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IMPROVED UNIFORM ERROR BOUNDS OF AN EXPONENTIAL WAVE INTEGRATOR METHOD FOR THE KLEIN-GORDON-SCHRÖDINGER EQUATION WITH THE SMALL COUPLING CONSTANT*

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Abstract. Recently, the long-time numerical simulation and error analysis of PDEs with weak nonlinearity (or small potentials) become an interesting topic. However, the existing results of long-time error analysis mostly focus on the single equations. In this paper, for the Klein-Gordon-Schrödinger equation (KGSE) with a small coupling constant $\varepsilon\in(0,1]$, we propose an exponential wave integrator Fourier pseudo-spectral (EWIFP) method by reformulating the KGSE into a coupled nonlinear Schrödinger system (CNLSS). Through careful and rigorous analysis, we establish improved error bounds for the numerical solution at $O(h^m+\varepsilon\tau^2)$ in the long-time domain up to $O(1/\varepsilon)$ where m is determined by the regularity conditions, h is the mesh size and τ is the time step, respectively. Compared with the existing results, our analysis shows the long-time errors of numerical solution for the KGSE. In error analysis, in addition to the classical tools such as energy method and cut-off technique, we also adopt the regularity compensation oscillation (RCO) technique which has been developed recently to analyze the accumulation of errors carefully. The numerical experiments support our error estimates and demonstrate the long-term stability of discrete mass and energy. To the best of our knowledge, there has not been any relevant long-time error analysis for the KGSE and any improved uniform error bounds for an exponential wave integrator. Our work is novel and provides a reference for analyzing the improved error bounds of the numerical methods for other coupled equations.

Keywords. Long-time dynamics; improved error bounds; small coupling constant; Klein-Gordon-Schrödinger equation; exponential wave integrator.

AMS subject classifications. 35L70; 65M12; 65M15; 65M70; 81-08.

1. Introduction

This paper focuses on the Klein-Gordon-Schrödinger equation (KGSE) as follows

$$\begin{aligned}
&\partial_{tt}u-\Delta u+u=\varepsilon|\Phi|^2, \quad (\mathbf{x},t)\in\mathbb{T}^d\times(0,+\infty),\\
&i\partial_t\Phi+\Delta\Phi+\varepsilon u\Phi=0, \quad (\mathbf{x},t)\in\mathbb{T}^d\times(0,+\infty),\\
&(u,\partial_t u,\Phi)(\mathbf{x},0)=(u^0,\dot{u}^0,\Phi^0)(\mathbf{x}), \quad \mathbf{x}\in\mathbb{T}^d,
\end{aligned} \tag{1.1}$$

which describes a neutron field $\Phi:=\Phi(\mathbf{x},t)\in\mathbb{C}$ interacting with a neutral scalar meson field $u:=u(\mathbf{x},t)\in\mathbb{R}$ through the Yukawa interaction [11, 20, 35, 39]. Here, d $(d=1,2,3)$ represents the dimension of the equation, $i=\sqrt{-1}$, $\varepsilon\in(0,1]$ denotes the coupling constant, $\mathbb{T}^d$ is a d-dimensional torus, $\mathbf{x}\in\mathbb{T}^d$ is the spatial coordinate and t is time. The KGSE (1.1) is time symmetric and conserves some invariants such as

$$\begin{aligned}
&M(t)=\|\Phi(\cdot,t)\|^2:=\int_{\mathbb{T}^d}|\Phi(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\equiv\|\Phi(\cdot,0)\|^2, \quad t\in[0,+\infty),\\
&E(t):=\int_{\mathbb{T}^d}\left[\frac{1}{2}\left(|\partial_t u|^2+|\nabla u|^2+|u|^2\right)+|\nabla\Phi|^2-\varepsilon u|\Phi|^2\right]\mathrm{d}\mathbf{x}\equiv E(0), \quad t\in[0,+\infty),
\end{aligned} \tag{1.2}$$

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which are called the mass and energy, respectively. Introducing new variables $w=\varepsilon u$ and $\Upsilon=\varepsilon\Phi$, we reformulate the KGSE (1.1) with small coupling constant as a KGSE with $O(\varepsilon)$ initial data:

$$\begin{aligned}&\partial_{tt}w-\Delta w+w=|\Upsilon|^2, \quad (\mathbf{x},t)\in\mathbb{T}^d\times(0,+\infty),\\ &i\partial_t\Upsilon+\Delta\Upsilon+w\Upsilon=0, \quad (\mathbf{x},t)\in\mathbb{T}^d\times(0,+\infty),\\ &(w,\partial_t w,\Upsilon)(\mathbf{x},0)=\varepsilon(u^0,\dot{u}^0,\Phi^0)(\mathbf{x}), \quad \mathbf{x}\in\mathbb{T}^d.\end{aligned} \tag{1.3}$$

Similarly, the KGSE (1.3) also has time symmetry and conserves the mass as well as energy as

$$\widetilde{M}(t)=\|\Upsilon(\cdot,t)\|^2:=\int_{\mathbb{T}^d}|\Upsilon(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\equiv\|\Upsilon(\cdot,0)\|^2, \quad t\in[0,+\infty),$$
$$\widetilde{E}(t):=\int_{\mathbb{T}^d}\left[\frac{1}{2}\left(|\partial_t w|^2+|\nabla w|^2+|w|^2\right)+|\nabla\Upsilon|^2-w|\Upsilon|^2\right]\mathrm{d}\mathbf{x}\equiv\widetilde{E}(0), \quad t\in[0,+\infty).$$

Due to the fact that the equations (1.1) and (1.3) are equivalent, next we only develop the numerical methods and give related analysis for the KGSE (1.1) with small coupling constant. For the equation (1.3), the formulation of the new methods and corresponding analysis process are completely similar.

For the case $\varepsilon=1$, there have been extensive mathematical and numerical studies for the KGSE (1.1) in the literature. Mathematically, the authors have considered the existence and uniqueness of global smooth solution for the KGSE in [19, 34]. For various properties of the equation, we refer to [12, 21, 24, 25, 36, 37, 40, 43]. Numerically, different kinds of methods have been carried out for the KGSE. The methods include the (pseudo-)spectral method [8, 44], the (multi-) symplectic method [27–29], the collocation method [13, 41, 45] and the finite difference method [42, 46]. Of course, each numerical method has its own advantages. Recently, uniformly accurate time integrators for the KGSE in the nonrelativistic limit regime and in the nonrelativistic and massless limit regime were constructed and analyzed [7, 9, 10].

Recently, the long-time numerical simulation of PDEs with weak nonlinearity (or small potentials) has become an interesting topic and got a lot of attention. For the Klein-Gordon equations (KGE) and Schrödinger/ nonlinear Schrödinger equations (NLSE), the long-time error analysis has been thoroughly studied in the literature [2, 4–6, 14–18, 33]. In a recent paper [3], in order to obtain improved uniform error bounds for time-splitting methods applied to the NLSE with small potential and weak nonlinearity, the authors proposed a new technique of regularity compensation oscillation (RCO). Specifically, they controlled the high frequency modes by regularity and analyzed the low frequency modes by phase cancellation. As a powerful tool, the RCO technique has been increasingly used for long-time error analysis of other types of PDEs [2, 6] and other kinds of numerical methods.

However, existing results of long-time error analysis mostly focus on the single equations and improved uniform error bounds are mainly about the splitting methods. As far as we know, there are few results about improved error bounds for the long-time dynamics of the coupled systems involving the KGSE (1.1) in the literature. Compared with the single equations, the difficulty of numerical analysis for the coupled systems comes from the coupling effect. Another noteworthy fact is that exponential wave integrator methods have been widely used to solve the highly oscillatory problems. Recently, for the Klein-Gordon-Dirac equation (KGDE) with weak coupling effect which is characterized by the small parameter $\varepsilon\in(0,1]$, the authors proposed two energy-preserving exponential wave integrator methods [32]. However, the authors can only prove that the methods achieve uniform error bounds $O(h^m+\tau^2)$ rather than improved uniform error bounds $O(h^m+\varepsilon\tau^2)$ up to the time at $O(1/\varepsilon)$. In addition, the methods are implicit, which means expensive computation cost because the calculation process often requires iteration.

In this paper, we propose an explicit exponential wave integrator Fourier pseudo-spectral (EWIFP) method for the KGSE (1.1). We establish the improved uniform error bounds at $O(h^m+\varepsilon\tau^2)$ up to the time at $O(1/\varepsilon)$. In error analysis, in addition to classical tools such as energy method and cut-off technique, we also adopt the regularity compensation oscillation (RCO) technique to analyze the accumulation of errors carefully. However, our proof of the error bounds is different from the papers [2, 6] in which, the authors first derived the error bounds of semi-discretization and then obtain the error bounds of full discretization by comparing them. In this paper, we will provide a direct proof of the error bounds without giving the error of semi-discretization. In addition, for the control of nonlinear terms, we use cut-off technique instead of mathematical induction as in the paper [2, 6]. The novelty of this paper is also reflected in that, there are few results of long-term error analysis for the KGSE and improved error bounds for the EWI method.

We remark here, in our recent manuscript [31], a time-splitting Fourier pseudo-spectral (TSFP) method has been proved to achieve improved error bounds up to the time at $O(1/\varepsilon)$ for solving the KGSE. However, as the paper [17] says, the EWIFP method and TSFP method are two completely different numerical methods, namely the former is based on Duhamel's principle while the latter uses the idea of time-splitting. In addition, we adopt different techniques to perform long-time convergence analysis. In the control of nonlinear terms, the cut-off technique will be used in EWIFP method and mathematical induction is used in TSFP method.

The next few sections are organized like this. In Section 2, we transform the KGSE (1.1) into a coupling nonlinear Schrödinger system (CNLSS) and then propose the EWIFP method for CNLSS. In Section 3, we establish improved error bounds and give rigorous proof. Section 4 presents the numerical experiments. Finally, we draw a conclusion in the last section. Throughout this paper, the notation $p\lesssim q$ always represents $|p|\le Cq$ where $C>0$ is independent of h, τ and ε.

2. Exponential wave integrator Fourier pseudo-spectral method

Here we only consider the one-dimensional problem of the KGSE (1.1). Higher dimensional problems can be treated similarly. For the one-dimensional problem, we take $\Omega=(a,b)$ and consider the periodic boundary conditions. In this case, the problem collapses to

$$\partial_{tt}u-\partial_{xx}u+u=\varepsilon|\Phi|^2,\quad (x,t)\in\Omega\times(0,\infty),\tag{2.1}$$

$$i\partial_t\Phi+\partial_{xx}\Phi+\varepsilon u\Phi=0,\quad (x,t)\in\Omega\times(0,\infty),\tag{2.2}$$

$$(u,\partial_x u,\Phi,\partial_x\Phi)(a,t)=(u,\partial_x u,\Phi,\partial_x\Phi)(b,t),\quad t\in[0,\infty),\tag{2.3}$$

$$(u,\partial_t u,\Phi)(x,0)=(u^0,\dot{u}^0,\Phi^0)(x),\quad x\in\bar{\Omega}.\tag{2.4}$$

We denote $H_p^s(\Omega)=\left\{u\in H^s(\Omega),\,\partial_x^l u(a)=\partial_x^l u(b),\,l=0,\cdots,s-1\right\}$ with integer $s\ge 0$ and define the H^s norm as

$$\|u\|_s^2=\sum_{l\in Z}(1+|\mu_l|^2)^s|\hat{u}_l|^2,\quad \text{for}\quad u(x)=\sum_{l\in Z}\hat{u}_l e^{i\mu_l(x-a)},\quad \mu_l=\frac{2\pi l}{b-a}.\tag{2.5}$$

Since we consider the periodic boundary conditions, the above space $H_p^s(\Omega)$ is suitable.

2.1. The equivalent coupled system. Firstly we introduce the operator

$$\langle\nabla\rangle=\sqrt{1-\Delta}, \tag{2.6}$$

which is defined as

$$\langle\nabla\rangle u(x)=\sum_{l\in Z}\sqrt{1+\mu_l^2}\,\widehat{u}_l e^{i\mu_l(x-a)}, \quad \text{for} \quad u(x)=\sum_{l\in Z}\widehat{u}_l e^{i\mu_l(x-a)}.$$

The inverse operator $\langle\nabla\rangle^{-1}$ can be defined as

$$\langle\nabla\rangle^{-1}u(x)=\sum_{l\in Z}\frac{\widehat{u}_l}{\sqrt{1+\mu_l^2}}e^{i\mu_l(x-a)}, \quad x\in[a,b]. \tag{2.7}$$

From the definition of norm, we have $\|\langle\nabla\rangle u\|_{s-1}=\|u\|_s=\left\|\langle\nabla\rangle^{-1}u\right\|_{s+1}$. Introducing $\dot{u}=\partial_t u$ and setting

$$\Psi=u-i\langle\nabla\rangle^{-1}\dot{u}, \tag{2.8}$$

the KGSE (2.1)-(2.4) could be reformulated into the coupled nonlinear Schrödinger system (CNLSS) as

$$\begin{aligned}
&\partial_t\Psi=i\langle\nabla\rangle\Psi-i\varepsilon\langle\nabla\rangle^{-1}|\Phi|^2, \quad (x,t)\in\Omega\times(0,\infty),\\
&\partial_t\Phi=i\Delta\Phi+\frac{1}{2}i\varepsilon(\Psi+\overline{\Psi})\Phi, \quad (x,t)\in\Omega\times(0,\infty),\\
&(\Psi,\partial_x\Psi,\Phi,\partial_x\Phi)(a,t)=(\Psi,\partial_x\Psi,\Phi,\partial_x\Phi)(b,t), \quad t\in[0,\infty),\\
&(\Psi,\Phi)(x,0)=(\Psi^0,\Phi^0)(x), \quad \Psi^0(x)=u^0(x)-i\langle\nabla\rangle^{-1}\dot{u}^0(x), \quad x\in\bar{\Omega}.
\end{aligned} \tag{2.9}$$

From (2.8), u and its derivative $\partial_t u$ of the KGSE (2.1)-(2.4) could be expressed by Ψ as

$$u=\mathrm{Re}\Psi=\frac{1}{2}\left(\Psi+\overline{\Psi}\right), \quad \partial_t u=-\langle\nabla\rangle\mathrm{Im}\Psi=\frac{i}{2}\langle\nabla\rangle\left(\Psi-\overline{\Psi}\right). \tag{2.10}$$

2.2. Semi-discretization by using the exponential wave integrator. Denoting

$$F(\Phi)=-i\langle\nabla\rangle^{-1}|\Phi|^2, \quad G(\Psi,\Phi)=\frac{i}{2}(\Psi+\overline{\Psi})\Phi, \tag{2.11}$$

we can express the CNLSS (2.9) as

$$\begin{aligned}
&\partial_t\Psi=i\langle\nabla\rangle\Psi+\varepsilon F(\Phi), \quad (x,t)\in\Omega\times(0,\infty),\\
&\partial_t\Phi=i\Delta\Phi+\varepsilon G(\Psi,\Phi), \quad (x,t)\in\Omega\times(0,\infty).
\end{aligned} \tag{2.12}$$

Let $\tau>0$ be the time step and define the time nodes as $t_n=n\tau$ for $n\geq 0$. Using Duhamel's principle as in [23, 26], we obtain

$$\begin{aligned}
&\Psi(t_{n+1})=e^{i\tau\langle\nabla\rangle}\Psi(t_n)+\varepsilon\int_0^\tau e^{i\langle\nabla\rangle(\tau-z)}F(\Phi(t_n+z))dz,\\
&\Phi(t_{n+1})=e^{i\tau\Delta}\Phi(t_n)+\varepsilon\int_0^\tau e^{i\Delta(\tau-z)}G\big(\Psi(t_n+z),\Phi(t_n+z)\big)dz.
\end{aligned} \tag{2.13}$$

Denote by $(\Psi^{[n]},\Phi^{[n]}):=(\Psi^{[n]},\Phi^{[n]})(x)$ the approximation of $(\Psi,\Phi)(x,t_n)$. Applying the trapezoidal formula to the integral terms of (2.13), we get the exponential wave integrator (EWI) for the CNLSS (2.9) as:

$$\begin{aligned}\Psi^{[n+1]}&=\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]}+\frac{\tau}{2}\varepsilon\left(\mathrm{e}^{i\tau\langle\nabla\rangle}F(\Phi^{[n]})+F(\Phi^{[n+1]})\right),\\ \Phi^{[n+1]}&=\mathrm{e}^{i\tau\Delta}\Phi^{[n]}+\frac{\tau}{2}\varepsilon\left(\mathrm{e}^{i\tau\Delta}G(\Psi^{[n]},\Phi^{[n]})+G(\Psi^{[n+1]},\Phi^{[n+1]})\right),\end{aligned}\tag{2.14}$$

with $\Psi^{[0]}=\Psi^0=u^0-i\langle\nabla\rangle^{-1}\dot{u}^0$ and $\Phi^{[0]}=\Phi^0$. From (2.10), we obtain $u^{[n]}$ and $\dot{u}^{[n]}$ for the KGSE (2.1)-(2.4) as

$$u^{[n]}=\mathrm{Re}\Psi^{[n]}=\frac{1}{2}\left(\Psi^{[n]}+\overline{\Psi^{[n]}}\right),\quad \dot{u}^{[n]}=-\langle\nabla\rangle\mathrm{Im}\Psi^{[n]}=\frac{i}{2}\langle\nabla\rangle\left(\Psi^{[n]}-\overline{\Psi^{[n]}}\right),\tag{2.15}$$

where $(u^{[n]},\dot{u}^{[n]}):=(u^{[n]},\dot{u}^{[n]})(x)$ are the approximations of $(u,\partial_t u)(x,t_n)$ $(n=0,1,2,\cdots)$.

