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ANALYSIS AND COMPUTATION FOR THE SCATTERING PROBLEM OF ELECTROMAGNETIC WAVES IN CHIRAL MEDIA*

GANG BAO† AND LEI ZHANG‡

Abstract. This paper considers an obstacle scattering problem in a chiral medium under circularly polarized oblique plane wave incidence, which can be represented as a combination of a left-circularly polarized plane wave and a right-circularly polarized one. We apply a reduced model problem with coupled oblique derivative boundary conditions, describing the cross-coupling effect of electric and magnetic fields. A novel boundary integral equation is constructed by introducing single-layer potential operators and the corresponding normal and tangential derivative operators. The corresponding properties are obtained by splitting techniques to overcome the singularity of integral operators. A numerical method for solving the boundary integral equation is developed, whose convergence is proved. Numerical results are presented to show the performance of the proposed method.

Keywords. Maxwell's equations; chiral medium; boundary integral equations; collocation method; convergence.

AMS subject classifications. 35Q61; 65N12; 65R20; 78A25.

1. Introduction

Chirality has played a critical role in studying optical activity [5], multiferroics [10], and superfluidity [25]. In recent years, it is also widely concerned in materials science and chemistry [26, 27]. The response of isotropic media, anisotropic media, and other nonchiral media to electric and magnetic fields is characterized by permittivity and permeability. However, chiral media need three fundamental quantities (permittivity, permeability, and chirality admittance) to control characteristics. The chirality admittance can produce a cross-coupling between the electric field and the magnetic field. Due to the electromagnetic characteristics of chiral media, the obstacle scattering in chiral media becomes more attractive, for example, the radiation characteristics for antennas in a chiral environment [6, 7, 28]. The cross-coupling effects that such media have on the polarization properties of the waves are characterized by constitutive relations. Various physical explanations for these constitutive relations describe the wave propagation in chiral media, see, e.g., [12, 18, 21, 22].

Several research methods have been developed including generalized Lorenz-Mie theories, analytical approximation, and numerical methods that study chiral media's electromagnetic scattering based on the electromagnetic field theory. There are some analytic methods for solving the boundary-value problems of the electromagnetic wave scattering for a cylinder with a circular (elliptical) section or a sphere, ellipsoid, see, e.g., [15]. An exact analytic solution is presented to the electromagnetic wave scattering by a chiral spheroid using the method of separation of variables [13]. The variational formulations and numerical analysis of the diffraction problem for chiral gratings have

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†School of Mathematical Sciences, Zhejiang University, Hangzhou, China (baog@zju.edu.cn). http://www.math.zju.edu.cn/bao/

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‡Corresponding author. Department of Mathematics, Zhejiang University of Technology, Hangzhou, China (zhanglei@zjut.edu.cn).

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been studied in [1, 4, 32]. To solve the exterior problem with variational methods, one needs to use the transmission boundary conditions to replace the radiation condition at infinity. We refer to [9] and the references cited therein for the recent advances. The boundary integral equation method is another practical approach for solving the electromagnetic scattering problems [11, 17, 31]. The authors investigated the solution for the scattering from a nonhomogeneous object in a chiral medium by using the boundary integral equation and asymptotic analysis method for the boundary value problem of Maxwell's equations [2, 3].

Fresnel interpreted the optical activity of a chiral medium as the different wave speeds of the left and right circularly polarized waves with different refractive indices [16]. Mathematically, the circular polarization can reduce the complexity of a Maxwell equation model with proper geometric structures. The left and right circular polarized waves satisfy Helmholtz equations at different wavenumbers with proper boundary conditions to describe the cross-coupling effect of electromagnetic fields [6,23]. This paper is concerned with an obliquely incident plane time-harmonic electromagnetic wave scattering by an infinite cylinder embedded in a three-dimensional chiral medium using circular polarization. Such polarization significantly reduces the complexity of the computation for the scattering problem. However, for an obliquely incident electromagnetic wave, boundary conditions of the scattering problem become more complicated because of the chirality. We construct a novel boundary integral equation and introduce a regularized system by splitting techniques in Sobolev spaces.

Throughout this paper, the permittivity ϵ and permeability μ of the chiral media are assumed to be either positive or negative corresponding to conventional or left-handed (double negative) media [29, 30]. Although the propagation properties of waves in these media are different, the dispersion relation is satisfied. The crucial point for our analysis is the constructive decompositions for the singular boundary integral operators in Sobolev spaces. We propose an effective numerical method for the cylinder scattering problem in a homogeneous isotropic chiral environment. Then we discretize the boundary integral system using the collocation method. The error estimates and the convergence are proved. Verification examples with the analytical solution are designed to demonstrate the feasibility and effectiveness of the proposed method.

The paper is organized as follows. We consider the plane wave propagation and cylinder scattering in chiral media in Section 2. Then Section 3 presents an operator form of the boundary integral equations and analyzes the integral operator's properties. The numerical method and convergence analysis are obtained in Section 4. Numerical results are presented to show the performance of the proposed method in Section 5, and the conclusions follow in Section 6.

2. Wave propagation in a chiral medium

Consider the propagation of electromagnetic wave in a chiral medium in $\mathbb{R}^3\backslash\bar{D}_\infty$, where D_∞ is a linear and isotropic infinite cylinder which is uniform along the x_3 axis with its cross section D (see Figure 2.1). The electromagnetic fields are governed by the time dependent Maxwell's equations:

$$\nabla\times\mathcal{E}+\frac{\partial\mathcal{B}}{\partial t}=0,\quad \nabla\times\mathcal{H}-\frac{\partial\mathcal{D}}{\partial t}=0,\quad \mathrm{div}\mathcal{D}=0,\quad \mathrm{div}\mathcal{B}=0, \tag{2.1}$$

and the Drude-Born-Fedorov (DBF) constitutive equations

$$\mathcal{D}=\epsilon(\mathcal{E}+\beta\nabla\times\mathcal{E}),\quad \mathcal{B}=\mu(\mathcal{H}+\beta\nabla\times\mathcal{H}), \tag{2.2}$$

where $\mathcal{E}=\mathcal{E}(\boldsymbol{x},t)$ is the electric field intensity, $\mathcal{H}=\mathcal{H}(\boldsymbol{x},t)$ is the mangetic field intensity, $\mathcal{D}=\mathcal{D}(\boldsymbol{x},t)$ is the electric flux density, $\mathcal{B}=\mathcal{B}(\boldsymbol{x},t)$ is the magntic flux density. The

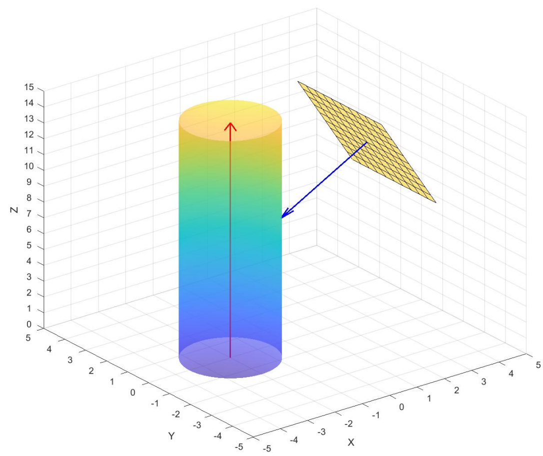

FIG. 2.1. *Diagram of the scattering problem with an obliquely incident plane wave.*

electric permittivity ϵ and magnetic permeability μ are assumed to be constants. $\beta \geq 0$ is the chirality admittance. Let $(\boldsymbol{E}, \boldsymbol{H})$ denote the time harmonic electromagnetic wave which satisfies

$$(\mathcal{E}, \mathcal{H}) = \Re\{(\boldsymbol{E}(\boldsymbol{x}), \boldsymbol{H}(\boldsymbol{x}))e^{-\mathrm{i}\omega t}\}, \tag{2.3}$$

where $\omega > 0$ is the angular frequency. Then, combining (2.1), (2.2) and (2.3), we have

$$(1-k^2\beta^2)\nabla \times \boldsymbol{E} = k^2\beta\boldsymbol{E} + \mathrm{i}\mu\omega\boldsymbol{H}, \qquad \mathrm{div}\boldsymbol{E} = 0, \tag{2.4}$$

$$(1-k^2\beta^2)\nabla \times \boldsymbol{H} = k^2\beta\boldsymbol{H} - \mathrm{i}\epsilon\omega\boldsymbol{E}, \qquad \mathrm{div}\boldsymbol{H} = 0, \tag{2.5}$$

where $k = \omega\sqrt{\epsilon\mu} > 0$. Throughout it is assumed that $0 \leq k\beta < 1$.

On the lateral surface of the cylinder, we apply Leontovich's impedance boundary condition

$$(\boldsymbol{\nu} \times \boldsymbol{E}) \times \boldsymbol{\nu} = \lambda(\boldsymbol{\nu} \times \boldsymbol{H}), \tag{2.6}$$

where $\boldsymbol{\nu}$ is the unit outward normal to ∂D, $\lambda = \sqrt{\mu_c/\epsilon_c} > 0$ is the impedance constant [24], ϵ_c and μ_c are the electric permittivity and magnetic permeability of the cylinder.

2.1. Circularly polarized plane wave. We are interested in the scattering problem under circularly polarized plane wave at oblique incidence [8]. The incident field is defined as

$$\boldsymbol{E}^i = [\boldsymbol{q}_L \exp(\mathrm{i}\tilde{\gamma}_L\boldsymbol{p}_L \cdot \boldsymbol{x}) + \boldsymbol{q}_R \exp(\mathrm{i}\tilde{\gamma}_R\boldsymbol{p}_R \cdot \boldsymbol{x})]\exp(-\mathrm{i}\alpha x_3), \tag{2.7}$$

$$\boldsymbol{H}^i = -\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}[\boldsymbol{q}_L \exp(\mathrm{i}\tilde{\gamma}_L\boldsymbol{p}_L \cdot \boldsymbol{x}) - \boldsymbol{q}_R \exp(\mathrm{i}\tilde{\gamma}_R\boldsymbol{p}_R \cdot \boldsymbol{x})]\exp(-\mathrm{i}\alpha x_3), \tag{2.8}$$

which satisfies the Equations (2.4) and (2.5), see Section A for more details. Here $\alpha = k\cos\theta$ is a constant depending on the obliquely incident angle θ between the incident direction and the positive x_3 axis, $\tilde{\gamma}_L = \frac{k}{1-k\beta} > 0$, $\tilde{\gamma}_R = \frac{k}{1+k\beta} > 0$. The vectors $\boldsymbol{p}_L = (p_{1,L}, p_{2,L}, 0)^\top$ and $\boldsymbol{p}_R = (p_{1,R}, p_{2,R}, 0)^\top$ satisfy

$$\boldsymbol{p}_L \cdot \boldsymbol{q}_L = \tilde{\gamma}_L^{-1}\alpha, \qquad \boldsymbol{p}_R \cdot \boldsymbol{q}_R = \tilde{\gamma}_R^{-1}\alpha, \tag{2.9}$$

$$\boldsymbol{p}_L \times \boldsymbol{q}_L = -\mathrm{i}A_L\boldsymbol{q}_L, \qquad \boldsymbol{p}_R \times \boldsymbol{q}_R = \mathrm{i}A_R\boldsymbol{q}_R, \tag{2.10}$$

$$\boldsymbol{p}_L \cdot \boldsymbol{p}_L = (1-\tilde{\gamma}_L^{-2}\alpha^2), \qquad \boldsymbol{p}_R \cdot \boldsymbol{p}_R = (1-\tilde{\gamma}_R^{-2}\alpha^2), \tag{2.11}$$

where $\boldsymbol{q}_L = (q_{1,L}, q_{2,L}, 1)^\top$, $\boldsymbol{q}_R = (q_{1,R}, q_{2,R}, 1)^\top$ and

$$A_L = \begin{pmatrix} 1 & -\mathrm{i}\tilde{\gamma}_L^{-1}\alpha & 0 \\ \mathrm{i}\tilde{\gamma}_L^{-1}\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & \mathrm{i}\tilde{\gamma}_R^{-1}\alpha & 0 \\ -\mathrm{i}\tilde{\gamma}_R^{-1}\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Obviously, combining (2.9), (2.10) and (2.11), we have

$$\nabla \times [\boldsymbol{q}_L \exp(\mathrm{i}\tilde{\gamma}_L \boldsymbol{p}_L \cdot \boldsymbol{x}) \exp(-\mathrm{i}\alpha x_3)] = \tilde{\gamma}_L [\boldsymbol{q}_L \exp(\mathrm{i}\tilde{\gamma}_L \boldsymbol{p}_L \cdot \boldsymbol{x}) \exp(-\mathrm{i}\alpha x_3)], \tag{2.13}$$

$$\nabla \times [\boldsymbol{q}_R \exp(\mathrm{i}\tilde{\gamma}_R \boldsymbol{p}_R \cdot \boldsymbol{x}) \exp(-\mathrm{i}\alpha x_3)] = -\tilde{\gamma}_R [\boldsymbol{q}_R \exp(\mathrm{i}\tilde{\gamma}_R \boldsymbol{p}_R \cdot \boldsymbol{x}) \exp(-\mathrm{i}\alpha x_3)], \tag{2.14}$$

this implies that the field $\boldsymbol{E}^i$ is a combination of left-circularly polarized plane wave and right-circularly polarized one.

Let an infinite cylinder be parallel to the x_3 axis in a chiral medium, and its cross-section D with C^2 boundary ∂D. Then the scattering problem for a chiral medium is to find the total field $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{E}^i + \boldsymbol{E}^s, \boldsymbol{H}^i + \boldsymbol{H}^s)$ such that the scattered field

$$(\boldsymbol{E}^s, \boldsymbol{H}^s) := (\boldsymbol{e}^s(x_1, x_2), \boldsymbol{h}^s(x_1, x_2)) \exp(-\mathrm{i}\alpha x_3) \tag{2.15}$$

satisfies (2.4)-(2.5). Here

$$\boldsymbol{e}^s(x_1, x_2) := (e_1^s(x_1, x_2), e_2^s(x_1, x_2), e_3^s(x_1, x_2))^\top, \tag{2.16}$$

$$\boldsymbol{h}^s(x_1, x_2) := (h_1^s(x_1, x_2), h_2^s(x_1, x_2), h_3^s(x_1, x_2))^\top. \tag{2.17}$$

