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STABILITY OF CONTACT LINES IN 2D STATIONARY BÉNARD CONVECTION*

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Abstract. We consider the evolution of contact lines for thermal convection of viscous fluids in a two-dimensional open-top vessel. The domain is bounded above by a free moving boundary and otherwise by the solid wall of a vessel. The dynamics of the fluid are governed by the incompressible Boussinesq approximation under the influence of gravity, and the interface between fluid and air is under the effect of capillary forces. Here we develop global well posedness theory in the framework of nonlinear energy methods for the initial data sufficiently close to equilibrium. Moreover, the solutions decay to equilibrium at an exponential rate. Our methods are mainly based on the elliptic analysis near corners and a priori estimates of a geometric formulation of the Boussinesq equations.

Keywords. free boundary problems; Bénard convection; global existence; capillarity.

AMS subject classifications. 35Q30; 35R35; 74G25; 76D45.

1. Introduction

1.1. Formulation of the problem in Eulerian coordinates. We consider a 2–dimensional open-top vessel as a bounded, connected open set $\mathcal{V}\subseteq\mathbb{R}^2$ which consists of two "almost" disjoint sections, i.e., $\mathcal{V}=\mathcal{V}_{top}\cup\mathcal{V}_{bot}$. The word almost means $\mathcal{V}_{top}\cap\mathcal{V}_{bot}$ is a set of measure 0 in $\mathbb{R}^2$. We assume that the "top" part $\mathcal{V}_{top}$ consists of a rectangular channel defined by

$$\mathcal{V}_{top}=\mathcal{V}\cap\mathbb{R}^2_+=\{y\in\mathbb{R}^2: -\ell<y_1<\ell,\ 0\le y_2<L\}$$

for some $\ell,\ L>0$, where $\mathbb{R}^2_+$ is the half-plane $\mathbb{R}^2_+=\{y\in\mathbb{R}^2: y_2\ge 0\}$. Similarly, we write the "bottom" part as

$$\mathcal{V}_{bot}=\mathcal{V}\cap\mathbb{R}^2_-=\mathcal{V}\cap\{y\in\mathbb{R}^2: y_2\le 0\}.$$

In addition, we also assume that the boundary $\partial\mathcal{V}$ of $\mathcal{V}$ is C^2 away from the points $(\pm\ell,L)$. We refer to Figure 1.1 for an example.

Now we consider a viscous incompressible fluid filling the $\mathcal{V}_{bot}$ entirely and $\mathcal{V}_{top}$ partially. More precisely, we assume that the fluid occupies the domain $\Omega(t)$ with an upper free surface,

$$\Omega(t)=\mathcal{V}_{bot}\cup\{y\in\mathbb{R}^2: -\ell<y_1<\ell,\ 0<y_2<\zeta(y_1,t)\},$$

where the free surface $\zeta(y_1,t)$ is assumed to be a graph of the function $\zeta:[-\ell,\ell]\times\mathbb{R}_+\to\mathbb{R}$ satisfying $0<\zeta(\pm\ell,t)\le L$ for all $t\in\mathbb{R}_+$, which means the fluid does not spill out of the top domain. For simplicity, we write the free surface as $\Sigma(t)=\{y_2=\zeta(y_1,t)\}$ and the interface between fluid and solid as $\Sigma_s(t)=\partial\Omega(t)\setminus\Sigma(t)$. We refer to Figure 1.2 for the description of the domain.

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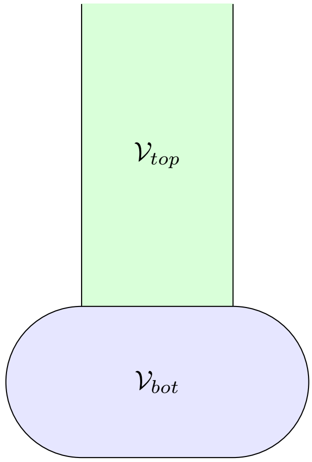

FIG. 1.1. *A vessel* $\mathcal{V}$.

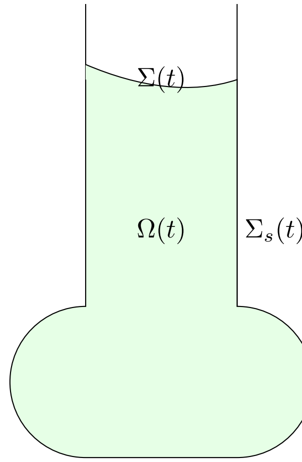

FIG. 1.2. *The domain* $\Omega(t)$.