PROPOSITION 2.1. *The EWI method* (2.14) *with* (2.15) *is equivalent to the following one*

$$\begin{aligned}u^{[n+1]}&=\cos(\tau\langle\nabla\rangle)u^{[n]}+\langle\nabla\rangle^{-1}\sin(\tau\langle\nabla\rangle)\dot{u}^{[n]}+\frac{1}{2}\varepsilon\tau\langle\nabla\rangle^{-1}\sin(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2,\\ \dot{u}^{[n+1]}&=-\langle\nabla\rangle\sin(\tau\langle\nabla\rangle)u^{[n]}+\cos(\tau\langle\nabla\rangle)\dot{u}^{[n]}+\frac{1}{2}\varepsilon\tau\left(\cos(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right),\\ \Phi^{[n+1]}&=\mathrm{e}^{i\tau\Delta}\Phi^{[n]}+\frac{i}{2}\varepsilon\tau\left(\mathrm{e}^{i\tau\Delta}u^{[n]}\Phi^{[n]}+u^{[n+1]}\Phi^{[n+1]}\right).\end{aligned}\tag{2.16}$$

Proof. Plugging F and G of (2.11) into (2.14), we have

$$\Psi^{[n+1]}=\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]}-\frac{1}{2}i\varepsilon\tau\langle\nabla\rangle^{-1}\left(\mathrm{e}^{i\tau\langle\nabla\rangle}|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right),\tag{2.17}$$

$$\Phi^{[n+1]}=\mathrm{e}^{i\tau\Delta}\Phi^{[n]}+\frac{1}{2}i\varepsilon\tau\left(\mathrm{e}^{i\tau\Delta}(\mathrm{Re}\Psi^{[n]})\Phi^{[n]}+(\mathrm{Re}\Psi^{[n+1]})\Phi^{[n+1]}\right).\tag{2.18}$$

Taking the real part and imaginary part of (2.17), respectively, we obtain

$$\begin{aligned}\mathrm{Re}\Psi^{[n+1]}&=\mathrm{Re}(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]})-\frac{1}{2}\varepsilon\tau\langle\nabla\rangle^{-1}\mathrm{Re}\left(i\mathrm{e}^{i\langle\nabla\rangle\tau}|\Phi^{[n]}|^2\right),\\ -\langle\nabla\rangle\mathrm{Im}\Psi^{[n+1]}&=-\langle\nabla\rangle\mathrm{Im}(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]})+\frac{1}{2}\varepsilon\tau\mathrm{Im}\left(i\left(\mathrm{e}^{i\langle\nabla\rangle\tau}|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right)\right).\end{aligned}\tag{2.19}$$

Using

$$\mathrm{e}^{ix}=\cos(x)+i\sin(x),\quad \mathrm{Re}(ix)=-\mathrm{Im}(x),\quad \mathrm{Im}(ix)=\mathrm{Re}(x),$$

we have

$$\begin{aligned}&\mathrm{Re}(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]})=\cos(\tau\langle\nabla\rangle)\mathrm{Re}\Psi^{[n]}-\sin(\tau\langle\nabla\rangle)\mathrm{Im}\Psi^{[n]},\\ &\mathrm{Im}(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]})=\sin(\tau\langle\nabla\rangle)\mathrm{Re}\Psi^{[n]}+\cos(\tau\langle\nabla\rangle)\mathrm{Im}\Psi^{[n]},\\ &\mathrm{Re}\left(i\mathrm{e}^{i\tau\langle\nabla\rangle}|\Phi^{[n]}|^2\right)=-\sin(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2,\\ &\mathrm{Im}\left(i\left(\mathrm{e}^{i\tau\langle\nabla\rangle}|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right)\right)=\cos(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2.\end{aligned}\tag{2.20}$$

Plugging (2.20) into (2.19), we obtain

$$\begin{aligned}\mathrm{Re}\Psi^{[n+1]}&=\cos(\tau\langle\nabla\rangle)\mathrm{Re}\Psi^{[n]}-\sin(\tau\langle\nabla\rangle)\mathrm{Im}\Psi^{[n]}+\frac{1}{2}\varepsilon\tau\langle\nabla\rangle^{-1}\sin(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2,\\-\langle\nabla\rangle\mathrm{Im}\Psi^{[n+1]}&=-\langle\nabla\rangle\left(\sin(\tau\langle\nabla\rangle)\mathrm{Re}\Psi^{[n]}+\cos(\tau\langle\nabla\rangle)\mathrm{Im}\Psi^{[n]}\right)\\&\quad+\frac{1}{2}\varepsilon\tau\left(\cos(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right).\end{aligned}\tag{2.21}$$

Plugging the relation (2.15) into (2.18) and (2.21), after a simple calculation, we obtain the method (2.16). ∎

Remark 2.1. In (2.14), we exchange $(n+1,\tau)$ with $(n,-\tau)$, and know that the resulting method is the same as (2.14). So the time semi-discretization method (2.14) is time symmetric. From the equivalence of (2.14) and (2.16), we obtain that (2.16) is time symmetric.

Remark 2.2. In the actual calculation, we can apply the method (2.16) in an explicit way as

$$\begin{aligned}u^{[n+1]}&=\cos(\tau\langle\nabla\rangle)u^{[n]}+\langle\nabla\rangle^{-1}\sin(\tau\langle\nabla\rangle)\dot u^{[n]}+\frac{1}{2}\varepsilon\tau\langle\nabla\rangle^{-1}\sin(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2,\\\Phi^{[n+1]}&=\left(\mathrm{e}^{i\tau\Delta}\Phi^{[n]}+\frac{i}{2}\varepsilon\tau\mathrm{e}^{i\tau\Delta}u^{[n]}\Phi^{[n]}\right)\Big/\left(1-\frac{i\varepsilon\tau}{2}u^{[n+1]}\right),\\\dot u^{[n+1]}&=-\langle\nabla\rangle\sin(\tau\langle\nabla\rangle)u^{[n]}+\cos(\tau\langle\nabla\rangle)\dot u^{[n]}\\&\quad+\frac{1}{2}\varepsilon\tau\left(\cos(\tau\langle\nabla\rangle)|\Phi^{[n]}|^2+|\Phi^{[n+1]}|^2\right).\end{aligned}\tag{2.22}$$

Here the denominator $1-\frac{i\varepsilon\tau}{2}u^{[n+1]}$ is not equal to zero because $u^{[n+1]}$ is real. So the expression (2.22) always makes sense.

Remark 2.3. Actually (2.22) (*or* (2.16)) is exactly the same as the Deuflhard type exponential integrator when applied to the Duhamel formula for the original variables u and Φ in the KGSE (2.1)-(2.4) directly. Of course, with a similar analysis to [4], we can get the uniform error bounds at $O(\tau^2)$ up to the time at $O(1/\varepsilon)$. However, in order to get the improved error bounds, we need to do a more detailed analysis of the error accumulation process. It is more convenient to consider (2.14) than to consider (2.22) (*or* (2.16)) because of the properties of operators $\mathrm{e}^{it\langle\nabla\rangle}$ and $\mathrm{e}^{it\Delta}$. So in the following we carry out our error analysis based on (2.14) rather than (2.22) (*or* (2.16)).

2.3. Full-discretization by the Fourier pseudo-spectral method. For a positive even integer M, introduce $\Omega_M^0=\{0,1,\cdots,M\}$ and $\Omega_M=\{-\frac{M}{2},\cdots,\frac{M}{2}-1\}$. Choose $h=(b-a)/M$ as the mesh size and denote grid points as $x_j:=a+jh$ for $j\in\Omega_M^0$. Introduce the two spaces as

$$Y_M:=\mathrm{span}\left\{\mathrm{e}^{i\mu_l(x-a)}\,\middle|\,\mu_l=\frac{2\pi l}{b-a},\quad l\in\Omega_M\right\},$$
$$X_M:=\mathrm{span}\{v=(v_0,\cdots,v_M)|\ v_0=v_M\}\in C^{M+1}.$$

For any $v(x)$ on $[a,b]$ with $v(a)=v(b)$ and vector $v\in X_M$, define $P_M:L^2(\Omega)\to Y_M$ as the standard projection operator, $I_M:C(\Omega)\to Y_M$ and $I_M:X_M\to Y_M$ as the trigonometric interpolation operators [38], i.e.

$$(P_Mv)(x)=\sum_{l\in\Omega_M}\widehat{v}_l\mathrm{e}^{i\mu_l(x-a)},\quad(I_Mv)(x)=\sum_{l\in\Omega_M}\widetilde{v}_l\mathrm{e}^{i\mu_l(x-a)},\tag{2.23}$$

with the coefficients

$$\widehat{v}_l = \frac{1}{b-a}\int_a^b v(x)\mathrm{e}^{-i\mu_l(x-a)}\mathrm{d}x, \quad \widetilde{v}_l = \frac{1}{M}\sum_{j=0}^{M-1} v_j \mathrm{e}^{-i\mu_l(x_j-a)}, \tag{2.24}$$

respectively, where $v_j = v(x_j)$ for the functions $v(x)$.

Let (Ψ_j^n, Φ_j^n) $(n\geq 0, j\in\Omega_M^0)$ be the approximations to $(\Psi,\Phi)(x_j,t_n)$. Then an exponential wave integrator Fourier pseudo-spectral (EWIFP) method for the CNLSS (2.9) is

$$\begin{aligned}
&\widetilde{(\Psi^{n+1})}_l = \mathrm{e}^{i\tau\beta_l}\widetilde{(\Psi^n)}_l - \frac{i\varepsilon\tau}{2\beta_l}\left(\mathrm{e}^{i\tau\beta_l}\widetilde{f(\Phi^n)}_l + \widetilde{f(\Phi^{n+1})}_l\right),\\
&\widetilde{(\Phi^{n+1})}_l = \mathrm{e}^{-i\tau\mu_l^2}\widetilde{(\Phi^n)}_l + \frac{i\varepsilon\tau}{2}\left(\mathrm{e}^{-i\tau\mu_l^2}\widetilde{g(\Psi^n,\Phi^n)}_l + \widetilde{g(\Psi^{n+1},\Phi^{n+1})}_l\right),\\
&\Psi_j^{n+1} = \sum_{l\in\Omega_M}\widetilde{(\Psi^{n+1})}_l \mathrm{e}^{2ijl\pi/M}, \quad \Phi_j^{n+1} = \sum_{l\in\Omega_M}\widetilde{(\Phi^{n+1})}_l \mathrm{e}^{2ijl\pi/M},
\end{aligned} \tag{2.25}$$

where $\beta_l = \sqrt{1+\mu_l^2}$ for $l\in\Omega_M$ and

$$\begin{aligned}
&f(\Phi^n)_j = |\Phi_j^n|^2, \quad g(\Psi^n,\Phi^n)_j = \frac{1}{2}(\Psi_j^n + \overline{\Psi}_j^n)\Phi_j^n,\\
&\Psi_j^0 = u^0(x_j) - i\sum_{l\in\Omega_M}\frac{\widetilde{(\dot{u}^0)}_l}{\sqrt{1+\mu_l^2}}\mathrm{e}^{2ijl\pi/M}, \quad \Phi_j^0 = \Phi^0(x_j), \quad j\in\Omega_M^0.
\end{aligned} \tag{2.26}$$

Let $(u_j^n, \dot{u}_j^n)$ $(n\geq 0, j\in\Omega_M^0)$ be the numerical solution to $(u,\partial_t u)(x_j,t_n)$. Then from (2.25) and (2.10), we can obtain u_j^{n+1} and $\dot{u}_j^{n+1}$ for the KGSE (2.1)-(2.4)

$$\begin{aligned}
&u_j^{n+1} = \mathrm{Re}\Psi_j^{n+1} = \frac{1}{2}\left(\Psi_j^{n+1} + \overline{\Psi_j^{n+1}}\right),\\
&\dot{u}_j^{n+1} = -\sum_{l\in\Omega_M}\beta_l\widetilde{(\mathrm{Im}\Psi^{n+1})}_l \mathrm{e}^{2ijl\pi/M} = \frac{i}{2}\sum_{l\in\Omega_M}\beta_l\left(\widetilde{(\Psi^{n+1})}_l - \widetilde{(\overline{\Psi^{n+1}})}_l\right)\mathrm{e}^{2ijl\pi/M}.
\end{aligned} \tag{2.27}$$

The EWIFP method (2.25) with (2.27) is equivalent to the following one.

PROPOSITION 2.2. *Let $(u_j^n, \dot{u}_j^n, \Phi_j^n)$ $(n\geq 0, j\in\Omega_M^0)$ be the approximations to $(u,\partial_t u,\Phi)(x_j,t_n)$. Choose $(u_j^0,\dot{u}_j^0,\Phi_j^0) = (u^0,\dot{u}^0,\Phi^0)(x_j)$, then the EWIFP method solving the KGSE (2.1)-(2.4) is*

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = \cos(\tau\beta_l)\widetilde{(u^n)}_l + \beta_l^{-1}\sin(\tau\beta_l)\widetilde{(\dot{u}^n)}_l + \frac{\tau}{2}\varepsilon\beta_l^{-1}\sin(\tau\beta_l)\widetilde{f(\Phi^n)}_l,\\
&\widetilde{(\dot{u}^{n+1})}_l = -\beta_l\sin(\tau\beta_l)\widetilde{(u^n)}_l + \cos(\tau\beta_l)\widetilde{(\dot{u}^n)}_l + \frac{\tau}{2}\varepsilon\left(\cos(\tau\beta_l)\widetilde{f(\Phi^n)}_l + \widetilde{f(\Phi^{n+1})}_l\right),\\
&\widetilde{(\Phi^{n+1})}_l = \mathrm{e}^{-i\tau\mu_l^2}\widetilde{(\Phi^n)}_l + \frac{i\tau}{2}\varepsilon\left(\mathrm{e}^{-i\tau\mu_l^2}\widetilde{gg(u^n,\Phi^n)}_l + \widetilde{gg(u^{n+1},\Phi^{n+1})}_l\right),\\
&u_j^{n+1} = \sum_{l\in\Omega_M}\widetilde{(u^{n+1})}_l \mathrm{e}^{2ijl\pi/M}, \quad \dot{u}_j^{n+1} = \sum_{l\in\Omega_M}\widetilde{(\dot{u}^{n+1})}_l \mathrm{e}^{2ijl\pi/M},\\
&\Phi_j^{n+1} = \sum_{l\in\Omega_M}\widetilde{(\Phi^{n+1})}_l \mathrm{e}^{2ijl\pi/M},
\end{aligned} \tag{2.28}$$

where $\beta_l = \sqrt{1+\mu_l^2}$ for $l\in\Omega_M$, $f(\Phi^n)_j = |\Phi_j^n|^2$ and $gg(u^n,\Phi^n)_j = u_j^n\Phi_j^n$.

