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GLOBAL STRONG SOLUTIONS TO THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH SLIP BOUNDARY CONDITIONS IN A 3D EXTERIOR DOMAIN*

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Abstract. In this paper we study the initial-boundary-value problem for the barotropic compressible magnetohydrodynamic system with slip boundary conditions in three-dimensional exterior domain. We establish the global existence and uniqueness of classical solutions to the exterior domain problem with the regular initial data that are of small energy but possibly large oscillations with constant state as far field which could be either vacuum or nonvacuum. In particular, the initial density of such a classical solution is allowed to have large oscillations and can contain vacuum states. Moreover, the large-time behavior of the solution is also shown.

Keywords. Compressible magnetohydrodynamic equations; global existence; exterior domain; slip boundary condition; vacuum.

AMS subject classifications. 35Q55; 35K65; 76N10; 76W05.

1. Introduction

We consider the viscous barotropic compressible magnetohydrodynamic (MHD) equations for isentropic flows in a domain $\Omega\subset\mathbb{R}^3$, which can be written as

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)+\nabla P=\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u+(\nabla\times H)\times H,\\ H_t-\nabla\times(u\times H)=-\nu\nabla\times(\nabla\times H),\\ \operatorname{div}H=0,\end{cases}\tag{1.1}$$

where $(x,t)\in\Omega\times(0,T]$, $t\geq 0$ is time, and $x=(x_1,x_2,x_3)$ is the spatial coordinate. The unknown functions $\rho, u=(u^1,u^2,u^3), P=P(\rho)$, and $H=(H^1,H^2,H^3)$ denote the fluid density, velocity, pressure, and magnetic field, respectively. Here we consider the barotropic flows with γ-law pressure $P(\rho)=a\rho^\gamma$ ($a>0$ and $\gamma>1$). The physical constants μ, λ and ν are shear viscosity, bulk coefficients and resistivity coefficient respectively satisfying $\mu>0$, $2\mu+3\lambda\geq 0$ and $\nu>0$.

In this paper, we are concerned with the global existence of classical solutions of (1.1) in the exterior domain of bounded region with slip boundary condition in $\mathbb{R}^3$, which can be regarded as a continuation of our work in [7]. Throughout this paper, let $\mathbb{D}$ be a simply connected bounded domain in $\mathbb{R}^3$ with smooth boundary and contained in the ball $B_R\triangleq\{x\in\mathbb{R}^3||x|<R\}$ for the fixed $R>0$. Let $\Omega\subset\mathbb{R}^3$ be the exterior domain to $\mathbb{D}$, i.e., $\Omega=\mathbb{R}^3\setminus\bar{\mathbb{D}}$, that is an unbounded domain with smooth boundary $\partial\Omega$. In addition, this paper concerns the problem (1.1) with the initial data

$$(\rho,\rho u,H)\big|_{t=0}=(\rho_0,\rho_0u_0,H_0),\quad \text{in } \Omega,\tag{1.2}$$

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and the far-field behavior

$$(\rho,u,H)\to(\rho_\infty,0,0),\qquad \text{as}\,|x|\to+\infty, \tag{1.3}$$

where $\rho_\infty\geq 0$ is a given constant. The boundary condition is supposed by

$$u\cdot n=0,\ \mathrm{curl}u\times n=0,\qquad \text{on }\partial\Omega, \tag{1.4}$$
$$H\cdot n=0,\ \mathrm{curl}H\times n=0,\qquad \text{on }\partial\Omega, \tag{1.5}$$

where $n=(n^1,n^2,n^3)$ is the unit outward normal vector to $\partial\Omega$. The boundary condition (1.4) for the velocity presented in this paper can be regarded as a Navier-type slip boundary condition (see e.g., [4]). For the magnetic field, the boundary condition (1.5) describes that the boundary $\partial\Omega$ is a perfect conductor (see e.g., [9]).

The compressible MHD system (1.1) is known to be one of the mathematical models describing the motion of electrically conducting media (cf. gases) in an electromagnetic field and a lot of literature has been devoted to the analysis of the well-posedness and dynamic behavior to the solutions of the system, see, for example, [5, 6, 9–12, 14–19, 21–23, 25, 26, 28–30, 32–34] and their references. Now, we briefly recall some results concerned with well-posedness of solutions for multi-dimensional compressible MHD equations which are related with our problem. Vol'pert-Hudjaev [30] and Fan-Yu [12] obtained the local existence of classical solutions to the 3D compressible MHD equations where the initial density is strictly positive or could contain vacuum, respectively. Lv-Huang [25] obtained the local existence of classical solutions in $\mathbb{R}^2$ with vacuum as far field density. Tang-Gao [29] obtained the local strong solutions to the compressible MHD equations in a 3D bounded domain with the Navier-slip condition. For global existence, Kawashima [18] first established the global smooth solutions to the general electro-magneto-fluid equations in two dimensions with non-vacuum. Hu-Wang [17] proved the global existence of renormalized solutions for general large initial data, also see [11, 15] for the non-isentropic compressible MHD equations. Recently, Li et al. [19] established the global existence and uniqueness of classical solutions with constant state as far field in $\mathbb{R}^3$ with large oscillations and vacuum. Hong et al. [14] generalized the result for large initial data when $\gamma-1$ and ν^{-1} are suitably small. Lv et al. [26] got the global existence of unqiue classical solutions in 2D case. Recently, we obtained the global classical solutions with vacuum and small energy but possibly large oscillations in a 3D bounded domain with slip boundary condition in [7]. Very recently, Liu et al. [21] established the global existence of smooth solutions and the explicit decay rate near a given constant state for 3-D compressible full MHD with the boundary conditions of Navier-slip for the velocity field and perfect conduction for the magnetic field in exterior domains. However, there are no works about the global existence of the strong (classical) solution to the initial-boundary-value problem (1.1)-(1.5) in a 3D exterior domain with initial density containing vacuum, at least to the best of our knowledge.

The main purpose of this paper is to establish the global well-posedness of classical solutions of compressible MHD system (1.1)-(1.5) in a 3D exterior domain Ω. Since Ω is no longer bounded, it is distinguishable from our former work [7]. Fortunately, we have the Gagliardo-Nirenberg-type inequality in the exterior domain (see Lemma 2.1). Moreover, thanks to [27], the elliptic regularity for the Neumann problem in exterior domain leads us to derive the estimates for the gradient of the effective viscous flux (see Lemma 2.9), which plays an important role in our analysis. Besides, we also apply the L^p-theory for the div-curl system for exterior domains to control ∇u by means of $\mathrm{div}u$ and $\mathrm{curl}u$ (Lemmas 2.3-2.5). In addition, the difficulties caused by the slip boundary still exist and cannot be dealt with by the methods in [7] directly. To deal with this difficulty, we adapt the idea of [3] to obtain the boundary estimates (see Lemma 2.10), which are used frequently to control the boundary terms in this paper. Furthermore, in order to estimate the derivatives of the solutions, we recall the similar Beale-Kato-Majda-type inequality in the exterior domain (see Lemma 2.6) to prove the important estimates on the gradients of the density and velocity. Based on the above analysis, we will establish the well-posedness of classical solutions to the initial-boundary-value problem of compressible MHD system in a 3D exterior domain with large oscillations and vacuum.

Before formulating our main result, we first explain the notation and conventions used throughout the paper. We start with the definition of simply connected domains.

Definition 1.1. *Let Ω be a domain in $\mathbb{R}^3$. If the first Betti number of Ω vanishes, namely, any simple closed curve in Ω can be contracted to a point, we say that Ω is simply connected. If the second Betti number of Ω is zero, we say that Ω has no holes.*

For integer $k\geq 1$ and $1\leq q<+\infty$, the standard Sobolev spaces are denoted as follows:

$$D^{k,q}(\Omega)=\{u\in L^1_{loc}(\Omega):\|\nabla^k u\|_{L^q(\Omega)}<+\infty\},\quad \|\nabla u\|_{D^{k,q}(\Omega)}\triangleq\|\nabla^k u\|_{L^q(\Omega)};$$

$$W^{k,q}(\Omega)=L^q(\Omega)\cap D^{k,q}(\Omega),\text{ with the norm }\|u\|_{W^{k,q}(\Omega)}\triangleq\left(\sum_{|m|\leq k}\|\nabla^m u\|^q_{L^q(\Omega)}\right)^{\frac{1}{q}}.$$

We denote $D^k(\Omega)\triangleq D^{k,2}(\Omega), H^k(\Omega)\triangleq W^{k,2}(\Omega)$. For simplicity, we denote $L^q(\Omega)$, $W^{k,q}(\Omega)$, $H^k(\Omega)$ and $D^k(\Omega)$ by L^q, $W^{k,q}$, H^k and D^k respectively, and set

$$\int f dx\triangleq\int_\Omega f dx,\quad \int_0^T\int f dxdt\triangleq\int_0^T\int_\Omega f dxdt.$$

For two 3×3 matrices $A=\{a_{ij}\}$, $B=\{b_{ij}\}$, the symbol $A\colon B$ represents the trace of AB^*, where B^* is the transpose of B, that is,

$$A\colon B\triangleq\operatorname{tr}(AB^*)=\sum_{i,j=1}^3 a_{ij}b_{ij}.$$

Finally, for $v=(v^1,v^2,v^3)$, we denote $\nabla_i v\triangleq(\partial_i v^1,\partial_i v^2,\partial_i v^3)$ for $i=1,2,3$, and the material derivative of v by $\dot{v}\triangleq v_t+u\cdot\nabla v$.

The initial total energy of (1.1) is defined as

$$C_0=\int_\Omega\left(\frac{1}{2}\rho_0|u_0|^2+G(\rho_0)+\frac{1}{2}|H_0|^2\right)dx, \tag{1.6}$$

where

$$G(\rho)\triangleq\rho\int_{\rho_\infty}^\rho\frac{P(s)-P_\infty}{s^2}ds,\quad P_\infty\triangleq P(\rho_\infty).$$

Now we can state our main result, Theorem 1.1, concerning existence of global classical solutions to the problem (1.1)-(1.5).

THEOREM 1.1. *Let Ω be the exterior domain of a simply connected bounded region $\mathbb{D}$ in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $q\in(3,6)$ and some given constants $M_1,M_2>0$, and $\bar\rho\geq\rho_\infty+1$, the initial data (ρ_0,u_0,H_0) satisfy the boundary conditions (1.4)-(1.5) and*

$$0\leq\rho_0\leq\bar\rho,\quad (\rho_0-\rho_\infty,P(\rho_0)-P_\infty)\in H^2\cap W^{2,q}, \tag{1.7}$$

$$(u_0,H_0)\in D^1\cap D^2,\quad \rho_0|u_0|^2+G(\rho_0)+|H_0|^2\in L^1,\quad \operatorname{div}H_0=0, \tag{1.8}$$

$$\|\nabla u_0\|_{L^2}\leq M_1,\quad \|\nabla H_0\|_{L^2}\leq M_2, \tag{1.9}$$

and the compatibility condition

$$-\mu\triangle u_0-(\mu+\lambda)\nabla\operatorname{div}u_0+\nabla P(\rho_0)-(\nabla\times H_0)\times H_0=\rho_0^{\frac{1}{2}}g, \tag{1.10}$$

for some $g\in L^2$. Moreover, $\rho\in L^{3/2}$ when $\rho_\infty=0$. Then there exists a positive constant ε depending only on μ, λ, ν, γ, a, ρ_∞, $\bar\rho$, M_1 and M_2 such that for the initial energy C_0 as in (1.6) if

$$C_0\leq\varepsilon,$$

then the system (1.1)-(1.5) has a unique global classical solution (ρ,u,H) in $\Omega\times(0,\infty)$ satisfying

$$0\leq\rho(x,t)\leq2\bar\rho,\quad (x,t)\in\Omega\times(0,\infty), \tag{1.11}$$

$$\begin{cases}(\rho-\rho_\infty,P-P_\infty)\in C([0,\infty);H^2\cap W^{2,q}),\\ \nabla u\in C([0,\infty);H^1)\cap L^\infty_{\rm loc}(0,\infty;H^2\cap W^{2,q}),\\ u_t\in L^\infty_{\rm loc}(0,\infty;D^1\cap D^2)\cap H^1_{\rm loc}(0,\infty;D^1),\\ H\in C([0,\infty);H^2)\cap L^\infty_{\rm loc}(0,\infty;H^4),\\ H_t\in C([0,\infty);L^2)\cap H^1_{\rm loc}(0,\infty;H^1)\cap L^\infty_{\rm loc}(0,\infty;H^2).\end{cases} \tag{1.12}$$

Furthermore, for all $r\in(2,\infty)$ if $\rho_\infty>0$ and $r\in(\gamma,\infty)$ if $\rho_\infty=0$, we have the following large-time behavior

$$\lim_{t\to\infty}\left(\|\rho(\cdot,t)-\rho_\infty\|_{L^r}+\|(\rho^{\frac{1}{8}}u)(\cdot,t)\|_{L^4}+\|\nabla u(\cdot,t)\|_{L^2}+\|\nabla H(\cdot,t)\|_{L^2}\right)=0. \tag{1.13}$$

Then, when $\rho_\infty>0$ and the initial density contains vacuum state, we can deduce the following large-time behavior of the gradient of the density.

THEOREM 1.2. *Under the conditions of Theorem 1.1, assume further that $\rho_\infty>0$ and there exists some point $x_0\in\Omega$ such that $\rho_0(x_0)=0$. Then the unique global classical solution (ρ,u,H) to the problem (1.1)-(1.5) obtained in Theorem 1.1 satisfies that for any $r_1>3$,*

$$\lim_{t\to\infty}\|\nabla\rho(\cdot,t)\|_{L^{r_1}}=\infty. \tag{1.14}$$

REMARK 1.1. From Sobolev embedding theorem and $(1.12)_1$ with $q>3$, the solution obtained in Theorem 1.1 becomes a classical one away from the initial time. As far as we know, this is the first result concerning the global existence of classical solutions to the compressible MHD system in a 3D exterior domain with large oscillations and vacuum.

REMARK 1.2. When we consider the following generalized slip boundary for the velocity field:

$$u\cdot n=0,\ \mathrm{curl}u\times n=-Au \text{ on } \partial\Omega,$$

and assume that the 3×3 symmetric matrix A is smooth and positive semi-definite, and even if the restriction on A is relaxed to $A\in W^{2,6}$ and the negative eigenvalues of A (if they exist) are small enough, in particular, set $A=B-2D(n)$, where $B\in W^{2,6}$ is a positive semi-definite 3×3 symmetric matrix, Theorems 1.1 and 1.2 will still hold provided that $2\mu+3\lambda>0$. This can be achieved by a similar way as in [4,7].

REMARK 1.3. For the magnetic field, we also can subject to the Dirichlet condition

$$H=0,\ \text{on } \partial\Omega,$$

or the insulating boundary condition (see [13])

$$H\times n=0,\ \text{on } \partial\Omega.$$

After some slight modification of the proof in this paper, Theorems 1.1 and 1.2 will still hold.

The rest of the paper is organized as follows. In Section 2, we review some known lemmas and derive the elementary energy estimates and some key a priori estimates that we use intensively in this paper. Section 3 is devoted to deriving the necessary time-independent lower-order estimates and time-dependent higher-order estimates, which can guarantee the local classical solution to be a global classical one. Finally, the proof of Theorems 1.1-1.2 will be completed in Section 4.

2. Preliminaries

In this section, we list some known facts and elementary inequalities that are used extensively in this paper. We also derive the elementary energy estimates for the system (1.1)-(1.5) and some key a priori estimates.

2.1. Some basic inequalities and lemmas. We first state the following Gagliardo-Nirenberg-type inequality in the exterior domain (see [8]).

LEMMA 2.1. *Assume that Ω is an exterior domain of some simply connected domain $\mathbb{D}$ in $\mathbb{R}^3$. For $p\in[2,6]$, $q\in(1,\infty)$, and $r\in(3,\infty)$, there exist two generic constants $C>0$ which may depend on p, q and r such that for any $f\in H^1(\Omega)$ and $g\in L^q(\Omega)\cap D^{1,r}(\Omega)$, such that*

$$\|f\|_{L^p(\Omega)}\le C\|f\|_{L^2}^{(6-p)/(2p)}\|\nabla f\|_{L^2}^{(3p-6)/(2p)}, \tag{2.1}$$

$$\|g\|_{C(\overline{\Omega})}\le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

Next, we give the following Zlotnik inequality (see [35]), which will be used to get the uniform (in time) upper bound of the density.

LEMMA 2.2. *Suppose the function y satisfies*

$$y'(t)=g(y)+b'(t) \text{ on } [0,T],\quad y(0)=y^0,$$

with $g\in C(R)$ and $y,b\in W^{1,1}(0,T)$. If $g(\infty)=-\infty$ and

$$b(t_2)-b(t_1)\le N_0+N_1(t_2-t_1) \tag{2.3}$$

for all $0\le t_1<t_2\le T$ *with some* $N_0\ge 0$ *and* $N_1\ge 0$, *then*

$$y(t)\le \max\{y^0,\overline{\zeta}\}+N_0<\infty \ \ on\ [0,T],$$

where $\overline{\zeta}$ *is a constant such that*

$$g(\zeta)\le -N_1 \quad for \quad \zeta\ge\overline{\zeta}. \tag{2.4}$$

The following two lemmas are given in [31, Theorem 3.2] and [24, Theorem 5.1].