2.2. The reduced model problem. Based on the above form of electromagnetic waves, we derive the following reduced model problem with oblique derivative boundary conditions (the detailed derivation is available in Section B):

$$\begin{cases} \Delta u^s + \gamma_L^2 u^s = 0 & \text{in} \quad \mathbb{R}^2 \setminus \bar{D}, \\ \Delta v^s + \gamma_R^2 v^s = 0 & \text{in} \quad \mathbb{R}^2 \setminus \bar{D}, \\ a_1 \dfrac{\partial u^s}{\partial \boldsymbol{\nu}} + a_2 \dfrac{\partial u^s}{\partial \boldsymbol{\tau}} + a_3 u^s + a_4 v^s = f_1 & \text{on} \quad \partial D, \\ a_5 \dfrac{\partial v^s}{\partial \boldsymbol{\nu}} + a_6 \dfrac{\partial v^s}{\partial \boldsymbol{\tau}} + a_7 v^s + a_8 u^s = f_2 & \text{on} \quad \partial D, \\ \lim\limits_{\boldsymbol{r}\to\infty} \sqrt{\boldsymbol{r}} \left(\frac{\partial u^s}{\partial \boldsymbol{r}} - \mathrm{i}\gamma_L u^s \right) = 0, & \boldsymbol{r} = |\boldsymbol{x}| \\ \lim\limits_{\boldsymbol{r}\to\infty} \sqrt{\boldsymbol{r}} \left(\frac{\partial v^s}{\partial \boldsymbol{r}} - \mathrm{i}\gamma_R v^s \right) = 0, & \boldsymbol{r} = |\boldsymbol{x}|, \end{cases} \tag{2.18}$$

where

$$f_1 = -a_1 \frac{\partial u^i}{\partial \boldsymbol{\nu}} - a_2 \frac{\partial u^i}{\partial \boldsymbol{\tau}} - a_3 u^i - a_4 v^i, \quad f_2 = -a_5 \frac{\partial v^i}{\partial \boldsymbol{\nu}} - a_6 \frac{\partial v^i}{\partial \boldsymbol{\tau}} - a_7 v^i - a_8 u^i, \tag{2.19}$$

$$u^i := u^i(x_1, x_2) = \exp(\mathrm{i}\tilde{\gamma}_L \boldsymbol{p}_L \cdot \boldsymbol{x}), \qquad v^i := v^i(x_1, x_2) = \exp(\mathrm{i}\tilde{\gamma}_R \boldsymbol{p}_R \cdot \boldsymbol{x}), \tag{2.20}$$

and

$$
\begin{aligned}
&\gamma_L^2=(\beta_1+\beta_2\sqrt{\epsilon\mu})^2-\alpha^2=\tilde{\gamma}_L^2-\alpha^2,\ \gamma_R^2=(\beta_1-\beta_2\sqrt{\epsilon\mu})^2-\alpha^2=\tilde{\gamma}_R^2-\alpha^2,\\
&\beta_1=\frac{k^2\beta}{1-k^2\beta^2},\quad \beta_2=\frac{\omega}{1-k^2\beta^2},\\
&a_1=-2\mathrm{i}(\beta_1+\beta_2\sqrt{\epsilon\mu}),\qquad a_2=2\alpha,\\
&a_3=(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\eta+\frac{1}{\lambda\eta}\right)+2\beta_1\beta_2\left(\lambda\epsilon+\frac{\mu}{\lambda}\right),\\
&a_4=-\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\eta-\frac{1}{\lambda\eta}\right)+2\beta_1\beta_2\left(\lambda\epsilon-\frac{\mu}{\lambda}\right)\right],\\
&a_5=-2\mathrm{i}(\beta_1-\beta_2\sqrt{\epsilon\mu}),\qquad a_6=2\alpha,\\
&a_7=-\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\eta+\frac{1}{\lambda\eta}\right)-2\beta_1\beta_2\left(\lambda\epsilon+\frac{\mu}{\lambda}\right)\right],\\
&a_8=(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\eta-\frac{1}{\lambda\eta}\right)-2\beta_1\beta_2\left(\lambda\epsilon-\frac{\mu}{\lambda}\right),
\end{aligned}
$$

here $\eta=\sqrt{\epsilon/\mu}$. As we already know that $\lambda=\sqrt{\mu_c/\epsilon_c}$. Physically, the material of the cylinder and the physical properties of the medium are different. Hence, we study the scattering problem for the case $\lambda\eta\neq 1$ in this paper. Mathematically, for the case $\lambda\eta=1$, we have $a_4=a_8=0$ implying that the electromagnetic field on the boundary becomes decoupled.

3. The boundary integral equations

Denote the fundamental solution to the Helmholtz equation in two dimensions by $G(\boldsymbol{x},\boldsymbol{y};\gamma_\sigma)=\frac{\mathrm{i}}{4}H_0^{(1)}(\gamma_\sigma|\boldsymbol{x}-\boldsymbol{y}|)$, where $\sigma=L$ or R and $H_0^{(1)}$ is the Hankel function of the first kind with order zero. For $\boldsymbol{x}\in\Gamma:=\partial D$, we introduce the following integral operators

$$(\mathbf{S}_\sigma\phi)(\boldsymbol{x})=2\int_\Gamma G(\boldsymbol{x},\boldsymbol{y};\gamma_\sigma)\phi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, \tag{3.1}$$

$$(\mathbf{K}_\sigma^{(*)}\phi)(\boldsymbol{x})=2\int_\Gamma \frac{\partial G(\boldsymbol{x},\boldsymbol{y};\gamma_\sigma)}{\partial\boldsymbol{\nu}(\boldsymbol{x})}\phi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, \tag{3.2}$$

$$(\mathbf{H}_\sigma\phi)(\boldsymbol{x})=2\int_\Gamma \frac{\partial G(\boldsymbol{x},\boldsymbol{y};\gamma_\sigma)}{\partial\boldsymbol{\tau}(\boldsymbol{x})}\phi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}. \tag{3.3}$$

Let the solution of the boundary value problem (2.18) be given by the single-layer potentials with densities ϕ_1,ϕ_2:

$$\begin{cases} u^s(\boldsymbol{x})=\int_\Gamma G(\boldsymbol{x},\boldsymbol{y};\gamma_L)\phi_1(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, & \boldsymbol{x}\in\mathbb{R}^2\setminus\bar{D},\\ v^s(\boldsymbol{x})=\int_\Gamma G(\boldsymbol{x},\boldsymbol{y};\gamma_R)\phi_2(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, & \boldsymbol{x}\in\mathbb{R}^2\setminus\bar{D}.\end{cases} \tag{3.4}$$

Then, from (3.4), using the jump relation of layer potentials and the boundary condition on Γ, we deduce for $\boldsymbol{x}\in\Gamma$ that

$$-\phi_1(\boldsymbol{x})+(\mathbf{K}_L^{(*)}\phi_1)(\boldsymbol{x})+\tilde{a}_2(\mathbf{H}_L\phi_1)(\boldsymbol{x})+\tilde{a}_3(\mathbf{S}_L\phi_1)(\boldsymbol{x})+\tilde{a}_4(\mathbf{S}_R\phi_2)(\boldsymbol{x})=\tilde{a}_1f_1(\boldsymbol{x}), \tag{3.5}$$

$$-\phi_2(\boldsymbol{x})+(\mathbf{K}_R^{(*)}\phi_2)(\boldsymbol{x})+\tilde{a}_6(\mathbf{H}_R\phi_2)(\boldsymbol{x})+\tilde{a}_7(\mathbf{S}_R\phi_2)(\boldsymbol{x})+\tilde{a}_8(\mathbf{S}_L\phi_1)(\boldsymbol{x})=\tilde{a}_5f_2(\boldsymbol{x}), \tag{3.6}$$

where $\tilde{a}_1=\frac{2}{a_1},\tilde{a}_j=\frac{a_j}{a_1},j=2,3,4$ and $\tilde{a}_5=\frac{2}{a_5},\tilde{a}_j=\frac{a_j}{a_5},j=6,7,8$.

We assume that the boundary possesses a regular analytic and 2π-periodic parametric representation of the form $\Gamma=\{\boldsymbol{x}(t)=(x_1(t),x_2(t))^\top:\ 0\le t<2\pi\}$ in counterclockwise orientation satisfying $|\boldsymbol{x}'(t)|^2>0$ for all t. Multiplying (3.5) and (3.6) by $-|\boldsymbol{x}'(t)|$, we obtain the following coupled boundary integral equations

$$\psi_1(t)-\big[(\mathbf{K}_L^{(*)}\psi_1)(t)+\tilde{a}_2(\mathbf{H}_L\psi_1)(t)+\tilde{a}_3|\boldsymbol{x}'(t)|(\mathbf{S}_L\psi_1)(t) \\ +\tilde{a}_4|\boldsymbol{x}'(t)|(\mathbf{S}_R\psi_2)(t)\big]=\tilde{a}_1\tilde{f}_1(t), \tag{3.7}$$

$$\psi_2(t)-\big[(\mathbf{K}_R^{(*)}\psi_2)(t)+\tilde{a}_6(\mathbf{H}_R\psi_2)(t)+\tilde{a}_7|\boldsymbol{x}'(t)|(\mathbf{S}_R\psi_2)(t) \\ +\tilde{a}_8|\boldsymbol{x}'(t)|(\mathbf{S}_L\psi_1)(t)\big]=\tilde{a}_5\tilde{f}_2(t), \tag{3.8}$$

where $\tilde{f}_j(t)=-|\boldsymbol{x}'(t)|f_j(\boldsymbol{x}(t))$, $\psi_j(t)=|\boldsymbol{x}'(t)|\phi_j(\boldsymbol{x}(t)), j=1,2$, and

$$(\mathbf{S}_\sigma\psi)(t)=\int_0^{2\pi}\left[\frac{\mathrm{i}}{2}H_0^{(1)}(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|)\right]\psi(\xi)\mathrm{d}\xi \\ :=\int_0^{2\pi}S_\sigma(t,\xi)\psi(\xi)\mathrm{d}\xi, \tag{3.9}$$

$$(\mathbf{K}_\sigma^{(*)}\psi)(t)=-\frac{\mathrm{i}\gamma_\sigma}{2}\int_0^{2\pi}\frac{\boldsymbol{n}(t)\cdot(\boldsymbol{x}(t)-\boldsymbol{x}(\xi))}{|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|}H_1^{(1)}(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|)\psi(\xi)\mathrm{d}\xi \\ :=\int_0^{2\pi}K_\sigma^{(*)}(t,\xi)\psi(\xi)\mathrm{d}\xi, \tag{3.10}$$

$$(\mathbf{H}_\sigma\psi)(t)=-\frac{\mathrm{i}\gamma_\sigma}{2}\int_0^{2\pi}\frac{\boldsymbol{n}^\perp(t)\cdot(\boldsymbol{x}(t)-\boldsymbol{x}(\xi))}{|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|}H_1^{(1)}(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|)\psi(\xi)\mathrm{d}\xi \\ :=\int_0^{2\pi}H_\sigma(t,\xi)\psi(\xi)\mathrm{d}\xi, \tag{3.11}$$

here $\boldsymbol{\nu}(\boldsymbol{x}(t))=\frac{\boldsymbol{n}(t)}{|\boldsymbol{x}'(t)|}$, $\boldsymbol{\tau}(\boldsymbol{x}(t))=\frac{\boldsymbol{n}^\perp(t)}{|\boldsymbol{x}'(t)|}$, $\boldsymbol{n}(t)=(x_2'(t),-x_1'(t))^\top$, $\boldsymbol{n}^\perp(t)=(x_1'(t),x_2'(t))^\top$, $\boldsymbol{x}'(t)=(x_1'(t),x_2'(t))^\top$, $|\boldsymbol{x}'(t)|=\sqrt{[x_1'(t)]^2+[x_2'(t)]^2}$, $\boldsymbol{x}''(t)=(x_1''(t),x_2''(t))^\top$, $\psi(\xi)=\phi(\boldsymbol{x}(\xi))|\boldsymbol{x}'(\xi)|$, and $\sigma=L,R$.

From (3.7) and (3.8), we get

$$\boldsymbol{\psi}-\mathcal{A}\boldsymbol{\psi}=\mathbf{b}, \tag{3.12}$$

where $\boldsymbol{\psi}=(\psi_1,\psi_2)^\top$, $\mathbf{b}=(\tilde{a}_1\tilde{f}_1,\tilde{a}_5\tilde{f}_2)^\top$, and

$$\mathcal{A}=\begin{bmatrix}\tilde{a}_2\mathbf{H}_L & 0\\ 0 & \tilde{a}_6\mathbf{H}_R\end{bmatrix}+\begin{bmatrix}\mathbf{K}_L^{(*)} & 0\\ 0 & \mathbf{K}_R^{(*)}\end{bmatrix}+\begin{bmatrix}\tilde{a}_3|\boldsymbol{x}'|\mathbf{S}_L & \tilde{a}_4|\boldsymbol{x}'|\mathbf{S}_R\\ \tilde{a}_8|\boldsymbol{x}'|\mathbf{S}_L & \tilde{a}_7|\boldsymbol{x}'|\mathbf{S}_R\end{bmatrix}. \tag{3.13}$$

Note that the kernel $S_\sigma(t,\xi)$ can be written as

$$S_\sigma(t,\xi)=S_{\sigma,1}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right)+S_{\sigma,2}(t,\xi),\qquad t,\xi\in[0,2\pi], \tag{3.14}$$

where

$$S_{\sigma,1}(t,\xi)=\begin{cases}-\frac{1}{2\pi}J_0(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|), & \text{if } t\neq\xi,\\ -\frac{1}{2\pi}, & \text{if } t=\xi,\end{cases} \tag{3.15}$$

$$S_{\sigma,2}(t,\xi)=\begin{cases}S_\sigma(t,\xi)-S_{\sigma,1}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right), & \text{if } t\neq\xi,\\ \frac{\mathrm{i}}{2}\left[1+\frac{\mathrm{i}}{\pi}\ln\left(\frac{\mathbf{Eu}\gamma_\sigma|\boldsymbol{x}'(t)|}{2}\right)^2\right], & \text{if } t=\xi.\end{cases} \tag{3.16}$$

Here $\mathbf{Eu}=1.78107$ is the Euler constant. Hence, $\mathbf{S}_\sigma\psi$ can be equivalently rewritten as

$$(\mathbf{S}_\sigma\psi)(t)=\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)S_{\sigma,1}(t,\xi)\psi(\xi)\mathrm{d}\xi+\int_0^{2\pi}S_{\sigma,2}(t,\xi)\psi(\xi)\mathrm{d}\xi,$$
$$:=(\mathbf{S}_{\sigma,1}\psi)(t)+(\mathbf{S}_{\sigma,2}\psi)(t). \tag{3.17}$$

Similarly, the kernel $K_\sigma^{(*)}(t,\xi)$ can be written as

$$K_\sigma^{(*)}(t,\xi)=K_{\sigma,1}^{(*)}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right)+K_{\sigma,2}^{(*)}(t,\xi),\qquad t,\xi\in[0,2\pi], \tag{3.18}$$

where

$$K_{\sigma,1}^{(*)}(t,\xi)=\begin{cases}\frac{\gamma_\sigma}{2\pi}[\boldsymbol{n}(t)\cdot(\boldsymbol{x}(t)-\boldsymbol{x}(\xi))]\frac{J_1(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|)}{|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|}, & \text{if } t\neq\xi,\\ 0, & \text{if } t=\xi,\end{cases} \tag{3.19}$$

$$K_{\sigma,2}^{(*)}(t,\xi)=\begin{cases}K_\sigma^{(*)}(t,\xi)-K_{\sigma,1}^{(*)}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right), & \text{if } t\neq\xi,\\ \frac{\boldsymbol{n}(t)\cdot\boldsymbol{x}''(t)}{2\pi|\boldsymbol{x}'(t)|^2}, & \text{if } t=\xi.\end{cases} \tag{3.20}$$

Hence, $\mathbf{K}_\sigma^{(*)}\psi$ can be equivalently rewritten as

$$(\mathbf{K}_\sigma^{(*)}\psi)(t)=\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)K_{\sigma,1}^{(*)}(t,\xi)\psi(\xi)\mathrm{d}\xi+\int_0^{2\pi}K_{\sigma,2}^{(*)}(t,\xi)\psi(\xi)\mathrm{d}\xi,$$
$$:=(\mathbf{K}_{\sigma,1}^{(*)}\psi)(t)+(\mathbf{K}_{\sigma,2}^{(*)}\psi)(t). \tag{3.21}$$

Next, we split the kernel $H_\sigma(t,\xi)$, $t,\xi\in[0,2\pi]$ into the following form

$$H_\sigma(t,\xi)=\hat{H}_1(t,\xi)+\tilde{H}_1(t,\xi)+H_{\sigma,1}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right)+\tilde{H}_{\sigma,2}(t,\xi), \tag{3.22}$$

where

$$\hat{H}_1(t,\xi)=\frac{1}{2\pi}\left(\cot\frac{\xi-t}{2}+\mathrm{i}\right), \tag{3.23}$$

$$\tilde{H}_1(t,\xi)=\begin{cases}\left\{-\frac{1}{\pi}[\boldsymbol{n}^\perp(t)\cdot(\boldsymbol{x}(t)-\boldsymbol{x}(\xi))]\frac{\tan\frac{\xi-t}{2}}{|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|^2}-\frac{1}{2\pi}\right\}\cot\frac{\xi-t}{2}, & \text{if } t\neq\xi,\\ 0, & \text{if } t=\xi,\end{cases} \tag{3.24}$$

$$H_{\sigma,1}(t,\xi)=\begin{cases}\frac{\gamma_\sigma}{2\pi}[\boldsymbol{n}^\perp(t)\cdot(\boldsymbol{x}(t)-\boldsymbol{x}(\xi))]\frac{J_1(\gamma_\sigma|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|)}{|\boldsymbol{x}(t)-\boldsymbol{x}(\xi)|}, & \text{if } t\neq\xi,\\ 0, & \text{if } t=\xi,\end{cases} \tag{3.25}$$

$$\tilde{H}_{\sigma,2}(t,\xi)=\begin{cases}H_\sigma(t,\xi)-\hat{H}_1(t,\xi)-\tilde{H}_1(t,\xi)-H_{\sigma,1}(t,\xi)\ln\left(4\sin^2\frac{t-\xi}{2}\right), & \text{if } t\neq\xi,\\ -\frac{\mathrm{i}}{2\pi}, & \text{if } t=\xi.\end{cases} \tag{3.26}$$