REMARK 2.4. In the actual calculation, we can apply the method (2.28) in an explicit way as

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l=\cos(\tau\beta_l)\widetilde{(u^n)}_l+\beta_l^{-1}\sin(\tau\beta_l)\widetilde{(\dot u^n)}_l+\frac{\tau}{2}\varepsilon\beta_l^{-1}\sin(\tau\beta_l)\widetilde{f(\Phi^n)}_l,\\
&u_j^{n+1}=\sum_{l\in\Omega_M}\widetilde{(u^{n+1})}_l e^{2ijl\pi/M},\quad \widetilde{(\Phi^{n+1,*})}_l=e^{-i\tau\mu_l^2}\widetilde{(\Phi^n)}_l+\frac{i\tau}{2}\varepsilon e^{-i\tau\mu_l^2}\widetilde{gg(u^n,\Phi^n)}_l,\\
&\Phi_j^{n+1,*}=\sum_{l\in\Omega_M}\widetilde{(\Phi^{n+1,*})}_l e^{2ijl\pi/M},\quad \Phi_j^{n+1}=\Phi_j^{n+1,*}\Big/\left(1-\frac{i\varepsilon\tau}{2}u_j^{n+1}\right),\\
&\widetilde{(\dot u^{n+1})}_l=-\beta_l\sin(\tau\beta_l)\widetilde{(u^n)}_l+\cos(\tau\beta_l)\widetilde{(\dot u^n)}_l+\frac{\tau}{2}\varepsilon\left(\cos(\tau\beta_l)\widetilde{f(\Phi^n)}_l+\widetilde{f(\Phi^{n+1})}_l\right),\\
&\dot u_j^{n+1}=\sum_{l\in\Omega_M}\widetilde{(\dot u^{n+1})}_l e^{2ijl\pi/M},\quad f(\Phi^n)_j=|\Phi_j^n|^2,\quad gg(u^n,\Phi^n)_j=u_j^n\Phi_j^n.
\end{aligned}\tag{2.29}$$

From Remark 2.4 and Proposition 2.2 we know that the EWIFP method (2.25) with (2.27) is essentially explicit and efficient thanks to the fast Fourier transform (FFT).

3. Error estimates

Next we focus on the improved error bounds of the EWIFP method (2.25)-(2.27) for solving the KGSE (2.1)-(2.4).

3.1. Main results.

For the KGSE (2.1)-(2.4), we assume:

$$\begin{aligned}
&u\in L^\infty\left([0,T_0/\varepsilon];H_p^{m+2}(\Omega)\right),\quad \partial_t u,\Phi\in L^\infty\left([0,T_0/\varepsilon];H_p^{m+1}(\Omega)\right),\\
&\|u\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+2}(\Omega))}\lesssim 1,\quad \|\partial_t u\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+1}(\Omega))}\lesssim 1,\\
&\|\Phi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+1}(\Omega))}\lesssim 1
\end{aligned}\tag{A}$$

where $m\geq 4$. Then the following conclusion is true.

THEOREM 3.1. *Under the assumption (A), for $\alpha\in(0,1)$, there exist sufficiently small constants $h_0>0$ and $0<\tau_0<1$ which are independent of ε, when $0<h\leq h_0$ and*

$$0<\tau\leq 2\alpha\pi\Big/\left(\sqrt{1+\frac{4\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}}+\frac{8\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}\right),$$

we have:

$$\begin{aligned}
&\|u(\cdot,t_n)-I_M u^n\|_2+\|\partial_t u(\cdot,t_n)-I_M\dot u^n\|_1\\
&\qquad+\|\Phi(\cdot,t_n)-I_M\Phi^n\|_1\lesssim h^m+\tau_0^m+\varepsilon\tau^2.
\end{aligned}\tag{3.1}$$

In particular, if $u,\partial_t u,\Phi$ are sufficiently smooth, the last term τ_0^m could be ignored practically for small enough τ_0, where the improved uniform error bounds for sufficiently small τ could be

$$\|u(\cdot,t_n)-I_M u^n\|_2+\|\partial_t u(\cdot,t_n)-I_M\dot u^n\|_1+\|\Phi(\cdot,t_n)-I_M\Phi^n\|_1\lesssim h^m+\varepsilon\tau^2.$$

REMARK 3.1. In Theorem 3.1 and the other results in this paper for the one-dimensional problem, we prove the error bounds for $u(x,t)$ in H^2-norm and for $(\partial_t u(x,t),\Psi(x,t))$ in H^1-norm due to the fact that H^m is an algebra for $m>1/2$. Similarly, for d-dimensional problems ($d=2,3$), H^m is an algebra for $m>d/2$ and the corresponding estimates (3.1) can be obtained for $u(x,t)$ in H^{m^*+1}-norm and for

$(\partial_t u(x,t), \Psi(x,t))$ in H^{m^*}-norm with $m^* = \frac{d}{2} + \delta$, $\delta > 0$ for the two and three dimensional cases if given the required regularity condition. The numerical results in Subsection 4.2 will confirm this conclusion.

Next we will prove Theorem 3.1. Note that our proof is different from that in [2,3,6]. In those papers, the authors first derived the error bounds of semi-discretization, and then obtained the error bounds of full discretization by comparing them. In this process, the bounds $\|\Psi^{[n]}\|_{m+2}$ and $\|\Phi^{[n]}\|_{m+1}$ were required. Here we provide a direct proof without giving the errors of semi-discretization. In addition, for the control of nonlinear terms, we use cut-off technique instead of mathematical induction as in the papers [2,3,6].

3.2. Preliminary estimates. Define two operators

$$F_t : \Phi \to \mathrm{e}^{-it\langle\nabla\rangle} F(\mathrm{e}^{it\Delta}\Phi), \quad G_t : (\Psi,\Phi) \to \mathrm{e}^{-it\Delta} G(\mathrm{e}^{it\langle\nabla\rangle}\Psi, \mathrm{e}^{it\Delta}\Phi), \quad t\in\mathbb{R}, \tag{3.2}$$

where F and G are given as in (2.11). Then we have the properties for F_t and G_t [31].

PROPOSITION 3.1.
(i) *For $s > 1/2$ and any $t \in \mathbb{R}$, we have*

$$\begin{aligned}
&\|F_t(\Phi)\|_{s+1} \le C\|\Phi\|_s^2, \quad \|G_t(\Psi,\Phi)\|_s \le C\|\Psi\|_s\|\Phi\|_s,\\
&\|F_t'(\Phi)(\gamma)\|_{s+1} \le C\|\Phi\|_s\|\gamma\|_s, \quad \|\partial_\Psi G_t(\Psi,\Phi)(\gamma)\|_s \le C\|\Phi\|_s\|\gamma\|_s,\\
&\|\partial_\Phi G_t(\Psi,\Phi)(\gamma)\|_s \le C\|\Psi\|_s\|\gamma\|_s,\\
&\|F_t''(\Phi)(\gamma,\delta)\|_{s+1} \le C\|\delta\|_s\|\gamma\|_s, \quad \|\partial_{\Psi\Phi} G_t(\Psi,\Phi)(\gamma,\delta)\|_s \le C\|\delta\|_s\|\gamma\|_s,\\
&\|\partial_{\Phi\Psi} G_t(\Psi,\Phi)(\gamma,\delta)\|_s \le C\|\delta\|_s\|\gamma\|_s, \quad \|\partial_{\Psi\Psi} G_t(\Psi,\Phi)(\gamma,\delta)\|_s = 0,\\
&\|\partial_{\Phi\Phi} G_t(\Psi,\Phi)(\gamma,\delta)\|_s = 0.
\end{aligned} \tag{3.3}$$

(ii) *For $s \ge 1$ and any $t \in \mathbb{R}$, we have*

$$\begin{aligned}
&\|\partial_t F_t(\Phi)\|_s \le C\|\Phi\|_{s+1}^2, \quad \|\partial_t G_t(\Psi,\Phi)\|_{s-1} \le C\|\Psi\|_{s+1}\|\Phi\|_{s+1},\\
&\|\partial_{tt} F_t(\Phi)\|_s \le C\|\Phi\|_{s+3}^2, \quad \|\partial_{tt} G_t(\Psi,\Phi)\|_{s-1} \le C\|\Psi\|_{s+3}\|\Phi\|_{s+3},\\
&\|\partial_t F_t'(\Phi)(\gamma)\|_s \le C\|\Phi\|_{s+1}\|\gamma\|_{s+1}, \quad \|\partial_t\partial_\Psi G_t(\Psi,\Phi)(\gamma)\|_{s-1} \le C\|\Phi\|_{s+1}\|\gamma\|_{s+1},\\
&\|\partial_t\partial_\Phi G_t(\Psi,\Phi)(\gamma)\|_{s-1} \le C\|\Psi\|_{s+1}\|\gamma\|_{s+1}.
\end{aligned} \tag{3.4}$$

3.3. Proof of Theorem 3.1. From the regularity condition (A), we know that

$$\begin{aligned}
&\Psi \in L^\infty\Big([0,T_0/\varepsilon]; H_p^{m+2}(\Omega)\Big), \quad \Phi \in L^\infty\Big([0,T_0/\varepsilon]; H_p^{m+1}(\Omega)\Big).\\
&\|\Psi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+2}(\Omega))} \lesssim 1, \quad \|\Phi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+1}(\Omega))} \lesssim 1
\end{aligned} \tag{A'}$$

We will prove the following error bounds for the EWIFP (2.25).

THEOREM 3.2. *Under the condition (A'), for $\alpha \in (0,1)$, there exist sufficiently small constants $h_0 > 0$ and $0 < \tau_0 < 1$ which are independent of ε, when $0 < h \le h_0$ and*

$$0 < \tau \le 2\alpha\pi \Bigg/ \left(\sqrt{1 + \frac{4\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}} + \frac{8\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}\right),$$

the global errors of the EWIFP (2.25) *satisfy*

$$\begin{aligned}&\|\Psi(\cdot,t_n)-I_M\Psi^n\|_2\lesssim\tau^2+\tau_0^m+h^m,\quad \|I_M\Psi^n\|_2\le 1+M_\Psi,\\&\|\Phi(\cdot,t_n)-I_M\Phi^n\|_1\lesssim\tau^2+\tau_0^m+h^m,\quad \|I_M\Phi^n\|_1\le 1+M_\Phi,\quad 0\le n\le T_0\varepsilon^{-1}/\tau,\end{aligned}\tag{3.5}$$

where $M_\Psi=\|\Psi\|_{L^\infty([0,T_0/\varepsilon];H_p^2(\Omega))}$ *and* $M_\Phi=\|\Phi\|_{L^\infty([0,T_0/\varepsilon];H_p^1(\Omega))}$.

The key of proving Theorem 3.2 is to show $\|I_M\Phi^n\|_1\lesssim 1$. To do this, we adapt the cut-off technique [1,30]. Firstly we choose $\rho(\theta)\in C_0^\infty(\mathbb{R}^+)$ which satisfies

$$\rho(\theta)=\begin{cases}1, & \theta\in[0,1],\\ \in[0,1], & \theta\in[1,2],\\ 0, & \theta\in[2,+\infty).\end{cases}\tag{3.6}$$

Denote $B=1+M_\Phi$ and define

$$T_B(\Phi)=\rho\left(\frac{\|\Phi\|_1}{B}\right)\Phi.\tag{3.7}$$

Let $(\breve{\Psi}_j^n,\breve{\Phi}_j^n)$ $(n\ge 0, j\in\Omega_M^0)$ be another approximation to $(\Psi,\Phi)(x_j,t_n)$. Choose $(\breve{\Psi}_j^0,\breve{\Phi}_j^0)=(\Psi_j^0,\Phi_j^0)$, then the modified EWIFP (MEWIFP) method solving the CNLSS (2.9) is constructed as

$$\begin{aligned}&\widetilde{(\breve{\Psi}^{n+1})}_l=e^{i\tau\beta_l}\widetilde{(\breve{\Psi}^n)}_l-\frac{i\varepsilon\tau}{2\beta_l}\left(e^{i\tau\beta_l}\widetilde{f(T_B(I_M\breve{\Phi}^n))}_l+\widetilde{f(T_B(I_M\breve{\Phi}^{n+1}))}_l\right),\\&\widetilde{(\breve{\Phi}^{n+1})}_l=e^{-i\tau\mu_l^2}\widetilde{(\breve{\Phi}^n)}_l\\&\qquad+\frac{i\varepsilon\tau}{2}\left(e^{-i\tau\mu_l^2}\widetilde{g(I_M\breve{\Psi}^n,T_B(I_M\breve{\Phi}^n))}_l+\widetilde{g(I_M\breve{\Psi}^{n+1},T_B(I_M\breve{\Phi}^{n+1}))}_l\right),\\&\breve{\Psi}_j^{n+1}=\sum_{l\in\Omega_M}\widetilde{(\breve{\Psi}^{n+1})}_l e^{2ijl\pi/M},\quad \breve{\Phi}_j^{n+1}=\sum_{l\in\Omega_M}\widetilde{(\breve{\Phi}^{n+1})}_l e^{2ijl\pi/M},\end{aligned}\tag{3.8}$$

where f and g are defined in (2.26). From the MEWIFP method (3.8), we obtain

$$\begin{aligned}I_M\breve{\Psi}^{n+1}=&e^{i\tau\langle\nabla\rangle}I_M\breve{\Psi}^n+\frac{1}{2}\varepsilon\tau I_M\left(e^{i\tau\langle\nabla\rangle}F(T_B(I_M\breve{\Phi}^n))+F(T_B(I_M\breve{\Phi}^{n+1}))\right),\\I_M\breve{\Phi}^{n+1}=&e^{i\tau\Delta}I_M\breve{\Phi}^n\\&+\frac{1}{2}\varepsilon\tau I_M\left(e^{i\tau\Delta}G(I_M\breve{\Psi}^n,T_B(I_M\breve{\Phi}^n))+G(I_M\breve{\Psi}^{n+1},T_B(I_M\breve{\Phi}^{n+1}))\right),\end{aligned}\tag{3.9}$$

where F and G are defined in (2.11). Next we are going to start with the error analysis for MEWIFP method (3.8). Firstly, we introduce the error functions

$$e_\Psi^n:=\Psi(\cdot,t_n)-I_M\breve{\Psi}^n,\quad e_\Phi^n:=\Phi(\cdot,t_n)-I_M\breve{\Phi}^n,\quad n=0,1,\cdots,\frac{T_0/\varepsilon}{\tau}.\tag{3.10}$$

Then we can prove the following result.