LEMMA 2.3. *Assume that* Ω *is an exterior domain of some simply connected domain* $\mathbb{D}$ *in* $\mathbb{R}^3$ *with* $C^{1,1}$ *boundary. For* $v\in D^{1,q}(\Omega)$ *with* $v\cdot n=0$ *on* $\partial\Omega$, *it holds that*

$$\|\nabla v\|_{L^q(\Omega)}\le C(\|\mathrm{div}v\|_{L^q(\Omega)}+\|\mathrm{curl}v\|_{L^q(\Omega)})\ for\ any\ 1<q<3,$$

and

$$\|\nabla v\|_{L^q(\Omega)}\le C(\|\mathrm{div}v\|_{L^q(\Omega)}+\|\mathrm{curl}v\|_{L^q(\Omega)}+\|\nabla v\|_{L^2(\Omega)})\ for\ any\ 3\le q<+\infty.$$

LEMMA 2.4. *Assume that* Ω *is an exterior domain of some simply connected domain* $\mathbb{D}$ *in* $\mathbb{R}^3$ *with* $C^{1,1}$ *boundary. For any* $v\in W^{1,q}(\Omega)\ (1<q<+\infty)$ *with* $v\times n=0$ *on* $\partial\Omega$, *it holds that*

$$\|\nabla v\|_{L^q(\Omega)}\le C(\|v\|_{L^q(\Omega)}+\|\mathrm{div}v\|_{L^q(\Omega)}+\|\mathrm{curl}v\|_{L^q(\Omega)}).$$

Moreover, we have the following conclusion (see [3]).

LEMMA 2.5. *Assume that* Ω *is an exterior domain of some simply connected domain* $\mathbb{D}$ *in* $\mathbb{R}^3$ *with* $C^{1,1}$ *boundary. For* $v\in D^{k+1,p}(\Omega)\cap D^{1,2}(\Omega)$ *for some* $k\ge 0$, $p\in[2,6]$ *with* $v\cdot n=0$ *or* $v\times n=0$ *on* $\partial\Omega$ *and* $v(x,t)\to 0$ *as* $|x|\to\infty$, *there exists a positive constant* $C=C(q,k,\mathbb{D})$ *such that*

$$\|\nabla v\|_{W^{k,p}(\Omega)}\le C(\|\mathrm{div}v\|_{W^{k,p}(\Omega)}+\|\mathrm{curl}v\|_{W^{k,p}(\Omega)}+\|\nabla v\|_{L^2(\Omega)}).$$

Besides, similar to [1], we need a Beale-Kato-Majda-type inequality with respect to the slip boundary condition (1.4) which can be found in [4].

LEMMA 2.6. *For* $3<q<\infty$, *assume that* $u\cdot n=0$ *and* $\mathrm{curl}u\times n=0$ *on* $\partial\Omega$, $\nabla u\in W^{1,q}$, *then there is a constant* $C=C(q)$ *such that the following estimate holds*

$$\|\nabla u\|_{L^\infty}\le C(\|\mathrm{div}u\|_{L^\infty}+\|\mathrm{curl}u\|_{L^\infty})\ln(e+\|\nabla^2u\|_{L^q})+C\|\nabla u\|_{L^2}+C.$$

Finally, we have the following local existence of classical solution of (1.1)-(1.5), which can be proven in a similar manner as that in [12, 29].

LEMMA 2.7. *Let* Ω *be as in Theorem 1.1 and assume that the initial data* (ρ_0,u_0,H_0) *satisfies the conditions* (1.7), (1.8) *and* (1.10). *Then there exist a small time* $T_0>0$ *and a unique classical solution* (ρ,u,H) *of the system* (1.1)-(1.5) *in* $\Omega\times(0,T_0]$, *satisfying that* $\rho\ge 0$, *and that for* $\tau\in(0,T_0)$,

$$\begin{cases}(\rho-\rho_\infty,P-P_\infty)\in C([0,T_0];H^2\cap W^{2,q}),\\ \nabla u\in C([0,T_0);H^1)\cap L^\infty(\tau,T_0;H^2\cap W^{2,q}),\\ u_t\in L^\infty(\tau,T_0;D^1\cap D^2)\cap H^1(\tau,T_0;D^1),\\ H\in C([0,T_0);H^2)\cap L^\infty(\tau,T_0;H^4),\\ H_t\in C([0,T_0);L^2)\cap H^1(\tau,T_0;H^1)\cap L^\infty(\tau,T_0;H^2).\end{cases} \tag{2.5}$$

2.2. Elementary energy estimates. In the following, let $T>0$ be a fixed time and (ρ,u,H) be a smooth solution to (1.1)-(1.5) on $\Omega\times(0,T]$. We derive the elementary energy estimates for the system (1.1)-(1.5) and some key a priori estimates which are frequently applied later. First, we rewrite (1.1) in the following form:

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ \rho u_t+\rho u\cdot\nabla u-(\lambda+2\mu)\nabla\operatorname{div}u+\mu\nabla\times\omega+\nabla(P-P_\infty)=H\cdot\nabla H-\nabla\frac{|H|^2}{2},\\ H_t+u\cdot\nabla H-H\cdot\nabla u+H\operatorname{div}u=-\nu\nabla\times\operatorname{curl}H,\\ \operatorname{div}H=0,\end{cases}\tag{2.6}$$

where $\omega\triangleq\nabla\times u,\operatorname{curl}H\triangleq\nabla\times H$. In view of (1.3), (1.4) and (1.5), multiplying $(2.6)_1$ by $G'(\rho)$, $(2.6)_2$ by u and $(2.6)_3$ by H respectively, integrating by parts over Ω, summing them up, we obtain

$$\begin{aligned}&\left(\int\left(G(\rho)+\frac{1}{2}\rho|u|^2+\frac{1}{2}|H|^2\right)dx\right)_t+(\lambda+2\mu)\int(\operatorname{div}u)^2dx\\&+\mu\int|\omega|^2dx+\nu\int|\operatorname{curl}H|^2dx=0,\end{aligned}\tag{2.7}$$

which, integrated over $(0,T)$, leads to the following elementary energy estimates.

LEMMA 2.8. *Let (ρ,u,H) be a smooth solution of (1.1)-(1.5) on $\Omega\times(0,T]$. Then*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\frac{1}{2}\|\sqrt{\rho}u\|_{L^2}^2+\|G(\rho)\|_{L^1}+\frac{1}{2}\|H\|_{L^2}^2\right)\\&+\int_0^T(\lambda+2\mu)\|\operatorname{div}u\|_{L^2}^2+\mu\|\omega\|_{L^2}^2+\nu\|\operatorname{curl}H\|_{L^2}^2)dt\le C_0.\end{aligned}\tag{2.8}$$

REMARK 2.1. According to Lemma 2.3, it follows from (1.4) and (1.5) that

$$\|\nabla u\|_{L^2}\le C(\|\operatorname{div}u\|_{L^2}+\|\omega\|_{L^2}),\tag{2.9}$$
$$\|\nabla H\|_{L^2}\le C\|\operatorname{curl}H\|_{L^2}.\tag{2.10}$$

Besides, it is easy to check that

$$\begin{cases}G(\rho)=\frac{1}{\gamma-1}P(\rho), & \text{if}\quad \rho_\infty=0,\\ C^{-1}(\rho-\rho_\infty)^2\le G(\rho)\le C(\rho-\rho_\infty)^2, & \text{if}\quad \rho_\infty>0,0\le\rho\le2\bar{\rho}.\end{cases}\tag{2.11}$$

Then (2.8) together with (2.9)-(2.11), for $0\le\rho\le2\bar{\rho}$, yields

$$\sup_{0\le t\le T}\|\rho-\rho_\infty\|_{L^r}^r+\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)dt\le CC_0,\tag{2.12}$$

where $r\in[2,\infty)$ if $\rho_\infty>0$ and $r\in[\gamma,\infty)$ if $\rho_\infty=0$.

Similarly to the compressible Navier-Stokes equations, the effective viscous flux

$$F\triangleq(\lambda+2\mu)\operatorname{div}u-(P-P_\infty)-\frac{1}{2}|H|^2,\tag{2.13}$$

plays an important role in our following analysis. More precisely, we derive some priori estimates for F, ω and ∇u, which will be frequently applied later.

LEMMA 2.9. *Let (ρ,u,H) be a smooth solution of (1.1)-(1.5) on $\Omega\times(0,T]$. Then for any $p\in[2,6]$, there exists a positive constant C depending only on p, μ and λ such that*

$$\|\nabla F\|_{L^p}\le C(\|\rho\dot u\|_{L^p}+\|H\cdot\nabla H\|_{L^p}), \tag{2.14}$$

$$\|\nabla\omega\|_{L^p}\le C(\|\rho\dot u\|_{L^p}+\|H\cdot\nabla H\|_{L^p}+\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|\nabla u\|_{L^2}), \tag{2.15}$$

$$\|F\|_{L^p}\le C(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2})^{\frac{3p-6}{2p}}(\|\nabla u\|_{L^2}+\|P-P_\infty\|_{L^2}+\|H\|_{L^4}^2)^{\frac{6-p}{2p}}, \tag{2.16}$$

$$\|\omega\|_{L^p}\le C(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2})^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}+C\|\nabla u\|_{L^2}, \tag{2.17}$$

$$\|\nabla u\|_{L^p}\le C(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|P-P_\infty\|_{L^6}+\||H|^2\|_{L^6})^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}+C\|\nabla u\|_{L^2}. \tag{2.18}$$

Proof. First, $(2.6)_2$ and (2.13) yields that

$$\rho\dot u=\nabla F-\mu\nabla\times\omega+H\cdot\nabla H. \tag{2.19}$$

By (1.4), one can find that the viscous flux F satisfies

$$\int\nabla F\cdot\nabla\eta dx=\int(\rho\dot u-H\cdot\nabla H)\cdot\nabla\eta dx,\ \forall\eta\in C^\infty(\mathbb{R}^3).$$

It follows from [27, Lemma 4.27], for $q\in(1,\infty)$, that

$$\|\nabla F\|_{L^q}\le C(\|\rho\dot u\|_{L^q}+\|H\cdot\nabla H\|_{L^q}), \tag{2.20}$$

which gives (2.14). Besides, for any integer $k\ge1$,

$$\|\nabla F\|_{W^{k,q}}\le C(\|\rho\dot u\|_{W^{k,q}}+\|H\cdot\nabla H\|_{W^{k,q}}). \tag{2.21}$$

Also, notice that $\omega\times n=0$ on $\partial\Omega$ and $\mathrm{div}\omega=0$, by Lemmas 2.4-2.5 and (2.19)-(2.21), we get

$$\|\nabla\omega\|_{L^q}\le C(\|\nabla\times\omega\|_{L^q}+\|\omega\|_{L^q})\le C(\|\rho\dot u\|_{L^q}+\|H\cdot\nabla H\|_{L^q}+\|\omega\|_{L^q}), \tag{2.22}$$

and for any integer $k\ge1$,

$$\begin{aligned}\|\nabla\omega\|_{W^{k,q}}&\le C(\|\nabla\times\omega\|_{W^{k,q}}+\|\omega\|_{L^2})\\&\le C(\|\rho\dot u\|_{W^{k,q}}+\|H\cdot\nabla H\|_{W^{k,q}}+\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|\nabla u\|_{L^2}).\end{aligned} \tag{2.23}$$

By Sobolev's inequality and (2.22), for $p\in[2,6]$, it follows that

$$\|\nabla\omega\|_{L^p}\le C(\|\rho\dot u\|_{L^p}+\|H\cdot\nabla H\|_{L^p}+\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|\omega\|_{L^2}),$$

which implies (2.15).

Now, in view of the Gagliardo-Nirenberg-type inequality (2.1) and (2.14), one can deduce that for $p\in[2,6]$,

$$\begin{aligned}\|F\|_{L^p}&\le C\|F\|_{L^2}^{\frac{6-p}{2p}}\|\nabla F\|_{L^2}^{\frac{3p-6}{2p}}\\&\le C(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2})^{\frac{3p-6}{2p}}(\|\nabla u\|_{L^2}+\|P-P_\infty\|_{L^2}+\|H\|_{L^4}^2)^{\frac{6-p}{2p}},\end{aligned} \tag{2.24}$$

similarly, by (2.1) and (2.22),

$$\begin{aligned}\|\omega\|_{L^p}&\le C\|\omega\|_{L^2}^{\frac{6-p}{2p}}\|\nabla\omega\|_{L^2}^{\frac{3p-6}{2p}}\\&\le C(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|\nabla u\|_{L^2})^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}+C\|\nabla u\|_{L^2},\end{aligned} \tag{2.25}$$

and we arrive at (2.16) and (2.17). Finally, combining (2.1), (2.9), (2.16) and (2.17) yields (2.18) holds and the proof is finished. ∎

REMARK 2.2. From Lemma 2.5 and Lemma 2.9, we can get the higher order estimates of ∇u, which will be devoted to giving higher order estimates in Section 3.2. More precisely, we can get the estimates of $\|\nabla^2 u\|_{L^p}$ and $\|\nabla^3 u\|_{L^p}$ for $p\in[2,6]$ by Lemma 2.5, (2.20) and (2.21), for $p\in[2,6]$,

$$\begin{aligned}&\|\nabla^2 u\|_{L^p}\le C(\|\mathrm{div}u\|_{W^{1,p}}+\|\omega\|_{W^{1,p}}+\|\nabla u\|_{L^2})\\ \le&\, C(\|\rho\dot u\|_{L^p}+\|H\cdot\nabla H\|_{L^p}+\|\nabla P\|_{L^p}+\|P-P_\infty\|_{L^p}+\||H|^2\|_{L^p}+\|\nabla H\cdot H\|_{L^p}\\ &+\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|P-P_\infty\|_{L^2}+\|H\|_{L^4}^2+\|\nabla u\|_{L^2}),\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}&\|\nabla^3 u\|_{L^p}\le C(\|\mathrm{div}u\|_{W^{2,p}}+\|\omega\|_{W^{2,p}}+\|\nabla u\|_{L^2})\\ \le&\, C(\|\rho\dot u\|_{W^{1,p}}+\|H\cdot\nabla H\|_{W^{1,p}}+\|P-P_\infty\|_{W^{2,p}}+\||H|^2\|_{W^{2,p}}\\ &+\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2}+\|P-P_\infty\|_{L^2}+\|H\|_{L^4}^2+\|\nabla u\|_{L^2}).\end{aligned}\tag{2.27}$$

Moreover, notice that $H\cdot n=0,\mathrm{curl}H\times n=0$ on $\partial\Omega$, by Lemma 2.5, for any integer $k\ge1$, $p\in[2,6]$, we obtain

$$\begin{aligned}\|\nabla H\|_{W^{k,p}}&\le C(\|\mathrm{curl}H\|_{W^{k,p}}+\|\nabla H\|_{L^2})\\ &\le C(\|\mathrm{curl}^2 H\|_{W^{k-1,p}}+\|\mathrm{curl}H\|_{L^p}+\|\nabla H\|_{L^2}),\end{aligned}\tag{2.28}$$

where $\mathrm{curl}^2 H\triangleq\mathrm{curlcurl}H$ and we have used the fact $\mathrm{divcurl}H=0$.

To this end, we give the following boundary estimates which will be used frequently later.