Consequently,

$$(\mathbf{H}_\sigma\psi)(t)=\int_0^{2\pi}\hat{H}_1(t,\xi)\psi(\xi)\mathrm{d}\xi+\int_0^{2\pi}\tilde{H}_1(t,\xi)\psi(\xi)\mathrm{d}\xi$$
$$+\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)H_{\sigma,1}(t,\xi)\psi(\xi)\mathrm{d}\xi+\int_0^{2\pi}\tilde{H}_{\sigma,2}(t,\xi)\psi(\xi)\mathrm{d}\xi,$$
$$:=(\hat{\mathbf{H}}_1\psi)(t)+(\tilde{\mathbf{H}}_1\psi)(t)+(\mathbf{H}_{\sigma,1}\psi)(t)+(\tilde{\mathbf{H}}_{\sigma,2}\psi)(t). \tag{3.27}$$

3.1. Operator splitting. From (3.13), (3.17), (3.21) and (3.27), we split $\mathcal{A}$ into

$$\mathcal{A}=\hat{\mathcal{H}}+\mathcal{W}+\mathcal{S}, \tag{3.28}$$

where $\mathcal{W}=\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3$, $\mathcal{S}=\mathcal{S}_1+\mathcal{S}_2$ and

$$\hat{\mathcal{H}}=\begin{bmatrix}\tilde{a}_2\hat{\mathbf{H}}_1 & 0\\ 0 & \tilde{a}_6\hat{\mathbf{H}}_1\end{bmatrix},\qquad \mathcal{W}_1=\begin{bmatrix}\tilde{a}_2\tilde{\mathbf{H}}_1 & 0\\ 0 & \tilde{a}_6\tilde{\mathbf{H}}_1\end{bmatrix}, \tag{3.29}$$

$$\mathcal{W}_2=\begin{bmatrix}\mathbf{K}_{L,1}^{(*)}+\tilde{a}_2\mathbf{H}_{L,1} & 0\\ 0 & \mathbf{K}_{R,1}^{(*)}+\tilde{a}_6\mathbf{H}_{R,1}\end{bmatrix}, \tag{3.30}$$
$$\mathcal{W}_3=\begin{bmatrix}\mathbf{K}_{L,2}^{(*)}+\tilde{a}_2\tilde{\mathbf{H}}_{L,2} & 0\\ 0 & \mathbf{K}_{R,2}^{(*)}+\tilde{a}_6\tilde{\mathbf{H}}_{R,2}\end{bmatrix},$$

$$\mathcal{S}_1=\begin{bmatrix}\tilde{a}_3\mathbf{FS}_{L,1} & \tilde{a}_4\mathbf{FS}_{R,1}\\ \tilde{a}_8\mathbf{FS}_{L,1} & \tilde{a}_7\mathbf{FS}_{R,1}\end{bmatrix},\quad \mathcal{S}_2=\begin{bmatrix}\tilde{a}_3\mathbf{FS}_{L,2} & \tilde{a}_4\mathbf{FS}_{R,2}\\ \tilde{a}_8\mathbf{FS}_{L,2} & \tilde{a}_7\mathbf{FS}_{R,2}\end{bmatrix}, \tag{3.31}$$

where

$$\mathbf{F}\psi:=|\boldsymbol{x}'|\psi.$$

For $p\geq 0$, let $H^p[0,2\pi]$ denote the Sobolev space of 2π-periodic functions $g:\mathbb{R}\to\mathbb{C}$ with the norm

$$\|g\|_p^2:=\sum_{m=-\infty}^{\infty}(1+m^2)^p|\hat{g}_m|^2<\infty, \tag{3.32}$$

where

$$\hat{g}_m:=\frac{1}{2\pi}\int_0^{2\pi}g(t)e^{-imt}\mathrm{d}t,\qquad m=0,\pm1,\pm2,\cdots, \tag{3.33}$$

are the Fourier coefficients of g. Define Sobolev spaces

$$H^p[0,2\pi]^2=\Big\{\boldsymbol{g}=(g_1,g_2)^\top\,\Big|\, g_j(t)\in H^p[0,2\pi], j=1,2\Big\}, \tag{3.34}$$

with the norm

$$\|\boldsymbol{g}\|_p=\|g_1\|_p+\|g_2\|_p. \tag{3.35}$$

THEOREM 3.1. *For all $p\geq 0$, the operator $\mathbf{F}:H^p[0,2\pi]\to H^p[0,2\pi]$ is bounded.*

Proof. For $\psi\in H^p[0,2\pi]$, recalling $\mathbf{F}\psi=f\psi$, where $f=f(t)=|\boldsymbol{x}'(t)|>0$ is analytic, in particular, $f\in C_{2\pi}^l$ $(l\geq p\geq 0)$. Then, by Corollary 8.8 of [20], $\mathbf{F}\psi\in H^p[0,2\pi]$ and

$$\|\mathbf{F}\psi\|_p=\|f\psi\|_p\leq C\big(\|f\|_\infty+\||f^{(l)}\|_\infty\big)\|\psi\|_p,\qquad 0\leq p\leq l, \tag{3.36}$$

for some constant C depending on p. □

THEOREM 3.2. *For all $p \geq 0$, the operator $\hat{\mathcal{H}}: H^p[0,2\pi]^2 \to H^p[0,2\pi]^2$ is bounded and has a bounded inverse*

$$\hat{\mathcal{H}}^{-1} = \mathcal{M}_1 \hat{\mathcal{H}}, \tag{3.37}$$

furthermore, the operator $\mathcal{I} - \hat{\mathcal{H}}$ has a bounded inverse as

$$\big(\mathcal{I} - \hat{\mathcal{H}}\big)^{-1} = \mathcal{M}_2(\mathcal{I} + \hat{\mathcal{H}}), \tag{3.38}$$

where

$$\mathcal{M}_1 = \begin{bmatrix} -\frac{1}{\tilde{a}_2^2} & 0 \\ 0 & -\frac{1}{\tilde{a}_6^2} \end{bmatrix}, \qquad \mathcal{M}_2 = \begin{bmatrix} \frac{1}{1+\tilde{a}_2^2} & 0 \\ 0 & \frac{1}{1+\tilde{a}_6^2} \end{bmatrix}. \tag{3.39}$$

Proof. For $\psi \in H^p[0,2\pi]$,

$$(\hat{\mathbf{H}}_1\psi)(t) = \int_0^{2\pi} \hat{H}_1(t,\xi)\psi(\xi)\mathrm{d}\xi = \frac{1}{2\pi}\int_0^{2\pi}\left(\cot\frac{\xi - t}{2} + \mathrm{i}\right)\psi(\xi)\mathrm{d}\xi. \tag{3.40}$$

Note that the singular integral operator $\hat{\mathbf{H}}_1 : H^p[0,2\pi] \to H^p[0,2\pi]$ is bounded and has a bounded inverse $\hat{\mathbf{H}}_1^{-1} = -\hat{\mathbf{H}}_1$ for all $p \geq 0$, see e.g. [14,20]. It follows from $\gamma_\sigma^2 > 0$ ($\sigma = L,R$) that $1+\tilde{a}_2^2 \neq 0$ and $1+\tilde{a}_6^2 \neq 0$. Then,

$$(\mathbf{I} - \tilde{a}_j\hat{\mathbf{H}}_1)\left[\frac{1}{1+\tilde{a}_j^2}(\mathbf{I} + \tilde{a}_j\hat{\mathbf{H}}_1)\right] = \frac{1}{1+\tilde{a}_j^2}(\mathbf{I} + \tilde{a}_j^2\mathbf{I}) = \mathbf{I}, \qquad j = 2,6. \tag{3.41}$$

Thus, from (3.41), we get

$$\big(\mathcal{I} - \hat{\mathcal{H}}\big)\begin{bmatrix} \frac{1}{1+\tilde{a}_2^2}(\mathbf{I} + \tilde{a}_2\hat{\mathbf{H}}_1) & 0 \\ 0 & \frac{1}{1+\tilde{a}_6^2}(\mathbf{I} + \tilde{a}_6\hat{\mathbf{H}}_1) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}. \tag{3.42}$$

Combining (3.42) and the boundedness of operator $\hat{\mathbf{H}}_1 : H^p[0,2\pi] \to H^p[0,2\pi]$, we find $\mathcal{I} - \hat{\mathcal{H}}$ has a bounded inverse as (3.38). Similarly, the corresponding inverse operator for $\hat{\mathcal{H}}$ is also easily obtained and is given by (3.37). ∎

THEOREM 3.3. *For all $p \geq 0$, the operator $\mathcal{W}: H^p[0,2\pi]^2 \to H^{p+1}[0,2\pi]^2$ is bounded. Furthermore, the operator $\mathcal{W}: H^p[0,2\pi]^2 \to H^p[0,2\pi]^2$ is compact.*

Proof. For $\sigma = L,R$, noting that the operators $\mathbf{K}_{\sigma,1}^{(*)}$ and $\mathbf{H}_{\sigma,1}$ have logarithmic singularities and are given by

$$\int_0^{2\pi}\left[\ln\left(4\sin^2\frac{t-\xi}{2}\right)K(t,\xi)\right]\psi(\xi)\mathrm{d}\xi, \qquad 0 \leq t \leq 2\pi, \tag{3.43}$$

where $K(t,\xi) = K_{\sigma,1}^{(*)}(t,\xi)$ or $H_{\sigma,1}(t,\xi)$ are infinitely differentiable and 2π-periodic with respect to both variables. Applying Theorem 12.15 and Corollary 12.16 in [20], we obtain that $\mathbf{K}_{\sigma,1}^{(*)}, \mathbf{H}_{\sigma,1} : H^p[0,2\pi] \to H^{p+1}[0,2\pi]$ are bounded and consequently are compact from $H^p[0,2\pi] \to H^p[0,2\pi]$ for all $p \geq 0$. Hence, $\mathcal{W}_2 : H^p[0,2\pi]^2 \to H^{p+1}[0,2\pi]^2$ is bounded and consequently is compact from $H^p[0,2\pi]^2 \to H^p[0,2\pi]^2$.

Since the kernel functions $\tilde{H}_1$, $K_{\sigma,2}^{(*)}$ and $\tilde{H}_{\sigma,2}$ are analytic, it follows from Theorem A.45 in [19] and Theorem 8.13 in [20] that the operators $\tilde{\mathbf{H}}_1, \mathbf{K}_{\sigma,2}^{(*)}, \tilde{\mathbf{H}}_{\sigma,2} : H^p[0,2\pi] \to$

$H^{p+r}[0,2\pi]$ are bounded for all integers $r\geq 0$ and arbitrary $p\geq 0$. Then the operators $\mathcal{W}_1,\mathcal{W}_3: H^p[0,2\pi]^2\to H^{p+r}[0,2\pi]^2$ are bounded for all integers $r\geq 0$ and arbitrary $p\geq 0$. In particular, for all $p\geq 0$, the operators $\mathcal{W}_1,\mathcal{W}_3: H^p[0,2\pi]^2\to H^{p+1}[0,2\pi]^2$ are bounded and consequently are compact from $H^p[0,2\pi]^2\to H^p[0,2\pi]^2$. □

THEOREM 3.4. *For all $p\geq 0$, the operator $\mathcal{S}: H^p[0,2\pi]^2\to H^{p+1}[0,2\pi]^2$ is bounded. Furthermore, the operator $\mathcal{S}: H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ is compact.*

Proof. For $\sigma=L,R$ and $p\geq 0$, similar to the proof of Theorem 3.3, we obtain that the operators $\mathbf{S}_{\sigma,1},\mathbf{S}_{\sigma,2}: H^p[0,2\pi]\to H^{p+1}[0,2\pi]$ are bounded, hence, compact from $H^p[0,2\pi]\to H^p[0,2\pi]$. Then, from Theorem 3.1, the operators $\mathcal{S}_1,\mathcal{S}_2: H^p[0,2\pi]^2\to H^{p+1}[0,2\pi]^2$ are bounded, and consequently compact from $H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ for all $p\geq 0$. □

We rewrite the operator Equation (3.12) in the form

$$\boldsymbol{\psi}-(\hat{\mathcal{H}}+\mathcal{W}+\mathcal{S})\boldsymbol{\psi}=\mathbf{b}. \tag{3.44}$$

i.e.,

$$\mathcal{H}\boldsymbol{\psi}-\mathcal{K}\boldsymbol{\psi}=\mathbf{b}, \tag{3.45}$$

where $\mathcal{H}=\mathcal{I}-\hat{\mathcal{H}}$ and $\mathcal{K}=\mathcal{W}+\mathcal{S}$. Using Theorem 3.2, we have $\mathcal{H}: H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ is a bijective bounded linear operator mapping which has a bounded inverse (3.38). By Theorem 3.3 and Theorem 3.4 we have

THEOREM 3.5. *For all $p\geq 0$, the operator $\mathcal{K}: H^p[0,2\pi]^2\to H^{p+1}[0,2\pi]^2$ is bounded. Furthermore, the operator $\mathcal{K}: H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ is compact.*

This follows immediately from the fact that we can transform the Equation (3.45) into the equivalent form

$$\boldsymbol{\psi}-\tilde{\mathcal{A}}\boldsymbol{\psi}=\mathcal{H}^{-1}\mathbf{b}=\mathcal{M}_2(\mathcal{I}+\hat{\mathcal{H}})\mathbf{b}, \tag{3.46}$$

where

$$\tilde{\mathcal{A}}=\mathcal{H}^{-1}\mathcal{K}=\mathcal{M}_2(\mathcal{K}+\hat{\mathcal{H}}\mathcal{K}). \tag{3.47}$$

Then, by Theorem 2.21 in [20] and Theorem 3.2, Theorem 3.5 , we have

THEOREM 3.6. *For all $p\geq 0$, the operator $\tilde{\mathcal{A}}: H^p[0,2\pi]^2\to H^{p+1}[0,2\pi]^2$ is bounded. Furthermore, the operator $\tilde{\mathcal{A}}: H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ is compact.*

Obviously, $\mathcal{I}-\tilde{\mathcal{A}}$ is surjective, then, using Theorem 3.4 in [20] and Theorem 3.6, we obtain the following theorem

THEOREM 3.7. *The operator $\mathcal{I}-\tilde{\mathcal{A}}$ is injective, and the inverse operator $(\mathcal{I}-\tilde{\mathcal{A}})^{-1}: H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ is bounded and there exists a unique solution to the Equation (3.46).*

4. Convergence analysis

In this section, we study the convergence of the discretization of integral Equations (3.46) by using the collocation method. Denote by T_n the interpolation operator, which maps 2π-periodic scalar function $g\in H^p[0,2\pi]$ into a unique trigonometric polynomial $T_n g$ at the equidistant interpolation points $\xi_j^{(n)}=\frac{\pi j}{n}$, $j=0,\ldots,2n-1$. Given the values

$g(\xi_j^{(n)})$, $j=0,\dots,2n-1$, then, there exists a unique trigonometric polynomial of the form

$$(T_n g)(t) = \frac{\alpha_0}{2} + \sum_{l=1}^{n-1}\left[\alpha_l \cos lt + \beta_l \sin lt\right] + \frac{\alpha_n}{2}\cos nt, \qquad 0\le t\le 2\pi, \tag{4.1}$$

with the interpolation property $(T_n g)(\xi_j^{(n)}) = g(\xi_j^{(n)})$, $j=0,\dots,2n-1$. Moreover, the coefficients are given by

$$\alpha_l = \frac{1}{n}\sum_{j=0}^{2n-1} g(\xi_j^{(n)})\cos(l\xi_j^{(n)}), \quad l=0,\dots,n, \tag{4.2}$$

$$\beta_l = \frac{1}{n}\sum_{j=0}^{2n-1} g(\xi_j^{(n)})\sin(l\xi_j^{(n)}), \quad l=1,\dots,n-1. \tag{4.3}$$

Let $X_n \subset H^p[0,2\pi]$ be the space of trigonometric polynomials of degree less than or equal to n of the form (4.1). Then, $T_n : H^p[0,2\pi]\to X_n$ is a bounded linear operator. Let $X_n^2 := \{\boldsymbol{g}=(g_1,g_2)^\top : g_1\in X_n,\ g_2\in X_n\}$ and define the interpolation operator $\mathcal{T}_n : H^p[0,2\pi]^2 \to X_n^2$ by

$$\mathcal{T}_n \boldsymbol{g} = (T_n g_1, T_n g_2)^\top, \qquad \forall\ \boldsymbol{g}=(g_1,g_2)^\top \in H^p[0,2\pi]^2. \tag{4.4}$$

THEOREM 4.1 ([20]). *For the trigonometric interpolation,*

$$\|T_n g - g\|_q \le \frac{C}{n^{p-q}}\|g\|_p, \qquad 0\le q\le p,\ \frac{1}{2}<p, \tag{4.5}$$

for all $g\in H^p[0,2\pi]$ and some constant C depending on p and q.