THEOREM 3.3. *Under the condition (A'), for* $\alpha\in(0,1)$, *there exist sufficiently small constants* $h_0>0$ *and* $0<\tau_0<1$ *which are independent of* ε, *when* $0<h\le h_0$ *and*

$$0<\tau\le 2\alpha\pi\Big/\left(\sqrt{1+\frac{4\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}}+\frac{8\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}\right),\tag{3.11}$$

the global errors of the MEWIFP (3.8) *satisfy*

$$\begin{aligned}&\left\|\Psi(\cdot,t_n)-I_M\breve{\Psi}^n\right\|_2\lesssim\tau^2+\tau_0^m+h^m,\quad \left\|I_M\breve{\Psi}^n\right\|_2\le 1+M_\Psi,\\&\left\|\Phi(\cdot,t_n)-I_M\breve{\Phi}^n\right\|_1\lesssim\tau^2+\tau_0^m+h^m,\quad \left\|I_M\breve{\Phi}^n\right\|_1\le 1+M_\Phi,\quad 0\le n\le T_0\varepsilon^{-1}/\tau,\end{aligned}\tag{3.12}$$

where $M_\Psi=\|\Psi\|_{L^\infty([0,T_0/\varepsilon];H_p^2(\Omega))}$ *and* $M_\Phi=\|\Phi\|_{L^\infty([0,T_0/\varepsilon];H_p^1(\Omega))}$.

In the following, we use $\Psi(t_n)$ and $\Phi(t_n)$ to refer to $\Psi(\cdot,t_n)$ and $\Phi(\cdot,t_n)$, respectively, for convenience. For the function $T_B(\Phi)$ of (3.7), we have the following lemma.

LEMMA 3.1. *The following inequality is true*

$$\|T_B(\Phi_1)-T_B(\Phi_2)\|_1\le C_B\|\Phi_1-\Phi_2\|_1,\tag{3.13}$$

where $C_B=1+2\max_{\theta\in[0,2]}|\rho'(\theta)|$.

Proof. If $\|\Phi_1\|_1\ge 2B$ and $\|\Phi_2\|_1\ge 2B$, we immediately get (3.13). For $\|\Phi_1\|_1<2B$ and $\|\Phi_2\|_1\ge 2B$, we obtain

$$\begin{aligned}&\|T_B(\Phi_1)-T_B(\Phi_2)\|_1\\&=\left\|\rho\left(\frac{\|\Phi_1\|_1}{B}\right)\Phi_1-\rho\left(\frac{\|\Phi_2\|_1}{B}\right)\Phi_2\right\|_1=\left\|\left(\rho\left(\frac{\|\Phi_1\|_1}{B}\right)-\rho\left(\frac{\|\Phi_2\|_1}{B}\right)\right)\Phi_1\right\|_1\\&=\left\|\rho'(\theta)\left(\frac{\|\Phi_1\|_1}{B}-\frac{\|\Phi_2\|_1}{B}\right)\Phi_1\right\|_1\le 2\max_{\theta\in[0,2]}|\rho'(\theta)|\,\|\Phi_2-\Phi_1\|_1.\end{aligned}\tag{3.14}$$

Similarly, for $\|\Phi_1\|_1\ge 2B$ and $\|\Phi_2\|_1<2B$, the result also holds. For $\|\Phi_1\|_1<2B$ and $\|\Phi_2\|_1<2B$, we have

$$\begin{aligned}&\|T_B(\Phi_1)-T_B(\Phi_2)\|_1=\left\|\rho\left(\frac{\|\Phi_1\|_1}{B}\right)\Phi_1-\rho\left(\frac{\|\Phi_2\|_1}{B}\right)\Phi_2\right\|_1\\&\le\left\|\rho\left(\frac{\|\Phi_1\|_1}{B}\right)(\Phi_1-\Phi_2)\right\|_1+\left\|\left(\rho\left(\frac{\|\Phi_1\|_1}{B}\right)-\rho\left(\frac{\|\Phi_2\|_1}{B}\right)\right)\Phi_2\right\|_1\\&\le\|\Phi_1-\Phi_2\|_1+\left\|\rho'(\theta)\left(\frac{\|\Phi_1\|_1}{B}-\frac{\|\Phi_2\|_1}{B}\right)\Phi_2\right\|_1\\&\le\left(1+2\max_{\theta\in[0,2]}|\rho'(\theta)|\right)\|\Phi_2-\Phi_1\|_1.\end{aligned}\tag{3.15}$$

Lemma 3.1 has been proved. ∎

LEMMA 3.2. *Under the assumptions (A'), we have*

$$\begin{aligned}&\left\|F(\Phi(t_n))-F(T_B(I_M\breve{\Phi}^n))\right\|_2\le C_F\|e_\Phi^n\|_1,\\&\left\|G(\Psi(t_n),\Phi(t_n))-G(I_M\breve{\Psi}^n,T_B(I_M\breve{\Phi}^n))\right\|_1\le C_G\left(\|e_\Psi^n\|_1+\|e_\Phi^n\|_1\right),\end{aligned}\tag{3.16}$$

where $C_F=C_B\left(2B+M_\Phi\right)$ *and* $C_G=2B+M_\Psi C_B$ *with* $C_B=1+2\max_{\theta\in[0,2]}|\rho'(\theta)|$.

Proof. According to Lemma 3.1 and the properties of $T_B(\Phi)$, we have

$$\begin{aligned}
&\left\|F(\Phi(t_n))-F(T_B(I_M\breve{\Phi}^n))\right\|_2\\
=&\left\|\overline{\Phi(t_n)}\Phi(t_n)-\overline{T_B(I_M\breve{\Phi}^n)}T_B(I_M\breve{\Phi}^n)\right\|_1\\
\le&\left\|\overline{\Phi(t_n)}\left(T_B(\Phi(t_n))-T_B(I_M\breve{\Phi}^n)\right)\right\|_1+\left\|\overline{\left(T_B(\Phi(t_n))-T_B(I_M\breve{\Phi}^n)\right)}T_B(I_M\breve{\Phi}^n)\right\|_1\\
\le&\left\|\left(T_B(\Phi(t_n))-T_B(I_M\breve{\Phi}^n)\right)\right\|_1\left(\|\Phi(t_n)\|_1+\left\|T_B(I_M\breve{\Phi}^n)\right\|_1\right).
\end{aligned}\tag{3.17}$$

The assumption (A') and Lemma 3.1 in (3.17) give the first result. Similarly, we obtain

$$\begin{aligned}
&\left\|G(\Psi(t_n),\Phi(t_n))-G(I_M\breve{\Psi}^n,T_B(I_M\breve{\Phi}^n))\right\|_1\\
=&\frac{1}{2}\left\|\left(\Psi(t_n)+\overline{\Psi}(t_n)\right)\Phi(t_n)-\left(I_M\breve{\Psi}^n+I_M\overline{\breve{\Psi}^n}\right)T_B(I_M\breve{\Phi}^n)\right\|_1\\
\le&\frac{1}{2}\left\|\left(\Psi(t_n)+\overline{\Psi}(t_n)\right)\left(T_B(\Phi(t_n))-T_B(I_M\breve{\Phi}^n)\right)\right\|_1+\frac{1}{2}\left\|(e_\Psi^n+\overline{e}_\Psi^n)T_B(I_M\breve{\Phi}^n)\right\|_1.
\end{aligned}\tag{3.18}$$

Considering the assumptions (A') and using Lemma 3.1 in (3.18), we get the second result. ∎

By plugging the solution of (2.9) into (2.14), we introduce local truncation errors (LTEs) ξ_Ψ^n and ξ_Φ^n $(1\le n\le T_0\varepsilon^{-1}/\tau-1)$ as

$$\begin{aligned}
\Psi(t_{n+1})&=\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi(t_n)+\frac{\tau}{2}\varepsilon\left(\mathrm{e}^{i\tau\langle\nabla\rangle}F(\Phi(t_n))+F(\Phi(t_{n+1}))\right)+\xi_\Psi^n,\\
\Phi(t_{n+1})&=\mathrm{e}^{i\tau\Delta}\Phi(t_n)+\frac{\tau}{2}\varepsilon\left(\mathrm{e}^{i\tau\Delta}G(\Psi(t_n),\Phi(t_n))+G(\Psi(t_{n+1}),\Phi(t_{n+1}))\right)+\xi_\Phi^n.
\end{aligned}\tag{3.19}$$

For the LTEs, we have the following result.

LEMMA 3.3. *The local errors of the EWIFP method (2.25) can be written as*

$$\begin{aligned}
\xi_\Psi^n&=\mathcal{F}(\Phi(t_n))+\mathcal{R}_\Psi^n,\quad n=0,1,\cdots\\
\xi_\Phi^n&=\mathcal{G}(\Psi(t_n),\Phi(t_n))+\mathcal{R}_\Phi^n,\quad n=0,1,\cdots
\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}
\mathcal{F}(\Phi(t_n))&=\varepsilon\mathrm{e}^{i\langle\nabla\rangle\tau}\left(\int_0^\tau F_z(\Phi(t_n))\mathrm{d}z-\frac{1}{2}\tau\left(F_0(\Phi(t_n))+F_\tau(\Phi(t_n))\right)\right),\\
\mathcal{G}(\Psi(t_n),\Phi(t_n))&=\varepsilon\mathrm{e}^{i\Delta\tau}\Bigg(\int_0^\tau G_z(\Psi(t_n),\Phi(t_n))\mathrm{d}z\\
&\qquad-\frac{1}{2}\tau\left(G_0(\Psi(t_n),\Phi(t_n))+G_\tau(\Psi(t_n),\Phi(t_n))\right)\Bigg),
\end{aligned}\tag{3.21}$$

with the bounds under the assumption (A) for $m\ge4$,

$$\begin{aligned}
&\|\mathcal{F}(\Phi(t_n))\|_2\lesssim\varepsilon\tau^3,\quad\|\mathcal{R}_\Psi^n\|_2\lesssim\varepsilon^2\tau^3,\\
&\|\mathcal{G}(\Psi(t_n),\Phi(t_n))\|_1\lesssim\varepsilon\tau^3,\quad\|\mathcal{R}_\Phi^n\|_1\lesssim\varepsilon^2\tau^3.
\end{aligned}\tag{3.22}$$

Proof. Denote $\Psi^n(t)=\Psi(t_n+t)$, $\Phi^n(t)=\Phi(t_n+t)$, $\Psi_n=\Psi(t_n)$ and $\Phi_n=\Phi(t_n)$. Applying the Taylor expansion to the first expression of (2.13), we get

$$\begin{aligned}\Psi(t_{n+1})=&\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n+\varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F\left(\mathrm{e}^{i\theta\Delta}\Phi_n+\varepsilon\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G(\Psi^n(s),\Phi^n(s))\mathrm{d}s\right)\mathrm{d}\theta\\
=&\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n+\varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F\left(\mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta\\
&+\varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F'\left(\mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G(\Psi^n(s),\Phi^n(s))\mathrm{d}s\right)\mathrm{d}\theta+\varepsilon^3E_{\Psi1}\\
=&\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n+\varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F\left(\mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta\\
&+\varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F'\left(\mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G\left(\mathrm{e}^{is\langle\nabla\rangle}\Psi_n,\mathrm{e}^{is\Delta}\Phi_n\right)\mathrm{d}s\right)\mathrm{d}\theta\\
&+\varepsilon^3(E_{\Psi1}+E_{\Psi2}+E_{\Psi3}),\end{aligned}$$

where $\varepsilon^3E_{\Psi j}$ with $j=1,2,3$ are the remainders of Taylor's expansion and satisfy $\|E_{\Psi j}\|_2\lesssim\tau^3, j=1,2,3$. Using the Definition (3.2) yields

$$\begin{aligned}\Psi(t_{n+1})=&\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n+\varepsilon\mathrm{e}^{i\tau\langle\nabla\rangle}\int_0^\tau F_\theta(\Phi_n)\mathrm{d}\theta\\
&+\varepsilon^2\mathrm{e}^{i\tau\langle\nabla\rangle}\int_0^\tau\int_0^\theta F'_\theta(\Phi_n)G_s(\Psi_n,\Phi_n)\mathrm{d}s\mathrm{d}\theta+\varepsilon^3(E_{\Psi1}+E_{\Psi2}+E_{\Psi3}).\end{aligned}\tag{3.23}$$

Applying the Taylor expansion, we obtain

$$\begin{aligned}F(\Phi(t_{n+1}))=&F\left(\mathrm{e}^{i\tau\Delta}\Phi_n+\varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}G(\Psi^n(\theta),\Phi^n(\theta))\mathrm{d}\theta\right)\\
=&F(\mathrm{e}^{i\tau\Delta}\Phi_n)+\varepsilon F'(\mathrm{e}^{i\tau\Delta}\Phi_n)\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}G(\Psi^n(\theta),\Phi^n(\theta))\mathrm{d}\theta+\varepsilon^2R_{\Psi1}\\
=&F(\mathrm{e}^{i\tau\Delta}\Phi_n)+\varepsilon F'(\mathrm{e}^{i\tau\Delta}\Phi_n)\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta\\
&+\varepsilon^2(R_{\Psi1}+R_{\Psi2}+R_{\Psi3}),\end{aligned}\tag{3.24}$$

where $\varepsilon^2R_{\Psi j}$ with $j=1,2,3$ are the remainders of Taylor's expansion and satisfy $\|R_{\Psi j}\|_2\lesssim\tau^2, j=1,2,3$. Using the Definition (3.2), we obtain

$$\begin{aligned}F(\Phi(t_{n+1}))=&\mathrm{e}^{i\tau\langle\nabla\rangle}F_\tau(\Phi_n)+\varepsilon\mathrm{e}^{i\tau\langle\nabla\rangle}F'_\tau(\Phi_n)\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta\\
&+\varepsilon^2(R_{\Psi1}+R_{\Psi2}+R_{\Psi3}).\end{aligned}\tag{3.25}$$

From (3.19), (3.23) and (3.25), we obtain

$$\xi_\Psi^n:=\varepsilon\mathrm{e}^{i\tau\langle\nabla\rangle}r_{\Psi1}+\varepsilon^2\mathrm{e}^{i\tau\langle\nabla\rangle}r_{\Psi2}+\varepsilon^3r_{\Psi3},\tag{3.26}$$

where

$$\begin{aligned} r_{\Psi 1} &= \int_0^\tau F_\theta(\Phi_n)\mathrm{d}\theta - \frac{\tau}{2}\left(F_0(\Phi_n)+F_\tau(\Phi_n)\right), \\ r_{\Psi 2} &= \int_0^\tau \int_0^\theta F'_\theta(\Phi_n)G_s(\Psi_n,\Phi_n)\mathrm{d}s\mathrm{d}\theta - \frac{\tau}{2}F'_\tau(\Phi_n)\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta, \\ r_{\Psi 3} &= E_{\Psi 1}+E_{\Psi 2}+E_{\Psi 3} - \frac{\tau}{2}(R_{\Psi 1}+R_{\Psi 2}+R_{\Psi 3}). \end{aligned} \tag{3.27}$$

For the term $r_{\Psi 1}$, we get

$$\|r_{\Psi 1}\|_2 := \left\| -\frac{1}{2}\int_0^\tau \theta(\tau-\theta)\partial_{\theta\theta}F_\theta(\Phi_n)\mathrm{d}\theta \right\|_2 \le \frac{1}{2}\|\Phi_n\|_5^2 \int_0^\tau \theta(\tau-\theta)\mathrm{d}\theta \lesssim \tau^3. \tag{3.28}$$