LEMMA 2.10. *Assume that Ω is an exterior domain of the simply connected domain $\mathbb{D}$ in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and $\bar{\mathbb{D}}\subset B_R$. Let $u\in D^1$ with $u\cdot n=0$ on $\partial\Omega$. It holds for $f\in D^1$,*

$$\int_{\partial\Omega} fu\cdot\nabla u\cdot n dS\le C\|\nabla f\|_{L^2}\|\nabla u\|_{L^2}^2,\tag{2.29}$$

$$\int_{\partial\Omega} u\cdot\nabla f dS\le C\|\nabla f\|_{L^2}\|\nabla u\|_{L^2}.\tag{2.30}$$

Moreover, for $\nabla u\in L^2\cap L^4, f\in D^1\cap D^2$,

$$\int_{\partial\Omega} fu\cdot\nabla u\cdot\nabla n\cdot u dS\le C\|\nabla f\|_{L^6}\|\nabla u\|_{L^2}^3+C\|\nabla f\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2.\tag{2.31}$$

Proof. We adapt the ideas due to Cai-Li [4]. Since $u\cdot n=0$ on $\partial\Omega$, it follows that

$$u\cdot\nabla u\cdot n=-u\cdot\nabla n\cdot u,\quad \text{on } \partial\Omega.\tag{2.32}$$

Furthermore, in order to get the boundary estimates, one can extend the unit normal vector n to Ω such that $n\in C^3(\Omega)$ and $n\equiv0$ outside B_{2R}. Denote $\tilde\Omega\triangleq B_{2R}\cap\Omega$, it follows that

$$\begin{aligned}&\int_{\partial\Omega} fu\cdot\nabla u\cdot n dS=-\int_{\partial\Omega} fu\cdot\nabla n\cdot u dS\\ \le&\, C(\|fu\cdot\nabla n\cdot u\|_{L^1}+\|\nabla(fu\cdot\nabla n\cdot u)\|_{L^1})\\ \le&\, C(\|f\|_{L^6(\tilde\Omega)}\|u\|_{L^6(\tilde\Omega)}^2+\|\nabla f\|_{L^2(\tilde\Omega)}\|u\|_{L^6(\tilde\Omega)}^2+\|f\|_{L^6(\tilde\Omega)}\|\nabla u\|_{L^2(\tilde\Omega)}\|u\|_{L^6(\tilde\Omega)})\\ \le&\, C\|\nabla f\|_{L^2}\|\nabla u\|_{L^2}^2,\end{aligned}$$

which yields (2.29). Similarly,

$$\begin{aligned}&\int_{\partial\Omega} f^2 u\cdot\nabla u\cdot n dS = -\int_{\partial\Omega} f^2 u\cdot\nabla n\cdot u dS\\&\le C(\|f^2 u\cdot\nabla n\cdot u\|_{L^1} + \|\nabla(f^2 u\cdot\nabla n\cdot u)\|_{L^1})\\&\le C(\|f\|^2_{L^6(\tilde\Omega)}\|u\|^2_{L^6(\tilde\Omega)} + \|f\|_{L^6(\tilde\Omega)}\|\nabla f\|_{L^2(\tilde\Omega)}\|u\|^2_{L^6(\tilde\Omega)} + \|f\|^2_{L^6(\tilde\Omega)}\|\nabla u\|_{L^2(\tilde\Omega)}\|u\|_{L^6(\tilde\Omega)})\\&\le C\|\nabla f\|^2_{L^2}\|\nabla u\|^2_{L^2}.\end{aligned}\tag{2.33}$$

Moreover, if $f\in L^\infty$, it is easy to check that

$$\int_{\partial\Omega} f u\cdot\nabla u\cdot n dS \le C\|f\|_{L^\infty}\|\nabla u\|^2_{L^2}.\tag{2.34}$$

Next, $u\cdot n = 0$ on $\partial\Omega$ implies

$$u = u^\perp\times n, \quad \text{on } \partial\Omega,\tag{2.35}$$

where $u^\perp \triangleq -u\times n$ on $\partial\Omega$. Noticing that $\mathrm{div}(\nabla f\times v) = \nabla\times v\cdot\nabla f$, we have

$$\begin{aligned}&\int_{\partial\Omega} u\cdot\nabla f dS = \int_{\partial\Omega} u^\perp\times n\cdot\nabla f dS = \int_{\partial\Omega}\nabla f\times u^\perp\cdot n dS\\&= \int \mathrm{div}(\nabla f\times u^\perp)dx = \int\nabla\times u^\perp\cdot\nabla f dx \le C\|\nabla f\|_{L^2}\|\nabla u\|_{L^2},\end{aligned}$$

which yields (2.30). Similarly,

$$\begin{aligned}&\int_{\partial\Omega} f u\cdot\nabla u\cdot\nabla n\cdot u dS = \int_{\partial\Omega} f(u^\perp\times n)\cdot\nabla u\cdot\nabla n\cdot u dS\\&= \int_{\partial\Omega} f(\nabla u\cdot\nabla n\cdot u)\times u^\perp\cdot n dS\\&= \int \mathrm{div}(f(\nabla u\cdot\nabla n\cdot u)\times u^\perp)dx\\&= \int((\nabla u\cdot\nabla n\cdot u)\times u^\perp)\cdot\nabla f dx + \int f(\nabla\times u^\perp)\cdot\nabla u\cdot\nabla n\cdot u dx\\&\le C\|\nabla f\|_{L^6}\|\nabla u\|^3_{L^2} + \|\nabla f\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|^2_{L^4},\end{aligned}$$

which yields (2.31) and finishes the proof of Lemma 2.10. □

3. A priori estimates

In this section, we will establish the necessary time-independent lower-order estimates and time-dependent higher-order estimates, which can guarantee the local classical solution to be a global classical one. Let $T>0$ be a fixed time and (ρ,u,H) be a smooth solution to (1.1)-(1.5) on $\Omega\times(0,T]$ with the initial data (ρ_0,u_0,H_0) satisfying (1.7)-(1.9).

3.1. Time-independent lower order estimates.

In this subsection, we will derive the time-independent a priori bounds of the solutions of the problem (1.1)-(1.5).

Set $\sigma = \sigma(t) \triangleq \min\{1,t\}$, we define

$$A_1(T) \triangleq \sup_{0\le t\le T}\sigma\left(\|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2}\right) + \int_0^T \sigma(\|\sqrt{\rho}\dot u\|^2_{L^2} + \|\mathrm{curl}^2 H\|^2_{L^2} + \|H_t\|^2_{L^2})dt,\tag{3.1}$$

$$A_2(T)\triangleq\sup_{0\le t\le T}\sigma^2(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2+\|H_t\|_{L^2}^2)+\int_0^T\sigma^2(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)dt, \quad (3.2)$$

$$A_3(T)\triangleq\sup_{0\le t\le T}\|H\|_{L^3}^3, \quad (3.3)$$

$$A_4(T)\triangleq\sup_{0\le t\le T}\sigma^{\frac14}(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)+\int_0^T\sigma^{\frac14}(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2+\|H_t\|_{L^2}^2)dt, \quad (3.4)$$

$$A_5(T)\triangleq\sup_{0\le t\le T}\|\rho^{\frac13}u\|_{L^3}^3, \quad (3.5)$$

where $\dot{v}=v_t+u\cdot\nabla v$ is the material derivative.

Now we will give the following key a priori estimates in this section, which guarantee the existence of a global classical solution of (1.1)–(1.5).

PROPOSITION 3.1. *Assume that initial data (ρ_0,u_0,H_0) satisfies (1.7)-(1.9). Let (ρ,u,H) is a smooth solution of (1.1)-(1.5) on $\Omega\times(0,T]$ satisfying*

$$\begin{cases}\sup\limits_{\Omega\times[0,T]}\rho\le 2\bar{\rho}, & A_1(T)+A_2(T)\le 2C_0^{\frac12},\\ A_3(T)\le 2C_0^{\frac19}, & A_4(\sigma(T))+A_5(\sigma(T))\le 2C_0^{\frac19},\end{cases} \quad (3.6)$$

then there exists a positive constant ε depending only on μ, λ, ν, a, γ, ρ_∞, $\bar{\rho}$, M_1 and M_2 such that

$$\begin{cases}\sup\limits_{\Omega\times[0,T]}\rho\le \frac{7\bar{\rho}}{4}, & A_1(T)+A_2(T)\le C_0^{\frac12},\\ A_3(T)\le C_0^{\frac19}, & A_4(\sigma(T))+A_5(\sigma(T))\le C_0^{\frac19},\end{cases} \quad (3.7)$$

provided $C_0\le\varepsilon$.

Proof. Proposition 3.1 is a consequence of Lemmas 3.1, 3.5-3.7 below. □

In what follows, we denote by C or C_i $(i=1,2,\cdots)$ the generic positive constants which may depend on μ,λ, ν,γ,a, $\rho_\infty,\bar{\rho}$, M_1 and M_2 but are independent of $T>0$. We also use $C(\alpha)$ to emphasize that C depends on α.

REMARK 3.1. Under the Assumption (3.6), it is easy to show that if $C_0\le 1$, there is a positive constant C such that

$$\int_0^T(\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^4)dt\le CC_0^{\frac29}, \quad (3.8)$$

which will be used frequently later.

The following Lemmas 3.1-3.7 will be proven under the same assumptions as in Proposition 3.1. First, we give the estimate of $A_3(T)$.

LEMMA 3.1. Under the conditions of Proposition 3.1, there is a positive constant ε_1 depending on μ, λ, ν, a, γ, ρ_∞, $\bar{\rho}$ and M_2 such that

$$A_3(T)\le C_0^{\frac19}, \quad (3.9)$$

provided $C_0\le\varepsilon_1$.

Proof. Multiplying (2.6)$_3$ by $3|H|H$ and integrating by parts over Ω, we have

$$\begin{aligned}\frac{d}{dt}\|H\|_{L^3}^3 &= -3\nu\int \mathrm{curl}H\cdot\mathrm{curl}(|H|H)dx+3\int |H|H\cdot\nabla u\cdot Hdx-2\int |H|^3\mathrm{div}udx\\ &\le C\|H\|_{L^\infty}\|\nabla H\|_{L^2}^2+C\|\nabla u\|_{L^2}\|H\|_{L^6}^3\\ &\le C\|\nabla H\|_{L^2}^{5/2}\|\mathrm{curl}^2H\|_{L^2}^{\frac12}+C\|\nabla H\|_{L^2}^2+C\|\nabla H\|_{L^2}^4+C\|\nabla u\|_{L^2}^4,\end{aligned} \tag{3.10}$$

which together with (3.6) and (3.8) indicates that

$$\begin{aligned}&\sup_{0\le t\le T}\|H\|_{L^3}^3\\ &\le \|H_0\|_{L^3}^3+C\int_0^T\|\nabla H\|_{L^2}^{5/2}\|\mathrm{curl}^2H\|_{L^2}^{\frac12}dt+CC_0+CC_0^{\frac29}\\ &\le \|H_0\|_{L^3}^3+C\int_0^{\sigma(T)}\left(\sigma^{\frac14}\|\nabla H\|_{L^2}^2\right)^{5/4}\left(\sigma^{\frac14}\|\mathrm{curl}^2H\|_{L^2}^2\right)^{1/4}\sigma^{-\frac38}dt\\ &\quad +C\sup_{t\in[\sigma(T),T]}\|\nabla H\|_{L^2}\int_{\sigma(T)}^T\left(\|\nabla H\|_{L^2}^2\right)^{3/4}\left(\sigma\|\mathrm{curl}^2H\|_{L^2}^2\right)^{1/4}dt+CC_0+CC_0^{\frac29}\\ &\le C_1C_0^{\frac16}\end{aligned} \tag{3.11}$$

where in the last inequality we have used the simple fact

$$\|H_0\|_{L^3}^3\le C\|H_0\|_{L^2}^{\frac32}\|\nabla H_0\|_{L^2}^{\frac32}\le C(M_2)C_0^{\frac34}. \tag{3.12}$$

Thus it follows from (3.12) that (3.9) holds provided $C_0\le\varepsilon_1\triangleq\min\{1,C_1^{-18}\}$. The proof of Lemma 3.1 is completed. ◻

LEMMA 3.2. *Under the conditions of Proposition 3.1, it holds*

$$A_1(T)\le CC_0+C\int_0^T\sigma\|\nabla u\|_{L^3}^3dt. \tag{3.13}$$

Proof. The proof of this lemma proceeds in two steps. First, let us consider the short-time estimate of H. Multiplying (2.6)$_3$ by H and integrating by parts over Ω, by (1.5), (2.1), (2.10), we have

$$\left(\frac12\|H\|_{L^2}^2\right)_t+\nu\|\mathrm{curl}H\|_{L^2}^2\le\|\nabla u\|_{L^2}\|H\|_{L^4}^2\le\frac{\nu}{2}\|\mathrm{curl}H\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|H\|_{L^2}^2,$$

which together with (2.10), (3.8) and Gronwall inequality gives

$$\sup_{0\le t\le T}\|H\|_{L^2}^2+\int_0^T\|\nabla H\|_{L^2}^2dt\le C\|H_0\|_{L^2}^2. \tag{3.14}$$

By Lemma 2.9, one easily deduces from (2.6)$_3$ and (1.5) that

$$\left(\frac{\nu}{2}\|\mathrm{curl}H\|_{L^2}^2\right)_t+\nu^2\|\mathrm{curl}^2H\|_{L^2}^2+\|H_t\|_{L^2}^2\le C(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^4)\|\nabla H\|_{L^2}^2, \tag{3.15}$$

using (2.10), (3.8) and Gronwall inequality, we get

$$\sup_{0\le t\le T}\|\nabla H\|_{L^2}^2+\int_0^T\left(\|\mathrm{curl}^2H\|_{L^2}^2+\|H_t\|_{L^2}^2\right)dt\le C\|\nabla H_0\|_{L^2}^2. \tag{3.16}$$

Besides, multiplying (3.15) by σ and integrating it over $(0,T)$, by (3.8) and (3.14), we obtain

$$\sup_{0\le t\le T}\left(\sigma\|\nabla H\|_{L^2}^2\right)+\int_0^T\sigma\left(\|\mathrm{curl}^2H\|_{L^2}^2+\|H_t\|_{L^2}^2\right)dt\le CC_0. \tag{3.17}$$

Next, it remains to show the estimate of u and we follow the same plan as used in [7,19]. We focus on the boundary terms and give the sketch of the proof. Let $m\ge 0$ be a real number which will be determined later. Multiplying $(2.6)_2$ by $\sigma^m\dot u$ and then integrating the resulting equality over Ω lead to

$$\begin{aligned}\int\sigma^m\rho|\dot u|^2dx=&-\int\sigma^m\dot u\cdot\nabla Pdx+(\lambda+2\mu)\int\sigma^m\nabla\mathrm{div}u\cdot\dot udx\\&-\mu\int\sigma^m\nabla\times\omega\cdot\dot udx+\int\sigma^m(H\cdot\nabla H-\nabla|H|^2/2)\cdot\dot udx\\\triangleq&\, I_1+I_2+I_3+I_4.\end{aligned}\tag{3.18}$$

First, by $(1.1)_1$ and Lemma 2.9, a direct calculation gives

$$\begin{aligned}I_1=&-\int\sigma^m u_t\cdot\nabla(P-P_\infty)dx-\int\sigma^m u\cdot\nabla u\cdot\nabla Pdx\\=&\left(\int\sigma^m(P-P_\infty)\mathrm{div}u\,dx\right)_t-m\sigma^{m-1}\sigma'\int(P-P_\infty)\mathrm{div}u\,dx\\&+\int\sigma^mP\nabla u:\nabla udx+(\gamma-1)\int\sigma^mP(\mathrm{div}u)^2dx-\int_{\partial\Omega}\sigma^mPu\cdot\nabla u\cdot nds\\\le&\left(\int\sigma^m(P-P_\infty)\mathrm{div}u\,dx\right)_t+C\|\nabla u\|_{L^2}^2+Cm\sigma^{m-1}\sigma'C_0,\end{aligned}\tag{3.19}$$

where we have used (2.34) with $f=P$ to deal with the boundary term in the second equality. Similarly, by (2.32), it indicates that

$$\begin{aligned}I_2=&(\lambda+2\mu)\int_{\partial\Omega}\sigma^m\mathrm{div}u\,(\dot u\cdot n)ds-(\lambda+2\mu)\int\sigma^m\mathrm{div}u\,\mathrm{div}\dot udx\\=&(\lambda+2\mu)\int_{\partial\Omega}\sigma^m\mathrm{div}u\,(u\cdot\nabla u\cdot n)ds-\frac{\lambda+2\mu}{2}\left(\int\sigma^m(\mathrm{div}u)^2dx\right)_t\\&+\frac{\lambda+2\mu}{2}\int\sigma^m(\mathrm{div}u)^3dx-(\lambda+2\mu)\int\sigma^m\mathrm{div}u\nabla u:\nabla udx\\&+\frac{m(\lambda+2\mu)}{2}\sigma^{m-1}\sigma'\int(\mathrm{div}u)^2dx.\end{aligned}\tag{3.20}$$

For the boundary term on the right-hand side of (3.20), applying Lemma 2.10 and (2.9), we obtain

$$\begin{aligned}&(\lambda+2\mu)\int_{\partial\Omega}\sigma^m\mathrm{div}u\,(u\cdot\nabla u\cdot n)ds\\=&\int_{\partial\Omega}\sigma^mFu\cdot\nabla u\cdot nds+\int_{\partial\Omega}\sigma^m(P-P_\infty)u\cdot\nabla u\cdot nds+\int_{\partial\Omega}\sigma^m\frac{|H|^2}{2}u\cdot\nabla u\cdot nds\\\le&C\sigma^m(\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2\|\nabla u\|_{L^2}^2)\\\le&C\sigma^m(\|\rho\dot u\|_{L^2}+\|H\cdot\nabla H\|_{L^2})\|\nabla u\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^2}^2+C\sigma^m\|\nabla H\|_{L^2}^2\|\nabla u\|_{L^2}^2\end{aligned}$$

$$\leq \frac{1}{2}\sigma^m\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\sigma^m\|\mathrm{curl}^2H\|_{L^2}^2+C\sigma^m(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+1), \quad (3.21)$$

where we have used

$$\|H\cdot\nabla H\|_{L^2}\leq C\|H\|_{L^3}\|\nabla H\|_{L^6}\leq CC_0^{\frac{1}{27}}(\|\mathrm{curl}^2H\|_{L^2}+\|\nabla H\|_{L^2}). \quad (3.22)$$