Obviously, by Theorem 4.1, the operators $T_n : H^p[0,2\pi]\to H^p[0,2\pi]$ are uniformly bounded. For the integral equations of the second kind (3.46), its projected equation is

$$\boldsymbol{\psi}^{(n)} - \mathcal{T}_n\tilde{\mathcal{A}}\boldsymbol{\psi}^{(n)} = \mathcal{T}_n\mathcal{M}_2(\mathcal{I}+\hat{\mathcal{H}})\mathbf{b}, \tag{4.6}$$

where $\boldsymbol{\psi}^{(n)} = (\psi_1^{(n)},\psi_2^{(n)})^\top \in X_n^2$ is the approximation of the solution $\boldsymbol{\psi}$ by a trigonometric polynomial.

THEOREM 4.2. *For sufficiently large n, the approximate Equation (4.6) is uniquely solvable and with an error estimate*

$$\|\boldsymbol{\psi}^{(n)} - \boldsymbol{\psi}\|_p \le M_1\|\mathcal{T}_n\boldsymbol{\psi} - \boldsymbol{\psi}\|_p, \qquad p\ge 0, \tag{4.7}$$

for some positive constant M_1 depending on $\tilde{\mathcal{A}}$.

Proof. For the solution $\boldsymbol{\psi}\in H^p[0,2\pi]^2$ of Equations (3.46), by Theorem 3.6, Theorem 3.7 and Theorem 4.1, we deduce that

$$\|\mathcal{T}_n\tilde{\mathcal{A}}\boldsymbol{\psi} - \tilde{\mathcal{A}}\boldsymbol{\psi}\|_p \le \frac{C}{n}\|\tilde{\mathcal{A}}\boldsymbol{\psi}\|_{p+1} \le \frac{C}{n}\|\boldsymbol{\psi}\|_p, \qquad p\ge 0, \tag{4.8}$$

which implies

$$\|\mathcal{T}_n\tilde{\mathcal{A}} - \tilde{\mathcal{A}}\|_p \to 0 \qquad \text{as } n\to\infty,\ p\ge 0. \tag{4.9}$$

Then, by Theorem 13.10 in [20], the proof is complete. ∎

Next, we deduce that the Lagrange basis for the trigonometric interpolation has the form

$$L_j(t)=\frac{1}{2n}\left\{1+2\sum_{l=1}^{n-1}\cos l(t-\xi_j^{(n)})+\cos n(t-\xi_j^{(n)})\right\}, \tag{4.10}$$

for $t\in[0,2\pi]$ and $j=0,\dots,2n-1$. Then, we find an approximation $\tilde{\boldsymbol{\psi}}^{(n)}\in X_n^2$ of the solution $\boldsymbol{\psi}$ by

$$\tilde{\boldsymbol{\psi}}^{(n)}=(\tilde{\psi}_1^{(n)},\tilde{\psi}_2^{(n)})^\top=\left(\sum_{j=0}^{2n-1}\tilde{\psi}_1^{(n)}(\xi_j^{(n)})L_j(t),\sum_{j=0}^{2n-1}\tilde{\psi}_2^{(n)}(\xi_j^{(n)})L_j(t)\right)^\top, \tag{4.11}$$

which satisfies

$$\tilde{\boldsymbol{\psi}}^{(n)}-\mathcal{T}_n\tilde{\mathcal{A}}_n\tilde{\boldsymbol{\psi}}^{(n)}=\mathcal{T}_n\mathcal{M}_2(\mathcal{I}+\hat{\mathcal{H}}_n)\mathbf{b}. \tag{4.12}$$

Here $\tilde{\mathcal{A}}_n=\mathcal{M}_2(\mathcal{K}_n+\hat{\mathcal{H}}_n\mathcal{K}_n)$, $\hat{\mathcal{H}}_n=\hat{\mathcal{H}}\mathcal{T}_n$ and $\mathcal{K}_n=\mathcal{W}_n+\mathcal{S}_n$.

For $\forall\boldsymbol{\eta}\in X_n^2$, $\hat{\mathcal{H}}_n,\mathcal{W}_n,\mathcal{S}_n$ have the following interpolatory quadrature operators of the form

$$\hat{\mathcal{H}}_n\boldsymbol{\eta}=\begin{bmatrix}\tilde{a}_2\hat{\mathbf{H}}_1T_n\eta_1\\ \tilde{a}_6\hat{\mathbf{H}}_1T_n\eta_2\end{bmatrix}, \tag{4.13}$$

$$\mathcal{W}_n\boldsymbol{\eta}=\begin{bmatrix}\tilde{a}_2\tilde{\mathbf{H}}_{1n}\eta_1\\ \tilde{a}_6\tilde{\mathbf{H}}_{1n}\eta_2\end{bmatrix}+\begin{bmatrix}(\mathbf{K}_{L,1n}^{(*)}+\tilde{a}_2\mathbf{H}_{L,1n})\eta_1\\ (\mathbf{K}_{R,1n}^{(*)}+\tilde{a}_6\mathbf{H}_{R,1n})\eta_2\end{bmatrix}+\begin{bmatrix}(\mathbf{K}_{L,2n}^{(*)}+\tilde{a}_2\tilde{\mathbf{H}}_{L,2n})\eta_1\\ (\mathbf{K}_{R,2n}^{(*)}+\tilde{a}_6\tilde{\mathbf{H}}_{R,2n})\eta_2\end{bmatrix}, \tag{4.14}$$

$$\mathcal{S}_n\boldsymbol{\eta}=\begin{bmatrix}\tilde{a}_3\mathbf{FS}_{L,1n}\eta_1+\tilde{a}_4\mathbf{FS}_{R,1n}\eta_2\\ \tilde{a}_8\mathbf{FS}_{L,1n}\eta_1+\tilde{a}_7\mathbf{FS}_{R,1n}\eta_2\end{bmatrix}+\begin{bmatrix}\tilde{a}_3\mathbf{FS}_{L,2n}\eta_1+\tilde{a}_4\mathbf{FS}_{R,2n}\eta_2\\ \tilde{a}_8\mathbf{FS}_{L,2n}\eta_1+\tilde{a}_7\mathbf{FS}_{R,2n}\eta_2\end{bmatrix}, \tag{4.15}$$

where

$$(\hat{\mathbf{H}}_1T_n\eta_k)(t)=\int_0^{2\pi}\hat{H}_1(t,\xi)[T_n\eta_k](\xi)\mathrm{d}\xi, \tag{4.16}$$

$$(\tilde{\mathbf{H}}_{1n}\eta_k)(t)=\int_0^{2\pi}[T_n(\tilde{H}_1(t,\cdot)\eta_k)](\xi)\mathrm{d}\xi, \tag{4.17}$$

$$\begin{aligned}&[(\mathbf{K}_{\sigma,1n}^{(*)}+\tilde{a}_j\mathbf{H}_{\sigma,1n})\eta_k](t)\\ =&\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)\left\{T_n\left[\left(K_{\sigma,1}^{(*)}(t,\cdot)+\tilde{a}_jH_{\sigma,2}(t,\cdot)\right)\eta_k\right]\right\}(\xi)\mathrm{d}\xi,\quad j=2,6,\end{aligned} \tag{4.18}$$

$$[(\mathbf{K}_{\sigma,2n}^{(*)}+\tilde{a}_j\tilde{\mathbf{H}}_{\sigma,2n})\eta_k](t)=\int_0^{2\pi}\left\{T_n\left[\left(K_{\sigma,2}^{(*)}(t,\cdot)+\tilde{a}_j\tilde{H}_{\sigma,2}(t,\cdot)\right)\eta_k\right]\right\}(\xi)\mathrm{d}\xi, j=2,6, \tag{4.19}$$

$$(\mathbf{FS}_{\sigma,1n}\eta_k)(t)=|x'(t)|\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)\left[T_n\left(S_{\sigma,1}(t,\cdot)\eta_k\right)\right](\xi)\mathrm{d}\xi, \tag{4.20}$$

$$(\mathbf{FS}_{\sigma,2n}\eta_k)(t)=|x'(t)|\int_0^{2\pi}\left[T_n\left(S_{\sigma,2}(t,\cdot)\eta_k\right)\right](\xi)\mathrm{d}\xi, \tag{4.21}$$

with $k=1,2$, $\sigma=L,R$.

THEOREM 4.3. *Assume that $p\geq 1$. For sufficiently large n, the approximate Equation (4.12) is uniquely solvable and with an error estimate*

$$\|\tilde{\boldsymbol{\psi}}^{(n)}-\boldsymbol{\psi}\|_p\leq M_2\big\{\|\mathcal{T}_n\boldsymbol{\psi}-\boldsymbol{\psi}\|_p+\|\mathcal{T}_n(\tilde{\mathcal{A}}_n-\tilde{\mathcal{A}})\boldsymbol{\psi}\|_p+\|\mathcal{T}_n(\hat{\mathcal{H}}_n\mathbf{b}-\hat{\mathcal{H}}\mathbf{b})\|_p\big\}, \tag{4.22}$$

for some positive constant M_2.

Proof. Using Theorem 3.3 and Theorem 3.4, for all $\boldsymbol{g}\in H^p[0,2\pi]^2$, setting $q=p-1\geq 0$ in Theorem 12.18 of [20], we have

$$\|\mathcal{W}_n\boldsymbol{g}-\mathcal{W}\boldsymbol{g}\|_p\leq\frac{C}{n}\|\boldsymbol{g}\|_p,\qquad \|\mathcal{S}_n\boldsymbol{g}-\mathcal{S}\boldsymbol{g}\|_p\leq\frac{C}{n}\|\boldsymbol{g}\|_p, \tag{4.23}$$

for some constant C depending on p. Then, the operators $\mathcal{W}_n,\mathcal{W}_n-\mathcal{W}$ and $\mathcal{S}_n,\mathcal{S}_n-\mathcal{S}$ are uniformly bounded from $H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ for $p\geq 1$.

Hence, we get $\mathcal{K}_n,\mathcal{K}_n-\mathcal{K}$ are uniformly bounded from $H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ for $p\geq 1$, and satisfy

$$\|\mathcal{K}_n\boldsymbol{g}-\mathcal{K}\boldsymbol{g}\|_p\leq\|\mathcal{W}_n\boldsymbol{g}-\mathcal{W}\boldsymbol{g}\|+\|\mathcal{S}_n\boldsymbol{g}-\mathcal{S}\boldsymbol{g}\|_p\leq\frac{C}{n}\|\boldsymbol{g}\|_p. \tag{4.24}$$

It follows from Theorem 3.2 and Theorem 4.1 that

$$\|(\hat{\mathcal{H}}_n-\hat{\mathcal{H}})\boldsymbol{g}\|_p=\|\hat{\mathcal{H}}(\mathcal{T}_n\boldsymbol{g}-\boldsymbol{g})\|_p\leq\tilde{C}_1\|\mathcal{T}_n\boldsymbol{g}-\boldsymbol{g}\|_p\leq\tilde{C}\|\boldsymbol{g}\|_p, \tag{4.25}$$

for some constant $\tilde{C}$ depending on p and $\hat{\mathcal{H}}$. Then, $\hat{\mathcal{H}}_n,\hat{\mathcal{H}}_n-\hat{\mathcal{H}}$ are uniformly bounded from $H^p[0,2\pi]\to H^p[0,2\pi]$ for $p>\frac{1}{2}$.

Therefore, from $\hat{\mathcal{H}}_n\boldsymbol{\phi}=\hat{\mathcal{H}}\boldsymbol{\phi}$ for $\boldsymbol{\phi}\in X_n^2$, (4.24) and (4.25), by Theorem 3.2, Theorem 3.5, Theorem 4.1 and the uniform boundedness of the operator $\mathcal{T}_n:H^p[0,2\pi]^2\to H^p[0,2\pi]^2$, we obtain

$$\begin{aligned}
&\|\hat{\mathcal{H}}_n\mathcal{K}_n\boldsymbol{g}-\hat{\mathcal{H}}\mathcal{K}\boldsymbol{g}\|_p\\
\leq&\|\hat{\mathcal{H}}_n\mathcal{K}_n\boldsymbol{g}-\hat{\mathcal{H}}_n\mathcal{K}\boldsymbol{g}\|_p+\|\hat{\mathcal{H}}_n\mathcal{K}\boldsymbol{g}-\hat{\mathcal{H}}\mathcal{K}\boldsymbol{g}\|_p\\
\leq&\|\hat{\mathcal{H}}_n(\mathcal{K}_n-\mathcal{K})\boldsymbol{g}\|_p+\|(\hat{\mathcal{H}}_n-\hat{\mathcal{H}})(\mathcal{K}\boldsymbol{g}-\mathcal{T}_n\mathcal{K}\boldsymbol{g})\|_p+\|(\hat{\mathcal{H}}_n-\hat{\mathcal{H}})\mathcal{T}_n\mathcal{K}\boldsymbol{g}\|_p\\
=&\|\hat{\mathcal{H}}\mathcal{T}_n(\mathcal{K}_n-\mathcal{K})\boldsymbol{g}\|_p+\|(\hat{\mathcal{H}}_n-\hat{\mathcal{H}})(\mathcal{K}\boldsymbol{g}-\mathcal{T}_n\mathcal{K}\boldsymbol{g})\|_p\\
\leq&\hat{C}_1\|(\mathcal{K}_n-\mathcal{K})\boldsymbol{g}\|_p+\tilde{C}\|\mathcal{K}\boldsymbol{g}-\mathcal{T}_n\mathcal{K}\boldsymbol{g}\|_p\leq\hat{C}_1\frac{C}{n}\|\boldsymbol{g}\|_p+\tilde{C}\frac{C}{n}\|\mathcal{K}\boldsymbol{g}\|_{p+1}\\
\leq&\hat{C}_1\frac{C}{n}\|\boldsymbol{g}\|_p+\tilde{C}\frac{C}{n}\|\boldsymbol{g}\|_p=\frac{\hat{C}_2}{n}\|\boldsymbol{g}\|_p.
\end{aligned} \tag{4.26}$$