For each $t \geq 0$, the fluid is described by its velocity, pressure and temperature $(u,P,\Theta):\Omega(t)\to\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}$, the dynamics of which is governed by the Boussinesq approximation [3] for $t>0$:

$$\begin{cases} \operatorname{div} S(P,u) + u\cdot\nabla u = \nabla P - \mu\Delta u + u\cdot\nabla u = g\alpha(\Theta-\Theta_0)e_{y_2} & \text{in } \Omega(t),\\ \operatorname{div} u = 0, \quad u\cdot\nabla\Theta - k\Delta\Theta = 0 & \text{in } \Omega(t),\\ S(P,u)\nu = g\zeta\nu - \sigma\mathcal{H}(\zeta)\nu, \quad \nu\cdot\nabla\Theta+\Theta = 1 & \text{on } \Sigma(t),\\ (S(P,u)\nu - \beta u)\cdot\tau = 0, \quad u\cdot\nu = 0, \quad \Theta = 1 & \text{on } \Sigma_s(t),\\ \partial_t\zeta = u\cdot\nu = u_2 - u_1\partial_1\zeta & \text{on } \Sigma(t),\\ \partial_t\zeta(\pm\ell,t) = \mathscr{V}\left([\![\gamma]\!] \mp \sigma \dfrac{\partial_1\zeta}{(1+|\partial_1\zeta|^2)^{1/2}}(\pm\ell,t)\right) \end{cases} \tag{1.1}$$

with the initial data $\zeta(y_1,t=0)=\zeta(0)$, $\partial_t\zeta(y_1,t=0)=\partial_t\zeta(0)$, and $\partial_t^2\zeta(y_1,t=0)=\partial_t^2\zeta(0)$.

In the above system (1.1), $S(p,u)$ is the viscous stress tensor determined by

$$S(P,u) = PI - \mu\mathbb{D}u,$$

where I is the 2×2 identity matrix, $\mu>0$ is the coefficient of viscosity, $\mathbb{D}u = \nabla u + \nabla^\top u$ is the symmetric gradient of u for $\nabla^\top u$ the transpose of the matrix ∇u, P is the difference between the full pressure and the hydrostatic pressure. Θ_0 is a reference temperature, which is chosen here to be the temperature on the solid wall, i.e. $\Theta_0=1$. $e_{y_2}=(0,1)$. $k>0$ is the thermal conduction coefficient. $\alpha>0$ is the coefficient of cubical expansion. ν is the outward unit normal. τ is the unit tangent. $\sigma>0$ is the coefficient of surface tension, and

$$\mathcal{H}(\zeta) = \partial_1\left(\frac{\partial_1\zeta}{(1+|\partial_1\zeta|^2)^{1/2}}\right)$$

is twice the mean curvature of the free surface. $\beta>0$ is the Navier slip friction coefficient on the vessel walls. The function $\mathscr{V}:\mathbb{R}\to\mathbb{R}$ is the contact point velocity response function which is a C^2 increasing diffeomorphism satisfying $\mathscr{V}(0)=0$. For more discussion about the choice of $\mathscr{V}$, we refer to [8, Section 1.3]. $[\![\gamma]\!] := \gamma_{sv}-\gamma_{sf}$ for γ_{sv}, $\gamma_{sf}\in\mathbb{R}$, where γ_{sv}, γ_{sf} are measures of the free-energy per unit length with respect to the solid-vapor and solid-fluid intersection, respectively. In addition, we assume that the Young relation [17] holds:

$$\frac{|[\![\gamma]\!]|}{\sigma}<1, \tag{1.2}$$

which is necessary for the existence of an equilibrium state. For convenience, we introduce the inverse function $\mathscr{W}=\mathscr{V}^{-1}$ and rewrite the final equation in (1.1) as

$$\mathscr{W}(\partial_t\zeta(\pm\ell,t))=[\![\gamma]\!]\mp\sigma\frac{\partial_1\zeta}{(1+|\partial_1\zeta|^2)^{1/2}}(\pm\ell,t). \tag{1.3}$$

The slip condition of fluids along the solid wall is introduced due to the the incompatibility of no-slip condition ($u=0$), and the kinematics of free boundary ($\partial_t\zeta=u\cdot\nu$) at contact points. In particular, fluid along the solid wall obeys the Navier-slip condition

$$u\cdot\nu=0, \quad \text{and} \quad (S(P,u)\nu-\beta u)\cdot\tau=0,$$

which is found in [13].

