde{B}_{32}^{-1}\| &= 0.159834, \\ \|\tilde{B}_{64}^{-1}\| &= 0.159834, \\ \|\tilde{B}_{128}^{-1}\| &= 0.159834. \end{aligned}$$

Hence $\|\tilde{B}_n^{-1}\| \le 0.16$ for all $n \ge 16$, from which for $n \ge 16$,

$$\sum_{i=-1}^{n-1} |c_i - c_{i-1}| \le 0.16 \|\tilde{b}\|. \tag{4.4}$$

The following inequality was shown in [16]. That is, for any function $f \in L^1(0,1)$ of bounded variation,

$$\|\tilde{b}\| \le 356 \bigvee_0^1 f. \tag{4.5}$$

Hence, using Lemma 4.2, (4.4), and (4.5), we have

$$\begin{aligned} \bigvee_0^1 Q_n f \le \sum_{i=-1}^{n-1} |c_i - c_{i-1}| &\le 0.16 \|\tilde{b}\| \\ &\le 0.16 \cdot 356 \bigvee_0^1 f \le 57 \bigvee_0^1 f. \end{aligned}$$

We summarize the above result in the following lemma, which shows the stability of the sequence $\{Q_n\}$.

LEMMA 4.3. *Let $f \in L^1(0,1)$ be of bounded variation. Then for all $n \ge 16$,*

$$\bigvee_0^1 Q_n f \le 57 \bigvee_0^1 f.$$

We are ready to prove the convergence of the quadratic spline projection method for a family of random maps consisting of the interval maps that satisfy the condition of the Lasota-Yorke theorem in [13]. Such a condition leads to a key variation inequality from the analysis of [11], which guarantees the existence of a stationary density of P_τ. We show that the Lasota-Yorke type of inequality is also sufficient for the convergence analysis of the numerical method.

Let $\tau = \{S_1, \ldots, S_k; p_1, \ldots, p_k\}$ be a random map such that each map S_i is piecewise C^2. We assume that each probability function p_i is continuously differentiable on $[0,1]$. For $f \in L^1(0,1)$ of bounded variation, by the Lasota-York inequality [13],

$$\bigvee_0^1 P_{S_i} f \le \alpha_i \bigvee_0^1 f + \beta_i \|f\| \quad i = 1, \ldots, k,$$

where $\alpha_i = 2/\inf|S_i|$ and β_i is a positive constant independent of the choice of f. Then, by a rigorous mathematical analysis in [11], there is the following variation inequality for the Markov operator P_τ:

$$\bigvee_0^1 P_\tau f \le \alpha \bigvee_0^1 f + C\|f\|, \tag{4.6}$$

where $\alpha = \max_{1 \le i \le k} \alpha_i$, $C = \alpha k K + \sum_{i=1}^k \beta_i \|p_i\|_\infty$, and K is a constant such that $|p_i'(x)| \le K$ for all $x \in [0,1]$ and $i = 1, \ldots, k$.

As usual for the sake of the convergence argument, we assume that f^* is a unique stationary density of P_τ in the following theorem.

THEOREM 4.1. *Suppose P_τ satisfies (4.6) with $\alpha < 1/57$. Then for the sequence $\{f_n\}$, where each f_n is a solution of (3.7) with $\|f_n\| = 1$,*

$$\lim_{n\to\infty} \|f_n - f^*\| = 0.$$

Proof. Since $f_n = Q_n P_\tau f_n$ by (3.9), Lemma 4.3 and (4.6) imply that

$$\begin{aligned}\bigvee_0^1 f_n = \bigvee_0^1 Q_n P_\tau f_n &\le 57 \bigvee_0^1 P_\tau f_n \\ &\le 57\left(\alpha \bigvee_0^1 f_n + C\|f_n\|\right) = 57\alpha \bigvee_0^1 f_n + 57C\end{aligned}$$

Since $57\alpha < 1$,

$$\bigvee_0^1 f_n \le \frac{57C}{1 - 57\alpha}, \; n \ge 16.$$

Therefore Helly's lemma assures the existence of a subsequence, for example, $\{f_{n_k}\}$ of $\{f_n\}$, that converges to some $g \in L^1(0,1)$. Clearly $\|g\| = 1$ (recall that $\|f_n\| \equiv 1$) and $\|Q_n P_\tau\| \le \|Q_n\|\|P_\tau\| \le 65$ by Lemma 4.1 (i) for all $n \ge 16$ and by (2.4). Hence, if $n_k \ge 16$,

$$\begin{aligned}\|g - P_\tau g\| &\le \|g - f_{n_k}\| + \|f_{n_k} - Q_{n_k} P_\tau f_{n_k}\| \\ &\quad + \|Q_{n_k} P_\tau f_{n_k} - Q_{n_k} P_\tau g\| + \|Q_{n_k} P_\tau g - P_\tau g\| \\ &\le \|g - f_{n_k}\| + 65\|f_{n_k} - g\| + \|Q_{n_k} P_\tau g - P_\tau g\| \\ &= 66\|g - f_{n_k}\| + \|Q_{n_k} P_\tau g - P_\tau g\|,\end{aligned}$$

which implies that $P_\tau g = g$ by Lemma 4.1 (ii) after taking the limit $k \to \infty$. Since $P_\tau g^+ = g^+$ and $P_\tau g^- = g^-$ by a property of P_τ mentioned in Section 2, and f^* is the unique stationary density of P_τ, we must have $g = f^*$ or $g = -f^*$. Without loss of generality we may assume that $g = f^*$. This proves the theorem since every convergent subsequence of $\{f_n\}$ converges to f^*. ∎