Therefore,

$$\begin{aligned} I_2 \leq & -\frac{\lambda+2\mu}{2}\left(\int\sigma^m(\mathrm{div}u)^2dx\right)_t+C\sigma^m\|\nabla u\|_{L^3}^3+\frac{1}{2}\sigma^m\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\sigma^m\|\mathrm{curl}^2H\|_{L^2}^2 \\ & +C\sigma^m(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)\|\nabla u\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2). \end{aligned} \quad (3.23)$$

Next, by (1.4), a straightforward computation shows that

$$\begin{aligned} I_3 = & -\frac{\mu}{2}\left(\int\sigma^m|\omega|^2dx\right)_t+\frac{\mu m}{2}\sigma^{m-1}\sigma'\int|\omega|^2dx \\ & -\mu\int\sigma^m(\nabla u^i\times\nabla_i u)\cdot\omega dx+\frac{\mu}{2}\int\sigma^m|\omega|^2\mathrm{div}u dx \\ \leq & -\frac{\mu}{2}\left(\int\sigma^m|\omega|^2dx\right)_t+Cm\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^3}^3. \end{aligned} \quad (3.24)$$

Finally, by (1.5) and (2.28), we have

$$\begin{aligned} I_4 = & \left(\int\sigma^m(H\cdot\nabla H-\nabla|H|^2/2)\cdot udx\right)_t-m\sigma^{m-1}\sigma'\int(H\cdot\nabla H-\nabla|H|^2/2)\cdot udx \\ & +\int\sigma^m\big((H\otimes H)_t:\nabla u-(|H|^2/2)_t\mathrm{div}u\big)dx+\int\sigma^m(H\cdot\nabla H-\nabla|H|^2/2)\cdot u\cdot\nabla udx \\ \leq & \left(\int\sigma^m(H\cdot\nabla H-\nabla|H|^2/2)\cdot udx\right)_t+C(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^2) \\ & +C\sigma^m(\|H_t\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)+C\sigma^m\|\nabla u\|_{L^3}^3+C\sigma^m\|\nabla H\|_{L^2}^2\|\nabla u\|_{L^2}^2 \\ & +C\sigma^m\|\nabla H\|_{L^2}^2(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4). \end{aligned} \quad (3.25)$$

Making use of the results (3.19), (3.23), (3.24) and (3.25), it follows from (3.18) that

$$\begin{aligned} & \left((\lambda+2\mu)\int\sigma^m(\mathrm{div}u)^2dx+\mu\int\sigma^m|\omega|^2dx\right)_t+\int\sigma^m\rho|\dot{u}|^2dx \\ \leq & \left(\int\sigma^m(P-P_\infty)\mathrm{div}udx\right)_t+\left(\int\sigma^m(H\cdot\nabla H-\nabla|H|^2/2)\cdot udx\right)_t \\ & +Cm\sigma^{m-1}\sigma'C_0+C(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+1)+C\sigma^m\|\nabla u\|_{L^3}^3 \\ & +C\sigma^m(\|H_t\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)+C\sigma^m\|\nabla H\|_{L^2}^2(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4), \end{aligned} \quad (3.26)$$

integrating over $(0,T]$, by (2.9), Lemma 2.8 and Young's inequality, we conclude that for any $m>0$,

$$\begin{aligned} & \sigma^m\|\nabla u\|_{L^2}^2+\int_0^T\int\sigma^m\rho|\dot{u}|^2dxdt \\ \leq & C\int_0^T\sigma^m(\|H_t\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)dt+C\int_0^T\sigma^m\|\nabla H\|_{L^2}^2(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4)dt \end{aligned}$$

$$+CC_0+C\int_0^T\sigma^m\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)dt+C\int_0^T\sigma^m\|\nabla u\|_{L^3}^3dt. \tag{3.27}$$

Choose $m=1$, together with (3.6), (3.8) and (3.17), we obtain (3.13). The proof of Lemma 3.2 is completed. □

LEMMA 3.3. *Under the conditions of Proposition 3.1, there is a positive constant ε_2 depending only on μ, λ, ν, a, γ, ρ_∞ and $\bar\rho$ such that if $C_0\le\varepsilon_2$,*

$$A_2(T)\le CC_0^{\frac{31}{54}}+CA_1(T)+C\int_0^T\sigma^2\|\nabla u\|_{L^4}^4dt. \tag{3.28}$$

Proof. Operating $\sigma^m\dot u^j[\partial/\partial t+\operatorname{div}(u\cdot)]$ to $(2.19)^j$, summing with respect to j, and integrating over Ω, together with $(1.1)_1$, we get

$$\begin{aligned}
&\left(\frac{\sigma^m}{2}\int\rho|\dot u|^2dx\right)_t-\frac{m}{2}\sigma^{m-1}\sigma'\int\rho|\dot u|^2dx\\
&=-\int\sigma^m\dot u^j\operatorname{div}(\rho\dot u^ju)dx-\frac12\int\sigma^m\rho_t|\dot u|^2dx+\int\sigma^m(\dot u\cdot\nabla F_t+\dot u^j\operatorname{div}(u\partial_jF))dx\\
&\quad+\mu\int\sigma^m(-\dot u\cdot\nabla\times\omega_t-\dot u^j\operatorname{div}((\nabla\times\omega)^ju))dx\\
&\quad+\int\sigma^m(\dot u\cdot(\operatorname{div}(H\otimes H))_t+\dot u^j\operatorname{div}((\operatorname{div}(H\otimes H^j)u))dx\\
&=\int\sigma^m(\dot u\cdot\nabla F_t+\dot u^j\operatorname{div}(u\partial_jF))dx\\
&\quad+\mu\int\sigma^m(-\dot u\cdot\nabla\times\omega_t-\dot u^j\operatorname{div}((\nabla\times\omega)^ju))dx\\
&\quad+\int\sigma^m(\dot u\cdot(\operatorname{div}(H\otimes H))_t+\dot u^j\operatorname{div}((\operatorname{div}(H\otimes H^j)u))dx\\
&\triangleq J_1+J_2+J_3.
\end{aligned}\tag{3.29}$$

Let us estimate J_1,J_2 and J_3. By (1.4) and $(2.6)_1$, a direct computation yields

$$\begin{aligned}
J_1&=\int_{\partial\Omega}\sigma^mF_t\dot u\cdot nds-\int\sigma^mF_t\operatorname{div}\dot udx-\int\sigma^mu\cdot\nabla\dot u^j\partial_jFdx\\
&=\int_{\partial\Omega}\sigma^mF_t\dot u\cdot nds-(\lambda+2\mu)\int\sigma^m(\operatorname{div}\dot u)^2dx+(\lambda+2\mu)\int\sigma^m\operatorname{div}\dot u\nabla u:\nabla udx\\
&\quad-\gamma\int\sigma^mP\operatorname{div}\dot u\operatorname{div}udx+\int\sigma^m\operatorname{div}\dot u u\cdot\nabla Fdx-\int\sigma^mu\cdot\nabla\dot u^j\partial_jFdx\\
&\quad+\int\sigma^m\operatorname{div}\dot uH\cdot H_tdx+\int\sigma^m\operatorname{div}\dot uu\cdot\nabla H\cdot Hdx\\
&\le\int_{\partial\Omega}\sigma^mF_t\dot u\cdot nds-(\lambda+2\mu)\int\sigma^m(\operatorname{div}\dot u)^2dx+\frac{\delta}{12}\sigma^m\|\nabla\dot u\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^4}^4\\
&\quad+C\sigma^m\big(\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^3}^2+C_0^{\frac{2}{27}}\|\nabla H_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2\|\nabla H\|_{L^6}^2\big),
\end{aligned}\tag{3.30}$$

where in the second equality we have used

$$F_t=(2\mu+\lambda)\operatorname{div}\dot u-(2\mu+\lambda)\nabla u:\nabla u-u\cdot\nabla F+u\cdot\nabla H\cdot H+\gamma P\operatorname{div}u-H\cdot H_t.$$

For the boundary term on the right-hand side of (3.30), using Lemma 2.10 with $f=F$, we have

$$
\begin{aligned}
&\int_{\partial\Omega}\sigma^m F_t\dot{u}\cdot n ds=-\int_{\partial\Omega}\sigma^m F_t(u\cdot\nabla n\cdot u)ds\\
=&-\left(\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)Fds\right)_t+m\sigma^{m-1}\sigma'\int_{\partial\Omega}(u\cdot\nabla n\cdot u)Fds\\
&+\int_{\partial\Omega}\sigma^m\left(F\dot{u}\cdot\nabla n\cdot u+Fu\cdot\nabla n\cdot\dot{u}\right)ds-\int_{\partial\Omega}\sigma^m\left(F(u\cdot\nabla)u\cdot\nabla n\cdot u+Fu\cdot\nabla n\cdot(u\cdot\nabla)u\right)ds\\
\le&-\left(\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)Fds\right)_t+Cm\sigma^{m-1}\sigma'\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^2}+\frac{\delta}{12}\sigma^m\|\nabla\dot{u}\|_{L^2}^2\\
&+C\sigma^m(\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^2}^2+\|\nabla u\|_{L^2}^3\|\nabla F\|_{L^2}+\|\nabla F\|_{L^6}\|\nabla u\|_{L^2}^3+\|\nabla u\|_{L^4}^4). \qquad (3.31)
\end{aligned}
$$

From Lemma 2.9 and (3.22), we have

$$
\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^2}^2\le C(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2, \qquad (3.32)
$$

$$
\begin{aligned}
\|\nabla F\|_{L^6}\|\nabla u\|_{L^2}^3\le&\frac{\delta}{12}\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+C\|\mathrm{curl}^2H\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&+C\|\nabla H\|_{L^2}^2+C\|\nabla H\|_{L^2}^4, \qquad (3.33)
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^3}^2\le&\frac{\delta}{12}\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C\|\sqrt{\rho}\dot{u}\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&+C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2\|\mathrm{curl}^2H\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^4. \qquad (3.34)
\end{aligned}
$$

Putting (3.31)-(3.34) into (3.30), we have

$$
\begin{aligned}
J_1\le&-(\lambda+2\mu)\int\sigma^m(\mathrm{div}\dot{u})^2dx-\left(\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)Fds\right)_t\\
&+\frac{\delta}{3}\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C\sigma^mC_0^{\frac{2}{27}}\|\nabla H_t\|_{L^2}^2+C\sigma^m(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4)\|\mathrm{curl}^2H\|_{L^2}^2\\
&+C\sigma^m(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)\|\nabla u\|_{L^2}^2+C\sigma^m\|\sqrt{\rho}\dot{u}\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&+C\sigma^m\|\nabla u\|_{L^4}^4+C\sigma^m(\|\nabla u\|_{L^2}^2+1)(\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^4)+C\sigma^m\|\nabla H\|_{L^2}^2\\
&+Cm\sigma^{m-1}\sigma'(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2+\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^2). \qquad (3.35)
\end{aligned}
$$

Next, notice $\omega_t=\mathrm{curl}\dot{u}-u\cdot\nabla\omega-\nabla u^i\times\partial_i u$ and (1.4), it follows

$$
\begin{aligned}
J_2=&-\mu\int\sigma^m|\mathrm{curl}\dot{u}|^2dx+\mu\int\sigma^m\mathrm{curl}\dot{u}\cdot(\nabla u^i\times\nabla_i u)dx\\
&-\mu\int\sigma^m\mathrm{div}u(\omega\cdot\mathrm{curl}\dot{u})dx-\mu\int\sigma^m(\omega\times\nabla u^i)\cdot\nabla_i\dot{u}dx\\
\le&-\mu\int\sigma^m|\mathrm{curl}\dot{u}|^2dx+\frac{\delta}{3}\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^4}^4. \qquad (3.36)
\end{aligned}
$$

Finally, a direct computation shows that

$$
\begin{aligned}
J_3=&-\int\sigma^m\nabla\dot{u}:(H\otimes H)_tdx-\mu\int\sigma^m H\cdot\nabla H^j u\cdot\nabla\dot{u}^jdx\\
\le&C\sigma^m(\|\nabla\dot{u}\|_{L^2}\|H\|_{L^3}\|H_t\|_{L^6}+\|\nabla\dot{u}\|_{L^2}\|H\|_{L^6}\|\nabla H\|_{L^6}\|u\|_{L^6})
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\delta}{3}\sigma^m\|\nabla\dot u\|_{L^2}^2+C\sigma^m(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4)\|\mathrm{curl}^2 H\|_{L^2}^2\\
&\quad +C\sigma^m C_0^{\frac{2}{27}}\|\nabla H_t\|_{L^2}^2+C\sigma^m\|\nabla H\|_{L^2}^4\|\nabla u\|_{L^2}^2. \qquad (3.37)
\end{aligned}
$$

Combining (3.35), (3.36) with (3.37), we deduce from (3.29) that

$$
\begin{aligned}
&\left(\frac{\sigma^m}{2}\|\sqrt{\rho}\dot u\|_{L^2}^2\right)_t+(\lambda+2\mu)\sigma^m\|\mathrm{div}\dot u\|_{L^2}^2+\mu\sigma^m\|\mathrm{curl}\dot u\|_{L^2}^2\\
&\le -\left(\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)F ds\right)_t+\delta\sigma^m\|\nabla\dot u\|_{L^2}^2+C\sigma^m C_0^{\frac{2}{27}}\|\nabla H_t\|_{L^2}^2\\
&\quad +C\sigma^m\|\nabla u\|_{L^4}^4+C\sigma^m(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4)\|\mathrm{curl}^2 H\|_{L^2}^2\\
&\quad +C\sigma^m(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2)\|\nabla u\|_{L^2}^2+C\sigma^m\|\sqrt{\rho}\dot u\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&\quad +C\sigma^m(\|\nabla u\|_{L^2}^2+1)(\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^4)+C\sigma^m\|\nabla H\|_{L^2}^2\\
&\quad +Cm\sigma^{m-1}\sigma'(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2+\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^2). \qquad (3.38)
\end{aligned}
$$

As observed in [4], it follows from (2.35) that

$$
(\dot u-(u\cdot\nabla n)\times u^{\perp})\cdot n=0, \qquad (3.39)
$$

which together with Lemma 2.3 yields

$$
\begin{aligned}
\|\nabla\dot u\|_{L^2}&\le C(\|\mathrm{div}\dot u\|_{L^2}+\|\mathrm{curl}\dot u\|_{L^2}+\|\nabla[(u\cdot\nabla n)\times u^{\perp}]\|_{L^2})\\
&\le C(\|\mathrm{div}\dot u\|_{L^2}+\|\mathrm{curl}\dot u\|_{L^2}+\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^4}^2). \qquad (3.40)
\end{aligned}
$$

By (3.40) and Lemma 2.8, choosing δ small enough, and integrating (3.38) over $(0,T]$, for $m>0$, we get

$$
\begin{aligned}
&\sigma^m\|\sqrt{\rho}\dot u\|_{L^2}^2+\int_0^T\sigma^m\|\nabla\dot u\|_{L^2}^2dt\\
&\le -\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)F ds+CC_0^{\frac{2}{27}}\int_0^T\sigma^m\|\nabla H_t\|_{L^2}^2dt\\
&\quad +C\int_0^T\sigma^m\|\nabla u\|_{L^4}^4dt+CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\mathrm{curl}^2 H\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2)\\
&\quad +CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+CC_0\sup_{0\le t\le \sigma(T)}\sigma^{m-1}\|\nabla u\|_{L^2}^2\\
&\quad +C\int_0^{\sigma(T)}m\sigma^{m-1}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2)dt+CC_0. \qquad (3.41)
\end{aligned}
$$

For the boundary term in the right-hand side of (3.41) , using Lemma 2.10 again, we have

$$
\begin{aligned}
&\int_{\partial\Omega}(u\cdot\nabla n\cdot u)F ds\le C\|\nabla u\|_{L^2}^2\|\nabla F\|_{L^2}\\
&\le \frac{1}{2}\|\sqrt{\rho}\dot u\|_{L^2}^2+CC_0^{\frac{2}{27}}\|\mathrm{curl}^2 H\|_{L^2}^2+C(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^4). \qquad (3.42)
\end{aligned}
$$

Therefore,

$$
\sigma^m\|\sqrt{\rho}\dot u\|_{L^2}^2+\int_0^T\sigma^m\|\nabla\dot u\|_{L^2}^2dt-C_2C_0^{\frac{2}{27}}\int_0^T\sigma^m\|\nabla H_t\|_{L^2}^2dt
$$

$$
\begin{aligned}
&\le C\int_0^T \sigma^m\|\nabla u\|_{L^4}^4dt+CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\mathrm{curl}^2H\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2)\\
&\quad+CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+CC_0\sup_{0\le t\le \sigma(T)}\sigma^{m-1}\|\nabla u\|_{L^2}^2\\
&\quad+C\int_0^{\sigma(T)} m\sigma^{m-1}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2)dt+CC_0\\
&\quad+CC_0^{\frac{2}{27}}\sigma^m\|\mathrm{curl}^2H\|_{L^2}^2+C\sigma^m(\|\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^4).
\end{aligned}\tag{3.43}
$$