Denoting

$$B_1(\theta,s) = F'_\theta(\Phi_n)G_s(\Psi_n,\Phi_n), \quad B_2(\theta) = G_\theta(\Psi_n,\Phi_n), \tag{3.29}$$

we get

$$\begin{aligned} \|r_{\Psi 2}\|_2 \le & \left\| \int_0^\tau \int_0^\theta F'_\theta(\Phi_n)G_s(\Psi_n,\Phi_n)\mathrm{d}s\mathrm{d}\theta - \frac{\tau^2}{2}F'_\tau(\Phi_n)G_\tau(\Psi_n,\Phi_n) \right\|_2 \\ & + \left\| \frac{\tau^2}{2}F'_\tau(\Phi_n)G_\tau(\Psi_n,\Phi_n) - \frac{\tau}{2}F'_\tau(\Phi_n)\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta \right\|_2 \\ \le & \left\| \int_0^\tau \int_0^\theta B_1(\theta,s)\mathrm{d}s\mathrm{d}\theta - \frac{\tau^2}{2}B_1(\tau,\tau) \right\|_2 \\ & + \left\| \frac{\tau}{2}F'_\tau(\Phi_n) \right\|_2 \left\| \int_0^\tau B_2(\theta)\mathrm{d}\theta - \tau B_2(\tau) \right\|_2 \\ \lesssim & \tau^3 \max_{0\le\theta,s\le\tau}\left(\|\partial_\theta B_1\|_2 + \|\partial_s B_1\|_2\right) + \tau^3 \max_{0\le\theta\le\tau}\|\partial_\theta B_2\|_2 \lesssim \tau^3. \end{aligned} \tag{3.30}$$

So we obtain the first results of (3.20), (3.21) and (3.22). Similarly, we get

$$\begin{aligned} \Phi(t_{n+1}) = & \mathrm{e}^{i\tau\Delta}\Phi_n + \varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta} G\Bigg(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n + \varepsilon\int_0^\theta \mathrm{e}^{i(\theta-s)\langle\nabla\rangle}F(\Phi^n(s))\mathrm{d}s, \\ & \qquad \mathrm{e}^{i\theta\Delta}\Phi_n + \varepsilon\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G(\Psi^n(s),\Phi^n(s))\mathrm{d}s \Bigg)\mathrm{d}\theta \\ = & \mathrm{e}^{i\tau\Delta}\Phi_n + \varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta \\ & + \varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}\partial_\mu G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\langle\nabla\rangle}F(\Phi^n(s))\mathrm{d}s\right)\mathrm{d}\theta \\ & + \varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}\partial_v G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G(\Psi^n(s),\Phi^n(s))\mathrm{d}s\right)\mathrm{d}\theta + \varepsilon^3 E_{\Phi 1} \\ = & \mathrm{e}^{i\tau\Delta}\Phi_n + \varepsilon\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta \\ & + \varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}\partial_\mu G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\langle\nabla\rangle}F\left(\mathrm{e}^{is\Delta}\Phi_n\right)\mathrm{d}s\right)\mathrm{d}\theta \\ & + \varepsilon^2\int_0^\tau \mathrm{e}^{i(\tau-\theta)\Delta}\partial_v G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, \mathrm{e}^{i\theta\Delta}\Phi_n\right)\left(\int_0^\theta \mathrm{e}^{i(\theta-s)\Delta}G\left(\mathrm{e}^{is\langle\nabla\rangle}\Psi_n, \mathrm{e}^{is\Delta}\Phi_n\right)\mathrm{d}s\right)\mathrm{d}\theta \\ & + \varepsilon^3(E_{\Phi 1}+E_{\Phi 2}+E_{\Phi 3}+E_{\Phi 4}), \end{aligned} \tag{3.31}$$

where $\mu=\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n, v=\mathrm{e}^{i\theta\Delta}\Phi_n$, $\varepsilon^3 E_{\Phi j}$ with $j=1,2,3,4$ are the remainders of Taylor's expansion and satisfy $\|E_{\Phi j}\|_1\lesssim\tau^3, j=1,2,3,4$. From the Definition (3.2) and (3.31), we obtain

$$\begin{aligned}\Phi(t_{n+1})=&\mathrm{e}^{i\tau\Delta}\Phi_n+\varepsilon\mathrm{e}^{i\tau\Delta}\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta\\&+\varepsilon^2\mathrm{e}^{i\tau\Delta}\int_0^\tau\int_0^\theta\partial_\Psi G_\theta(\Psi_n,\Phi_n)F_s(\Phi_n)\mathrm{d}s\mathrm{d}\theta\\&+\varepsilon^2\mathrm{e}^{i\tau\Delta}\int_0^\tau\int_0^\theta\partial_\Phi G_\theta(\Psi_n,\Phi_n)G_s(\Psi_n,\Phi_n)\mathrm{d}s\mathrm{d}\theta\\&+\varepsilon^3(E_{\Phi1}+E_{\Phi2}+E_{\Phi3}+E_{\Phi4}).\end{aligned}\tag{3.32}$$

Applying the Taylor expansion, we obtain

$$\begin{aligned}&G(\Psi(t_{n+1}),\Phi(t_{n+1}))\\=&G\Bigg(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n+\varepsilon\int_0^\tau\mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F(\Phi^n(\theta))\mathrm{d}\theta,\\&\qquad\mathrm{e}^{i\tau\Delta}\Phi_n+\varepsilon\int_0^\tau\mathrm{e}^{i(\tau-\theta)\Delta}G(\Psi^n(\theta),\Phi^n(\theta))\mathrm{d}\theta\Bigg)\\=&G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)+\varepsilon\partial_\mu G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)\int_0^\tau\mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F(\Phi^n(\theta))\mathrm{d}\theta\\&+\varepsilon\partial_v G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)\int_0^\tau\mathrm{e}^{i(\tau-\theta)\Delta}G(\Psi^n(\theta),\Phi^n(\theta))\mathrm{d}\theta+\varepsilon^2R_{\Phi1}\\=&G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)+\varepsilon\partial_\mu G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)\int_0^\tau\mathrm{e}^{i(\tau-\theta)\langle\nabla\rangle}F\left(\mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta\\&+\varepsilon\partial_v G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)\int_0^\tau\mathrm{e}^{i(\tau-\theta)\Delta}G\left(\mathrm{e}^{i\theta\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\theta\Delta}\Phi_n\right)\mathrm{d}\theta\\&+\varepsilon^2(R_{\Phi1}+R_{\Phi2}+R_{\Phi3}+R_{\Phi4}),\end{aligned}\tag{3.33}$$

where $\varepsilon^2R_{\Phi j}$ with $j=1,2,3,4$ are the remainders of Taylor's expansion and satisfy $\|R_{\Phi j}\|_1\lesssim\tau^2, j=1,2,3,4$. From the Definition (3.2) and (3.33), we have

$$\begin{aligned}&G(\Psi(t_{n+1}),\Phi(t_{n+1}))\\=&G\left(\mathrm{e}^{i\tau\langle\nabla\rangle}\Psi_n,\mathrm{e}^{i\tau\Delta}\Phi_n\right)+\varepsilon\mathrm{e}^{i\tau\Delta}\partial_\Psi G_\tau(\Psi_n,\Phi_n)\int_0^\tau F_\theta(\Phi_n)\mathrm{d}\theta\\&+\varepsilon\mathrm{e}^{i\tau\Delta}\partial_\Phi G_\tau(\Psi_n,\Phi_n)\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta+\varepsilon^2(R_{\Phi1}+R_{\Phi2}+R_{\Phi3}+R_{\Phi4}).\end{aligned}\tag{3.34}$$

From (3.19), (3.32) and (3.34), we have

$$\xi_\Phi^n:=\varepsilon\mathrm{e}^{i\tau\Delta}r_{\Phi1}+\varepsilon^2\mathrm{e}^{i\tau\Delta}r_{\Phi2}+\varepsilon^2\mathrm{e}^{i\tau\Delta}r_{\Phi3}+\varepsilon^3r_{\Phi4},\tag{3.35}$$

where

$$\begin{aligned}r_{\Phi1}=&\int_0^\tau G_\theta(\Psi_n,\Phi_n)\mathrm{d}\theta-\frac{\tau}{2}(G_0(\Psi_n,\Phi_n)+G_\tau(\Psi_n,\Phi_n)),\\r_{\Phi2}=&\int_0^\tau\int_0^\theta\partial_\Psi G_\theta(\Psi_n,\Phi_n)F_s(\Phi_n)\mathrm{d}s\mathrm{d}\theta-\frac{\tau}{2}\partial_\Psi G_\tau(\Psi_n,\Phi_n)\int_0^\tau F_\theta(\Phi_n)\mathrm{d}\theta,\end{aligned}$$

$$r_{\Phi 3} = \int_0^\tau \int_0^\theta \partial_\Phi G_\theta(\Psi_n, \Phi_n) G_s(\Psi_n, \Phi_n) \mathrm{d}s\mathrm{d}\theta - \frac{\tau}{2} \partial_\Phi G_\tau(\Psi_n, \Phi_n) \int_0^\tau G_\theta(\Psi_n, \Phi_n) \mathrm{d}\theta,$$
$$r_{\Phi 4} = E_{\Phi 1} + E_{\Phi 2} + E_{\Phi 3} + E_{\Phi 4} - \frac{\tau}{2}(R_{\Phi 1} + R_{\Phi 2} + R_{\Phi 3} + R_{\Phi 4}). \tag{3.36}$$

For each term of (3.35), similar to (3.28) and (3.30), we obtain the estimates

$$\|r_{\Phi j}\|_1 \lesssim \tau^3, \quad j = 1,2,3,4. \tag{3.37}$$

Combining (3.35), (3.36) and (3.37), we get the second results of (3.20), (3.21) and (3.22). So we can see that the lemma holds. □

Proof of Theorem 3.3. For $n=0$, (3.12) is obvious due to that

$$\begin{aligned}
&\|e_\Psi^0\|_2 = \left\|\Psi(0) - I_M \breve{\Psi}^0\right\|_2 \le Ch^m \|\Psi(0)\|_{m+2} \lesssim h^m, \\
&\left\|I_M \breve{\Psi}^0\right\|_2 \le \|e_\Psi^0\|_2 + \|\Psi(0)\|_2 \le Ch^m \|\Psi(0)\|_{m+2} + M_\Psi \le 1 + M_\Psi, \\
&\|e_\Phi^0\|_1 = \left\|\Phi(0) - I_M \breve{\Phi}^0\right\|_1 \le Ch^m \|\Phi(0)\|_{m+1} \lesssim h^m, \\
&\left\|I_M \breve{\Phi}^0\right\|_1 \le \|e_\Phi^0\|_1 + \|\Phi(0)\|_1 \le Ch^m \|\Phi(0)\|_{m+1} + M_\Phi \le 1 + M_\Phi,
\end{aligned} \tag{3.38}$$

when $0 < h \le h_1$ for sufficiently small value $h_1 > 0$. Subtracting (3.9) from (3.19), we have the following error equation

$$e_\Psi^{n+1} = \mathrm{e}^{i\tau\langle\nabla\rangle} e_\Psi^n + W_\Psi^n + \xi_\Psi^n, \quad e_\Phi^{n+1} = \mathrm{e}^{i\tau\Delta} e_\Phi^n + W_\Phi^n + \xi_\Phi^n, \quad n = 0,1,\cdots,\frac{T_0/\varepsilon}{\tau} - 1, \tag{3.39}$$

where W_Ψ^n and W_Φ^n are given by

$$\begin{aligned}
W_\Psi^n = &\frac{1}{2}\varepsilon\tau \Big(\mathrm{e}^{i\tau\langle\nabla\rangle} \Big[F(\Phi(t_n)) - I_M F(T_B(I_M \breve{\Phi}^n)) \Big] \\
&+ \Big[F(\Phi(t_{n+1})) - I_M F(T_B(I_M \breve{\Phi}^{n+1})) \Big] \Big), \\
W_\Phi^n = &\frac{1}{2}\varepsilon\tau \Big[\mathrm{e}^{i\tau\Delta} \Big(G(\Psi(t_n), \Phi(t_n)) - I_M G(I_M \breve{\Psi}^n, T_B(I_M \breve{\Phi}^n)) \Big) \\
&+ G(\Psi(t_{n+1}), \Phi(t_{n+1})) - I_M G(I_M \breve{\Psi}^{n+1}, T_B(I_M \breve{\Phi}^{n+1})) \Big].
\end{aligned} \tag{3.40}$$

For the convenience of further analysis, we express the formula (3.40) as

$$W_\Psi^n = \frac{1}{2}\varepsilon\tau \left(\mathrm{e}^{i\tau\langle\nabla\rangle} W_{\Psi 1}^n + W_{\Psi 2}^n \right), \quad W_\Phi^n = \frac{1}{2}\varepsilon\tau \left(\mathrm{e}^{i\tau\Delta} W_{\Phi 1}^n + W_{\Phi 2}^n \right), \tag{3.41}$$

where

$$\begin{aligned}
&W_{\Psi 1}^n = F(\Phi(t_n)) - I_M F(T_B(I_M \breve{\Phi}^n)), \\
&W_{\Psi 2}^n = F(\Phi(t_{n+1})) - I_M F(T_B(I_M \breve{\Phi}^{n+1})), \\
&W_{\Phi 1}^n = G(\Psi(t_n), \Phi(t_n)) - I_M G(I_M \breve{\Psi}^n, T_B(I_M \breve{\Phi}^n)), \\
&W_{\Phi 2}^n = G(\Psi(t_{n+1}), \Phi(t_{n+1})) - I_M G(I_M \breve{\Psi}^{n+1}, T_B(I_M \breve{\Phi}^{n+1})).
\end{aligned} \tag{3.42}$$

From (3.41), we obtain

$$\|W_\Psi^n\|_2 \le \frac{1}{2}\varepsilon\tau(\|W_{\Psi 1}^n\|_2+\|W_{\Psi 2}^n\|_2), \quad \|W_\Phi^n\|_1 \le \frac{1}{2}\varepsilon\tau(\|W_{\Phi 1}^n\|_1+\|W_{\Phi 2}^n\|_1). \tag{3.43}$$

For $W_{\Psi 1}^n$, we have

$$\begin{aligned}\|W_{\Psi 1}^n\|_2 &\le \left\|I_M F(\Phi(t_n)) - I_M F(T_B(I_M \breve{\Phi}^n))\right\|_2 + Ch^m \|F(\Phi(t_n))\|_{m+2} \\ &\le C\left\|F(\Phi(t_n)) - F(T_B(I_M \breve{\Phi}^n))\right\|_2 + Ch^m \|F(\Phi(t_n))\|_{m+2}.\end{aligned} \tag{3.44}$$

Under the assumption (A’), from (3.44) and Lemma 3.2, we have

$$\|W_{\Psi 1}^n\|_2 \lesssim \|e_\Phi^n\|_1 + h^m. \tag{3.45}$$

Similarly, we have

$$\begin{aligned}&\|W_{\Psi 2}^n\|_2 \lesssim \left\|e_\Phi^{n+1}\right\|_1 + h^m, \quad \|W_{\Phi 1}^n\|_1 \lesssim \|e_\Psi^n\|_1 + \|e_\Phi^n\|_1 + h^m, \\ &\|W_{\Phi 2}^n\|_1 \lesssim \left\|e_\Psi^{n+1}\right\|_1 + \left\|e_\Phi^{n+1}\right\|_1 + h^m.\end{aligned} \tag{3.46}$$

Plugging (3.45) and (3.46) into (3.43), we have

$$\begin{aligned}&\|W_\Psi^n\|_2 \lesssim \varepsilon\tau\left(\|e_\Phi^n\|_1 + \left\|e_\Phi^{n+1}\right\|_1\right) + \varepsilon\tau h^m, \\ &\|W_\Phi^n\|_1 \lesssim \varepsilon\tau\left(\|e_\Psi^n\|_2 + \left\|e_\Psi^{n+1}\right\|_2 + \|e_\Phi^n\|_1 + \left\|e_\Phi^{n+1}\right\|_1\right) + \varepsilon\tau h^m.\end{aligned} \tag{3.47}$$