Furthermore, from (4.24) and (4.26), we have

$$\begin{aligned}
\|\mathcal{T}_n\tilde{\mathcal{A}}_n\boldsymbol{g}-\mathcal{T}_n\tilde{\mathcal{A}}\boldsymbol{g}\|_p&=\|\mathcal{T}_n\mathcal{M}_2(\mathcal{K}_n+\hat{\mathcal{H}}_n\mathcal{K}_n)\boldsymbol{g}-\mathcal{T}_n\mathcal{M}_2(\mathcal{K}+\hat{\mathcal{H}}\mathcal{K})\boldsymbol{g}\|_p\\
&\leq\|(\mathcal{K}_n+\hat{\mathcal{H}}_n\mathcal{K}_n)\boldsymbol{g}-(\mathcal{K}+\hat{\mathcal{H}}\mathcal{K})\boldsymbol{g}\|_p\\
&\leq\|\mathcal{K}_n\boldsymbol{g}-\mathcal{K}\boldsymbol{g}\|_p+\|\hat{\mathcal{H}}_n\mathcal{K}_n\boldsymbol{g}-\hat{\mathcal{H}}\mathcal{K}\boldsymbol{g}\|_p\leq\frac{\hat{C}}{n}\|\boldsymbol{g}\|_p,
\end{aligned} \tag{4.27}$$

for some constant $\hat{C}$ depending on p and $\mathcal{K},\hat{\mathcal{H}}$. Similarly, for $\boldsymbol{\phi}\in X_n^2$, we have

$$\|(\mathcal{T}_n\tilde{\mathcal{A}}_n-\mathcal{T}_n\tilde{\mathcal{A}})\boldsymbol{\phi}\|_p\leq\frac{\hat{C}}{n}\|\boldsymbol{\phi}\|_p. \tag{4.28}$$

The proof is complete by using Theorem 3.6, Theorem 3.7, the uniform boundedness of the operators $\mathcal{T}_n:H^p[0,2\pi]^2\to H^p[0,2\pi]^2$ and Corollary 13.11 in [20]. ∎

5. Numerical experiments

Since the Equation (3.44) is equivalent to (3.46), we only need to solve the equivalent full-discrete equation of (3.44), i.e.,

$$\tilde{\psi}^{(n)} - \mathcal{T}_n\mathcal{A}_n\tilde{\psi}^{(n)} = \tilde{\psi}^{(n)} - \mathcal{T}_n(\hat{\mathcal{H}}_n + \mathcal{W}_n + \mathcal{S}_n)\tilde{\psi}^{(n)} = \mathcal{T}_n\mathbf{b}. \tag{5.1}$$

Let $\mathbf{M}_{\mathcal{I}-\tilde{\mathcal{A}}}$ and $\mathbf{M}_{\mathcal{I}-\mathcal{A}}$ denote the coefficient matrix of the full-discrete Equation (4.12) and (5.1), respectively. By Theorem 4.3, we know that (4.12) is uniquely solvable which implies that the matrix $\mathbf{M}_{\mathcal{I}-\tilde{\mathcal{A}}}$ is invertible. Noting that $\mathbf{M}_{\mathcal{I}-\mathcal{A}} = \mathbf{M}_{\mathcal{H}} - \mathbf{M}_{\mathcal{K}} = \mathbf{M}_{\mathcal{H}}\mathbf{M}_{\mathcal{I}-\tilde{\mathcal{A}}}$ with the aid of $\mathbf{M}_{\mathcal{H}}$ being invertible, we find that $\mathbf{M}_{\mathcal{I}-\mathcal{A}}$ is invertible. Hence, (5.1) is uniquely solvable.

For the integrals, the following formulations (Lemma 8.23 in [20]) are used

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left\{\cot\left(\frac{\xi-t}{2}\right)[T_n\psi](\xi)\right\}\mathrm{d}\xi &= \sum_{j=0}^{2n-1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\cot\left(\frac{\xi-t}{2}\right)L_j(\xi)\mathrm{d}\xi\right\}\psi(\xi_j^{(n)})\\ &\approx \sum_{j=0}^{2n-1} C_j^{(n)}(t)\psi(\xi_j^{(n)}),\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\int_0^{2\pi}\left\{\ln\left(4\sin^2\frac{t-\xi}{2}\right)[T_nK(t,\cdot)\psi](\xi)\right\}\mathrm{d}\xi\\ &= \sum_{j=0}^{2n-1}\left\{\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\xi}{2}\right)L_j(\xi)\mathrm{d}\xi\right\}K(t,\xi_j^{(n)})\psi(\xi_j^{(n)})\\ &\approx \sum_{j=0}^{2n-1} S_j^{(n)}(t)K(t,\xi_j^{(n)})\psi(\xi_j^{(n)}),\end{aligned} \tag{5.3}$$

and

$$\int_0^{2\pi}\left\{[T_nN(t,\cdot)\psi](\xi)\right\}\mathrm{d}\xi \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} N(t,\xi_j^{(n)})\psi(\xi_j^{(n)}), \tag{5.4}$$

where $K(\cdot,\cdot), N(\cdot,\cdot)$ are analytic functions, and

$$C_j^{(n)}(t) = \frac{1}{2n}\left[1-\cos n(\xi_j^{(n)}-t)\right]\cot\left(\frac{\xi_j^{(n)}-t}{2}\right), \tag{5.5}$$

$$S_j^{(n)}(t) = -\frac{\pi}{n}\left[2\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-\xi_j^{(n)}) + \frac{1}{n}\cos n(t-\xi_j^{(n)})\right]. \tag{5.6}$$

Using (5.2)–(5.4), the fully discrete collocation method for (3.44) leads to the linear system as follows

$$\psi_{1,i}^{(n)} - \sum_{j=0}^{2n-1} D_{i,j,L}^{(n)}\psi_{1,j}^{(n)} - \sum_{j=0}^{2n-1} Q_{i,j,R}^{(n)}\psi_{2,j}^{(n)} = b_{1,i}, \tag{5.7}$$

$$\psi_{2,i}^{(n)} - \sum_{j=0}^{2n-1} D_{i,j,R}^{(n)}\psi_{2,j}^{(n)} - \sum_{j=0}^{2n-1} Q_{i,j,L}^{(n)}\psi_{1,j}^{(n)} = b_{2,i}, \tag{5.8}$$

where

$$D^{(n)}_{i,j,L} = \tilde{a}_2\Bigg[C^{(n)}_j(\xi^{(n)}_i) + S^{(n)}_j(\xi^{(n)}_i)H_{L,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}\left(\tilde{H}_1(\xi^{(n)}_i,\xi^{(n)}_j) + \tilde{H}_{L,2}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\mathrm{i}}{2\pi}\right)\Bigg] + \left[S^{(n)}_j(\xi^{(n)}_i)K^{(*)}_{L,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}K^{(*)}_{L,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right] + \tilde{a}_3\left[|\boldsymbol{x}'(\xi^{(n)}_i)|S^{(n)}_j(\xi^{(n)}_i)S_{L,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}|\boldsymbol{x}'(\xi^{(n)}_i)|S_{L,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right], \tag{5.9}$$

$$Q^{(n)}_{i,j,R} = \tilde{a}_4\left[|\boldsymbol{x}'(\xi^{(n)}_i)|S^{(n)}_j(\xi^{(n)}_i)S_{R,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}|\boldsymbol{x}'(\xi^{(n)}_i)|S_{R,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right], \tag{5.10}$$

$$D^{(n)}_{i,j,R} = \tilde{a}_6\Bigg[C^{(n)}_j(\xi^{(n)}_i) + S^{(n)}_j(\xi^{(n)}_i)H_{R,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}\left(\tilde{H}_1(\xi^{(n)}_i,\xi^{(n)}_j) + \tilde{H}_{R,2}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\mathrm{i}}{2\pi}\right)\Bigg] + \left[S^{(n)}_j(\xi^{(n)}_i)K^{(*)}_{R,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}K^{(*)}_{R,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right] + \tilde{a}_7\left[|\boldsymbol{x}'(\xi^{(n)}_i)|S^{(n)}_j(\xi^{(n)}_i)S_{R,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}|\boldsymbol{x}'(\xi^{(n)}_i)|S_{R,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right], \tag{5.11}$$

$$Q^{(n)}_{i,j,L} = \tilde{a}_8\left[|\boldsymbol{x}'(\xi^{(n)}_i)|S^{(n)}_j(\xi^{(n)}_i)S_{L,1}(\xi^{(n)}_i,\xi^{(n)}_j) + \frac{\pi}{n}|\boldsymbol{x}'(\xi^{(n)}_i)|S_{L,2}(\xi^{(n)}_i,\xi^{(n)}_j)\right], \tag{5.12}$$

and

$$b_{1,i} = \tilde{a}_1\tilde{f}_1(\xi^{(n)}_i), \qquad b_{2,i} = \tilde{a}_5\tilde{f}_2(\xi^{(n)}_i). \tag{5.13}$$

Two numerical examples are presented to show the excellent performance of the proposed method. The numerical tests are implemented using Matlab on a PC with an Intel i9-9980H processor. We consider the cylinder scattering problems with different cross-sections in a chiral environment. Table 5.1 shows the parametric equations of Apple-shaped and Peanut-shaped boundary curves Γ.

Shaped	**Parametrization**
Apple − shaped	$\boldsymbol{x}(t) = \frac{0.5(1+0.8\cos t+0.2\sin 2t)}{1+0.7\cos t}(\cos t,\sin t), \quad t\in[0,2\pi]$
Peanut − shaped	$\boldsymbol{x}(t) = 0.25\sqrt{3\cos^2 t+1}\ (\cos t,\sin t), \quad t\in[0,2\pi]$

TABLE 5.1. *The smooth boundary curve Γ of cross-sections D.*

Numerical example 1. For the cross-sections D with the smooth boundary curve Γ in Table 5.1, we construct an exact solution by two incident point sources

$$u^i(\boldsymbol{x}) = \frac{\mathrm{i}}{4}H^{(1)}_0(\gamma_L|\boldsymbol{x}-\boldsymbol{x_p}|), \quad v^i(\boldsymbol{x}) = \frac{\mathrm{i}}{4}H^{(1)}_0(\gamma_L|\boldsymbol{x}-\boldsymbol{x_p}|), \qquad \boldsymbol{x}\in\mathbb{R}^2\setminus\bar{D}, \tag{5.14}$$

	Apple − shaped		**Peanut − shaped**	
n	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$
8	0.0078	0.0078	$5.2031e-04$	$5.2034e-04$
16	0.0015	0.0015	$5.1380e-06$	$5.1385e-06$
32	$1.7858e-05$	$1.7860e-05$	$9.4580e-10$	$9.4566e-10$
64	$2.5342e-08$	$2.5344e-08$	$5.2788e-14$	$5.2667e-14$
128	$9.6322e-14$	$3.2206e-14$	$2.6587e-14$	$2.6160e-14$
256	$1.5843e-14$	$1.6188e-14$	$1.2161e-14$	$1.4096e-14$
512	$8.5748e-15$	$7.6641e-15$	$3.1390e-15$	$1.0181e-14$
1024	$3.2045e-15$	$5.1075e-15$	$1.2374e-15$	$7.0294e-15$

TABLE 5.2. *The errors norm for the Apple-shaped and Peanut-shaped D with $\beta = 1, \omega = 2\pi \times 10^3$.*

	Apple − shaped		**Peanut − shaped**	
n	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$
8	0.0076	0.0080	$4.9905e-04$	$5.4196e-04$
16	0.0015	0.0016	$4.9253e-06$	$5.3570e-06$
32	$1.7105e-05$	$1.8609e-05$	$9.0760e-10$	$9.8519e-10$
64	$2.4274e-08$	$2.6404e-08$	$5.6482e-13$	$4.9115e-13$
128	$4.0593e-13$	$2.3285e-13$	$2.8255e-13$	$2.4533e-13$
256	$1.7201e-13$	$1.5010e-13$	$1.4038e-13$	$1.2381e-13$
512	$8.6641e-14$	$7.4462e-14$	$6.7410e-14$	$6.5140e-14$
1024	$4.2289e-14$	$3.8398e-14$	$3.2051e-14$	$3.4338e-14$

TABLE 5.3. *The errors norm for the Apple-shaped and Peanut-shaped D with $\beta = 100, \omega = 2\pi \times 10^4$.*

	Apple − shaped		**Peanut − shaped**	
n	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$	$\frac{\\|u_N^s - u_*^s\\|_2}{\\|u_*^s\\|_2}$	$\frac{\\|v_N^s - v_*^s\\|_2}{\\|v_*^s\\|_2}$
8	0.0059	0.0089	$2.8737e-04$	$7.2872e-04$
16	$9.6924e-04$	0.0021	$2.6529e-06$	$7.3137e-06$
32	$9.4503e-06$	$2.4919e-05$	$6.7341e-10$	$1.3008e-09$
64	$1.3508e-08$	$3.5290e-08$	$1.0385e-10$	$2.5774e-11$
128	$6.2237e-11$	$1.5811e-11$	$5.1926e-11$	$1.2887e-11$
256	$3.1106e-11$	$7.9492e-12$	$2.5963e-11$	$6.4448e-12$
512	$1.5553e-11$	$3.9739e-12$	$1.2980e-11$	$3.2265e-12$
1024	$7.7758e-12$	$1.9885e-12$	$6.4890e-12$	$1.6160e-12$

TABLE 5.4. *The errors norm for the Apple-shaped and Peanut-shaped D with $\beta = 10, \omega = 2\pi \times 10^6$.*

	$\lambda=10^3$		$\lambda=10^6$	
n	$\frac{\|u_N^s-u_*^s\|_2}{\|u_*^s\|_2}$	$\frac{\|v_N^s-v_*^s\|_2}{\|v_*^s\|_2}$	$\frac{\|u_N^s-u_*^s\|_2}{\|u_*^s\|_2}$	$\frac{\|v_N^s-v_*^s\|_2}{\|v_*^s\|_2}$
64	$2.8751e-06$	$3.3207e-06$	$8.1322e-07$	$7.6346e-07$
128	$7.1960e-07$	$8.3455e-07$	$1.7839e-07$	$1.6028e-07$
256	$1.8039e-07$	$2.0945e-07$	$2.8666e-08$	$2.3365e-08$
512	$4.5157e-08$	$5.2458e-08$	$4.0874e-09$	$3.2824e-09$
1024	$1.1298e-08$	$1.3125e-08$	$9.9293e-10$	$1.0851e-09$

TABLE 5.5. *The errors norm for the drop-shaped D with $\beta=1,\omega=2\pi\times10^6$.*

	$\beta=10$		$\beta=1$	
n	$\frac{\|u_N^s-u_*^s\|_2}{\|u_*^s\|_2}$	$\frac{\|v_N^s-v_*^s\|_2}{\|v_*^s\|_2}$	$\frac{\|u_N^s-u_*^s\|_2}{\|u_*^s\|_2}$	$\frac{\|v_N^s-v_*^s\|_2}{\|v_*^s\|_2}$
64	$5.9354e-06$	$5.5674e-06$	$3.3771e-06$	$3.3275e-06$
128	$1.4875e-06$	$1.4200e-06$	$8.4119e-07$	$8.3858e-07$
256	$3.7008e-07$	$3.5664e-07$	$2.1021e-07$	$2.1079e-07$
512	$9.2227e-08$	$8.9305e-08$	$5.2559e-08$	$5.2807e-08$
1024	$2.3044e-08$	$2.2326e-08$	$1.3163e-08$	$1.3192e-08$

TABLE 5.6. *The errors norm for the drop-shaped D with $\lambda=10,\omega=2\pi\times10^6$.*

which located at $\boldsymbol{x_p}=(0.2,0.1)^\top\in D$. Thus, by enforcing the following boundary conditions on Γ:

$$f_1=a_1\frac{\partial u^i}{\partial\nu}+a_2\frac{\partial u^i}{\partial\tau}+a_3u^i+a_4v^i,\quad f_2=a_5\frac{\partial v^i}{\partial\nu}+a_6\frac{\partial v^i}{\partial\tau}+a_7v^i+a_8u^i, \tag{5.15}$$

the exact solution $(u_*^s,v_*^s)=(u^i,v^i)$ of (2.18) can be constructed explicitly by (5.14). Taking the observation points $\{\boldsymbol{x}(\frac{\pi j}{\tilde n})\}_{j=0}^{2\tilde n-1}$ on the circle $\partial B=\{\boldsymbol{x}\in\mathbb{R}^2:|\boldsymbol{x}|=3\}$, where $\tilde n=16$. The electric permittivity and magnetic permeability of a vacuum are denoted as ϵ_0 and μ_0, respectively, the other parameters are chosen as $\epsilon=2\epsilon_0,\mu=2\mu_0,\lambda=10^3,\theta=\frac{\pi}{3}$. Tables 5.2 and 5.3 show the numerical errors between the numerical solution and the corresponding exact solution with $L^2(\partial B)$ norm for the apple-shaped D and peanut-shaped D when $\beta=1,\omega=2\pi\times10^3$ and $\beta=100,\omega=2\pi\times10^4$, respectively. Furthermore, in experiment 1, for the high-frequency $\omega=2\pi\times10^6$ case, we show the numerical results in Table 5.4. The numerical results also show that permittivity, permeability, chirality, and angular frequency affect the convergence rate. We can get highly accurate results by increasing the number of interpolation points.