The system of (1.1) is semi-stationary. The dynamics of viscous thermal convection is stationary for each time, while the domain $\Omega(t)$ is time-dependent and the free surface is deformable. Here, for simplicity, we neglect the Marangoni effect, that means we still assume that the coefficient of surface tension σ has no relationship with temperature.

1.2. Known results. The Bénard convection is a classical problem in fluids and will creates chaos, which has attracted many famous mathematicians, see [5] and references therein. The free boundary problem of global well-posedness and stability for Bénard convection was first proved in [9] for 2D domains. The global well-posedness and stability for Bénard convection with a free surface for 3D domains was proved by [12]. Both of these results employ parabolic regularity theory in a functional framework of [2]. They assume that the surface tension is under the effect of Marangoni, and the domains are horizontally periodic. Both the results of [9, 12] are inspired by the idea in [2]. In the case without surface tension, [18] proved the local well-posedness for the 3D horizontally periodic domains using the energy estimates inspired by ideas of [6, 7]. All of these results are considered the Navier-Stokes equations coupled with the evolution heat equations.

For the framework of contact lines in bounded domains, [19] proved the local well-posedness for Stokes equations. [8] proved the global well-posedness and stability of Stokes equations. The series of [19] and [8] is the first complete theory for contact lines in Stokes equations which allow both dynamic contact points and dynamic contact angles.

In this paper, we consider the stationary Bénard convection including the dynamic contact points and contact angles. We first give the elliptic analysis for evolutionary heat equations near corner points, then construct a priori estimates for Boussinesq equations. Finally, we establish the linear solutions for steady heat equations evolving in time, which coupled with the a priori estimates yields our main results. In the following papers, we will present the full non-stationary Bénard convection.

1.3. Reformulation around equilibrium. A steady-state equilibrium solution to (1.1) corresponds to $u=0$, $P(y,t)=P_0(y)$, $\zeta(y_1,t)=\zeta_0(y)$ and $\Theta=1$. These satisfy

$$\begin{cases} \nabla P_0 = 0 & \text{in } \Omega(0), \\ P_0 = g\zeta_0 - \sigma\mathcal{H}(\zeta_0) & \text{on } (-\ell,\ell), \\ \sigma\dfrac{\partial_1\zeta_0}{\sqrt{1+|\partial_1\zeta_0|^2}}(\pm\ell) = \pm[\![\gamma]\!]. \end{cases} \tag{1.4}$$

It is well known (see, for instance, the discussion in the introduction of [8]) that there exists a smooth solution $\zeta_0 : [-\ell,\ell] \to (0,L)$.

In order to work in the fixed domain formed by equilibrium, we follow the path of [19]. Let $\zeta_0 \in C^\infty[-\ell,\ell]$ be the equilibrium surface given by (1.4). We then define the equilibrium domain $\Omega \subset \mathbb{R}^2$ by

$$\Omega := \mathcal{V}_b \cup \{x \in \mathbb{R}^2 \,|\, -\ell < x_1 < \ell, 0 < x_2 < \zeta_0(x_1)\}.$$

The boundary $\partial\Omega$ of the equilibrium Ω is defined by

$$\partial\Omega := \Sigma \sqcup \Sigma_s,$$

where

$$\Sigma := \{x \in \mathbb{R}^2 \,|\, -\ell < x_1 < \ell, x_2 = \zeta_0(x_1)\}, \quad \Sigma_s = \partial\Omega \setminus \Sigma.$$

Here Σ is the equilibrium free surface. The corner angle $\omega \in (0,\pi)$ of Ω is the contact angle formed by the fluid and solid. We will view the function $\zeta(y_1,t)$ of the free surface as the perturbation of $\zeta_0(y_1)$:

$$\zeta(y_1,t) = \zeta_0(y_1) + \eta(y_1,t). \tag{1.5}$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi(z) = 0$ for $z \le \frac{1}{4}\min\zeta_0$ and $\phi(z) = z$ for $z \ge \frac{1}{2}\min\zeta_0$. Now we define the mapping $\Phi : \Omega \mapsto \Omega(t)$ by