Next, we need to estimate the term $\|\nabla H_t\|_{L^2}$. Noticing that

$$
\begin{cases}
H_{tt}-\nu\nabla\times(\mathrm{curl}H_t)=(H\cdot\nabla u-u\cdot\nabla H-H\mathrm{div}u)_t, & \text{in}\quad \Omega,\\
H_t\cdot n=0,\quad \mathrm{curl}H_t\times n=0, & \text{on}\quad \partial\Omega,
\end{cases}\tag{3.44}
$$

and after direct computations we obtain

$$
\begin{aligned}
&\left(\frac{\sigma^m}{2}\|H_t\|_{L^2}^2\right)_t+\sigma^m\|\mathrm{curl}H_t\|_{L^2}^2-\frac{m}{2}\sigma^{m-1}\sigma'\|H_t\|_{L^2}^2\\
&=\int\sigma^m(H_t\cdot\nabla u-u\cdot\nabla H_t-H_t\mathrm{div}u)\cdot H_tdx\\
&\quad+\int\sigma^m(H\cdot\nabla\dot u-\dot u\cdot\nabla H-H\mathrm{div}\dot u)\cdot H_tdx\\
&\quad-\int\sigma^m(H\cdot\nabla(u\cdot\nabla u)-(u\cdot\nabla u)\cdot\nabla H-H\mathrm{div}(u\cdot\nabla u))\cdot H_tdx\\
&\triangleq K_1+K_2+K_3.
\end{aligned}\tag{3.45}
$$

By Lemma 2.1 and Lemma 2.9, a direct calculation leads to

$$
\begin{aligned}
K_1&\le C\sigma^m(\|H_t\|_{L^3}\|H_t\|_{L^6}\|\nabla u\|_{L^2}+\|u\|_{L^6}\|H_t\|_{L^3}\|\nabla H_t\|_{L^2})\\
&\le\frac{\delta}{4}\sigma^m\|\nabla H_t\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^2}^4\|H_t\|_{L^2}^2.
\end{aligned}\tag{3.46}
$$

Similarly,

$$
\begin{aligned}
K_2&\le C\sigma^m\|H\|_{L^3}\|H_t\|_{L^6}\|\nabla\dot u\|_{L^2}-\int_{\partial\Omega}\sigma^m(\dot u\cdot n)(H\cdot H_t)ds\\
&\quad+\int\sigma^m\mathrm{div}\dot u H\cdot H_tdx+\int\sigma^m\dot u\cdot\nabla H_t\cdot Hdx\\
&\le\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)(H\cdot H_t)ds+CC_0^{\frac{1}{27}}\sigma^m(\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2).
\end{aligned}\tag{3.47}
$$

For the boundary term in the last inequality, we use the similar method as that used in Lemma 2.10 to get that

$$
\begin{aligned}
&\int_{\partial\Omega}\sigma^m(u\cdot\nabla n\cdot u)(H\cdot H_t)ds\\
&\le C\sigma^m(\|u\|_{L^6}\|\nabla u\|_{L^2}\|H\|_{L^6}\|H_t\|_{L^6}+\|u\|_{L^6}^2\|\nabla H\|_{L^2}\|H_t\|_{L^6}\\
&\qquad+\|u\|_{L^6}^2\|\nabla H_t\|_{L^2}\|H\|_{L^6}+\|u\|_{L^6}^2\|H\|_{L^6}\|H_t\|_{L^6})\\
&\le\frac{\delta}{4}\sigma^m\|\nabla H_t\|_{L^2}^2+C\sigma^m\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2.
\end{aligned}\tag{3.48}
$$

Combining (3.47) and (3.48), we have

$$K_2 \le \frac{\delta}{4}\sigma^m\|\nabla H_t\|_{L^2}^2 + C\sigma^m(\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2 + C^{\frac{1}{27}}(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)). \tag{3.49}$$

Similarly, by (3.48), a direct computation yields

$$\begin{aligned} K_3 \le& \frac{\delta}{2}\sigma^m\|\nabla H_t\|_{L^2}^2 + C\sigma^m(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2) \\ &+ C\sigma^m\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1). \end{aligned} \tag{3.50}$$

Putting (3.46), (3.49) and (3.50) into (3.45), choosing δ small enough, we have

$$\begin{aligned} &\left(\sigma^m\|H_t\|_{L^2}^2\right)_t + \sigma^m\|\nabla H_t\|_{L^2}^2 - CC_0^{\frac{1}{27}}\sigma^m(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) \\ \le& C\sigma^m(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \\ &+ C\sigma^m\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1) + Cm\sigma^{m-1}\sigma'\|H_t\|_{L^2}^2. \end{aligned} \tag{3.51}$$

Integrating over $(0,T]$, then by Lemma 2.8, for $m>0$, we get

$$\begin{aligned} &\sigma^m\|H_t\|_{L^2}^2 + \int_0^T \sigma^m\|\nabla H_t\|_{L^2}^2 dt - C_3C_0^{\frac{1}{27}}\int_0^T \sigma^m(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)dt \\ \le& CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) + CC_0\sup_{0\le t\le T}\sigma^m\|\nabla u\|_{L^2}^2 \\ &+ CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^m(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + C\int_0^{\sigma(T)} m\sigma^{m-1}\|H_t\|_{L^2}^2 dt. \end{aligned} \tag{3.52}$$

Now take $m=2$ in (3.43) and (3.52), we deduce after adding them together that

$$\begin{aligned} &\sigma^2(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^T \sigma^2(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)dt \\ &\quad - C_2C_0^{\frac{2}{27}}\int_0^T \sigma^2\|\nabla H_t\|_{L^2}^2 dt - C_3C^{\frac{1}{27}}\int_0^T \sigma^2(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)dt \\ \le& C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4 dt + CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^2(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \\ &\quad + CC_0^{\frac{2}{9}}\sup_{0\le t\le T}\sigma^2(\|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + CC_0\sup_{0\le t\le T}\sigma\|\nabla u\|_{L^2}^2 \\ &\quad + C\int_0^{\sigma(T)} \sigma(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2)dt + CC_0 \\ &\quad + CC_0^{\frac{2}{27}}\sigma^2\|\mathrm{curl}^2 H\|_{L^2}^2 + C\sigma^2(\|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^4) \\ \le& C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4 dt + CC_0^{\frac{13}{18}} + CA_1(T) + CC_0 + CC_0^{\frac{31}{54}}. \end{aligned} \tag{3.53}$$

Thus we have

$$\begin{aligned} &\sup_{0\le t\le T}\sigma^2(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^T \sigma^2(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)dt \\ \le& C\int_0^T \sigma^2\|\nabla u\|_{L^4}^4 dt + CA_1(T) + CC_0^{\frac{31}{54}}, \end{aligned} \tag{3.54}$$

provided that C_0 is chosen to satisfy

$$C_0 \le \varepsilon_2 \triangleq \min\{\varepsilon_1, (4C_2)^{-\frac{27}{2}}, (4C_3)^{-27}\}.$$

Finally, by Lemma 2.1 and $(1.1)_3$, it holds

$$\begin{aligned}\|\mathrm{curl}^2 H\|_{L^2} &\le C(\|H_t\|_{L^2} + \|\mathrm{curl}^2 H\|_{L^2}^{\frac{1}{2}} \|\nabla H\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2} + \|\nabla H\|_{L^2}\|\nabla u\|_{L^2})\\ &\le \frac{1}{2}\|\mathrm{curl}^2 H\|_{L^2} + C(\|H_t\|_{L^2} + \|\nabla H\|_{L^2}\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}\|\nabla u\|_{L^2}). \quad (3.55)\end{aligned}$$

Thus, by (3.6) and (3.55), we have

$$\sup_{0\le t\le T} \sigma^2 \|\mathrm{curl}^2 H\|_{L^2}^2 \le C\int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt + CA_1(T) + CC_0^{\frac{31}{54}}. \quad (3.56)$$

Combining (3.54) and (3.56), we give (3.28) and complete the proof of Lemma 3.3. □

LEMMA 3.4. *Under the conditions of Proposition 3.1, there exist positive constants $\tilde{C} = C(\bar{\rho}, M_1, M_2)$ and ε_3 depending only on μ, λ, ν, γ, a, ρ_∞, $\bar{\rho}$, M_1 and M_2 such that if $C_0 < \varepsilon_3$,*

$$\sup_{0\le t\le \sigma(T)} \|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 dt \le \tilde{C}, \quad (3.57)$$

$$\sup_{0\le t\le \sigma(T)} t(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^{\sigma(T)} t(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \le \tilde{C}. \quad (3.58)$$

Proof. As we have done in the proof of Lemma 2.9, multiplying $(2.6)_2$ by u_t and integrating over Ω, using (3.6), Sobolev's and Young's inequalities leads to

$$\begin{aligned}&\left(\frac{\lambda+2\mu}{2}\int (\mathrm{div} u)^2 dx + \frac{\mu}{2}\int |\mathrm{curl} u|^2 dx - \int (P - P_\infty - \frac{|H|^2}{2})\mathrm{div} u dx\right)_t + \int \rho|\dot{u}|^2 dx\\ &= \left(\int (H\cdot\nabla H)\cdot u dx\right)_t + \int \rho\dot{u}\cdot(u\cdot\nabla u)dx - \int P_t \mathrm{div} u dx\\ &\quad - \int (H\cdot\nabla H - \nabla H\cdot H)_t \cdot u dx\\ &\triangleq \frac{d}{dt}L_0 + L_1 + L_2 + L_3. \end{aligned} \quad (3.59)$$

By (2.1) and (3.9), we have

$$L_0 \le C\|H\|_{L^3}\|\nabla H\|_{L^2}\|u\|_{L^6} \le \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + CC_0^{\frac{2}{27}}\|\nabla H\|_{L^2}^2. \quad (3.60)$$

Using Lemma 2.1, Lemma 2.9, (3.6) and (3.22) yields

$$\begin{aligned}L_1 &= \int \rho\dot{u}\cdot(u\cdot\nabla u)dx\\ &\le C\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}\|\rho^{1/3}u\|_{L^3}\|\nabla u\|_{L^6}\\ &\le CC_0^{\frac{1}{27}}\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|P-P_\infty\|_{L^6}^2 + \|H\cdot\nabla H\|_{L^2}^2)\\ &\le C_4 C_0^{\frac{1}{27}}(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2 + \|\mathrm{curl}^2 H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + C(\|\nabla u\|_{L^2}^2 + \|P-P_\infty\|_{L^6}^2). \end{aligned} \quad (3.61)$$

Next, by $(2.6)_1$, (2.13), (3.6), Lemma 2.9, Sobolev's and Young's inequalities leads to

$$\begin{aligned}L_2=&-\frac{1}{\lambda+2\mu}\int(P-P_\infty)(F\mathrm{div}u+\nabla F\cdot u)dx\\&-\frac{1}{2(\lambda+2\mu)}\int(P-P_\infty)^2\mathrm{div}udx+\gamma\int P(\mathrm{div}u)^2dx\\\le&C(\|\nabla u\|_{L^2}\|F\|_{L^2}+\|P-P_\infty\|_{L^3}\|\nabla F\|_{L^2}\|u\|_{L^6})+\|P-P_\infty\|_{L^2}\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^2)\\\le&C\|\nabla u\|_{L^2}(\|\sqrt{\rho}\dot u\|_{L^2}+\|\nabla u\|_{L^2}+\|P-P_\infty\|_{L^2}+C_0^{\frac{1}{27}}(\|\mathrm{curl}^2H\|_{L^2}+\|\nabla H\|_{L^2}))\\\le&\frac14\|\sqrt{\rho}\dot u\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|P-P_\infty\|_{L^2}^2+C_0^{\frac{2}{27}}(\|\mathrm{curl}^2H\|_{L^2}^2+\|\nabla H\|_{L^2}^2)).\end{aligned}\tag{3.62}$$

Using Lemma 2.1 and (3.17), a direct computation yields

$$\begin{aligned}L_3=&-\int(H_t\cdot\nabla H-\nabla H\cdot H_t)\cdot udx-\int(H\cdot\nabla H_t-\nabla H_t\cdot H)\cdot udx\\\le&C(\|H_t\|_{L^2}\|\nabla H\|_{L^3}\|\nabla u\|_{L^2}+\|H_t\|_{L^2}\|H\|_{L^3}\|\nabla u\|_{L^6})\\\le&C_5C_0^{\frac{1}{27}}(\|\rho^{\frac12}\dot u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2+\|\nabla H\|_{L^2}^2+\|P-P_\infty\|_{L^6}^2)\\&+C\|H_t\|_{L^2}^2+C(\|\nabla H\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}\|\nabla H\|_{L^2})\|\nabla u\|_{L^2}^2.\end{aligned}\tag{3.63}$$

Putting (3.60)-(3.63) into (3.59), we obtain

$$\begin{aligned}&\left((\lambda+2\mu)\|\mathrm{div}u\|_{L^2}^2+\mu\|\mathrm{curl}u\|_{L^2}^2-2\int(P-P_\infty-\frac{|H|^2}{2})\mathrm{div}udx\right)_t+\int\rho|\dot u|^2dx\\\le&(\frac{\mu}{4}\|\nabla u\|_{L^2}^2+CC_0^{\frac{2}{27}}\|\nabla H\|_{L^2}^2)_t+C(1+\|\nabla H\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}\|\nabla H\|_{L^2})\|\nabla u\|_{L^2}^2\\&+C\left(\|P-P_\infty\|_{L^6}^2+\|P-P_\infty\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2+\|\nabla H\|_{L^2}^2\right),\end{aligned}\tag{3.64}$$

provided that $C_0<\hat\varepsilon_1\triangleq(4C_4+4C_5)^{-27}$. By Gronwall's inequality, (3.16) and Lemmas 2.3, 2.8, one has

$$\sup_{0\le t\le\sigma(T)}\|\nabla u\|_{L^2}^2+\int_0^{\sigma(T)}\int\rho|\dot u|^2dxdt\le C(\|\nabla u_0\|_{L^2}^2+\|\nabla H_0\|_{L^2}^2)+CC_0^{1/3},\tag{3.65}$$

which yields (3.57).

It remains to prove (3.58). Taking $m=2-s$ in (3.38), (3.52), and integrating over $(0,\sigma(T)]$ instead of $(0,T]$, in a similar way as we have gotten (3.53), we obtain

$$\begin{aligned}&\sup_{0\le t\le\sigma(T)}\sigma(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2)+\int_0^{\sigma(T)}\sigma(\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)dt\\\le&C\int_0^{\sigma(T)}\sigma\|\nabla u\|_{L^4}^4dt+\tilde C,\end{aligned}\tag{3.66}$$

where we have taken advantage of (3.55) and (3.57). Furthermore, by (2.18), (3.22) and (3.57), for $s\in(1/2,1]$, we have

$$\begin{aligned}&\int_0^{\sigma(T)}t\|\nabla u\|_{L^4}^4dt\\\le&C\int_0^{\sigma(T)}t(\|\sqrt{\rho}\dot u\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3+\|P-P_\infty\|_{L^6}^3+\|\nabla H\|_{L^2}^3)\|\nabla u\|_{L^2}dt+C\int_0^{\sigma(T)}t\|\nabla u\|_{L^2}^4dt\end{aligned}$$

$$
\begin{aligned}
&\le C+C\int_0^{\sigma(T)} t^{-\frac{1}{4}}(t^{\frac{1}{4}}\|\nabla u\|_{L^2}^2)^{\frac{1}{2}}(t^{\frac{1}{4}}\|\sqrt{\rho}\dot u\|_{L^2}^2)^{\frac{1}{2}}(t\|\sqrt{\rho}\dot u\|_{L^2}^2)dt\\
&\quad +C\int_0^{\sigma(T)} t^{-\frac{1}{4}}(t^{\frac{1}{4}}\|\nabla u\|_{L^2}^2)^{\frac{1}{2}}(t^{\frac{1}{4}}\|\mathrm{curl}^2 H\|_{L^2}^2)^{\frac{1}{2}}(t\|\mathrm{curl}^2 H\|_{L^2}^2)dt\\
&\le CC_0^{\frac{1}{9}}\sup_{0\le t\le\sigma(T)}(t(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2))+C. \qquad (3.67)
\end{aligned}
$$

Besides, from (3.55) and (3.57), we have

$$
\sup_{0\le t\le\sigma(T)} t\|\mathrm{curl}^2 H\|_{L^2}^2 \le C\sup_{0\le t\le\sigma(T)} t\|H_t\|_{L^2}^2+\tilde C. \qquad (3.68)
$$

Then combining this with (3.66) and (3.67), we have

$$
\begin{aligned}
&\sup_{0\le t\le\sigma(T)}\sigma(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2)+\int_0^{\sigma(T)}\sigma(\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)dt\\
\le& C_6C_0^{\frac{1}{9}}\sup_{0\le t\le\sigma(T)}\sigma(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2)+\tilde C. \qquad (3.69)
\end{aligned}
$$

Therefore, if we choose C_0 to be such that $C_0\le\varepsilon_3\triangleq\min\{\hat\varepsilon_1,(2C_6)^{-9}\}$, (3.69) and (3.68) implies (3.58). The proof of Lemma 3.4 is completed. ∎