5. Numerical results

In this section we present some numerical experiment results on the performance of the quadratic spline projection method (denoted as D2 in the tables) and compare them with the linear spline projection method (denoted as D1 in the tables) for two examples of tested random maps, which involve the interval maps

$$S_1(x) = \begin{cases} \frac{2x}{1-x^2} & 0 \le x \le \sqrt{2}-1 \\ \frac{1-x^2}{2x} & \sqrt{2}-1 \le x \le 1 \end{cases},$$
$$S_2(x) = 1 - \sqrt{|2x-1|},$$
$$S_3(x) = \begin{cases} \frac{2x}{1-x} & 0 \le x \le \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \le x \le 1. \end{cases}$$

For the comparison of the errors we used

$$e_n \equiv \|f_n - f^*\| = \int_0^1 |f_n(x) - f^*(x)| dx.$$

where $n = 2^k$ for $k = 2, 3, \ldots, 10$.

Example 1. Let $\tau_1 = \{S_1, S_2; p_1, p_2\}$, where

$$p_1(x) = \frac{4}{4 + \pi(1-x)(1+x^2)},$$
$$p_2(x) = \frac{\pi(1-x)(1+x^2)}{4 + \pi(1-x)(1+x^2)}.$$

The stationary density of P_{τ_1} [2] is given by

$$f^*_{\tau_1}(x) = \frac{2}{3}\left[\frac{4}{\pi(1+x^2)} + 1 - x\right].$$

Example 2. Let $\tau_2 = \{S_1, S_2, S_3; p_1, p_2, p_3\}$, where

$$p_1(x) = \frac{2(1+x)^2}{2(1+x)^2 + \pi(1-x^4)(1+x) + \pi(1+x^2)},$$
$$p_2(x) = \frac{\pi(1-x^4)(1+x)}{2(1+x)^2 + \pi(1-x^4)(1+x) + \pi(1+x^2)},$$
$$p_3(x) = \frac{\pi(1+x^2)}{2(1+x)^2 + \pi(1-x^4)(1+x) + \pi(1+x^2)}.$$

The stationary density of P_{τ_2} [2] is given by

$$f^*_{\tau_2}(x) = \frac{2}{3}\left[\frac{2}{\pi(1+x^2)} + 1 - x + \frac{1}{(1+x)^2}\right].$$

n	D1	D2
4	1.67×10^{-3}	2.79×10^{-4}
8	3.81×10^{-4}	3.08×10^{-5}
16	9.54×10^{-5}	3.43×10^{-6}
32	2.35×10^{-5}	4.31×10^{-7}
64	6.17×10^{-6}	4.68×10^{-8}
128	1.48×10^{-6}	5.83×10^{-9}
256	3.67×10^{-7}	7.80×10^{-10}

TABLE 5.1. *L^1-norm errors comparison for Example 1.*

n	D1	D2
4	1.99×10^{-3}	1.79×10^{-4}
8	5.09×10^{-4}	2.58×10^{-5}
16	1.28×10^{-4}	3.48×10^{-6}
32	3.13×10^{-5}	4.46×10^{-7}
64	7.88×10^{-6}	5.74×10^{-8}
128	1.97×10^{-6}	7.21×10^{-9}
256	4.91×10^{-7}	9.22×10^{-10}

TABLE 5.2. *L^1-norm errors comparison for Example 2.*

	D1	D2
e_4/e_8	4.38	9.64
e_8/e_{16}	3.99	8.98
e_{16}/e_{32}	4.06	7.96
e_{32}/e_{64}	3.81	9.21
e_{64}/e_{128}	4.17	8.03
e_{128}/e_{256}	4.03	7.47

TABLE 5.3. *Ratios comparison for Example 1.*

	D1	D2
e_4/e_8	3.91	6.94
e_8/e_{16}	3.98	7.41
e_{16}/e_{32}	4.09	7.80
e_{32}/e_{64}	3.97	7.77
e_{64}/e_{128}	4.00	7.96
e_{128}/e_{256}	4.01	7.82

TABLE 5.4. *Ratios comparison for Example 2.*

The numerical results in Table 5.1 and Table 5.2 represent the errors for Example 1 and Example 2, respectively. These two tables show that the quadratic spline projection method for random maps has a faster convergence rate than the linear spline projection method for the random maps for all n-values. Table 5.3 and Table 5.4 show the ratios of $e_4/e_8, \ldots, e_{128}/e_{256}$ for Example 1 and Example 2, respectively. We observe that the convergence order for the quadratic spline projection method is three compared to order two for the linear spline method.

6. Conclusions

We have proposed a quadratic spline projection method for computing stationary densities for random maps with position-dependent probabilities and proved its L^1-norm convergence for random maps composed of the Lasota-Yorke class of interval maps, using the standard variation inequality technique. This improves the convergence order of the previous linear spline projection method for random maps in the literature. A future research will be toward a theoretical convergence rate analysis for the method, which can confirm the observed convergence order from the numerical experiments.

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