$$\Phi(x_1,x_2,t) = \left(x_1, x_2 + \frac{\phi(x_2)}{\zeta_0(x_1)}\bar{\eta}(x_1,x_2,t)\right) = (\Phi_1(x_1,x_2,t), \Phi_2(x_1,x_2,t)) = (y_1,y_2) \in \Omega(t) \tag{1.6}$$

with $\bar{\eta}$ defined by

$$\bar{\eta}(x_1,x_2,t) := \mathcal{P}E\eta(x_1, x_2 - \zeta_0(x_1), t), \tag{1.7}$$

where $E : H^s(-\ell,\ell) \mapsto H^s(\mathbb{R})$ is a bounded extension operator for all $0 \le s \le 3$ and $\mathcal{P}$ is the lower Poisson extension given by

$$\mathcal{P}f(x_1,x_2) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi|\xi|x_2} e^{2\pi i x_1 \xi} \, \mathrm{d}\xi.$$

If η is sufficiently small (in appropriate Sobolev spaces), the mapping Φ is a C^1 diffeomorphism of Ω onto $\Omega(t)$ that maps the components of $\partial\Omega$ to the corresponding components of $\partial\Omega(t)$.

We have the Jacobian matrix $\nabla\Phi$ and the transform matrix $\mathcal{A}$ of Φ,

$$\nabla\Phi = \begin{pmatrix} 1 & 0 \\ A & J \end{pmatrix}, \quad \mathcal{A} = (\nabla\Phi)^{-\top} = \begin{pmatrix} 1 & -AK \\ 0 & K \end{pmatrix} \tag{1.8}$$

for

$$A=\frac{\phi}{\zeta_0}\partial_1\bar\eta-\frac{\phi}{\zeta_0^2}\partial_1\zeta_0\bar\eta,\quad J=1+\frac{\phi'}{\zeta_0}\bar\eta+\frac{\phi}{\zeta_0}\partial_2\bar\eta,\quad K=\frac{1}{J}. \tag{1.9}$$

We define the transformed differential operators as follows:

$$(\nabla_{\mathcal A}f)_i:=\mathcal{A}_{ij}\partial_j f,\quad \operatorname{div}_{\mathcal A}X:=\mathcal{A}_{ij}\partial_j X_i,\quad \Delta_{\mathcal A}f:=\operatorname{div}_{\mathcal A}\nabla_{\mathcal A}f$$

for appropriate f and X. We write the stress tensor as

$$S_{\mathcal A}(P,u)=PI-\mu\mathbb{D}_{\mathcal A}u,$$

where I is the 2×2 identity matrix and $(\mathbb{D}_{\mathcal A}u)_{ij}=\mathcal{A}_{ik}\partial_k u_j+\mathcal{A}_{jk}\partial_k u_i$ is the symmetric $\mathcal{A}$–gradient. Note that if we extend $\operatorname{div}_{\mathcal A}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal A}S_{\mathcal A}(P,u)=-\mu\Delta_{\mathcal A}u+\nabla_{\mathcal A}P$ for vector fields satisfying $\operatorname{div}_{\mathcal A}u=0$.

We assume that Φ is a diffeomorphism (actually this will be proved by the local well-posedness). Then we can transform the problem (1.1) into the equilibrium domain Ω for $t\geq 0$. In the new coordinates, (1.1) becomes the $\mathcal{A}$–Stokes problem

$$\begin{cases}\operatorname{div}_{\mathcal A}S_{\mathcal A}(P,u)+u\cdot\nabla_{\mathcal A}u=-\mu\Delta_{\mathcal A}u+\nabla_{\mathcal A}P+u\cdot\nabla_{\mathcal A}u=-g\alpha(\Theta-1)\nabla_{\mathcal A}\Phi_2 & \text{in } \Omega,\\ \operatorname{div}_{\mathcal A}u=0,\quad u\cdot\nabla_{\mathcal A}\Theta-k\Delta_{\mathcal A}\Theta=0 & \text{in } \Omega,\\ S_{\mathcal A}(P,u)\mathcal{N}=g\zeta\mathcal{N}-\sigma\mathcal{H}(\zeta)\mathcal{N},\quad k\nabla_{\mathcal A}\Theta\cdot\mathcal{N}+\Theta=1 & \text{on } \Sigma,\\ (S_{\mathcal A}(P,u)\nu-\beta u)\cdot\tau=0,\quad u\cdot\nu=0,\quad \Theta=1 & \text{on } \Sigma_s,\\ \partial_t\zeta=u\cdot\mathcal{N} & \text{on } \Sigma,\\ \mathscr{W}(\partial_t\zeta(\pm\ell,t))=[\![\gamma]\!]\mp\sigma\dfrac{\partial_1\zeta}{\sqrt{1+|\zeta|^2}}(\pm\ell,t), & \\ \zeta(x_1,0)=\zeta_0(x_1)+\eta_0(x_1),\quad \partial_t\zeta(x_1,0)=\partial_t\eta(x_1,0),\quad \partial_t^2\zeta(x_1,0)=\partial_t^2\eta(x_1,0). & \end{cases} \tag{1.10}$$