LEMMA 3.5. *Under the conditions of Proposition 3.1, there exists a positive constant ε_4 depending only on $\mu,\lambda,\nu,\gamma,a,\rho_\infty,\bar\rho$, M_1 and M_2 such that if $C_0<\varepsilon_4$,*

$$
A_4(\sigma(T))+A_5(\sigma(T))\le C_0^{\frac{1}{9}}. \qquad (3.70)
$$

Proof. We begin with the estimate on $A_4(\sigma(T))$. Using (3.57), we have

$$
\begin{aligned}
A_4(\sigma(T))\le&\sup_{0\le t\le\sigma(T)}(\|\nabla u\|_{L^2}^2)^{\frac{3}{4}}\sup_{0\le t\le\sigma(T)}(t\|\nabla u\|_{L^2}^2)^{\frac{1}{4}}\\
&+\sup_{0\le t\le\sigma(T)}(\|\nabla H\|_{L^2}^2)^{\frac{3}{4}}\sup_{0\le t\le\sigma(T)}(t\|\nabla H\|_{L^2}^2)^{\frac{1}{4}}\\
&+\left(\int_0^{\sigma(T)}\|\rho^{\frac{1}{2}}\dot u\|_{L^2}^2dt\right)^{\frac{3}{4}}\left(\int_0^{\sigma(T)}t\|\rho^{\frac{1}{2}}\dot u\|_{L^2}^2dt\right)^{\frac{1}{4}}\\
&+\left(\int_0^{\sigma(T)}\|\mathrm{curl}^2 H\|_{L^2}^2dt\right)^{\frac{3}{4}}\left(\int_0^{\sigma(T)}t\|\mathrm{curl}^2 H\|_{L^2}^2dt\right)^{\frac{1}{4}}\\
\le& CA_1(T)^{\frac{1}{4}}\le CC_0^{\frac{1}{8}}\le C_0^{\frac{1}{9}}. \qquad (3.71)
\end{aligned}
$$

Next, it remains to estimate $A_5(\sigma(T))$. Multiplying $(1.1)_2$ by $3|u|u$, and integrating over Ω leads to

$$
\begin{aligned}
\left(\int\rho|u|^3dx\right)_t\le& C\int|u||\nabla u|^2dx+C\int|P-P_\infty||u||\nabla u|dx+C\int|H||\nabla H||u|^2dx\\
\le& C\|\nabla u\|_{L^2}^{\frac{5}{2}}(\|\rho\dot u\|_{L^2}^{\frac{1}{2}}+\|P-P_\infty\|_{L^2}^{\frac{1}{2}}+\|\mathrm{curl}^2 H\|_{L^2}^{\frac{1}{2}}+\|\nabla H\|_{L^2}^{\frac{1}{2}})\\
&+C\|\nabla u\|_{L^2}^3+CC_0^{\frac{1}{6}}\|\nabla u\|_{L^2}^2+C(\|\nabla H\|_{L^2}^4+\|\nabla u\|_{L^2}^4). \qquad (3.72)
\end{aligned}
$$

Hence, integrating (3.72) over $(0,\sigma(T))$ and using (3.6), (3.8), we get

$$\begin{aligned}\sup_{0\le t\le\sigma(T)}\int\rho|u|^3dx\le{}& C\int_0^{\sigma(T)}(t^{\frac14}\|\nabla u\|_{L^2}^2)^{\frac54}(t^{\frac14}(\|\rho\dot u\|_{L^2}^2+\|\mathrm{curl}^2H\|_{L^2}^2))^{\frac14}t^{-\frac38}dt\\&+C\int_0^{\sigma(T)}(t^{\frac14}\|\nabla u\|_{L^2}^2)^{\frac54}(t^{\frac14}\|\nabla H\|_{L^2}^2)^{\frac14}t^{-\frac38}dt\\&+CC_0+CC_0^{\frac29}+\int\rho_0|u_0|^3dx\\ \le{}& CC_0^{\frac16}+\int\rho_0|u_0|^3dx\le C_7C_0^{\frac16},\end{aligned}\tag{3.73}$$

where we have used the fact

$$\int\rho_0|u_0|^3dx\le C\|\rho_0^{\frac12}u_0\|_{L^2}^{\frac32}\|\nabla u_0\|_{L^2}^{\frac32}\le CC_0^{\frac32}.\tag{3.74}$$

Finally, set $\varepsilon_4\triangleq\min\{\varepsilon_3,(C_7)^{-18}\}$, we get $A_5(\sigma(T))\le C_0^{\frac19}$. The proof of Lemma 3.5 is completed. □

LEMMA 3.6. *Under the conditions of Proposition 3.1, there exists a positive constant ε_5 depending only on μ, λ, ν, γ, a, ρ_∞, $\bar\rho$, M_1 and M_2 such that*

$$A_1(T)+A_2(T)\le C_0^{\frac12},\tag{3.75}$$

provided $C_0\le\varepsilon_5$.

Proof. By (2.1) and (2.12), one can check that

$$\int_0^T\sigma\|\nabla u\|_{L^3}^3dt\le C\int_0^T\sigma\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2dt\le CC_0+C\int_0^T\sigma^2\|\nabla u\|_{L^4}^4dt,\tag{3.76}$$

which, along with (3.13) and (3.28) gives

$$A_1(T)+A_2(T)\le C(C_0^{\frac{31}{54}}+\int_0^T\sigma^2\|\nabla u\|_{L^4}^4dt).\tag{3.77}$$

So it reduces to estimating $\int_0^T\sigma^2\|\nabla u\|_{L^4}^4dt$. On the one hand, by (2.18), (3.22), (3.6) and Lemma 2.9 again, it indicates that

$$\begin{aligned}&\int_0^{\sigma(T)}t^2\|\nabla u\|_{L^4}^4dt\\&\le C\int_0^{\sigma(T)}t^2\big((\|\sqrt\rho\dot u\|_{L^2}^3+\|P-P_\infty\|_{L^6}^3+\|\nabla H\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^4\big)dt\\&\le C\int_0^{\sigma(T)}t^{-\frac14}(t^{\frac14}\|\nabla u\|_{L^2}^2)^{\frac12}(t^{\frac14}\|\sqrt\rho\dot u\|_{L^2}^2)^{\frac12}(t^2\|\sqrt\rho\dot u\|_{L^2}^2)dt+CC_0\\&\quad+C\int_0^{\sigma(T)}t^{-\frac14}(t^{\frac14}\|\nabla u\|_{L^2}^2)^{\frac12}(t^{\frac14}\|\mathrm{curl}^2H\|_{L^2}^2)^{\frac12}(t^2\|\mathrm{curl}^2H\|_{L^2}^2)dt\\&\le CC_0^{\frac{11}{18}}.\end{aligned}\tag{3.78}$$

On the other hand, by (3.6), (2.12) and Lemma 3.4, we have

$$
\begin{aligned}
&\int_{\sigma(T)}^{T}\sigma^2\|\nabla u\|_{L^4}^4dt\\
\le& C\int_{\sigma(T)}^{T}\sigma^2\big((\|\sqrt{\rho}\dot{u}\|_{L^2}^3+\|P-P_\infty\|_{L^6}^3+\|\nabla H\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^4\big)dt\\
\le& CC_0+C\int_{\sigma(T)}^{T}\sigma^2\|P-P_\infty\|_{L^4}^4dt. \qquad (3.79)
\end{aligned}
$$

Furthermore, it follows from $(1.1)_1$ and (2.13) that $P-P_\infty$ satisfies

$$
\begin{aligned}
&(P-P_\infty)_t+u\cdot\nabla(P-P_\infty)+\frac{\gamma}{2\mu+\lambda}(P-P_\infty)F\\
&\quad+\frac{\gamma}{2\mu+\lambda}(P-P_\infty)^2+\frac{\gamma}{2(2\mu+\lambda)}(P-P_\infty)|H|^2+\gamma P_\infty\mathrm{div}u=0. \qquad (3.80)
\end{aligned}
$$

Multiplying (3.80) by $3(P-P_\infty)^2$ and integrating over Ω, after using (2.16), we get

$$
\begin{aligned}
&\frac{3\gamma-1}{2\mu+\lambda}\|P-P_\infty\|_{L^4}^4\\
\le& -\big(\|P-P_\infty\|_{L^3}^3\big)_t+\delta\|P-P_\infty\|_{L^4}^4+C\|F\|_{L^4}^4+C\|\nabla H\|_{L^2}^6+C\|\nabla u\|_{L^2}^2\\
\le& -\big(\|P-P_\infty\|_{L^3}^3\big)_t+\delta\|P-P_\infty\|_{L^4}^4+C(\|\rho\dot{u}\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)(\|\nabla u\|_{L^2}+\|\nabla H\|_{L^2})\\
&+C(\|\rho\dot{u}\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)\|P-P_\infty\|_{L^2}+C\|\nabla H\|_{L^2}^6+C\|\nabla u\|_{L^2}^2. \qquad (3.81)
\end{aligned}
$$

Multiplying (3.81) by σ^2, then integrating over $(0,T]$, and choosing δ suitably small, by (2.12), (3.6) and (3.8), we obtain

$$
\begin{aligned}
&\int_0^T\sigma^2\|P-P_\infty\|_{L^4}^4dt\\
\le& C\sup_{0\le t\le T}\|P-P_\infty\|_{L^3}^3+C\int_0^{\sigma(T)}\|P-P_\infty\|_{L^3}^3dt\\
&+C\int_0^T\sigma^2(\|\rho\dot{u}\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)(\|\nabla u\|_{L^2}+\|\nabla H\|_{L^2})dt\\
&+C\int_0^T\sigma^2(\|\rho\dot{u}\|_{L^2}^3+\|\mathrm{curl}^2H\|_{L^2}^3)\|P-P_\infty\|_{L^2}dt+CC_0^{\frac{13}{18}}+CC_0\\
\le& C(\bar{\rho})C_0^{\frac{11}{18}}. \qquad (3.82)
\end{aligned}
$$

Combining (3.78), (3.79) and (3.82), it follows from (3.77) that

$$
A_1(T)+A_2(T)\le C_8C_0^{\frac{11}{18}}. \qquad (3.83)
$$

Set $\varepsilon_5\triangleq\min\{\varepsilon_4,(C_8^{-9}\}$, then (3.75) holds when $C_0<\varepsilon_5$. The proof of Lemma 3.6 is completed. ∎

We now proceed to prove the uniform (in time) upper bound for the density.

LEMMA 3.7. *Under the conditions of Proposition 3.1, there exists a positive constant ε_6 depending only on μ, λ, ν, γ, a, ρ_∞, $\bar{\rho}$, M_1 and M_2 such that*

$$
\sup_{0\le t\le T}\|\rho(t)\|_{L^\infty}\le\frac{7\bar{\rho}}{4}, \qquad (3.84)
$$

provided $C_0 \le \varepsilon_6$.

Proof. First, the equation of mass conservation $(1.1)_1$ can be equivalently rewritten in the form

$$D_t\rho = g(\rho) + b'(t), \tag{3.85}$$

where

$$D_t\rho \triangleq \rho_t + u\cdot\nabla\rho,\ g(\rho) \triangleq -\frac{\rho(P-P_\infty)}{2\mu+\lambda},\ b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \rho(F+\frac{|H|^2}{2})dt.$$

Naturally, we shall prove our conclusion by Lemma 2.2. It is sufficient to check that the function $b(t)$ must verify (2.3) with some suitable constants N_0, N_1.

For $t\in[0,\sigma(T)]$, one deduces from (2.1), (2.2), (2.14), (2.16), (3.6) and Lemmas 2.9, 3.4 that for δ_0 as in Proposition 3.1 and for all $0\le t_1\le t_2\le \sigma(T)$,

$$\begin{aligned}
&|b(t_2)-b(t_1)| = \frac{1}{\lambda+2\mu}\left|\int_{t_1}^{t_2}\rho(F+\frac{|H|^2}{2})dt\right| \\
&\le C\int_0^{\sigma(T)}(\|F\|_{L^\infty}+\|H\|_{L^\infty}^2)dt \\
&\le C\int_0^{\sigma(T)}\|F\|_{L^6}^{\frac12}\|\nabla F\|_{L^6}^{\frac12}dt + C\int_0^{\sigma(T)}\|H\|_{L^6}\|\nabla H\|_{L^6}dt \\
&\le C\int_0^{\sigma(T)}(\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac12}+\|\mathrm{curl}^2 H\|_{L^2}^{\frac12})\|\nabla\dot{u}\|_{L^2}^{\frac12}dt \\
&\quad + C\int_0^{\sigma(T)}(\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac12}+\|\mathrm{curl}^2 H\|_{L^2}^{\frac12})\|\nabla H\|_{L^2}^{\frac14}\|\mathrm{curl}^2 H\|_{L^2}^{\frac34}dt \\
&\quad + C\int_0^{\sigma(T)}(\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac12}+\|\mathrm{curl}^2 H\|_{L^2}^{\frac12})\|\nabla H\|_{L^2}dt \\
&\quad + C\int_0^{\sigma(T)}(\|\nabla H\|_{L^2}^{\frac12}\|\nabla\dot{u}\|_{L^2}^{\frac12}+\|\nabla H\|_{L^2}^{\frac34}\|\mathrm{curl}^2 H\|_{L^2}^{\frac34}+\|\nabla H\|_{L^2}^{\frac32})dt \\
&\quad + C\int_0^{\sigma(T)}\|\nabla H\|_{L^2}(\|\nabla H\|_{L^2}+\|\mathrm{curl}^2 H\|_{L^2})dt \triangleq \sum_{i=1}^5 B_i.
\end{aligned} \tag{3.86}$$

We have to estimate $B_i, i=1,2,\cdots,5$ one by one. A direct computation gives

$$\begin{aligned}
B_1 &\le C\int_0^{\sigma(T)}\left(t^{\frac14}(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2)\right)^{\frac14}\left(t\|\nabla\dot{u}\|_{L^2}^2\right)^{\frac14}t^{-\frac{5}{16}}dt \\
&\le CC_0^{\frac{1}{36}},
\end{aligned} \tag{3.87}$$

similarly,

$$\begin{aligned}
B_2 \le\ & C\int_0^{\sigma(T)}\left(t(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2)\right)^{\frac14}\left(t^{\frac14}\|\nabla H\|_{L^2}^2\right)^{\frac18} \\
&\times\left(t^{\frac14}\|\mathrm{curl}^2 H\|_{L^2}^2\right)^{\frac38}t^{-\frac38}dt \le CC_0^{\frac{1}{36}},
\end{aligned} \tag{3.88}$$

$$B_3 \le C\int_0^{\sigma(T)}\left(t^{\frac14}(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\mathrm{curl}^2 H\|_{L^2}^2)\right)^{\frac14}\left(t^{\frac14}\|\nabla H\|_{L^2}^2\right)^{\frac12}t^{-\frac38}dt \le CC_0^{\frac{1}{12}}, \tag{3.89}$$

$$
\begin{aligned}
B_4 \le & C\int_0^{\sigma(T)} \|\nabla H\|_{L^2}^{\frac12}\left(t\|\nabla\dot u\|_{L^2}^2\right)^{\frac14} t^{-\frac14}dt \\
& + C\int_0^{\sigma(T)} \left(t^{\frac14}\|\nabla H\|_{L^2}^2\right)^{\frac38}\left(t^{\frac14}\|\mathrm{curl}^2 H\|_{L^2}^2\right)^{\frac38} t^{-\frac{3}{16}}dt \\
& + C\int_0^{\sigma(T)} \left(t^{\frac14}\|\nabla H\|_{L^2}^2\right)^{\frac34} t^{-\frac{3}{16}}dt \le CC_0^{\frac{1}{12}}, \qquad (3.90)\\
B_5 \le & \int_0^{\sigma(T)} \left(t^{\frac14}\|\nabla H\|_{L^2}^2\right)^{\frac12}\left(t^{\frac14}\|\mathrm{curl}^2 H\|_{L^2}^2\right)^{\frac12} t^{-\frac14}dtdt + CC_0 \\
& \le CC_0^{\frac59}. \qquad (3.91)
\end{aligned}
$$

Putting (3.87)-(3.91) into (3.86), we have

$$
|b(t_2)-b(t_1)| \le C_9 C_0^{\frac{1}{36}}. \qquad (3.92)
$$

Combining (3.92) with (3.85) and choosing $N_1=0$, $N_0=C_9C_0^{\frac{1}{36}}$, $\bar\zeta=\rho_\infty$ in Lemma 2.2 give

$$
\sup_{t\in[0,\sigma(T)]}\|\rho\|_{L^\infty} \le \bar\rho + C_9C_0^{\frac{1}{36}} \le \frac{3\bar\rho}{2}, \qquad (3.93)
$$

provided $C_0\le\hat\varepsilon_6 \triangleq \min\{\varepsilon_5, \left(\frac{\bar\rho}{2C_9}\right)^{36}\}$.