Numerical example 2. Similar to the setting in Example 1, such as incident wave and the right-hand side, we test the accuracy of the numerical method for nonsmooth cross-section boundary $\Gamma=\{\boldsymbol{x}(t)=(\sin\frac{t}{2}-\frac{1}{2},-\frac{1}{2}\sin t),\ t\in[0,2\pi]\}$. Taking the observation points $\{\boldsymbol{x}(\frac{\pi j}{\tilde n})\}_{j=0}^{2\tilde n-1}$ on the circle $\partial B=\{\boldsymbol{x}\in\mathbb{R}^2:|\boldsymbol{x}|=3\}$, where $\tilde n=16$, the parameters are chosen as $\epsilon=2\epsilon_0,\mu=2\mu_0,\omega=2\pi\times10^6,\theta=\frac{\pi}{3},\boldsymbol{x_p}=(0.4,0.2)^\top$. Table 5.5 shows the numerical errors between the numerical solution and the corresponding exact solution with $L^2(\partial B)$ norm for the Drop-shaped D with $\lambda=10^3$ and $\lambda=10^6$. We note that the numerical and exact solutions coincide with the nonsmooth cross-sections with suitable interpolation points. Furthermore, in experiment 2, for $\beta=10$ and $\beta=1$, we also show the numerical results in Table 5.6.

	Apple − shaped		Peanut − shaped	
n	$\frac{\|u_N^s-u_{N*}^s\|_2}{\|u_{N*}^s\|_2}$	$\frac{\|v_N^s-v_{N*}^s\|_2}{\|v_{N*}^s\|_2}$	$\frac{\|u_N^s-u_{N*}^s\|_2}{\|u_{N*}^s\|_2}$	$\frac{\|v_N^s-v_{N*}^s\|_2}{\|v_{N*}^s\|_2}$
16	$2.5000e-03$	$2.5200e-02$	$5.4522e-06$	$2.6799e-05$
32	$3.9919e-05$	$1.7465e-04$	$1.1475e-09$	$4.6792e-09$
64	$5.4827e-08$	$2.4404e-07$	$3.8710e-10$	$2.9269e-10$
128	$2.8971e-10$	$2.4595e-10$	$1.8730e-10$	$1.4161e-10$
256	$1.3524e-10$	$1.1487e-10$	$8.7406e-11$	$6.6048e-11$
512	$5.7963e-11$	$4.9255e-11$	$3.7456e-11$	$2.8250e-11$
1024	$1.9326e-11$	$1.6439e-11$	$1.2483e-11$	$9.3735e-12$

TABLE 5.7. *The errors norm for the Apple-shaped and Peanut-shaped D with $\beta=10, \omega=2\pi\times 10^6$.*

Numerical example 3. For the scattering problem of the plane wave incidence which is given by (A.15) with (A.11) and (A.12), we calculate the values of compressional and shear scattered fields $u_{N^*}^s, v_{N^*}^s$ on $\partial B=\{\boldsymbol{x}\in\mathbb{R}^2:|\boldsymbol{x}|=3\}$ with $\tilde{n}=16$, $N^*=2048$. The electric permittivity and magnetic permeability of a vacuum are denoted as ϵ_0 and μ_0, respectively, the other parameters are chosen as $\epsilon=2\epsilon_0, \mu=2\mu_0, \lambda=10^3, \theta=\frac{\pi}{3}$. Tables 5.7 shows the numerical errors between the numerical solution and $u_{N^*}^s, v_{N^*}^s$ with $L^2(\partial B)$ norm for the apple-shaped D and peanut-shaped D when $\beta=10, \omega=2\pi\times 10^6$. We can also get highly accurate results by increasing the number of interpolation points with the plane wave incidence.

6. Concluding remarks

We have presented an effective algorithm for a cylinder scattering problem with obliquely incident electromagnetic waves in a chiral environment. For general cross-sectional cylinder geometric structures, both the left-circularly polarization and right-circularly polarization are employed to reduce the computational complexity for the boundary value problem of Maxwell's equations in chiral media. A novel integral equation method is developed for solving the scattering problem. The convergence of the method has been established. An interesting future direction is to develop a fast computational method for solving the related inverse obstacle scattering problem.

Appendix A. The electromagnetic plane wave. The time-harmonic electromagnetic plane wave

$$\tilde{\boldsymbol{E}}^i=(\tilde{e}_1^i,\tilde{e}_2^i,\tilde{e}_3^i)^\top=[\boldsymbol{q}_L\exp(\mathrm{i}\tilde{\gamma}_L\tilde{\boldsymbol{p}}_L\cdot\boldsymbol{x})+\boldsymbol{q}_R\exp(\mathrm{i}\tilde{\gamma}_R\tilde{\boldsymbol{p}}_R\cdot\boldsymbol{x})], \tag{A.1}$$

$$\tilde{\boldsymbol{H}}^i=(\tilde{h}_1^i,\tilde{h}_2^i,\tilde{h}_3^i)^\top=-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}[\boldsymbol{q}_L\exp(\mathrm{i}\tilde{\gamma}_L\tilde{\boldsymbol{p}}_L\cdot\boldsymbol{x})-\boldsymbol{q}_R\exp(\mathrm{i}\tilde{\gamma}_R\tilde{\boldsymbol{p}}_R\cdot\boldsymbol{x})], \tag{A.2}$$

is a combination of left-circularly polarized plane wave and right-circularly polarized one, which satisfies the isotropic DBF equations in $\mathbb{R}^3$ [3]. Here

$$\tilde{\gamma}_L=\frac{k}{1-k\beta}=\beta_1+\beta_2\sqrt{\epsilon\mu}>0, \quad \tilde{\gamma}_R=\frac{k}{1+k\beta}=-(\beta_1-\beta_2\sqrt{\epsilon\mu})>0, \tag{A.3}$$

the constants β_1 and β_2 are given by

$$\beta_1=k^2(1-k^2\beta^2)^{-1}\beta\geq 0, \qquad \beta_2=\omega(1-k^2\beta^2)^{-1}>0. \tag{A.4}$$

The complex vectors

$$\boldsymbol{q}_L=(q_{1,L},q_{2,L},q_{3,L})^\top,\quad \tilde{\boldsymbol{p}}_L=(p_{1,L},p_{2,L},p_{3,L})^\top, \tag{A.5}$$

$$\boldsymbol{q}_R=(q_{1,R},q_{2,R},q_{3,L})^\top,\quad \tilde{\boldsymbol{p}}_R=(p_{1,R},p_{2,R},,p_{3,R})^\top, \tag{A.6}$$

satisfy

$$\tilde{\boldsymbol{p}}_L\cdot\boldsymbol{q}_L=0,\quad \tilde{\boldsymbol{p}}_R\cdot\boldsymbol{q}_R=0,\quad \tilde{\boldsymbol{p}}_L\times\boldsymbol{q}_L=-\mathrm{i}\boldsymbol{q}_L,\quad \tilde{\boldsymbol{p}}_R\times\boldsymbol{q}_R=\mathrm{i}\boldsymbol{q}_R. \tag{A.7}$$

Let $p_{3,L}=-\tilde{\gamma}_L^{-1}\alpha$ and $\alpha=k\cos\theta$, where $\theta\in[\frac{\pi}{3},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{2\pi}{3}]$ is the obliquely incident angle between the incident direction and the negative x_3 axis. Considering now the case of $\tilde{\boldsymbol{p}}_L\cdot\tilde{\boldsymbol{p}}_L=1$, that is, $p_{1,L}^2+p_{2,L}^2=1-(-\tilde{\gamma}_L^{-1}\alpha)^2>0$, it follows from $\tilde{\boldsymbol{p}}_L\times\boldsymbol{q}_L=-\mathrm{i}\boldsymbol{q}_L$ that

$$\begin{pmatrix}\mathrm{i} & \tilde{\gamma}_L^{-1}\alpha & p_{2,L}\\ -\tilde{\gamma}_L^{-1}\alpha & \mathrm{i} & -p_{1,L}\\ -p_{2,L} & p_{1,L} & \mathrm{i}\end{pmatrix}\begin{pmatrix}q_{1,L}\\ q_{2,L}\\ q_{3,L}\end{pmatrix}:=\tilde{A}_L\begin{pmatrix}q_{1,L}\\ q_{2,L}\\ q_{3,L}\end{pmatrix}=0 \tag{A.8}$$

has nonzero solutions. For an example, choose

$$\tilde{\boldsymbol{p}}_L=(\sqrt{2(1-\tilde{\gamma}_L^{-2}\alpha^2)},\mathrm{i}\sqrt{1-\tilde{\gamma}_L^{-2}\alpha^2},-\tilde{\gamma}_L^{-1}\alpha)^\top$$

then

$$\boldsymbol{q}_L=\left(\frac{-1+\sqrt{2}\tilde{\gamma}_L^{-1}\alpha}{\sqrt{1-\tilde{\gamma}_L^{-2}\alpha^2}},\frac{-\mathrm{i}\sqrt{2}+\mathrm{i}\tilde{\gamma}_L^{-1}\alpha}{\sqrt{1-\tilde{\gamma}_L^{-2}\alpha^2}},1\right)^\top \tag{A.9}$$

is a nonzero solution of (A.8). Similarly, let $p_{3,R}=-\tilde{\gamma}_R^{-1}\alpha$, assume that $p_{1,R}^2+p_{2,R}^2=1-(-\tilde{\gamma}_R^{-1}\alpha)^2>0$, we can also obtain that

$$\begin{pmatrix}\mathrm{i} & -\tilde{\gamma}_R^{-1}\alpha & -p_{2,R}\\ \tilde{\gamma}_R^{-1}\alpha & \mathrm{i} & p_{1,R}\\ p_{2,R} & -p_{1,R} & \mathrm{i}\end{pmatrix}\begin{pmatrix}q_{1,R}\\ q_{2,R}\\ q_{3,R}\end{pmatrix}:=\tilde{A}_R\begin{pmatrix}q_{1,R}\\ q_{2,R}\\ q_{3,R}\end{pmatrix}=0 \tag{A.10}$$

has nonzero solutions. Moreover, without loss of generalities, taking $q_{3,L}=q_{3,R}=1$.

When considering the plane wave incidence in numerical experiments, we choose real $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$ in (2.20) as

$$\boldsymbol{p}_L=(\frac{\sqrt{3}}{2}\sqrt{1-\tilde{\gamma}_L^{-2}\alpha^2},\frac{1}{2}\sqrt{1-\tilde{\gamma}_L^{-2}\alpha^2},0)^\top, \tag{A.11}$$

$$\boldsymbol{p}_R=(\frac{1}{2}\sqrt{1-\tilde{\gamma}_R^{-2}\alpha^2},\frac{\sqrt{3}}{2}\sqrt{1-\tilde{\gamma}_R^{-2}\alpha^2},0)^\top. \tag{A.12}$$

The incident fields $\tilde{e}_3^i,\tilde{h}_3^i$ can been expressed as

$$\tilde{e}_3^i=(u^i(x_1,x_2)+v^i(x_1,x_2))\exp(-\mathrm{i}\alpha x_3), \tag{A.13}$$

$$\tilde{h}_3^i=-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}(u^i(x_1,x_2)-v^i(x_1,x_2))\exp(-\mathrm{i}\alpha x_3). \tag{A.14}$$

It follows from (A.1), (A.2) (A.13) and (A.14) that

$$u^i:=u^i(x_1,x_2)=\exp(\mathrm{i}\tilde{\gamma}_L\boldsymbol{p}_L\cdot\boldsymbol{x}),\qquad v^i:=v^i(x_1,x_2)=\exp(\mathrm{i}\tilde{\gamma}_R\boldsymbol{p}_R\cdot\boldsymbol{x}), \tag{A.15}$$

where $\boldsymbol{p}_L=(p_{1,L},p_{2,L},0)$ and $\boldsymbol{p}_R=(p_{1,R},p_{2,R},0)$.

From (A.15), the incident fields u^i and v^i satisfy the equations:

$$\Delta u^i+\gamma_L^2u^i=0, \qquad \Delta v^i+\gamma_R^2v^i=0, \tag{A.16}$$

where $\gamma_L^2:=\tilde{\gamma}_L^2-\alpha^2>0$ and $\gamma_R^2:=\tilde{\gamma}_R^2-\alpha^2>0$ for $\theta\in[\frac{\pi}{3},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{2\pi}{3}]$.

Appendix B. The reduced problem. For time-harmonic electromagnetic waves of the form

$$\boldsymbol{E}:=(\tilde{e}_1(x_1,x_2),\tilde{e}_2(x_1,x_2),\tilde{e}_3(x_1,x_2))^\top\exp(-\mathrm{i}\alpha x_3), \tag{B.1}$$

$$\boldsymbol{H}:=(\tilde{h}_1(x_1,x_2),\tilde{h}_2(x_1,x_2),\tilde{h}_3(x_1,x_2))^\top\exp(-\mathrm{i}\alpha x_3), \tag{B.2}$$

we deduce that $\partial_{x_3}\tilde{e}_l=-\mathrm{i}\alpha\tilde{e}_l,\ \partial_{x_3}\tilde{h}_l=-\mathrm{i}\alpha\tilde{h}_l\ (l=1,2,3)$.

From Maxwell's Equations (2.4) and (2.5), we have

$$\partial_{x_2}\tilde{e}_3+\mathrm{i}\alpha\tilde{e}_2=\beta_1\tilde{e}_1+\mathrm{i}\mu\beta_2\tilde{h}_1, \tag{B.3}$$

$$\mathrm{i}\alpha\tilde{e}_1+\partial_{x_1}\tilde{e}_3=-\beta_1\tilde{e}_2-\mathrm{i}\mu\beta_2\tilde{h}_2, \tag{B.4}$$

$$\partial_{x_1}\tilde{e}_2-\partial_{x_2}\tilde{e}_1=\beta_1\tilde{e}_3+\mathrm{i}\mu\beta_2\tilde{h}_3, \tag{B.5}$$

$$\partial_{x_2}\tilde{h}_3+\mathrm{i}\alpha\tilde{h}_2=\beta_1\tilde{h}_1-\mathrm{i}\epsilon\beta_2\tilde{e}_1, \tag{B.6}$$

$$\mathrm{i}\alpha\tilde{h}_1+\partial_{x_1}\tilde{h}_3=-\beta_1\tilde{h}_2+\mathrm{i}\epsilon\beta_2\tilde{e}_2, \tag{B.7}$$

$$\partial_{x_1}\tilde{h}_2-\partial_{x_2}\tilde{h}_1=\beta_1\tilde{h}_3-\mathrm{i}\epsilon\beta_2\tilde{e}_3. \tag{B.8}$$

Using $\mathrm{i}\epsilon\beta_2\times$ (B.3)+$\mathrm{i}\alpha\times$ (B.7) and $\mathrm{i}\alpha\times$ (B.3)+$\mathrm{i}\mu\beta_2\times$ (B.7), we get

$$(\alpha^2-\epsilon\mu\beta_2^2)\tilde{h}_1=\mathrm{i}\epsilon\beta_2(\partial_{x_2}\tilde{e}_3-\beta_1\tilde{e}_1)+\mathrm{i}\alpha(\partial_{x_1}\tilde{h}_3+\beta_1\tilde{h}_2), \tag{B.9}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\tilde{e}_2=\mathrm{i}\mu\beta_2(\partial_{x_1}\tilde{h}_3+\beta_1\tilde{h}_2)+\mathrm{i}\alpha(\partial_{x_2}\tilde{e}_3-\beta_1\tilde{e}_1). \tag{B.10}$$