Here we have still written $\mathcal{N}:=-\partial_1\zeta e_1+e_2$ for the normal to $\Sigma(t)$.

Since all terms in (1.10) are in terms of η, (1.10) is connected to the geometry of the free surface. This geometric structure is essential to control higher-order derivatives.

To this end we define new perturbed unknowns (u,p,θ,η) so that $u=0+u$, $P=P_0+p$, $\Theta=1+\theta$ and $\zeta=\zeta_0+\eta$. Then we will reformulate (1.10) in terms of the new unknowns following the path of [19] to the following perturbative form of the Stokes equations:

$$\begin{cases}\operatorname{div}_{\mathcal A}S_{\mathcal A}(p,u)+u\cdot\nabla_{\mathcal A}u=-\mu\Delta_{\mathcal A}u+\nabla_{\mathcal A}p+u\cdot\nabla_{\mathcal A}u=-g\alpha\theta\nabla_{\mathcal A}\Phi_2 & \text{in } \Omega,\\ \operatorname{div}_{\mathcal A}u=0,\quad u\cdot\nabla_{\mathcal A}\theta-k\Delta_{\mathcal A}\theta=0 & \text{in } \Omega,\\ S_{\mathcal A}(p,u)\mathcal{N}=g\eta\mathcal{N}-\sigma\partial_1\left(\dfrac{\partial_1\eta}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N}-\sigma\partial_1(\mathcal{R}(\partial_1\zeta_0,\partial_1\eta))\mathcal{N} & \text{on } \Sigma,\\ k\nabla_{\mathcal A}\theta\cdot\mathcal{N}=0 & \text{on } \Sigma,\\ (S_{\mathcal A}(p,u)\nu-\beta u)\cdot\tau=0,\quad u\cdot\nu=0,\quad \theta=0 & \text{on } \Sigma_s,\\ \partial_t\eta=u\cdot\mathcal{N} & \text{on } \Sigma,\\ \kappa\partial_t\eta(\pm\ell,t)+\kappa\hat{\mathscr{W}}(\partial_t\eta(\pm\ell,t))=\mp\sigma\left(\dfrac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\right)(\pm\ell,t) & \end{cases} \tag{1.11}$$

with the initial data $\eta(x_1,0)=\eta_0(x_1)$, $\partial_t\eta(x_1,0)$, and $\partial_t^2\eta(x_1,0)$. Here $\mathcal{A}$ and $\mathcal{N}$ are still determined in terms of $\zeta=\zeta_0+\eta$. In the following, we write $\mathcal{N}_0$ to be the non-unit-normal for the equilibrium surface Σ, and $\mathcal{N}=\mathcal{N}_0-\partial_1\eta e_1$.

1.4. Main theorems. In order to state our result, we need to explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k\geq 0$ to be the usual Sobolev spaces, and take $W^k_\delta(\Omega)$ and $W^k_\delta(\Sigma)$ for $k\geq 0$ and $\delta\in(0,1)$ to be the weighted Sobolev spaces defined in [19, Section 2]. We write norms $\|\partial_t^j u\|_k$ and $\|\partial_t^j p\|_k$ in the space $H^k(\Omega)$, and $\|\partial_t^j \eta\|_k$ in the space $H^k(\Sigma)$.