According to the error equation (3.39), we get

$$\begin{aligned}&e_\Psi^{n+1} = \mathrm{e}^{i(n+1)\tau\langle\nabla\rangle} e_\Psi^0 + \sum_{k=0}^{n} \mathrm{e}^{i(n-k)\tau\langle\nabla\rangle}\left(W_\Psi^k + \xi_\Psi^k\right), \\ &e_\Phi^{n+1} = \mathrm{e}^{i(n+1)\tau\Delta} e_\Phi^0 + \sum_{k=0}^{n} \mathrm{e}^{i(n-k)\tau\Delta}\left(W_\Phi^k + \xi_\Phi^k\right).\end{aligned} \tag{3.48}$$

From (3.20), (3.22), (3.38), (3.47) and (3.48), we have

$$\begin{aligned}&\left\|e_\Psi^{n+1}\right\|_2 \lesssim h^m + \varepsilon\tau^2 + \varepsilon\tau\sum_{k=0}^{n+1}\left\|e_\Phi^k\right\|_1 + \left\|\sum_{k=0}^{n}\mathrm{e}^{i(n-k)\tau\langle\nabla\rangle}\mathcal{F}(\Phi(t_k))\right\|_2, \\ &\left\|e_\Phi^{n+1}\right\|_1 \lesssim h^m + \varepsilon\tau^2 + \varepsilon\tau\sum_{k=0}^{n+1}\left(\left\|e_\Psi^k\right\|_2 + \left\|e_\Phi^k\right\|_1\right) + \left\|\sum_{k=0}^{n}\mathrm{e}^{i(n-k)\tau\Delta}\mathcal{G}(\Psi(t_k),\Phi(t_k))\right\|_1.\end{aligned} \tag{3.49}$$

Here we will use the RCO technique [3] for the last terms of (3.49). From the CNLSS (2.9), we find that

$$\partial_t\Psi - i\langle\nabla\rangle\Psi = \varepsilon F(\Phi) = O(\varepsilon), \quad \partial_t\Phi - i\Delta\Phi = \varepsilon G(\Psi,\Phi) = O(\varepsilon).$$

Thus, we introduce the twisted variables

$$\psi = \mathrm{e}^{-i\langle\nabla\rangle t}\Psi, \quad \phi = \mathrm{e}^{-i\Delta t}\Phi, \quad t > 0, \tag{3.50}$$

which satisfy the equations

$$\partial_t\psi = \varepsilon\mathrm{e}^{-it\langle\nabla\rangle}F(\mathrm{e}^{it\Delta}\phi), \quad \partial_t\phi = \varepsilon\mathrm{e}^{-it\Delta}G(\mathrm{e}^{it\langle\nabla\rangle}\psi, \mathrm{e}^{it\Delta}\phi). \tag{3.51}$$

Under the assumption (A'), the solution of (3.51) satisfies

$$\begin{aligned}&\|\psi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+2}(\Omega))}+\|\phi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+1}(\Omega))}\lesssim 1,\\&\|\partial_t\psi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+2}(\Omega))}+\|\partial_t\phi\|_{L^\infty([0,T_0/\varepsilon];H_p^{m+1}(\Omega))}\lesssim \varepsilon.\end{aligned} \tag{3.52}$$

As shown in [3], the RCO technique requires the following steps.

Step 1: Let $\tau_0\in(0,1)$ and choose $M_0=2\lceil 1/\tau_0\rceil\in Z^+$ ($\lceil\cdot\rceil$ is the ceiling function) with $1/\tau_0\le M_0/2<1+1/\tau_0$. From the assumption (A') and the properties of operators $\mathrm{e}^{it\langle\nabla\rangle}$, $\langle\nabla\rangle^{-1}$ and $\mathrm{e}^{it\Delta}$, we have

$$\begin{aligned}&\left\|P_{M_0}\mathcal{F}\left(\mathrm{e}^{it_k\Delta}P_{M_0}\phi(t_k)\right)-\mathcal{F}\left(\mathrm{e}^{it_k\Delta}\phi(t_k)\right)\right\|_2\le\varepsilon\tau\tau_0^m,\\&\left\|P_{M_0}\mathcal{G}\left(\mathrm{e}^{it_k\langle\nabla\rangle}P_{M_0}\psi(t_k),\mathrm{e}^{it_k\Delta}P_{M_0}\phi(t_k)\right)-\mathcal{G}\left(\mathrm{e}^{it_k\langle\nabla\rangle}\psi(t_k),\mathrm{e}^{it_k\Delta}\phi(t_k)\right)\right\|_1\le\varepsilon\tau\tau_0^m.\end{aligned} \tag{3.53}$$

Adding the two inequalities in (3.49) together and combining above estimates, we obtain

$$\|e_\Psi^{n+1}\|_2+\|e_\Phi^{n+1}\|_1\lesssim h^m+\tau_0^m+\varepsilon\tau^2+\varepsilon\tau\sum_{k=0}^{n+1}\left(\|e_\Psi^k\|_2+\|e_\Phi^k\|_1\right)+\|\mathcal{J}_\Psi^n\|_2+\|\mathcal{J}_\Phi^n\|_1, \tag{3.54}$$

where

$$\begin{aligned}&\mathcal{J}_\Psi^n=\sum_{k=0}^{n}\mathrm{e}^{-i(k+1)\tau\langle\nabla\rangle}P_{M_0}\mathcal{F}\left(\mathrm{e}^{it_k\Delta}P_{M_0}\phi(t_k)\right),\\&\mathcal{J}_\Phi^n=\sum_{k=0}^{n}\mathrm{e}^{-i(k+1)\tau\Delta}P_{M_0}\mathcal{G}\left(\mathrm{e}^{it_k\langle\nabla\rangle}P_{M_0}\psi(t_k),\mathrm{e}^{it_k\Delta}P_{M_0}\phi(t_k)\right).\end{aligned} \tag{3.55}$$

Step 2: Analyze the low Fourier mode terms $\mathcal{J}_\Psi^n$ and $\mathcal{J}_\Phi^n$. For $F(\Phi)$ and $G(\Psi,\Phi)$, we have

$$F(\Phi)=-i\langle\nabla\rangle^{-1}\overline{\Phi}\Phi,\quad G(\Psi,\Phi)=\sum_{q=1}^{2}G^{(q)}(\Psi,\Phi), \tag{3.56}$$

with

$$G^{(1)}(\Psi,\Phi)=\frac{i}{2}\Psi\Phi,\quad G^{(2)}(\Psi,\Phi)=\frac{i}{2}\overline{\Psi}\Phi. \tag{3.57}$$

For $t\in\mathbb{R}$ and $q=1,2$, defining $G_t^{(q)}(\Psi,\Phi)=\mathrm{e}^{-it\Delta}G^{(q)}(\mathrm{e}^{it\langle\nabla\rangle}\Psi,\mathrm{e}^{it\Delta}\Phi)$ and

$$\begin{aligned}\mathcal{G}^{(q)}(\Psi(t_n),\Phi(t_n))=&\varepsilon\mathrm{e}^{i\tau\Delta}\Bigg(\int_0^\tau G_z^{(q)}(\Psi(t_n),\Phi(t_n))\mathrm{d}z\\&-\frac{1}{2}\tau\Big(G_0^{(q)}(\Psi(t_n),\Phi(t_n))+G_\tau^{(q)}(\Psi(t_n),\Phi(t_n))\Big)\Bigg),\end{aligned} \tag{3.58}$$

recalling (3.2) and (3.21), we have

$$\mathcal{J}_\Phi^n=\sum_{q=1}^{2}\mathcal{J}_{\Phi,q}^n,\quad \mathcal{J}_{\Phi,q}^n=\sum_{k=0}^{n}\mathrm{e}^{-i(k+1)\tau\Delta}P_{M_0}\mathcal{G}^{(q)}\left(\mathrm{e}^{it_k\langle\nabla\rangle}P_{M_0}\psi(t_k),\mathrm{e}^{it_k\Delta}P_{M_0}\phi(t_k)\right). \tag{3.59}$$

Since the estimates on $\mathcal{J}^n_{\Phi,1}$ and $\mathcal{J}^n_{\Phi,2}$ are similar, we only give the bounds of $\mathcal{J}^n_{\Psi}, \mathcal{J}^n_{\Phi,1} (0 \le n \le T_0\varepsilon^{-1}/\tau - 1)$. For $l \in \Omega_{M_0}$, we define $\mathcal{I}^{M_0}_l$ associated to l as

$$\mathcal{I}^{M_0}_l = \{(l_1,l_2) | l_1 + l_2 = l, \quad l_1, l_2 \in \Omega_{M_0}\}. \tag{3.60}$$

In view of $P_{M_0}\psi(t_k) = \sum_{l\in\Omega_{M_0}} \widehat{\psi}_l(t_k) \mathrm{e}^{i\mu_l(x-a)}$, the following expansion holds

$$\begin{aligned}
&\mathrm{e}^{-i(k+1)\tau\langle\nabla\rangle} P_{M_0}\left(\mathrm{e}^{i\tau\langle\nabla\rangle} F_z(\mathrm{e}^{it_k\Delta} P_{M_0}\phi(t_k))\right) \\
&= -\sum_{l\in\Omega_{M_0}} \sum_{(l_1,l_2)\in\mathcal{I}^{M_0}_l} \frac{i}{\beta_l} \mathcal{H}^k_{l,l_1,l_2}(z) \mathrm{e}^{i\mu_l(x-a)}, \\
&\mathrm{e}^{-i(k+1)\tau\Delta} P_{M_0}\left(\mathrm{e}^{i\tau\Delta} G^{(1)}_z(\mathrm{e}^{it_k\langle\nabla\rangle} P_{M_0}\psi(t_k)), \mathrm{e}^{it_k\Delta} P_{M_0}\phi(t_k))\right) \\
&= \sum_{l\in\Omega_{M_0}} \sum_{(l_1,l_2)\in\mathcal{I}^{M_0}_l} \frac{i}{2} \grave{\mathcal{H}}^k_{l,l_1,l_2}(z) \mathrm{e}^{i\mu_l(x-a)},
\end{aligned} \tag{3.61}$$

where the coefficients $\mathcal{H}^k_{l,l_1,l_2}(z)$ and $\grave{\mathcal{H}}^k_{l,l_1,l_2}(z)$ are functions of $z \in \mathbb{R}$ defined as

$$\begin{aligned}
\mathcal{H}^k_{l,l_1,l_2}(z) &= \mathrm{e}^{-i(t_k+z)\delta_{l,l_1,l_2}} \hat{\bar{\phi}}_{l_1}(t_k) \hat{\phi}_{l_2}(t_k), \\
\grave{\mathcal{H}}^k_{l,l_1,l_2}(z) &= \mathrm{e}^{-i(t_k+z)\grave{\delta}_{l,l_1,l_2}} \hat{\psi}_{l_1}(t_k) \hat{\phi}_{l_2}(t_k),
\end{aligned} \tag{3.62}$$

with

$$\delta_{l,l_1,l_2} = \sqrt{1+\mu_l^2} + \mu_{l_1}^2 - \mu_{l_2}^2, \quad \grave{\delta}_{l,l_1,l_2} = \mu_l^2 - \sqrt{1+\mu_{l_1}^2} - \mu_{l_2}^2, \quad l \in \Omega_{M_0}. \tag{3.63}$$

Thus, we have

$$\begin{aligned}
\mathcal{J}^n_{\Psi} &= -\frac{i\varepsilon}{\beta_l} \sum_{k=0}^{n} \sum_{l\in\Omega_{M_0}} \sum_{(l_1,l_2)\in\mathcal{I}^{M_0}_l} \Lambda^k_{l,l_1,l_2} \mathrm{e}^{i\mu_l(x-a)}, \\
\mathcal{J}^n_{\Phi,1} &= \frac{i\varepsilon}{2} \sum_{k=0}^{n} \sum_{l\in\Omega_{M_0}} \sum_{(l_1,l_2)\in\mathcal{I}^{M_0}_l} \grave{\Lambda}^k_{l,l_1,l_2} \mathrm{e}^{i\mu_l(x-a)},
\end{aligned} \tag{3.64}$$

where

$$\begin{aligned}
\Lambda^k_{l,l_1,l_2} &= \int_0^\tau \mathcal{H}^k_{l,l_1,l_2}(z) \mathrm{d}z - \frac{\tau}{2}\left(\mathcal{H}^k_{l,l_1,l_2}(0) + \mathcal{H}^k_{l,l_1,l_2}(\tau)\right) = r_{l,l_1,l_2} \mathrm{e}^{-it_k\delta_{l,l_1,l_2}} c^k_{l,l_1,l_2}, \\
\grave{\Lambda}^k_{l,l_1,l_2} &= \int_0^\tau \grave{\mathcal{H}}^k_{l,l_1,l_2}(z) \mathrm{d}z - \frac{\tau}{2}\left(\grave{\mathcal{H}}^k_{l,l_1,l_2}(0) + \grave{\mathcal{H}}^k_{l,l_1,l_2}(\tau)\right) = \grave{r}_{l,l_1,l_2} \mathrm{e}^{-it_k\grave{\delta}_{l,l_1,l_2}} \grave{c}^k_{l,l_1,l_2},
\end{aligned} \tag{3.65}$$

with the coefficients c^k_{l,l_1,l_2}, r_{l,l_1,l_2}, $\grave{c}^k_{l,l_1,l_2}$ and $\grave{r}_{l,l_1,l_2}$ given by

$$\begin{aligned}
&c^k_{l,l_1,l_2} = \hat{\bar{\phi}}_{l_1}(t_k) \hat{\phi}_{l_2}(t_k), \quad \grave{c}^k_{l,l_1,l_2} = \hat{\psi}_{l_1}(t_k) \hat{\phi}_{l_2}(t_k), \\
&r_{l,l_1,l_2} = \int_0^\tau \mathrm{e}^{-iz\delta_{l,l_1,l_2}} \mathrm{d}z - \frac{\tau}{2}\left(1 + \mathrm{e}^{-i\tau\delta_{l,l_1,l_2}}\right) = O\left(\tau^3 (\delta_{l,l_1,l_2})^2\right), \\
&\grave{r}_{l,l_1,l_2} = \int_0^\tau \mathrm{e}^{-iz\grave{\delta}_{l,l_1,l_2}} \mathrm{d}z - \frac{\tau}{2}\left(1 + \mathrm{e}^{-i\tau\grave{\delta}_{l,l_1,l_2}}\right) = O\left(\tau^3 (\grave{\delta}_{l,l_1,l_2})^2\right).
\end{aligned} \tag{3.66}$$

If $\delta_{l,l_1,l_2}=0$ or $\grave{\delta}_{l,l_1,l_2}=0$, we obtain $r_{l,l_1,l_2}=0$ or $\grave{r}_{l,l_1,l_2}=0$. Next we only need to consider the case $\delta_{l,l_1,l_2}\neq 0$ and $\grave{\delta}_{l,l_1,l_2}\neq 0$. For $l\in\Omega_{M_0}$ and $(l_1,l_2)\in\mathcal{I}_l^{M_0}$, we get

$$\begin{aligned}\max\left\{|\delta_{l,l_1,l_2}|,|\grave{\delta}_{l,l_1,l_2}|\right\}&\leq\sqrt{1+\mu_{M_0/2}^2}+2\mu_{M_0/2}^2\\&<\sqrt{1+\frac{4\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}}+\frac{8\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}.\end{aligned}$$

We conclude that for given $0<\alpha<1$ when

$$0<\tau\leq 2\alpha\pi\Bigg/\left(\sqrt{1+\frac{4\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}}+\frac{8\pi^2(1+\tau_0)^2}{\tau_0^2(b-a)^2}\right),\tag{3.67}$$

there holds

$$\max\left\{\frac{\tau}{2}|\delta_{l,l_1,l_2}|,\frac{\tau}{2}|\grave{\delta}_{l,l_1,l_2}|\right\}\leq\alpha\pi.\tag{3.68}$$