From $\mathrm{i}\epsilon\beta_2\times$ (B.4) $-\mathrm{i}\alpha\times$ (B.6) and $\mathrm{i}\alpha\times$ (B.4)$-\mathrm{i}\mu\beta_2\times$ (B.6), we have

$$(\alpha^2-\epsilon\mu\beta_2^2)\tilde{h}_2=-\mathrm{i}\epsilon\beta_2(\partial_{x_1}\tilde{e}_3+\beta_1\tilde{e}_2)+\mathrm{i}\alpha(\partial_{x_2}\tilde{h}_3-\beta_1\tilde{h}_1), \tag{B.11}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\tilde{e}_1=-\mathrm{i}\mu\beta_2(\partial_{x_2}\tilde{h}_3-\beta_1\tilde{h}_1)+\mathrm{i}\alpha(\partial_{x_1}\tilde{e}_3+\beta_1\tilde{e}_2). \tag{B.12}$$

Then, from (B.10) and (B.12), we obtain

$$(\alpha^2-\epsilon\mu\beta_2^2)\partial_{x_1}\tilde{e}_2=\mathrm{i}\mu\beta_2(\partial_{x_1}^2\tilde{h}_3+\beta_1\partial_{x_1}\tilde{h}_2)+\mathrm{i}\alpha(\partial_{x_2,x_1}^2\tilde{e}_3-\beta_1\partial_{x_1}\tilde{e}_1), \tag{B.13}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\partial_{x_2}\tilde{e}_1=-\mathrm{i}\mu\beta_2(\partial_{x_2}^2\tilde{h}_3-\beta_1\partial_{x_2}\tilde{h}_1)+\mathrm{i}\alpha(\partial_{x_1,x_2}^2\tilde{e}_3+\beta_1\partial_{x_2}\tilde{e}_2). \tag{B.14}$$

Combining (B.5) and (B.13)$-$(B.14), we derive

$$\begin{aligned}&(\partial_{x_1}^2\tilde{h}_3+\partial_{x_2}^2\tilde{h}_3)+\beta_1(\partial_{x_1}\tilde{h}_2-\partial_{x_2}\tilde{h}_1)-\frac{\alpha\beta_1}{\mu\beta_2}(\partial_{x_1}\tilde{e}_1+\partial_{x_2}\tilde{e}_2)\\ =&-\mathrm{i}\frac{\alpha^2\beta_1}{\mu\beta_2}\tilde{e}_3+\mathrm{i}\epsilon\beta_1\beta_2\tilde{e}_3+(\alpha^2-\epsilon\mu\beta_2^2)\tilde{h}_3.\end{aligned} \tag{B.15}$$

Thus, from (B.8), (B.15) and div$\boldsymbol{E}=0$, we have

$$\begin{aligned}&(\partial_{x_1}^2\tilde{h}_3+\partial_{x_2}^2\tilde{h}_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)\tilde{h}_3-\mathrm{i}2\epsilon\beta_1\beta_2\tilde{e}_3\\ =&[(\partial_{x_1}^2h_3+\partial_{x_2}^2h_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)h_3-\mathrm{i}2\epsilon\beta_1\beta_2e_3]\exp(-\mathrm{i}\alpha x_3)=0.\end{aligned}\tag{B.16}$$

By (B.9) and (B.11), we derive

$$(\alpha^2-\epsilon\mu\beta_2^2)\partial_{x_2}\tilde{h}_1=\mathrm{i}\epsilon\beta_2(\partial_{x_2}^2\tilde{e}_3-\beta_1\partial_{x_2}\tilde{e}_1)+\mathrm{i}\alpha(\partial_{x_1,x_2}^2\tilde{h}_3+\beta_1\partial_{x_2}\tilde{h}_2),\tag{B.17}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\partial_{x_1}\tilde{h}_2=-\mathrm{i}\epsilon\beta_2(\partial_{x_1}^2\tilde{e}_3+\beta_1\partial_{x_1}\tilde{e}_2)+\mathrm{i}\alpha(\partial_{x_2,x_1}^2\tilde{h}_3-\beta_1\partial_{x_1}\tilde{h}_1).\tag{B.18}$$

Combining (B.8) and (B.17)–(B.18), we arrive at

$$\begin{aligned}&(\partial_{x_1}^2\tilde{e}_3+\partial_{x_2}^2\tilde{e}_3)+\beta_1(\partial_{x_1}\tilde{e}_2-\partial_{x_2}\tilde{e}_1)+\frac{\alpha\beta_1}{\epsilon\beta_2}(\partial_{x_1}\tilde{h}_1+\partial_{x_2}\tilde{h}_2)\\ =&\mathrm{i}\frac{\alpha^2\beta_1}{\epsilon\beta_2}\tilde{h}_3-\mathrm{i}\mu\beta_1\beta_2\tilde{h}_3+(\alpha^2-\epsilon\mu\beta_2^2)\tilde{e}_3.\end{aligned}\tag{B.19}$$

From (B.5) and (B.19) and div$\boldsymbol{H}=0$, we have

$$\begin{aligned}&(\partial_{x_1}^2\tilde{e}_3+\partial_{x_2}^2\tilde{e}_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)\tilde{e}_3+\mathrm{i}2\mu\beta_1\beta_2\tilde{h}_3\\ =&[(\partial_{x_1}^2e_3+\partial_{x_2}^2e_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)e_3+\mathrm{i}2\mu\beta_1\beta_2h_3]\exp(-\mathrm{i}\alpha x_3)=0.\end{aligned}\tag{B.20}$$

By (B.16) and (B.20), we get

$$(\partial_{x_1}^2e_3+\partial_{x_2}^2e_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)e_3+\mathrm{i}2\mu\beta_1\beta_2h_3=0,\tag{B.21}$$

$$(\partial_{x_1}^2h_3+\partial_{x_2}^2h_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)h_3-\mathrm{i}2\epsilon\beta_1\beta_2e_3=0.\tag{B.22}$$

We next derive the boundary conditions. It is convenient to introduce the following notations:

$$\boldsymbol{e_t}=\boldsymbol{\hat{x}_1}\tilde{e}_1+\boldsymbol{\hat{x}_2}\tilde{e}_2,\quad \boldsymbol{h_t}=\boldsymbol{\hat{x}_1}\tilde{h}_1+\boldsymbol{\hat{x}_2}\tilde{h}_2,\quad \nabla=\boldsymbol{\hat{x}_1}\partial_{x_1}+\boldsymbol{\hat{x}_2}\partial_{x_2},\tag{B.23}$$

where $\boldsymbol{\hat{x}_1}=(1,0,0)$, $\boldsymbol{\hat{x}_2}=(0,1,0)$ and $\boldsymbol{\hat{x}_3}=(0,0,1)$. Let $\boldsymbol{\nu}=(\nu_1,\nu_2,0)^\top$ and $\boldsymbol{\tau}=(-\nu_2,\nu_1,0)^\top$ denote the unit outward normal vector and tangential vector of Γ respectively.

Then, from $|\boldsymbol{\nu}|=|\boldsymbol{\tau}|=1$ and (2.6), we have

$$\begin{pmatrix}\nu_2\tilde{e}_3\\-\nu_1\tilde{e}_3\\\nu_1\tilde{e}_2-\nu_2\tilde{e}_1\end{pmatrix}\times\begin{pmatrix}\nu_1\\\nu_2\\0\end{pmatrix}=\lambda\begin{pmatrix}\nu_2\tilde{h}_3\\-\nu_1\tilde{h}_3\\\nu_1\tilde{h}_2-\nu_2\tilde{h}_1\end{pmatrix}.\tag{B.24}$$

Hence

$$-\boldsymbol{\tau}\cdot\boldsymbol{e_t}=-(-\nu_2,\nu_1,0)^\top\cdot(\tilde{e}_1,\tilde{e}_2,0)^\top=\nu_2\tilde{e}_1-\nu_1\tilde{e}_2=\lambda\tilde{h}_3,\tag{B.25}$$

$$\tilde{e}_3=\lambda(\nu_1\tilde{h}_2-\nu_2\tilde{h}_1)=\lambda(-\nu_2,\nu_1,0)^\top\cdot(\tilde{h}_1,\tilde{h}_2,0)^\top=\lambda\boldsymbol{\tau}\cdot\boldsymbol{h_t}.\tag{B.26}$$

Combining (B.25), (B.10) and (B.12), we obtain

$$-\lambda\tilde{h}_3=\frac{1}{(\alpha^2-\epsilon\mu\beta_2^2)}\big\{\mathrm{i}\mu\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\nu})+\mathrm{i}\mu\beta_1\beta_2\boldsymbol{\tau}\cdot\boldsymbol{h_t}+\mathrm{i}\alpha\boldsymbol{\tau}\cdot\nabla\tilde{e}_3-\mathrm{i}\alpha\beta_1(\boldsymbol{e_t}\cdot\boldsymbol{\nu})\big\}.\tag{B.27}$$

Combining (B.26), (B.9) and (B.11), we obtain

$$\frac{1}{\lambda}\tilde{e}_3=\frac{1}{(\alpha^2-\epsilon\mu\beta_2^2)}\big\{-\mathrm{i}\epsilon\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})-\mathrm{i}\epsilon\beta_1\beta_2\boldsymbol{\tau}\cdot\boldsymbol{e_t}+\mathrm{i}\alpha\boldsymbol{\tau}\cdot\nabla\tilde{h}_3-\mathrm{i}\alpha\beta_1(\boldsymbol{h_t}\cdot\boldsymbol{\nu})\big\}. \tag{B.28}$$

Similarly

$$\begin{aligned}\boldsymbol{\nu}\cdot\boldsymbol{e_t}&=(\nu_1,\nu_2,0)^\top\cdot(\tilde{e}_1,\tilde{e}_2,0)^\top=\nu_1\tilde{e}_1+\nu_2\tilde{e}_2\\&=\frac{-1}{(\alpha^2-\epsilon\mu\beta_2^2)}\big\{\mathrm{i}\mu\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\tau})-\mathrm{i}\mu\beta_1\beta_2\boldsymbol{\nu}\cdot\boldsymbol{h_t}-\mathrm{i}\alpha\boldsymbol{\nu}\cdot\nabla\tilde{e}_3-\mathrm{i}\alpha\beta_1(\boldsymbol{e_t}\cdot\boldsymbol{\tau})\big\}, \qquad (B.29)\\ \boldsymbol{\nu}\cdot\boldsymbol{h_t}&=(\nu_1,\nu_2,0)^\top\cdot(\tilde{h}_1,\tilde{h}_2,0)^\top=\nu_1\tilde{h}_1+\nu_2\tilde{h}_2\\&=\frac{1}{(\alpha^2-\epsilon\mu\beta_2^2)}\big\{\mathrm{i}\epsilon\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})-\mathrm{i}\epsilon\beta_1\beta_2\boldsymbol{\nu}\cdot\boldsymbol{e_t}+\mathrm{i}\alpha\boldsymbol{\nu}\cdot\nabla\tilde{h}_3+\mathrm{i}\alpha\beta_1(\boldsymbol{h_t}\cdot\boldsymbol{\tau})\big\}. \qquad (B.30)\end{aligned}$$

Therefore, we can rewrite (B.27), (B.28), (B.29) and (B.30) as, respectively,

$$(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{e_t}=\mathrm{i}\mu\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\nu})+\mathrm{i}\mu\beta_1\beta_2(\boldsymbol{\tau}\cdot\boldsymbol{h_t})+\mathrm{i}\alpha(\boldsymbol{\tau}\cdot\nabla\tilde{e}_3)-\mathrm{i}\alpha\beta_1(\boldsymbol{e_t}\cdot\boldsymbol{\nu}), \tag{B.31}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{h_t}=-\mathrm{i}\epsilon\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})-\mathrm{i}\epsilon\beta_1\beta_2(\boldsymbol{\tau}\cdot\boldsymbol{e_t})+\mathrm{i}\alpha(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)-\mathrm{i}\alpha\beta_1(\boldsymbol{h_t}\cdot\boldsymbol{\nu}), \tag{B.32}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\nu}\cdot\boldsymbol{e_t}=-\mathrm{i}\mu\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\tau})+\mathrm{i}\mu\beta_1\beta_2(\boldsymbol{\nu}\cdot\boldsymbol{h_t})+\mathrm{i}\alpha(\boldsymbol{\nu}\cdot\nabla\tilde{e}_3)+\mathrm{i}\alpha\beta_1(\boldsymbol{e_t}\cdot\boldsymbol{\tau}), \tag{B.33}$$

$$(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\nu}\cdot\boldsymbol{h_t}=\mathrm{i}\epsilon\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})-\mathrm{i}\epsilon\beta_1\beta_2(\boldsymbol{\nu}\cdot\boldsymbol{e_t})+\mathrm{i}\alpha(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3)+\mathrm{i}\alpha\beta_1(\boldsymbol{h_t}\cdot\boldsymbol{\tau}). \tag{B.34}$$

From $(\alpha^2-\epsilon\mu\beta_2^2)\times$(B.33)$+\mathrm{i}\mu\beta_1\beta_2\times$(B.34) and $(\alpha^2-\epsilon\mu\beta_2^2)\times$(B.32)$-\mathrm{i}\alpha\beta_1\times$(B.34), we get

$$\begin{aligned}&[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2]\boldsymbol{\nu}\cdot\boldsymbol{e_t}\\=&\mathrm{i}\alpha\beta_1(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{e_t}\cdot\boldsymbol{\tau})-\alpha\mu\beta_1^2\beta_2(\boldsymbol{h_t}\cdot\boldsymbol{\tau})-\mathrm{i}\mu\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{h}_3\cdot\boldsymbol{\tau})\\&+\mathrm{i}\alpha(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\nu}\cdot\nabla\tilde{e}_3)-\epsilon\mu\beta_1\beta_2^2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})-\alpha\mu\beta_1\beta_2(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3), \qquad (B.35)\\&[(\alpha^2-\epsilon\mu\beta_2^2)^2-\alpha^2\beta_1^2]\boldsymbol{\tau}\cdot\boldsymbol{h_t}\\=&-\mathrm{i}\epsilon\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\boldsymbol{e_t})-\alpha\epsilon\beta_1^2\beta_2(\boldsymbol{\nu}\cdot\boldsymbol{e_t})+\mathrm{i}\alpha(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)\\&-\mathrm{i}\epsilon\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})+\alpha\epsilon\beta_1\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})+\alpha^2\beta_1(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3). \qquad (B.36)\end{aligned}$$

From $\mathrm{i}\epsilon\beta_1\beta_2\times$(B.31)+(B.36) and $-\mathrm{i}\alpha\beta_1\times$(B.35)$+[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2]\times$(B.31), we have

$$\begin{aligned}&\mathrm{i}2\epsilon\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{e_t}\\=&-[(\alpha^2-\epsilon\mu\beta_2^2)^2-\alpha^2\beta_1^2+\epsilon\mu\beta_1^2\beta_2^2]\boldsymbol{\tau}\cdot\boldsymbol{h_t}+\beta_1(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3)\\&+\mathrm{i}\alpha(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)-\mathrm{i}\epsilon\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{e}_3\cdot\boldsymbol{\nu}), \qquad (B.37)\\&[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2-\alpha^2\beta_1^2](\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{e_t}\\=&\mathrm{i}\mu\beta_1\beta_2[\alpha^2\beta_1^2+((\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2)](\boldsymbol{h_t}\cdot\boldsymbol{\tau})\\&+\mathrm{i}\alpha(\alpha^2-\epsilon\mu\beta_2^2)^2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})\\&+\mathrm{i}\mu\beta_2[\alpha^2\beta_1^2+((\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2)](\nabla\tilde{h}_3\cdot\boldsymbol{\nu})\\&+\alpha^2\beta_1(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\nu}\cdot\nabla\tilde{e}_3)-\alpha\mu\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{h}_3\cdot\boldsymbol{\tau}). \qquad (B.38)\end{aligned}$$