On the other hand, for $t\in[\sigma(T),T]$, $\sigma(T)\le t_1\le t_2\le T$, it follows from (2.14), (3.6), and Lemma 2.9 that

$$
\begin{aligned}
& |b(t_2)-b(t_1)| \le C\int_{t_1}^{t_2} (\|F\|_{L^\infty}+\|H\|_{L^\infty}^2)dt \\
\le & \frac{a}{\lambda+2\mu}(t_2-t_1) + C\int_{t_1}^{t_2}\|F\|_{L^\infty}^{8/3}dt + C\int_{t_1}^{t_2}\|H\|_{L^\infty}^2dt \\
\le & \frac{a}{\lambda+2\mu}(t_2-t_1) + CC_0^{\frac16}\int_{\sigma(T)}^{T}(\|\nabla\dot u\|_{L^2}^2 + \|\nabla H\|_{L^2}\|\mathrm{curl}^2 H\|_{L^2}^3 + \|\nabla H\|_{L^2}^4)dt \\
& + CC_0 + C\int_{t_1}^{t_2}(\|\nabla H\|_{L^2}\|\mathrm{curl}^2 H\|_{L^2} + \|\nabla H\|_{L^2}^2)dt \\
\le & \frac{a}{\lambda+2\mu}(t_2-t_1) + C_{10}C_0^{2/3}. \qquad (3.94)
\end{aligned}
$$

Now we choose $N_0=C_{10}C_0^{2/3}$, $N_1=\frac{a}{\lambda+2\mu}$ in (2.3) and set $\bar\zeta=\frac{3\bar\rho}{2}$ in (2.4). Since for all $\zeta\ge\bar\zeta=\frac{3\bar\rho}{2}>\rho_\infty+1$,

$$
g(\zeta) = -\frac{a\zeta}{2\mu+\lambda}(\zeta^\gamma-\rho_\infty^\gamma) \le -\frac{a}{\lambda+2\mu} = -N_1.
$$

Together with (3.85) and (3.94), by Lemma 2.2, we have

$$
\sup_{t\in[\sigma(T),T]}\|\rho\|_{L^\infty} \le \frac{3\hat\rho}{2} + C_{10}C_0^{2/3} \le \frac{7\hat\rho}{4}, \qquad (3.95)
$$

provided $C_0\le\varepsilon_6\triangleq\min\{\hat\varepsilon_6, (\frac{\hat\rho}{4C_{10}})^{3/2}\}$. The combination of (3.93) with (3.95) completes the proof of Lemma 3.7. □

3.2. Time-dependent higher order estimates. In this subsection, we derive the time-dependent higher order estimates, which are necessary for the global existence of classical solutions. The procedure is similar as that in [3, 19, 20], and we sketch it here for completeness. From now on, assume that the initial energy $C_0\le\varepsilon_6$, and the positive constant C may depend on T, μ, λ, ν, a, γ, ρ_∞, $\bar\rho$, Ω, M_1, M_2, $\|\nabla u_0\|_{H^1}, \|\nabla H_0\|_{H^1}, \|\rho_0-\rho_\infty\|_{W^{2,q}}$, $\|P(\rho_0)-P_\infty\|_{W^{2,q}}$, $\|g\|_{L^2}$ for $q\in(3,6)$ where $g\in L^2(\Omega)$ is given by compatibility condition (1.10).

LEMMA 3.8. *There exists a positive constant C, such that*

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)+\int_0^T(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\nabla^2 H\|_{L^2}^2)dt\le C, \tag{3.96}$$

$$\sup_{0\le t\le T}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\nabla^2 H\|_{L^2}^2)+\int_0^T(\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)dt\le C, \tag{3.97}$$

$$\sup_{0\le t\le T}(\|\nabla\rho\|_{L^6\cap L^2}+\|\nabla u\|_{H^1})+\int_0^T(\|\nabla u\|_{L^\infty}+\|\nabla^2 u\|_{L^6}^2)dt\le C. \tag{3.98}$$

Proof. First, combining (2.28), (3.16) and (3.57) along with Proposition 3.1 gives (3.96). Then choosing $m=0$ in (3.38) and (3.51), integrating them over $(0,T)$, by (3.42), (3.96) and the compatibility condition (1.10), we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|H_t\|_{L^2}^2+\|\nabla^2 H\|_{L^2}^2)+\int_0^T(\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)dt\\&\le C+C\int_0^T(\|\sqrt{\rho}\dot u\|_{L^2}^3+\|\nabla^2 H\|_{L^2}^3+\|\nabla H\|_{L^2}^4\|\nabla^2 H\|_{L^2}^2)dt\\&\le C+\frac12\sup_{0\le t\le T}(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla^2 H\|_{L^2}^2),\end{aligned} \tag{3.99}$$

where we have also used Lemma 2.1, Lemma 2.9, (3.22) and (3.55), then we deduce (3.97) from (3.99).

Next we want to prove (3.98). For $2\le p\le 6$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t+\mathrm{div}(|\nabla\rho|^p u)+(p-1)|\nabla\rho|^p\mathrm{div}u\\&\quad+p|\nabla\rho|^{p-2}(\nabla\rho)^*\nabla u(\nabla\rho)+p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\mathrm{div}u=0.\end{aligned}$$

Thus, by (2.14), it follows

$$\begin{aligned}(\|\nabla\rho\|_{L^p})_t&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\nabla F\|_{L^p}+C\|\nabla H\cdot H\|_{L^p}\\&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\rho\dot u\|_{L^p}+C\|H\|_{L^\infty}\|\nabla H\|_{L^p}.\end{aligned} \tag{3.100}$$

We deduce from Gagliardo-Nirenberg's inequality, (2.14), (2.15) and (3.97) that

$$\begin{aligned}&\|\mathrm{div}u\|_{L^\infty}+\|\omega\|_{L^\infty}\\&\le C(\|F\|_{L^\infty}+\|P-P_\infty\|_{L^\infty}+\|H\|_{L^\infty}^2)+\|\omega\|_{L^\infty}\\&\le C(\|F\|_{L^2}+\|\nabla F\|_{L^6}+\|\omega\|_{L^2}+\|\nabla\omega\|_{L^6}+\|P-P_\infty\|_{L^\infty}+\|H\|_{L^6}\|\nabla H\|_{L^6})\\&\le C(\|\nabla\dot u\|_{L^2}+1),\end{aligned} \tag{3.101}$$

which together with Lemma 2.6 and (2.26) indicates that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C(\|\mathrm{div} u\|_{L^\infty}+\|\omega\|_{L^\infty})\ln(e+\|\nabla^2 u\|_{L^6})+C\|\nabla u\|_{L^2}+C\\ &\le C(1+\|\nabla\dot u\|_{L^2})\ln(e+\|\nabla\dot u\|_{L^2}+\|\nabla\rho\|_{L^6})\\ &\le C(1+\|\nabla\dot u\|_{L^2}^2)+C(1+\|\nabla\dot u\|_{L^2})\ln(e+\|\nabla\rho\|_{L^6}).\end{aligned} \tag{3.102}$$

Consequently, taking $p=6$ in (3.100) leads to

$$(e+\|\nabla\rho\|_{L^6})_t \le C[1+\|\nabla\dot u\|_{L^2}^2+(1+\|\nabla\dot u\|_{L^2})\ln(e+\|\nabla\rho\|_{L^6})](e+\|\nabla\rho\|_{L^6}),$$

which can be rewritten as

$$(\ln(e+\|\nabla\rho\|_{L^6}))_t \le C(1+\|\nabla\dot u\|_{L^2}^2)+C(1+\|\nabla\dot u\|_{L^2})\ln(e+\|\nabla\rho\|_{L^6}). \tag{3.103}$$

By Gronwall's inequality and (3.97), we derive

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^6}\le C. \tag{3.104}$$

Furthermore, by (3.102) and (2.26), together with (3.96) and (3.97), we have

$$\int_0^T \|\nabla u\|_{L^\infty}dt\le C. \tag{3.105}$$

Similarly, taking $p=2$ in (3.100), by Gronwall's inequality, together with (3.97) and (3.105), we obtain that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C. \tag{3.106}$$

Moreover, combining (3.96), (3.97), (3.104), (3.105) and (3.106) yields

$$\int_0^T \|\nabla^2 u\|_{L^6}^2 dt\le C,\qquad \sup_{0\le t\le T}\|u\|_{H^2}\le C. \tag{3.107}$$

This finishes the proof of Lemma 3.8. □

LEMMA 3.9. *There exists a positive constant C such that*

$$\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2+\int_0^T\int|\nabla u_t|^2dxdt\le C, \tag{3.108}$$

$$\sup_{0\le t\le T}(\|\rho-\rho_\infty\|_{H^2}+\|P-P_\infty\|_{H^2})\le C, \tag{3.109}$$

$$\sup_{0\le t\le T}(\|P-P_\infty\|_{H^2}+\|\rho_t\|_{H^1}+\|P_t\|_{H^1})+\int_0^T\left(\|\rho_{tt}\|_{L^2}^2+\|P_{tt}\|_{L^2}^2\right)dt\le C, \tag{3.110}$$

$$\sup_{0\le t\le T}\sigma(\|\nabla u_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2)+\int_0^T\sigma(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|H_{tt}\|_{L^2}^2)dt\le C. \tag{3.111}$$

Proof. Based on Lemma 3.8, (3.108)-(3.110) can be obtained by the same way as that in [4]. It remains to prove (3.111). Introduce the function

$$K(t)=(\lambda+2\mu)\int(\mathrm{div}u_t)^2dx+\mu\int|\omega_t|^2dx+\nu\int|\mathrm{curl}H_t|^2dx.$$

Since $u_t\cdot n=0, H_t\cdot n=0$ on $\partial\Omega$, by Lemma 2.3, we have

$$\|\nabla u_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2\le C(\Omega)K(t). \tag{3.112}$$

Differentiating $(2.6)_{2,3}$ with respect to t,

$$\rho u_{tt}-(\lambda+2\mu)\nabla\mathrm{div}u_t+\mu\nabla\times\omega_t=-\nabla P_t-\rho_t u_t-(\rho u\cdot\nabla u)_t+(H\cdot\nabla H-\nabla|H|^2/2)_t, \tag{3.113}$$

and

$$H_{tt}-\nu\nabla\times\mathrm{curl}H_t=(H\cdot\nabla u-u\cdot\nabla H-H\mathrm{div}u)_t, \tag{3.114}$$

then multiplying (3.113) by $2u_{tt}$, multiplying (3.114) by $2H_{tt}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}K(t)+2\int(\rho|u_{tt}|^2+|H_{tt}|^2)dx\\
=&\frac{d}{dt}\Big(-\int\rho_t|u_t|^2dx-2\int\rho_t u\cdot\nabla u\cdot u_t dx+2\int P_t\mathrm{div}u_t dx\\
&\quad-\int(2(H\otimes H)_t:\nabla u_t-|H|_t^2\mathrm{div}u_t dx)\Big)\\
&\quad+\int\rho_{tt}|u_t|^2dx+2\int(\rho_t u\cdot\nabla u)_t\cdot u_t dx-2\int\rho(u\cdot\nabla u)_t\cdot u_{tt}dx\\
&\quad-2\int P_{tt}\mathrm{div}u_t dx+\int(2(H\otimes H)_{tt}:\nabla u_t-|H|_{tt}^2\mathrm{div}u_t)dx\\
&\quad+2\int(H\cdot\nabla u-u\cdot\nabla H-H\mathrm{div}u)_t\cdot H_{tt}dx\\
\triangleq&\frac{d}{dt}K_0+\sum_{i=1}^6 K_i.
\end{aligned} \tag{3.115}$$

Let us estimate K_i, $i=0,1,\cdots,6$. We conclude from $(1.1)_1$, (3.97), (3.98), (3.108), (3.110), (3.112) and Sobolev's, Poincaré's inequalities that

$$\begin{aligned}
K_0\le&\left|\int\mathrm{div}(\rho u)\,|u_t|^2dx\right|+C\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^6}+C\|P_t\|_{L^2}\|\nabla u_t\|_{L^2}\\
&+C\|H\|_{L^\infty}\|H_t\|_{L^2}\|\nabla u_t\|_{L^2}\le\frac{1}{2}K(t)+C,
\end{aligned} \tag{3.116}$$

$$\begin{aligned}
K_1\le&\left|\int\rho_{tt}\,|u_t|^2dx\right|=\left|\int\mathrm{div}(\rho u)_t\,|u_t|^2dx\right|=2\left|\int(\rho_t u+\rho u_t)\cdot\nabla u_t\cdot u_t dx\right|\\
\le&C\|\nabla u_t\|_{L^2}^2K(t)+C\|\nabla u_t\|_{L^2}^2+C,
\end{aligned} \tag{3.117}$$

$$K_2+K_3+K_4\le C\|\rho_{tt}\|_{L^2}^2+C\|\nabla u_t\|_{L^2}^2+\|\rho^{\frac{1}{2}}u_{tt}\|_{L^2}^2+C\|P_{tt}\|_{L^2}^2+C, \tag{3.118}$$

$$K_5\le\frac{1}{2}\|H_{tt}\|_{L^2}^2+C\|H_t\|_{L^2}^2K(t)+C(\|\nabla H_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2), \tag{3.119}$$

$$K_6\le\frac{1}{2}\|H_{tt}\|_{L^2}^2+C(\|\nabla H_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2). \tag{3.120}$$

Consequently, multiplying (3.115) by σ, together with (3.117)-(3.120), we get

$$\begin{aligned}
&\frac{d}{dt}(\sigma K(t)-\sigma K_0)+\sigma\int(\rho|u_{tt}|^2+|H_{tt}|^2)dx\\
\le&C(1+\|\nabla u_t\|_{L^2}^2)\sigma K(t)+C(1+\|\nabla u_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2+\|\rho_{tt}\|_{L^2}^2+\|P_{tt}\|_{L^2}^2).
\end{aligned} \tag{3.121}$$

By Gronwall's inequality, (3.97), (3.108), (3.110) and (3.116), we derive that

$$\sup_{0\le t\le T}(\sigma K(t))+\int_0^T\sigma(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|H_{tt}\|_{L^2}^2)dt\le C. \tag{3.122}$$

As a result, by (3.112), we get (3.111). This finishes the proof . ◻

LEMMA 3.10. *There exists a positive constant C so that for any $q\in(3,6)$,*

$$\sup_{t\in[0,T]}(\|\rho-\rho_\infty\|_{W^{2,q}}+\|P-P_\infty\|_{W^{2,q}})\le C, \tag{3.123}$$

$$\begin{aligned}&\sup_{t\in[0,T]}\sigma(\|\nabla u\|_{H^2}^2+\|\nabla H\|_{H^2}^2)\\&\quad+\int_0^T\left(\|\nabla u\|_{H^2}^2+\|\nabla H\|_{H^2}^2+\|\nabla^2 u\|_{W^{1,q}}^{p_0}+\sigma\|\nabla u_t\|_{H^1}^2\right)dt\le C,\end{aligned} \tag{3.124}$$

where $p_0=\frac{9q-6}{10q-12}\in(1,\frac{7}{6})$.