Denoting

$$S_{l,l_1,l_2}^n=\sum_{k=0}^n \mathrm{e}^{-it_k\delta_{l,l_1,l_2}},\quad \grave{S}_{l,l_1,l_2}^n=\sum_{k=0}^n \mathrm{e}^{-it_k\grave{\delta}_{l,l_1,l_2}},\quad n=0,1,\cdots,\tag{3.69}$$

we then obtain from (3.68) that

$$\begin{aligned}&|S_{l,l_1,l_2}^n|\leq\frac{1}{|\sin(\tau\delta_{l,l_1,l_2}/2)|}\leq\frac{C}{|\tau\delta_{l,l_1,l_2}|},\\&|\grave{S}_{l,l_1,l_2}^n|\leq\frac{1}{|\sin(\tau\grave{\delta}_{l,l_1,l_2}/2)|}\leq\frac{C}{|\tau\grave{\delta}_{l,l_1,l_2}|},\quad C=\frac{2\alpha\pi}{\sin(\alpha\pi)}.\end{aligned}\tag{3.70}$$

Using summation by parts in (3.65) results in

$$\begin{aligned}&\sum_{k=0}^n\Lambda_{l,l_1,l_2}^k=r_{l,l_1,l_2}\left[\sum_{k=0}^{n-1}S_{l,l_1,l_2}^k(c_{l,l_1,l_2}^k-c_{l,l_1,l_2}^{k+1})+S_{l,l_1,l_2}^n c_{l,l_1,l_2}^n\right],\\&\sum_{k=0}^n\grave{\Lambda}_{l,l_1,l_2}^k=\grave{r}_{l,l_1,l_2}\left[\sum_{k=0}^{n-1}\grave{S}_{l,l_1,l_2}^k(\grave{c}_{l,l_1,l_2}^k-\grave{c}_{l,l_1,l_2}^{k+1})+\grave{S}_{l,l_1,l_2}^n \grave{c}_{l,l_1,l_2}^n\right],\end{aligned}\tag{3.71}$$

with

$$\begin{aligned}&c_{l,l_1,l_2}^k-c_{l,l_1,l_2}^{k+1}=\left(\hat{\bar{\phi}}_{l_1}(t_k)-\hat{\bar{\phi}}_{l_1}(t_{k+1})\right)\hat{\phi}_{l_2}(t_k)+\hat{\bar{\phi}}_{l_1}(t_{k+1})\left(\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})\right),\\&\grave{c}_{l,l_1,l_2}^k-\grave{c}_{l,l_1,l_2}^{k+1}=\left(\hat{\psi}_{l_1}(t_k)-\hat{\psi}_{l_1}(t_{k+1})\right)\hat{\phi}_{l_2}(t_k)+\hat{\psi}_{l_1}(t_{k+1})\left(\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})\right).\end{aligned}\tag{3.72}$$

From (3.66), (3.70), (3.71) and (3.72), we get

$$\begin{aligned}\left|\sum_{k=0}^n\Lambda_{l,l_1,l_2}^k\right|\lesssim&\tau^2|\delta_{l,l_1,l_2}|\sum_{k=0}^{n-1}\Big(|\hat{\bar{\phi}}_{l_1}(t_k)-\hat{\bar{\phi}}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)|\\&+|\hat{\bar{\phi}}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})|\Big)+\tau^2|\delta_{l,l_1,l_2}||\hat{\bar{\phi}}_{l_1}(t_n)||\hat{\phi}_{l_2}(t_n)|,\\\left|\sum_{k=0}^n\grave{\Lambda}_{l,l_1,l_2}^k\right|\lesssim&\tau^2|\grave{\delta}_{l,l_1,l_2}|\sum_{k=0}^{n-1}\Big(|\hat{\psi}_{l_1}(t_k)-\hat{\psi}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)|\\&+|\hat{\psi}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})|\Big)+\tau^2|\grave{\delta}_{l,l_1,l_2}||\hat{\psi}_{l_1}(t_n)||\hat{\phi}_{l_2}(t_n)|.\end{aligned}\tag{3.73}$$

For $l\in\Omega_{M_0}$ and $(l_1,l_2)\in\mathcal{I}_l^{M_0}$, the following inequalities are true

$$\begin{aligned}&|\delta_{l,l_1,l_2}|\le\sqrt{1+(\mu_{l_1}+\mu_{l_2})^2}+\mu_{l_1}^2+\mu_{l_2}^2\lesssim(1+\mu_{l_1}^2)(1+\mu_{l_2}^2),\\&|\grave{\delta}_{l,l_1,l_2}|\le(\mu_{l_1}+\mu_{l_2})^2+\sqrt{1+\mu_{l_1}^2}+\mu_{l_2}^2\lesssim(1+\mu_{l_1}^2)(1+\mu_{l_2}^2).\end{aligned}\tag{3.74}$$

Based on (3.64), (3.73) and (3.74), noticing $\beta_l=\sqrt{1+\mu_l^2}$, we have

$$\begin{aligned}\|\mathcal{J}_\Psi^n\|_2^2=&\varepsilon^2\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\left|\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}\sum_{k=0}^{n}\Lambda_{l,l_1,l_2}^k\right|^2\\\lesssim&\varepsilon^2\tau^4\Bigg\{n\sum_{k=0}^{n-1}\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\bar{\phi}}_{l_1}(t_k)-\hat{\bar{\phi}}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\\&+n\sum_{k=0}^{n-1}\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\bar{\phi}}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\\&+\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\bar{\phi}}_{l_1}(t_n)||\hat{\phi}_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\Bigg\},\end{aligned}\tag{3.75}$$

and

$$\begin{aligned}\|\mathcal{J}_{\Phi,1}^n\|_1^2=&\varepsilon^2\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\left|\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}\sum_{k=0}^{n}\grave{\Lambda}_{l,l_1,l_2}^k\right|^2\\\lesssim&\varepsilon^2\tau^4\Bigg\{n\sum_{k=0}^{n-1}\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\psi}_{l_1}(t_k)-\hat{\psi}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\\&+n\sum_{k=0}^{n-1}\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\psi}_{l_1}(t_{k+1})||\hat{\phi}_{l_2}(t_k)-\hat{\phi}_{l_2}(t_{k+1})|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\\&+\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\Bigg(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\psi}_{l_1}(t_n)||\hat{\phi}_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\Bigg)^2\Bigg\}.\end{aligned}\tag{3.76}$$

For the last terms of (3.75) and (3.76), we use the auxiliary functions

$$\begin{aligned}&\eta_\psi(x)=\sum_{l\in Z}(1+\mu_l^2)|\hat{\psi}_l(t_n)|\mathrm{e}^{i\mu_l(x-a)},\quad\eta_\phi(x)=\sum_{l\in Z}(1+\mu_l^2)|\hat{\phi}_l(t_n)|\mathrm{e}^{i\mu_l(x-a)},\\&\eta_{\bar{\phi}}(x)=\sum_{l\in Z}(1+\mu_l^2)|\hat{\bar{\phi}}_l(t_n)|\mathrm{e}^{i\mu_l(x-a)}.\end{aligned}\tag{3.77}$$

From the assumption (A') and (3.77), we get

$$\begin{aligned}&\|\eta_\psi(x)\|_s\lesssim\|\psi(t_n)\|_{s+2},\quad s\le m,\\&\|\eta_\phi(x)\|_s\lesssim\|\phi(t_n)\|_{s+2},\quad\|\eta_{\bar{\phi}}(x)\|_s\lesssim\|\phi(t_n)\|_{s+2},\quad s\le m-1,\end{aligned}$$

which means that $\eta_\psi(x)\in H^m(\Omega)$ and $\eta_\phi(x),\eta_{\bar\phi}(x)\in H^{m-1}(\Omega)$. Expanding

$$\eta_\psi(x)\eta_\phi(x)=\sum_{l\in Z}\sum_{l_1+l_2=l}|\hat\psi_{l_1}(t_n)||\hat\phi_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\mathrm{e}^{i\mu_l(x-a)},$$
$$\eta_{\bar\phi}(x)\eta_\phi(x)=\sum_{l\in Z}\sum_{l_1+l_2=l}|\hat{\bar\phi}_{l_1}(t_n)||\hat\phi_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\mathrm{e}^{i\mu_l(x-a)},$$

leads to the following results

$$\begin{aligned}&\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\left(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat{\bar\phi}_{l_1}(t_n)||\hat\phi_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\right)^2\\&\le\left\|\eta_{\bar\phi}(x)\eta_\phi(x)\right\|_1^2\le\left\|\eta_{\bar\phi}(x)\right\|_1^2\|\eta_\phi(x)\|_1^2\le\|\phi(t_n)\|_3^4,\\&\sum_{l\in\Omega_{M_0}}(1+\mu_l^2)\left(\sum_{(l_1,l_2)\in\mathcal{I}_l^{M_0}}|\hat\psi_{l_1}(t_n)||\hat\phi_{l_2}(t_n)|\prod_{j=1}^{2}(1+\mu_{l_j}^2)\right)^2\\&\le\|\eta_\psi(x)\eta_\phi(x)\|_1^2\le\|\eta_\psi(x)\|_1^2\|\eta_\phi(x)\|_1^2\le\|\psi(t_n)\|_3^2\|\phi(t_n)\|_3^2.\end{aligned}$$

Similarly, we could estimate each term in (3.75) and (3.76) as

$$\begin{aligned}\|\mathcal{J}_\Psi^n\|_2^2\lesssim&\varepsilon^2\tau^4\Bigg(n\sum_{k=0}^{n-1}\|\phi(t_k)-\phi(t_{k+1})\|_3^2\|\phi(t_k)\|_3^2\\&+n\sum_{k=0}^{n-1}\|\phi(t_{k+1})\|_3^2\|\phi(t_k)-\phi(t_{k+1})\|_3^2+\|\phi(t_n)\|_3^2\|\phi(t_n)\|_3^2\Bigg)\lesssim\varepsilon^2\tau^4,\\\left\|\mathcal{J}_{\Phi,1}^n\right\|_1^2\lesssim&\varepsilon^2\tau^4\Bigg(n\sum_{k=0}^{n-1}\|\psi(t_k)-\psi(t_{k+1})\|_3^2\|\phi(t_k)\|_3^2\\&+n\sum_{k=0}^{n-1}\|\psi(t_{k+1})\|_3^2\|\phi(t_k)-\phi(t_{k+1})\|_3^2+\|\psi(t_n)\|_3^2\|\phi(t_n)\|_3^2\Bigg)\lesssim\varepsilon^2\tau^4.\end{aligned}\tag{3.78}$$

In the same process, we have the estimates $\left\|\mathcal{J}_{\Phi,2}^n\right\|_1^2\lesssim\varepsilon^2\tau^4$. Substituting the estimates for $\mathcal{J}_\Psi^n,\mathcal{J}_{\Phi,1}^n,\mathcal{J}_{\Phi,2}^n$ into (3.54), we have

$$\left\|e_\Psi^{n+1}\right\|_2+\left\|e_\Phi^{n+1}\right\|_1\lesssim h^m+\tau_0^m+\varepsilon\tau^2+\varepsilon\tau\sum_{k=0}^{n+1}\left(\left\|e_\Psi^k\right\|_2+\left\|e_\Phi^k\right\|_1\right).\tag{3.79}$$

Defining $\mathcal{E}^n=\|e_\Psi^n\|_2+\|e_\Phi^n\|_1$ as the energy and then using discrete Gronwall lemma, we conclude that for sufficiently small value $\tau_1>0$, when $0<\tau_0\le\tau_1$,

$$\left\|e_\Psi^{n+1}\right\|_2+\left\|e_\Phi^{n+1}\right\|_1\lesssim h^m+\tau_0^m+\varepsilon\tau^2,\quad n=0,\cdots,T_0\varepsilon^{-1}/\tau-1,\tag{3.80}$$

where we need to note that τ is constrained by τ_0 in the condition (3.11) of the theorem. Then, it is easy to check that there exist sufficiently small values $h_2>0$ and $\tau_2>0$, when

$0 < h \le h_2$ and $0 < \tau_0 \le \tau_2$, the following result holds

$$\begin{aligned}\left\|I_M \breve{\Psi}^{n+1}\right\|_2 &\le \|e_\Psi^{n+1}\|_2 + \|\Psi(t_{n+1})\|_2 \lesssim h^m + \tau_0^m + \varepsilon\tau^2 + M_\Psi \le 1 + M_\Psi,\\ \left\|I_M \breve{\Phi}^{n+1}\right\|_1 &\le \|e_\Phi^{n+1}\|_1 + \|\Phi(t_{n+1})\|_1 \lesssim h^m + \tau_0^m + \varepsilon\tau^2 + M_\Phi \le 1 + M_\Phi.\end{aligned} \tag{3.81}$$

The choice of $h_0 = \min\{h_1, h_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$ completes the proof of Theorem 3.3.

Proof of Theorem 3.2. From (3.7), we know that for such τ_0 and h_0, the modified EWIFP method (3.8) collapses exactly to the EWIFP method (2.25), which implies that Theorem 3.2 is ture.

Proof of Theorem 3.1. From (3.5) and (2.27), we have

$$\begin{aligned}\|u(\cdot,t_n) - I_M u^n\|_2 &= \|\mathrm{Re}(\Psi(\cdot,t_n) - I_M\Psi^n)\|_2 \lesssim h^m + \tau_0^m + \varepsilon\tau^2,\\ \|\dot{u}(\cdot,t_n) - I_M \dot{u}^n\|_1 &= \|\langle\nabla\rangle \mathrm{Im}(\Psi(\cdot,t_n) - I_M\Psi^n)\|_1\\ &\le \|\mathrm{Im}(\Psi(\cdot,t_n) - I_M\Psi^n)\|_2 \lesssim h^m + \tau_0^m + \varepsilon\tau^2.\end{aligned} \tag{3.82}$$

Combining (3.80) and (3.82), we obtain Theorem 3.1.

Remark 3.2. From the analysis of this paper and existing literature, we know that the r_{l,l_1,l_2} and $\grave{r}_{l,l_1,l_2}$ in (3.66) are critical and necessary. Specifically, the r_{l,l_1,l_2} and $\grave{r}_{l,l_1,l_2}$ need to satisfy the conditions as

$$r_{l,l_1,l_2} = O\left(\tau^3(\delta_{l,l_1,l_2})^k\right), \quad \grave{r}_{l,l_1,l_2} = O\left(\tau^3(\grave{\delta}_{l,l_1,l_2})^k\right), \quad k \ge 1, \tag{3.83}$$

in order to obtain the improved uniform error bounds.

Remark 3.3. Actually the proposed scheme is the Deuflhard-type exponential integrator in time, which is exactly the same as the scheme obtained from the second-order time splitting method by choosing an appropriate splitting. Similarly, we can get the Gautschi-type exponential integrator for the CNLSS (2.9) as

$$\begin{aligned}\Psi^{[n+1]} &= \mathrm{e}^{i\tau\langle\nabla\rangle}\Psi^{[n]} + \frac{\tau}{2}\varepsilon\varphi_1(i\langle\nabla\rangle\tau)\left(F(\Phi^{[n]}) + F(\Phi^{[n+1]})\right),\\ \Phi^{[n+1]} &= \mathrm{e}^{i\tau\Delta}\Phi^{[n]} + \frac{\tau}{2}\varepsilon\varphi_1(i\Delta\tau)\left(G(\Psi^{[n]},\Phi^{[n]}) + G(\Psi^{[n+1]},\Phi^{[n+1]})\right),\end{aligned} \tag{3.84}$$

where $\varphi_1(z) = \int_0^1 \mathrm{e}^{(1-\xi)z}\mathrm{d}\xi$. However, the same improved error estimate (3.1) is no longer valid for the method (3.84) applied to the KGSE (2.1)-(2.4) because the conditions (3.83) no longer hold. More generally, for the various other variations or extensions of the Gautschi-type exponential integrator, improved error bounds also often no longer hold. In addition, the method (3.84) can not be applied in an explicit way as (2.22) in the actual calculation.