From $[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2-\alpha^2\beta_1^2]\times$(B.37)$-\mathrm{i}2\epsilon\beta_1\beta_2\times$(B.38), we have

$$\begin{aligned}&\{[(\alpha^2-\epsilon\mu\beta_2^2)^2-\alpha^2\beta_1^2-\epsilon\mu\beta_1^2\beta_2^2][(\alpha^2-\epsilon\mu\beta_2^2)^2-\alpha^2\beta_1^2+\epsilon\mu\beta_1^2\beta_2^2]\\&\quad-2\epsilon\mu\beta_1^2\beta_2^2[(\alpha^2-\epsilon\mu\beta_2^2)^2+\alpha^2\beta_1^2-\epsilon\mu\beta_1^2\beta_2^2]\}\boldsymbol{\tau}\cdot\boldsymbol{h_t}\\=&\{\beta_1(\alpha^2-\epsilon\mu\beta_2^2)[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2-\alpha^2\beta_1^2]\\&\qquad+2\epsilon\mu\beta_1\beta_2^2[(\alpha^2-\epsilon\mu\beta_2^2)^2+\alpha^2\beta_1^2-\epsilon\mu\beta_1^2\beta_2^2]\}(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3)\\&\quad+\{\mathrm{i}\alpha(\alpha^2-\epsilon\mu\beta_2^2)[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2-\alpha^2\beta_1^2]\\&\qquad+\mathrm{i}2\alpha\epsilon\mu\beta_1^2\beta_2^2(\alpha^2-\epsilon\mu\beta_2^2)\}(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)\\&\quad-\{\mathrm{i}\epsilon\beta_2(\alpha^2-\epsilon\mu\beta_2^2)[(\alpha^2-\epsilon\mu\beta_2^2)^2-\epsilon\mu\beta_1^2\beta_2^2-\alpha^2\beta_1^2]\\&\qquad+\mathrm{i}2\alpha^2\epsilon\beta_1^2\beta_2(\alpha^2-\epsilon\mu\beta_2^2)\}(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})+\mathrm{i}2\alpha\epsilon\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)^2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau}).\end{aligned}\tag{B.39}$$

With the help of $-\epsilon(\alpha^2-\epsilon\mu\beta_2^2)\times$(B.31), $\mathrm{i}\epsilon\mu\beta_1\beta_2\times$(B.32) and $\mathrm{i}\alpha\epsilon\beta_1\times$(B.33), we get

$$\begin{aligned}-\epsilon(\alpha^2-\epsilon\mu\beta_2^2)^2\boldsymbol{\tau}\cdot\boldsymbol{e_t}=&-\mathrm{i}\epsilon\mu\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{h}_3\cdot\boldsymbol{\nu})-\mathrm{i}\epsilon\mu\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\boldsymbol{h_t})\\&-\mathrm{i}\alpha\epsilon(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\nabla\tilde{e}_3)+\mathrm{i}\alpha\epsilon\beta_1(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{e_t}\cdot\boldsymbol{\nu}),\end{aligned}\tag{B.40}$$

$$\begin{aligned}-\epsilon^2\mu\beta_1^2\beta_2^2(\boldsymbol{\tau}\cdot\boldsymbol{e_t})=&-\mathrm{i}\epsilon\mu\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{h_t}+\epsilon^2\mu\beta_1\beta_2^2(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})\\&-\alpha\epsilon\mu\beta_1\beta_2(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)+\alpha\epsilon\mu\beta_1^2\beta_2(\boldsymbol{h_t}\cdot\boldsymbol{\nu}),\end{aligned}\tag{B.41}$$

$$\begin{aligned}\alpha^2\epsilon\beta_1^2(\boldsymbol{e_t}\cdot\boldsymbol{\tau})=&-\mathrm{i}\alpha\epsilon\beta_1(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\nu}\cdot\boldsymbol{e_t}+\alpha\epsilon\mu\beta_1\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\tau})\\&-\alpha\epsilon\mu\beta_1^2\beta_2(\boldsymbol{\nu}\cdot\boldsymbol{h_t})-\alpha^2\epsilon\beta_1(\boldsymbol{\nu}\cdot\nabla\tilde{e}_3),\end{aligned}\tag{B.42}$$

hence, by (B.40)+(B.41)+(B.42) that

$$\begin{aligned}&(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)\boldsymbol{\tau}\cdot\boldsymbol{e_t}\\=&-\mathrm{i}\mu\beta_2(\nabla\tilde{h}_3\cdot\boldsymbol{\nu})-\mathrm{i}2\mu\beta_1\beta_2(\boldsymbol{\tau}\cdot\boldsymbol{h_t})-\mathrm{i}\alpha(\boldsymbol{\tau}\cdot\nabla\tilde{e}_3)-\beta_1(\boldsymbol{\nu}\cdot\nabla\tilde{e}_3).\end{aligned}\tag{B.43}$$

With the help of $-\mathrm{i}\epsilon\mu\beta_1\beta_2\times$(B.31), $-\mu(\alpha^2-\epsilon\mu\beta_2^2)\times$(B.32) and $\mathrm{i}\alpha\mu\beta_1\times$(B.34), we get

$$\begin{aligned}-\epsilon\mu^2\beta_1^2\beta_2^2(\boldsymbol{\tau}\cdot\boldsymbol{h_t})=&\,\mathrm{i}\epsilon\mu\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\tau}\cdot\boldsymbol{e_t}+\epsilon\mu^2\beta_1\beta_2^2(\nabla\tilde{h}_3\cdot\boldsymbol{\nu})\\&+\alpha\epsilon\mu\beta_1\beta_2(\boldsymbol{\tau}\cdot\nabla\tilde{e}_3)-\alpha\epsilon\mu\beta_1^2\beta_2(\boldsymbol{e_t}\cdot\boldsymbol{\nu}),\end{aligned}\tag{B.44}$$

$$\begin{aligned}-\mu(\alpha^2-\epsilon\mu\beta_2^2)^2\boldsymbol{\tau}\cdot\boldsymbol{h_t}=&\,\mathrm{i}\epsilon\mu\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})+\mathrm{i}\epsilon\mu\beta_1\beta_2(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\boldsymbol{e_t})\\&-\mathrm{i}\alpha\mu(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3)+\mathrm{i}\alpha\mu\beta_1(\alpha^2-\epsilon\mu\beta_2^2)(\boldsymbol{h_t}\cdot\boldsymbol{\nu}),\end{aligned}\tag{B.45}$$

$$\begin{aligned}\alpha^2\mu\beta_1^2(\boldsymbol{h_t}\cdot\boldsymbol{\tau})=&-\mathrm{i}\alpha\mu\beta_1(\alpha^2-\epsilon\mu\beta_2^2)\boldsymbol{\nu}\cdot\boldsymbol{h_t}-\alpha\epsilon\mu\beta_1\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\tau})\\&+\alpha\epsilon\mu\beta_1^2\beta_2(\boldsymbol{\nu}\cdot\boldsymbol{e_t})-\alpha^2\mu\beta_1(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3),\end{aligned}\tag{B.46}$$

hence, by (B.44)+(B.45)+(B.46)

$$\begin{aligned}&(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)(\boldsymbol{\tau}\cdot\boldsymbol{h_t})\\=&\,\mathrm{i}2\epsilon\beta_1\beta_2\boldsymbol{\tau}\cdot\boldsymbol{e_t}-\beta_1(\boldsymbol{\nu}\cdot\nabla\tilde{h}_3)+\mathrm{i}\epsilon\beta_2(\nabla\tilde{e}_3\cdot\boldsymbol{\nu})-\mathrm{i}\alpha(\boldsymbol{\tau}\cdot\nabla\tilde{h}_3).\end{aligned}\tag{B.47}$$

Combining (B.25) and (B.43), we derive

$$\mathrm{i}\alpha\frac{\partial\tilde{e}_3}{\partial\boldsymbol{\tau}}+\beta_1\frac{\partial\tilde{e}_3}{\partial\boldsymbol{\nu}}+\mathrm{i}\mu\beta_2\frac{\partial\tilde{h}_3}{\partial\boldsymbol{\nu}}=(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)(\lambda\tilde{h}_3)-\mathrm{i}2\mu\beta_1\beta_2\left(\frac{\tilde{e}_3}{\lambda}\right).\tag{B.48}$$

Combining (B.26) and (B.47), we derive

$$\mathrm{i}\epsilon\beta_2\frac{\partial\tilde{e}_3}{\partial\boldsymbol{\nu}}-\beta_1\frac{\partial\tilde{h}_3}{\partial\boldsymbol{\nu}}-\mathrm{i}\alpha\frac{\partial\tilde{h}_3}{\partial\boldsymbol{\tau}}=\mathrm{i}2\epsilon\beta_1\beta_2(\lambda\tilde{h}_3)+(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)\left(\frac{\tilde{e}_3}{\lambda}\right). \tag{B.49}$$

Then, from (B.48), (B.49) and $\tilde{e}_3=e_3\exp(-\mathrm{i}\alpha x_3)$ and $\tilde{h}_3=h_3\exp(-\mathrm{i}\alpha x_3)$, we obtain the impedance boundary conditions

$$\beta_1\frac{\partial e_3}{\partial\boldsymbol{\nu}}+\mathrm{i}\alpha\frac{\partial e_3}{\partial\boldsymbol{\tau}}+\mathrm{i}\mu\beta_2\frac{\partial h_3}{\partial\boldsymbol{\nu}}=\lambda(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)h_3-\frac{\mathrm{i}2\mu\beta_1\beta_2}{\lambda}e_3\quad \text{on}\,\Gamma, \tag{B.50}$$

$$-\beta_1\frac{\partial h_3}{\partial\boldsymbol{\nu}}-\mathrm{i}\alpha\frac{\partial h_3}{\partial\boldsymbol{\tau}}+\mathrm{i}\epsilon\beta_2\frac{\partial e_3}{\partial\boldsymbol{\nu}}=\mathrm{i}2\epsilon\lambda\beta_1\beta_2h_3+\frac{(\beta_1^2+\epsilon\mu\beta_2^2-\alpha^2)}{\lambda}e_3\quad \text{on}\,\Gamma. \tag{B.51}$$

We introduce the Beltrami decomposition

$$e_3=u+v,\qquad h_3=-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\,(u-v), \tag{B.52}$$

where the total field (u,v) consists of the incident field (u^i,v^i) and the scattered field (u^s,v^s), i.e., $(u,v)=(u^i+u^s,v^i+v^s)$. Then, from (B.21) and (B.22), we have

$$\Delta(u+v)+(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)(u+v)+2\beta_1\beta_2\sqrt{\epsilon\mu}(u-v)=0, \tag{B.53}$$

$$\Delta(u-v)+(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)(u-v)+2\beta_1\beta_2\sqrt{\epsilon\mu}(u+v)=0. \tag{B.54}$$

With the help of (B.53)+(B.54) and (B.53)−(B.54), we find

$$\Delta u+\gamma_L^2u=0\quad \text{in}\,\mathbb{R}^2\setminus\bar{D}, \tag{B.55}$$

$$\Delta v+\gamma_R^2v=0\quad \text{in}\,\mathbb{R}^2\setminus\bar{D}, \tag{B.56}$$

where $\gamma_L^2=(\beta_1+\beta_2\sqrt{\epsilon\mu})^2-\alpha^2$ and $\gamma_R^2=(\beta_1-\beta_2\sqrt{\epsilon\mu})^2-\alpha^2$. Furthermore, from (B.50) and (B.51), we obtain

$$\begin{aligned}&\beta_1\frac{\partial(u+v)}{\partial\boldsymbol{\nu}}+\mathrm{i}\alpha\frac{\partial(u+v)}{\partial\boldsymbol{\tau}}+\mathrm{i}\beta_2\mu\left(-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\right)\frac{\partial(u-v)}{\partial\boldsymbol{\nu}}\\ &=\lambda(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\right)(u-v)-\frac{\mathrm{i}2\beta_1\beta_2\mu}{\lambda}(u+v)\qquad \text{on }\Gamma,\end{aligned} \tag{B.57}$$

$$\begin{aligned}&-\beta_1\left(-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\right)\frac{\partial(u-v)}{\partial\boldsymbol{\nu}}-\mathrm{i}\alpha\left(-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\right)\frac{\partial(u-v)}{\partial\boldsymbol{\tau}}+\mathrm{i}\beta_2\epsilon\frac{\partial(u+v)}{\partial\boldsymbol{\nu}}\\ &=\mathrm{i}2\beta_1\beta_2\epsilon\lambda\left(-\mathrm{i}\sqrt{\frac{\epsilon}{\mu}}\right)(u-v)+\frac{(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)}{\lambda}(u+v)\qquad \text{on }\Gamma.\end{aligned} \tag{B.58}$$

By $\mathrm{i}\times$(B.57) and $\sqrt{\frac{\mu}{\epsilon}}\times$(B.58), we find

$$\begin{aligned}&\mathrm{i}\beta_1\frac{\partial(u+v)}{\partial\boldsymbol{\nu}}+\mathrm{i}\beta_2\sqrt{\epsilon\mu}\frac{\partial(u-v)}{\partial\boldsymbol{\nu}}-\alpha\frac{\partial(u+v)}{\partial\boldsymbol{\tau}}\\ &=\lambda(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\sqrt{\frac{\epsilon}{\mu}}\,(u-v)+\frac{2\beta_1\beta_2\mu}{\lambda}(u+v)\qquad \text{on }\Gamma,\end{aligned} \tag{B.59}$$

$$\mathrm{i}\beta_1\frac{\partial(u-v)}{\partial\boldsymbol{\nu}}+\mathrm{i}\beta_2\sqrt{\epsilon\mu}\frac{\partial(u+v)}{\partial\boldsymbol{\nu}}-\alpha\frac{\partial(u-v)}{\partial\boldsymbol{\tau}}$$

$$=2\beta_1\beta_2\epsilon\lambda(u-v)+\frac{(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)}{\lambda}\sqrt{\frac{\mu}{\epsilon}}\,(u+v) \qquad \text{on } \Gamma. \tag{B.60}$$

From (B.59)+(B.60) and (B.59)−(B.60), we derive

$$\begin{aligned}
&-2\mathrm{i}(\beta_1+\beta_2\sqrt{\epsilon\mu})\frac{\partial u}{\partial \boldsymbol{\nu}}+2\alpha\frac{\partial u}{\partial \boldsymbol{\tau}}\\
&\quad+\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\sqrt{\frac{\epsilon}{\mu}}+\frac{1}{\lambda}\sqrt{\frac{\mu}{\epsilon}}\right)+2\beta_1\beta_2\left(\lambda\epsilon+\frac{\mu}{\lambda}\right)\right]u\\
&\quad-\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\sqrt{\frac{\epsilon}{\mu}}-\frac{1}{\lambda}\sqrt{\frac{\mu}{\epsilon}}\right)+2\beta_1\beta_2\left(\lambda\epsilon-\frac{\mu}{\lambda}\right)\right]v=0 \quad \text{on } \Gamma
\end{aligned} \tag{B.61}$$

and

$$\begin{aligned}
&-2\mathrm{i}(\beta_1-\beta_2\sqrt{\epsilon\mu})\frac{\partial v}{\partial \boldsymbol{\nu}}+2\alpha\frac{\partial v}{\partial \boldsymbol{\tau}}\\
&\quad-\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\sqrt{\frac{\epsilon}{\mu}}+\frac{1}{\lambda}\sqrt{\frac{\mu}{\epsilon}}\right)-2\beta_1\beta_2\left(\lambda\epsilon+\frac{\mu}{\lambda}\right)\right]v\\
&\quad+\left[(\beta_1^2+\beta_2^2\epsilon\mu-\alpha^2)\left(\lambda\sqrt{\frac{\epsilon}{\mu}}-\frac{1}{\lambda}\sqrt{\frac{\mu}{\epsilon}}\right)-2\beta_1\beta_2\left(\lambda\epsilon-\frac{\mu}{\lambda}\right)\right]u=0 \quad \text{on } \Gamma.
\end{aligned} \tag{B.62}$$

Therefore, the scalar functions u and v satisfy the following boundary conditions:

$$a_1\frac{\partial u}{\partial \boldsymbol{\nu}}+a_2\frac{\partial u}{\partial \boldsymbol{\tau}}+a_3u+a_4v=0 \quad \text{on } \Gamma, \tag{B.63}$$

$$a_5\frac{\partial v}{\partial \boldsymbol{\nu}}+a_6\frac{\partial v}{\partial \boldsymbol{\tau}}+a_7v+a_8u=0 \quad \text{on } \Gamma. \tag{B.64}$$

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