Proof. Let's start with (3.124). By Lemma 3.8 and Poincaré's, Sobolev's inequalities, one can check that

$$\begin{aligned}\|\nabla(\rho\dot{u})\|_{L^2}&\le\||\nabla\rho||u_t|\|_{L^2}+\|\rho\nabla u_t\|_{L^2}+\||\nabla\rho||u||\nabla u|\|_{L^2}+\|\rho|\nabla u|^2\|_{L^2}+\|\rho|u||\nabla^2u|\|_{L^2}\\&\le C+C\|\nabla u_t\|_{L^2}.\end{aligned} \tag{3.125}$$

Consequently, together with (3.110) and Lemma 3.8, it yields

$$\begin{aligned}\|\nabla^2 u\|_{H^1}&\le C(\|\rho\dot{u}\|_{H^1}+\|H\cdot\nabla H\|_{H^1}+\|P-P_\infty\|_{H^2}+\||H|^2\|_{H^2}+\|u\|_{L^2})\\&\le C+C\|\nabla u_t\|_{L^2}.\end{aligned} \tag{3.126}$$

It then follows from (3.126), (3.98), (3.108) and (3.111) that

$$\sup_{0\le t\le T}\sigma\|\nabla u\|_{H^2}^2+\int_0^T\|\nabla u\|_{H^2}^2dt\le C. \tag{3.127}$$

Next, from $(2.6)_3$, (2.28), it follows

$$\begin{aligned}\|\nabla^2 H\|_{H^1}&\le C(\|H_t\|_{H^1}+\|u\cdot\nabla H\|_{H^1}+\|H\cdot\nabla u\|_{H^1}+\|H\mathrm{div}u\|_{H^1}+\|\nabla H\|_{L^2})\\&\le C+C\|\nabla H_t\|_{L^2}.\end{aligned} \tag{3.128}$$

Similarly, from (3.125), (3.96) and (3.98), we obtain

$$\sup_{0\le t\le T}\sigma\|\nabla H\|_{H^2}^2+\int_0^T\|\nabla H\|_{H^2}^2dt\le C. \tag{3.129}$$

Next, we deduce from Lemma 3.8 and (3.110) that

$$\begin{aligned}\|\nabla^2 u_t\|_{L^2}&\le C(\|(\rho\dot{u})_t\|_{L^2}+\|\nabla P_t\|_{L^2}+\|((\nabla\times H)\times H)_t\|_{L^2}+\|u_t\|_{L^2})\\&\le C\|\rho^{\frac12}u_{tt}\|_{L^2}+C\|\nabla u_t\|_{L^2}+C\|\nabla H_t\|_{L^2}+C,\end{aligned} \tag{3.130}$$

where in the first inequality, we have utilized the L^p-estimate for the following elliptic system

$$\begin{cases}\mu\Delta u_t+(\lambda+\mu)\nabla\mathrm{div}u_t=(\rho\dot{u})_t+\nabla P_t+((\nabla\times H)\times H)_t & \text{in }\Omega,\\ u_t\cdot n=0\text{ and }\omega_t\times n=0 & \text{on }\partial\Omega.\end{cases} \tag{3.131}$$

Together with (3.130) and (3.111) yields

$$\int_0^T \sigma\|\nabla u_t\|_{H^1}^2 dt \le C. \tag{3.132}$$

By Sobolev's inequality, (3.98), (3.110) and (3.111), we get for any $q\in(3,6)$,

$$\begin{aligned}\|\nabla(\rho\dot{u})\|_{L^q} &\le C\|\nabla\rho\|_{L^q}(\|\nabla\dot{u}\|_{L^q}+\|\nabla\dot{u}\|_{L^2}+\|\nabla u\|_{L^2}^2)+C\|\nabla\dot{u}\|_{L^q}\\ &\le C\sigma^{-\frac{1}{2}}+C\|\nabla u\|_{H^2}+C\sigma^{-\frac{1}{2}}(\sigma\|\nabla u_t\|_{H^1}^2)^{\frac{3(q-2)}{4q}}+C.\end{aligned} \tag{3.133}$$

Integrating this inequality over $[0,T]$, by (3.97) and (3.132), we have

$$\int_0^T \|\nabla(\rho\dot{u})\|_{L^q}^{p_0} dt \le C. \tag{3.134}$$

On the other hand, notice that P satisfies

$$P_t+u\cdot\nabla P+\gamma P\operatorname{div}u=0.$$

Differentiating it twice with respect to x leads to

$$\nabla^2 P_t+u\cdot\nabla\nabla^2 P+2\nabla u\cdot\nabla^2 P+\nabla^2 u\cdot\nabla P+\gamma\nabla^2 P\operatorname{div}u+2\gamma\nabla P\nabla\operatorname{div}u+\gamma P\nabla^2\operatorname{div}u=0,$$

and by Lemma 3.8 and (3.110), one has

$$\begin{aligned}(\|\nabla^2 P\|_{L^q})_t &\le C\|\nabla u\|_{L^\infty}\|\nabla^2 P\|_{L^q}+C\|\nabla^2 u\|_{W^{1,q}}\\ &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla^2 P\|_{L^q}+C(1+\|\nabla u_t\|_{L^2})+C\|\nabla(\rho\dot{u})\|_{L^q},\end{aligned} \tag{3.135}$$

where in the last inequality we have used the following simple fact that

$$\|\nabla^2 u\|_{W^{1,q}} \le C(1+\|\nabla u_t\|_{L^2}+\|\nabla(\rho\dot{u})\|_{L^q}+\|\nabla^2 P\|_{L^q}), \tag{3.136}$$

due to (2.26), (2.27), (3.97) and (3.110).

Hence, applying Gronwall's inequality in (3.135), we deduce from (3.98), (3.108) and (3.134) that

$$\sup_{t\in[0,T]}\|\nabla^2 P\|_{L^q}\le C, \tag{3.137}$$

which along with (3.108), (3.110), (3.136) and (3.134) also gives

$$\sup_{t\in[0,T]}\|P-P_\infty\|_{W^{2,q}}+\int_0^T\|\nabla^2 u\|_{W^{1,q}}^{p_0}dt\le C. \tag{3.138}$$

Similarly, one has

$$\sup_{0\le t\le T}\|\rho-\rho_\infty\|_{W^{2,q}}\le C, \tag{3.139}$$

which together with (3.138) gives (3.123). The proof of Lemma 3.10 is finished. □

LEMMA 3.11. *There exists a positive constant C such that, for any $q\in(3,6)$,*

$$\begin{aligned}&\sup_{0\le t\le T}\sigma\Big(\|\rho^{\frac{1}{2}}u_{tt}\|_{L^2}+\|H_{tt}\|_{L^2}+\|\nabla u_t\|_{H^1}+\|\nabla H_t\|_{H^1}+\|\nabla^2 H\|_{H^2}+\|\nabla u\|_{W^{2,q}}\Big)\\ &\quad+\int_0^T\sigma^2(\|\nabla u_{tt}\|_2^2+\|\nabla H_{tt}\|_2^2)dt\le C.\end{aligned} \tag{3.140}$$

Proof. Differentiating (2.6)$_{2,3}$ with respect to t twice, multiplying them by $2u_{tt}$ and $2H_{tt}$ respectively, and integrating over Ω lead to

$$\begin{aligned}
&\frac{d}{dt}\int(\rho|u_{tt}|^2+|H_{tt}|^2)dx\\
&\quad+2(\lambda+2\mu)\int(\mathrm{div}u_{tt})^2dx+2\mu\int|\omega_{tt}|^2dx+2\nu\int|\mathrm{curl}H_{tt}|^2dx\\
&=-8\int\rho u_{tt}^i u\cdot\nabla u_{tt}^i dx-2\int(\rho u)_t\cdot[\nabla(u_t\cdot u_{tt})+2\nabla u_t\cdot u_{tt}]\,dx\\
&\quad-2\int(\rho_{tt}u+2\rho_t u_t)\cdot\nabla u\cdot u_{tt}dx-2\int(\rho u_{tt}\cdot\nabla u\cdot u_{tt}-P_{tt}\mathrm{div}u_{tt})dx\\
&\quad-2\int(H\cdot\nabla H-\nabla|H|^2/2)_{tt}u_{tt}dx+2\int(H\cdot\nabla u-u\cdot\nabla H-H\mathrm{div}u)_{tt}H_{tt}dx\\
&\triangleq\sum_{i=1}^{6}R_i.
\end{aligned}\tag{3.141}$$

Let us estimate R_i for $i=1,\cdots,6$. Hölder's inequality and (3.98) give

$$R_1\le C\|\sqrt{\rho}u_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\|u\|_{L^\infty}\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|\sqrt{\rho}u_{tt}\|_{L^2}^2.\tag{3.142}$$

By (3.97), (3.108), (3.110) and (3.111), we conclude that

$$R_2\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|\nabla u_t\|_{L^2}^3+C(\delta)\|\nabla u_t\|_{L^2}^2,\tag{3.143}$$
$$R_3\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|\rho_{tt}\|_{L^2}^2+C(\delta)\|\nabla u_t\|_{L^2}^2,\tag{3.144}$$
$$R_4\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C(\delta)\|P_{tt}\|_{L^2}^2,\tag{3.145}$$
$$R_5\le\delta\|\nabla u_{tt}\|_{L^2}^2+C(\delta)\|H_{tt}\|_{L^2}^2+C(\delta)\|\nabla H_t\|_{L^2}^3,\tag{3.146}$$
$$\begin{aligned}R_6\le&\,\delta(\|\nabla u_{tt}\|_{L^2}^2+\|\nabla H_{tt}\|_{L^2}^2)+C(\delta)\|H_{tt}\|_{L^2}^2\\&+C(\delta)(\|\nabla u_t\|_{L^2}\|\nabla H_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\|\nabla H_t\|_{L^2}^2).\end{aligned}\tag{3.147}$$

Substituting (3.142)-(3.147) into (3.141), utilizing the fact that

$$\|\nabla u_{tt}\|_{L^2}\le C(\|\mathrm{div}u_{tt}\|_{L^2}+\|\omega_{tt}\|_{L^2}),\quad \|\nabla H_{tt}\|_{L^2}\le C\|\mathrm{curl}H_{tt}\|_{L^2},\tag{3.148}$$

and then choosing δ small enough, we can get

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|H_{tt}\|_{L^2}^2)+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla H_{tt}\|_{L^2}^2\\
&\le C(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|H_{tt}\|_{L^2}^2+\|\rho_{tt}\|_{L^2}^2+\|P_{tt}\|_{L^2}^2+\|\nabla u_t\|_{L^2}^3+\|\nabla H_t\|_{L^2}^3)\\
&\quad+C(\|\nabla u_t\|_{L^2}\|\nabla H_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\|\nabla H_t\|_{L^2}^2),
\end{aligned}\tag{3.149}$$

which together with (3.110), (3.111), and by Gronwall's inequality yields that

$$\sup_{0\le t\le T}\sigma^2(\|\sqrt{\rho}u_{tt}\|_{L^2}^2+\|H_{tt}\|_{L^2}^2)+\int_0^T\sigma^2(\|\nabla u_{tt}\|_{L^2}^2+\|\nabla H_{tt}\|_{L^2}^2)dt\le C.\tag{3.150}$$

Furthermore, it follows from (2.28), (2.27), (3.130) and (3.111) that

$$\begin{aligned}
&\sup_{0\le t\le T}(\sigma\|\nabla^2u_t\|_{L^2}+\sigma\|\nabla^2H_t\|_{L^2})\\
&\le C\sigma(1+\|\rho^{\frac12}u_{tt}\|_{L^2}+\|H_{tt}\|_{L^2}+\|\nabla u_t\|_{L^2}+\|\nabla H_t\|_{L^2})\le C.
\end{aligned}\tag{3.151}$$

Finally, we deduce from (3.111), (3.123), (3.124), (3.128), (3.133), (3.136), (3.150) and (3.151) that

$$\begin{aligned}\sigma\|\nabla^2 u\|_{W^{1,q}} &\le C\sigma(1+\|\nabla u_t\|_{L^2}+\|\nabla H_t\|_{L^2}+\|\nabla(\rho\dot{u})\|_{L^q}+\|\nabla^2 P\|_{L^q})\\ &\le C(1+\sigma\|\nabla u\|_{H^2}+\sigma^{\frac12}(\sigma\|\nabla u_t\|_{H^1}^2)^{\frac{3(q-2)}{4q}})\le C+C\sigma^{\frac12}(\sigma^{-1})^{\frac{3(q-2)}{4q}}\le C, \end{aligned}\tag{3.152}$$

and

$$\sigma\|\nabla^2 H\|_{H^2}\le C\sigma(1+\|\nabla H_t\|_{H^1}+\|\nabla u\|_{H^2}\|\nabla H\|_{H^2})\le C, \tag{3.153}$$

together with (3.150) and (3.151) yields (3.140) and finishes the proof. □

4. Proof of Theorems 1.1-1.2

In this section, we are prepared to prove the main results of this paper. Based on the estimates in Section 3, we follow the procedure in [3, 20] to give the sketch of the proof.

Proof. **(Proof of Theorem 1.1.)** By Lemma 2.7, there exists a $T_*>0$ such that the system (1.1)-(1.5) has a unique classical solution (ρ,u,H) on $\Omega\times(0,T_*]$. One may use the a priori estimates, Proposition 3.1 and Lemmas 3.9-3.11 to extend the classical solution (ρ,u,H) globally in time.

First, by the definition of (3.1)-(3.5), the assumption of the initial data (1.8) and (3.74), one immediately checks that

$$0\le\rho_0\le\bar\rho,\ A_1(0)+A_2(0)=0,\ A_3(0)\le C_0^{\frac19},\ A_4(0)+A_5(0)\le C_0^{\frac19}. \tag{4.1}$$

Therefore, there exists a $T_1\in(0,T_*]$ such that

$$\begin{cases}0\le\rho_0\le 2\bar\rho,\ A_1(T)+A_2(T)\le 2C_0^{\frac12},\\ A_3(T)\le 2C_0^{\frac19},\ A_4(\sigma(T))+A_5(\sigma(T))\le 2C_0^{\frac19},\end{cases}\tag{4.2}$$

hold for $T=T_1$. Next, we set

$$T^*=\sup\{T\,|\,(4.2)\text{ holds}\}. \tag{4.3}$$

Then $T^*\ge T_1>0$. Hence, for any $0<\tau<T\le T^*$ with T finite, it follows from Lemmas 3.8-3.11 that

$$\begin{cases}\rho-\rho_\infty\in C([0,T];H^2\cap W^{2,q}),\\ (\nabla u,\nabla H)\in C([\tau,T];H^1),\quad (\nabla u_t,\nabla H_t)\in C([\tau,T];L^q);\end{cases}\tag{4.4}$$

where one has taken advantage of the standard embedding

$$L^\infty(\tau,T;H^1)\cap H^1(\tau,T;H^{-1})\hookrightarrow C([\tau,T];L^q),\qquad \text{for any } q\in[2,6).$$

Due to (3.108), (3.111), (3.140) and $(1.1)_1$, we obtain

$$\begin{aligned}&\int_\tau^T\left|\left(\int\rho|u_t|^2dx\right)_t\right|dt\le\int_\tau^T(\|\rho_t|u_t|^2\|_{L^1}+2\|\rho u_t\cdot u_{tt}\|_{L^1})\,dt\\ \le{}& C\int_\tau^T\left(\|\rho^{\frac12}|u_t|^2\|_{L^2}\|\nabla u\|_{L^\infty}+\|u\|_{L^6}\|\nabla\rho\|_{L^2}\|u_t\|_{L^6}^2+\|\sqrt{\rho}u_{tt}\|_{L^2}\right)dt\le C,\end{aligned}$$

which together with (4.4) yields

$$\rho^{\frac{1}{2}}u_t, \quad \rho^{\frac{1}{2}}\dot{u}\in C([\tau,T];L^2). \tag{4.5}$$

Finally, we claim that

$$T^*=\infty. \tag{4.6}$$

Otherwise, $T^*<\infty$. Then by Proposition 3.1, it holds that

$$\begin{cases} 0\le\rho\le\frac{7\bar{\rho}}{4},\, A_1(T^*)+A_2(T^*)\le C_0^{\frac{1}{2}},\\ A_3(T^*)\le C_0^{\frac{1}{9}},\, A_4(\sigma(T^*))+A_5(\sigma(T^*))\le C_0^{\frac{1}{9}}. \end{cases} \tag{4.7}$$

It follows from Lemmas 3.10, 3.11 and (4.5) that $(\rho(x,T^*),u(x,T^*),H(x,T^*))$ satisfies the initial data condition (1.7)-(1.8), (1.10), where $g(x)\triangleq\sqrt{\rho}\dot{u}(x,T^*)$, $x\in\Omega$. Thus, Lemma 2.7 implies that there exists some $T^{**}>T^*$ such that (4.2) holds for $T=T^{**}$, which contradicts the definition of T^*. As a result, (4.6) holds. By Lemmas 2.7 and 3.8-3.11, it indicates that (ρ,u,H) is in fact the unique classical solution defined on $\Omega\times(0,T]$ for any $0<T<T^*=\infty$.

Finally, with (2.7), (2.9), (3.14), (2.11), (3.15) and (3.26) at hand, (1.13) can be obtained by similar arguments as used in [3], and we omit the details. The proof of Theorem 1.1 is finished. □

Proof. **(Proof of Theorem 1.2.)** As is shown by [3], we sketch the proof for completeness. First, we show that, for $T>0$, the Lagrangian coordinates of the system are given by

$$\begin{cases} \frac{\partial}{\partial\tau}X(\tau;t,x)=u(X(\tau;t,x),\tau),\ 0\le\tau\le T\\ X(t;t,x)=x,\ 0\le t\le T, x\in\bar{\Omega}. \end{cases} \tag{4.8}$$

By (1.12), the transformation (4.8) is well-defined. In addition, by $(1.1)_1$, we have

$$\rho(x,t)=\rho_0(X(0;t,x))\exp\{-\int_0^t \mathrm{div}u(X(\tau;t,x),\tau)d\tau\}. \tag{4.9}$$

If there exists some point $x_0\in\Omega$ such that $\rho_0(x_0)=0$, then there is a point $x_0(t)\in\bar{\Omega}$ such that $X(0;t,x_0(t))=x_0$. Hence, by (4.9), $\rho(x_0(t),t)\equiv 0$ for any $t\ge 0$.

Now we will prove Theorem 1.2 by contradiction. Suppose there exist some positive constant C_1 and a subsequence t_{n_j}, $t_{n_j}\to\infty$ as $j\to\infty$ such that $\|\nabla\rho(\cdot,t_{n_j})\|_{L^{r_1}}<C_1$. Consequently, by (2.2) and the assumption $\rho_\infty>0$, we get that for $r_1\in(3,\infty)$ and $\theta=\frac{r_1-3}{2r_1-3}$,

$$\rho_\infty\le\|\rho(\cdot,t_{n_j})-\rho_\infty\|_{C(\overline{\Omega})}\le C\|\rho(\cdot,t_{n_j})-\rho_\infty\|_{L^3}^{\theta}\|\nabla\rho(\cdot,t_{n_j})\|_{L^{r_1}}^{1-\theta}, \tag{4.10}$$

which is in contradiction with (1.13). The proof is completed. □

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