4. Numerical experiment

This section presents the numerical results of the EWIFP method (2.25)-(2.27) to show our improved error bounds.

4.1. The long-time dynamics in 1D. In 1D, we take $T_0 = 1$, $(a,b) = (0, 2\pi)$ and

$$(u^0(x), \dot{u}^0(x), \Phi^0(x)) = \left(\frac{1}{2+\cos^2(x)}, \quad \frac{1}{1+\cos^2(x)}, \quad \frac{1}{1+\sin^2(x)}\right).$$

For the characterization of errors, we define

$$\|e(\cdot,t_n)\|_2 = \|u(\cdot,t_n) - I_M u^n\|_2 + \|\partial_t u(\cdot,t_n) - I_M \dot{u}^n\|_1 + \|\Phi(\cdot,t_n) - I_M \Phi^n\|_1 .$$

Table 4.1 shows the difference in the temporal errors for different τ and ε with $h=\pi/2^6$. Here, in order to show that the time error magnitude of the method is $O(\varepsilon\tau^2)$, we adopt $\|e(\cdot,t_n)\|_2/\varepsilon$ and just show that its magnitude is $O(\tau^2)$ in time. Table 4.2 shows the difference in the spatial error for different h and ε with $\tau=10^{-4}$.

$\|\|e(\cdot,1/\varepsilon)\|\|_2/\varepsilon$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
$\varepsilon_0=1$	6.50E−3	1.60E−3	3.87E−4	9.65E−5	2.41E−5	6.02E−6
order	–	2.03	2.04	2.01	2.00	2.00
$\varepsilon_0/2$	7.23E−3	1.70E−3	4.19E−4	1.04E−4	2.61E−5	6.51E−6
order	–	2.09	2.02	2.00	2.00	2.00
$\varepsilon_0/2^2$	5.51E−3	1.24E−3	2.98E−4	7.42E−5	1.85E−5	4.63E−6
order	–	2.15	2.06	2.00	2.00	2.00
$\varepsilon_0/2^3$	6.60E−3	1.40E−3	3.29E−4	8.19E−5	2.05E−5	5.11E−6
order	–	2.23	2.09	2.00	2.00	2.00
$\varepsilon_0/2^4$	7.85E−3	1.68E−3	4.05E−4	1.01E−4	2.52E−5	6.30E−6
order	–	2.22	2.06	2.00	2.00	2.00
$\varepsilon_0/2^5$	5.43E−3	1.33E−3	2.99E−4	7.46E−5	1.86E−5	4.65E−6
order	–	2.03	2.15	2.01	2.00	2.00

TABLE 4.1. *Temporal errors of EWIFP with different τ and ε.*

$\|\|e(\cdot,1/\varepsilon)\|\|_2$	$h_0=\pi/2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\varepsilon_0=1$	2.76E−1	3.24E−2	3.07E−3	7.44E−6	2.44E−11
$\varepsilon_0/2$	2.61E−1	4.72E−2	3.41E−3	6.11E−6	2.88E−11
$\varepsilon_0/2^2$	1.49E−1	4.97E−2	3.69E−3	7.85E−6	3.18E−11
$\varepsilon_0/2^3$	1.62E−1	2.14E−2	2.68E−3	1.71E−6	6.27E−11
$\varepsilon_0/2^4$	8.91E−2	4.25E−2	8.30E−4	3.37E−6	8.60E−11
$\varepsilon_0/2^5$	1.41E−1	2.79E−2	1.70E−3	5.21E−6	1.35E−10

TABLE 4.2. *Spatial errors of EWIFP for different h and ε.*

And we know that the time symmetric methods tend to show good properties in terms of structure-preservation. Because our method is time symmetric, next we will show the long-term stability of the discrete mass and energy of the EWIFP method. We take $h=\pi/2$ and $\tau=0.1$ and the long enough time interval $[0,1000]$. The errors of discrete mass and energy are shown in Figure 4.1. Here, the expressions of discrete mass and discrete energy are

$$M^n = \|\Phi^n\|_{l^2}^2 := h\sum_{j=0}^{M-1} |\Phi_j^n|^2, \quad n\geq 0,$$

$$E^n = \frac{1}{2}\Big(\|\dot{u}^n\|_{l^2}^2 + \|D_x u^n\|_{l^2}^2 + \|u^n\|_{l^2}^2\Big) + \|D_x\Phi^n\|_{l^2}^2 - \varepsilon h\sum_{j=0}^{M-1} u_j^n |\Phi_j^n|^2, \quad n\geq 0,$$

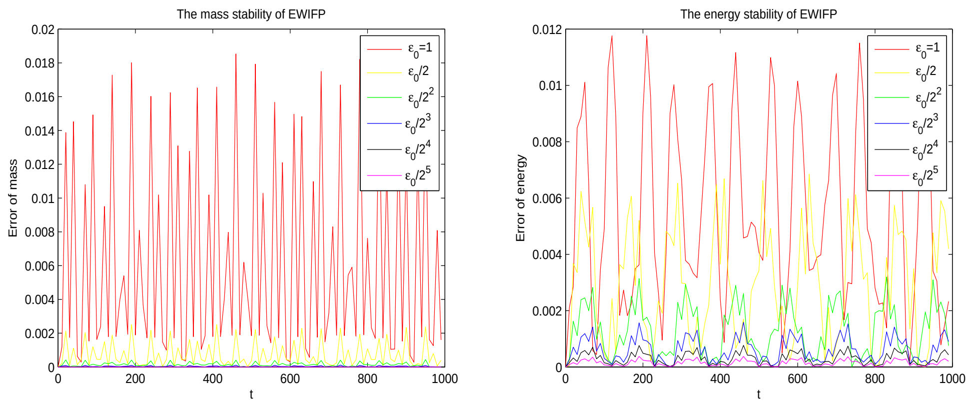

FIG. 4.1. *Long-term mass stability (left) and energy stability (right) for EWIFP.*

respectively, where $(D_x u^n)_j$ is given as

$$(D_x u^n)_j = \sum_{l\in\Omega_M} i\mu_l(\widetilde{u^n})_l e^{i\mu_l(x_j-a)} = \sum_{l\in\Omega_M} i\mu_l(\widetilde{u^n})_l e^{2ijl\pi/M}.$$

From Tables 4.1-4.2 and Figure 4.1, we can draw the following observations up to the long-time at $O(1/\varepsilon)$: (i) The temporal errors of the method behave like $O(\varepsilon\tau^2)$ (see Table 4.1). This means that the method has improved error bounds in time. (ii) In space, the errors of the method are at $O(h^m)$ for any $\varepsilon\in(0,1]$ (see Table 4.2). In other words, the method is uniformly accurate in space and has spectral order precision. (iii) The method exhibits good long-term stability of the discrete mass and energy because it is time symmetric. (iv) The improved error bounds of the method are always true without any CFL-type condition constraint. In summary, the results of numerical experiments support our error estimates and theoretical analysis.

REMARK 4.1. In the numerical results, the errors of the mass and energy look dependent on the small parameter ε. More numerical results not listed here show that for a very long time much larger than $O(1/\varepsilon)$, the the errors of the mass and energy behave $O(\varepsilon)$ in time for given τ and h. For explaining this phenomenon in detail, we may need to make use of some other tools such as modulated Fourier expansion (MFE) [22, 23]. The specific analysis is very tedious and we will not discuss it in detail due to the limitation of space because the main task of this paper is to study the improved error bounds.

REMARK 4.2. Generally speaking, if one combines the use of time splitting or Deuflhard-type exponential integrator in time and finite difference discretization in space, improved error bounds still hold in time. However, the spatial accuracy tends to worsen. For example, if approximating the Laplacian operator Δ by second-order central difference operator δ_x^2 as

$$\delta_x^2 u_j^n = \frac{1}{h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$

and then considering the Deuflhard-type exponential integrator for the resulting space semi-discrete system, we obtain the improved error bounds $O(\varepsilon^{-1}h^2+\varepsilon\tau^2)$. The detailed proof process is similar and omitted. Tables 4.3-4.4 show the temporal errors and spatial errors, respectively, which support our conclusion. Here we denote the scheme consisting of exponential wave integrator in time and finite difference discretization in space as EWIFD.

$\|\|e(\cdot,1/\varepsilon)\|\|_2/\varepsilon$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
$\varepsilon_0=1$	6.83E−3	1.55E−3	3.75E−4	9.35E−5	2.34E−5	5.83E−6
order	–	2.14	2.04	2.00	2.00	2.00
$\varepsilon_0/2$	7.65E−3	1.68E−3	4.08E−4	1.02E−4	2.54E−5	6.35E−6
order	–	2.19	2.04	2.00	2.00	2.00
$\varepsilon_0/2^2$	5.87E−3	1.49E−3	3.45E−4	8.57E−5	2.14E−5	5.34E−6
order	–	1.98	2.11	2.01	2.00	2.00
$\varepsilon_0/2^3$	6.94E−3	1.60E−3	3.68E−4	9.16E−5	2.29E−5	5.72E−6
order	–	2.12	2.12	2.00	2.00	2.00
$\varepsilon_0/2^4$	6.53E−3	1.41E−3	3.40E−4	8.46E−5	2.11E−5	5.28E−6
order	–	2.21	2.05	2.00	2.00	2.00
$\varepsilon_0/2^5$	5.19E−3	1.31E−3	2.97E−4	7.39E−5	1.84E−5	4.61E−6
order	–	1.99	2.14	2.01	2.00	2.00

TABLE 4.3. *Temporal errors of EWIFD with different τ and ε.*

$\|\|e(\cdot,1/\varepsilon)\|\|_2$	$h_0=\pi/2^7$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\varepsilon_0=1$	**3.48E−3**	8.73E−4	2.18E−4	5.46E−5	1.36E−5
order	–	2.00	2.00	2.00	2.00
$\varepsilon_0/4$	1.40E−2	**3.56E−3**	8.91E−4	2.23E−4	5.57E−5
order	–	**1.98**	2.00	2.00	2.00
$\varepsilon_0/4^2$	5.00E−2	1.39E−2	**3.53E−3**	8.84E−4	2.21E−4
order	–	1.85	**1.97**	2.00	2.00
$\varepsilon_0/4^3$	1.43E−1	4.99E−2	1.38E−2	**3.52E−3**	8.82E−4
order	–	1.52	1.85	**1.97**	2.00
$\varepsilon_0/4^4$	3.78E−1	1.43E−1	4.99E−2	1.38E−2	**3.52E−3**
order	–	1.40	1.52	1.85	**1.97**

TABLE 4.4. *Spatial errors of EWIFD for different h and ε.*

4.2. The long-time dynamics in 2D. Here we take $\Omega=(0,2\pi)\times(0,2\pi)$, $T_0=1$ and

$$u^0(x,y)=\frac{1}{2+\cos^2(x)+\cos^2(y)},\quad \dot{u}^0(x,y)=\frac{1}{1+\cos^2(x)+\cos^2(y)},$$
$$\Phi^0(x,y)=\frac{1}{1+\sin^2(x)+\sin^2(y)}.$$

Table 4.5 and Table 4.6 show the temporal and spatial errors of EWIFP, respectively.

$\|\|e(\cdot,1/\varepsilon)\|\|_2/\varepsilon$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
$\varepsilon_0=1$	5.22E−3	1.10E−3	2.60E−4	6.47E−5	1.61E−5	4.03E−6
order	–	2.25	2.08	2.01	2.00	2.00
$\varepsilon_0/2$	5.09E−3	1.19E−3	2.69E−4	6.70E−5	1.67E−5	4.18E−6
order	–	2.10	2.14	2.00	2.00	2.00
$\varepsilon_0/2^2$	5.38E−3	9.91E−4	2.30E−4	5.73E−5	1.43E−5	3.58E−6
order	–	2.44	2.10	2.01	2.00	2.00
$\varepsilon_0/2^3$	4.92E−3	9.26E−4	2.00E−4	4.98E−5	1.24E−5	3.10E−6
order	–	2.41	2.21	2.01	2.00	2.00
$\varepsilon_0/2^4$	5.43E−3	1.08E−3	2.58E−4	6.42E−5	1.60E−5	4.00E−6
order	–	2.33	2.06	2.00	2.00	2.00
$\varepsilon_0/2^5$	4.16E−3	8.90E−4	2.04E−4	5.08E−5	1.27E−5	3.17E−6
order	–	2.23	2.12	2.01	2.00	2.00

TABLE 4.5. *Long-time temporal errors of EWIFP for different τ and ε in 2D.*

$\|\|e(\cdot,1/\varepsilon)\|\|_2$	$h_0=\pi/2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\varepsilon_0=1$	3.42E−1	3.47E−2	2.22E−3	4.41E−6	3.17E−11
$\varepsilon_0/2$	2.84E−1	3.94E−2	2.27E−3	3.43E−6	4.75E−11
$\varepsilon_0/2^2$	2.80E−1	4.89E−2	2.53E−3	4.73E−6	4.55E−11
$\varepsilon_0/2^3$	2.42E−1	4.17E−2	2.02E−3	9.24E−7	8.29E−11
$\varepsilon_0/2^4$	1.60E−1	4.68E−2	8.29E−4	1.64E−6	8.57E−11
$\varepsilon_0/2^5$	1.90E−1	3.17E−2	1.39E−3	2.57E−6	1.27E−10

TABLE 4.6. *Long-time spatial errors of EWIFP for different h and ε in 2D.*

The long-term mass-stability and energy-stability of EWIFP in 2D are shown in Figure 4.2. From Tables 4.5-4.6 and Figure 4.2, we can be sure that our theoretical

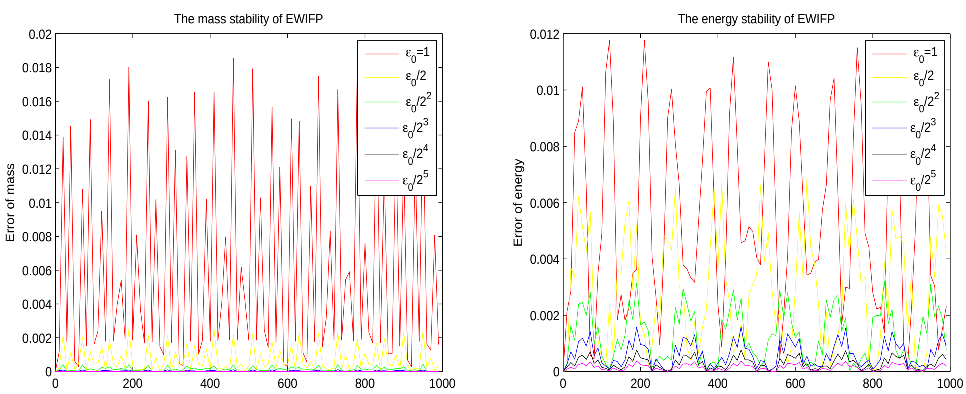

FIG. 4.2. *The errors of mass (left) and energy (right) for EWIFP in 2D.*

results apply to two-dimensional problems as well.

5. Conclusions and discussions

We applied an EWIFP method to the KGSE with $\varepsilon \in (0,1]$ and analyzed its long-time errors up to the long time at $O(1/\varepsilon)$. The method was proved to have time symmetry and is efficient thanks to the FFT. By rigorous error analysis, we established improved uniform error bounds of the method at $O(h^m + \varepsilon\tau^2)$ up to the time at $O(1/\varepsilon)$. Compared with the existing results, our work focuses on the long-time numerical error analysis for the KGSE. In error analysis, in addition to classical tools such as the energy method and cut-off technique, we also adopted the regularity compensation oscillation (RCO) technique which has been developed recently to analyze the accumulation of errors carefully. The numerical experiments were presented to support our long-time error estimates and demonstrate the long-term stability of discrete mass and energy.

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