Now, we define the energy and dissipation used in this paper. The energy is

$$\mathcal{E}(t)=\|u\|^2_{W^2_\delta}+\|\partial_t u\|_1^2+\|p\|^2_{\mathring{W}^1_\delta}+\|\partial_t p\|^2_{\mathring{H}^0}+\|\theta\|^2_{W^2_\delta}+\|\partial_t\theta\|_1^2 \\ +\|\eta\|^2_{W^{5/2}_\delta}+\|\partial_t\eta\|^2_{3/2}+\sum_{j=0}^{2}\|\partial_t^j\eta\|^2_{\mathring{H}^1}, \tag{1.12}$$

and the dissipation is

$$\mathcal{D}(t)=\sum_{j=0}^{1}\Big(\|\partial_t^j u\|^2_{W^2_\delta}+\|\partial_t^j p\|^2_{\mathring{W}^1_\delta}+\|\partial_t^j\theta\|^2_{W^2_\delta}+\|\partial_t^j\eta\|^2_{W^{5/2}_\delta}\Big) \\ +\sum_{j=0}^{2}\Big(\|\partial_t^j u\|_1^2+\|\partial_t^j u\|^2_{H^0(\Sigma_s)}+[\partial_t^j u\cdot\mathcal{N}]_\ell^2\Big) \\ +\sum_{j=0}^{2}\Big(\|\partial_t^j p\|_0^2+\|\partial_t^j\theta\|_1^2+\|\partial_t^j\eta\|^2_{3/2}\Big)+\|\partial_t^3\eta\|^2_{1/2}, \tag{1.13}$$

where $[f]_\ell^2$, $\mathring{H}^s((-\ell,\ell))$, and $\mathring{W}^k_\delta(\Omega)$ are defined in [19, Section 2].

Our main result is the global-in-time solutions and decay estimates for (1.1).

THEOREM 1.1. *Let $\omega\in(0,\pi)$ be the angle formed by ζ_0 at the corners, $\delta_\omega=\max\{0,2-\frac{\pi}{\omega}\}\in[0,1)$ and $\delta\in(\delta_\omega,1)$. Suppose that the initial data $(\eta_0,\partial_t\eta(0),\partial_t^2\eta(0))$ satisfy the compatibility condition (5.18) and (5.19) such that there exists a universal small parameter $\gamma_0>0$, and the initial energy satisfies*

$$\mathcal{E}(0)\leq\gamma_0. \tag{1.14}$$

Then there exist a universal constant $\lambda>0$ and a solution (u,p,θ,η) of (1.11) global in time such that

$$\sup_{t\geq 0}\Big[\mathcal{E}(t)+e^{\lambda t}\Big(\|u(t)\|_1^2+\|u(t)\cdot\tau\|^2_{L^2(\Sigma_s)}+[u(t)\cdot\mathcal{N}(t)]_\ell^2+\|p(t)\|_0^2+\|\theta(t)\|_1^2\Big)\Big] \\ +\int_0^\infty\mathcal{D}(t)\,\mathrm{d}t\leq C\mathcal{E}(0), \tag{1.15}$$

where C is a universal constant.

REMARK 1.1. Theorem 1.1 implies that for any given Rayleigh number $R_a\sim O\left(\frac{1}{\mu k}\right)$, the system (1.11) has a unique solution global in time. This is in contrast with the non-stationary case in [12] where they assume that the Rayleigh number is sufficiently small.

1.5. Notation and terminology. Now, we mention some definitions, notation, and conventions that we will use throughout this paper.

(1) Constants. The symbol $C>0$ will denote a universal constant that only depends on the parameters of the problem and Ω, but does not depend on the data, etc. It is allowed to change from line to line. We will write $C=C(z)$ to indicate that the constant C depends on z. We will write $a\lesssim b$ to mean that $a\leq Cb$ for a universal constant $C>0$.

(2) Norms. We will write H^k for $H^k(\Omega)$ for $k\geq 0$, and $H^s(\Sigma)$ with $s\in\mathbb{R}$ for the usual Sobolev spaces. We will typically write $H^0=L^2$, though we will also use $L^2([0,T];H^k)$ (or $L^2([0,T];H^s(\Sigma))$) to denote the space of temporal square–integrable functions with values in H^k (or $H^s(\Sigma)$). Sometimes we will write $\|\cdot\|_k$ instead of $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_{H^k(\Sigma)}$. We also will write $\|\cdot\|_{L^2H^k}$ instead of $\|\cdot\|_{L^2([0,T];H^k(\Omega))}$ or $\|\cdot\|_{L^2([0,T];H^k(\Sigma))}$. When we do this it will be clear from the context on which set the norm is evaluated and the argument of the norm.

2. Functional setting and basic estimates

2.1. Functional spaces. Throughout this paper, we use the functional spaces introduced in [19, Section 2]. But for convenience, we still list them here. In addition, the proof of almost all estimates in this section might be found in [19, Section 2], so we omit the details here. For the details of usual Sobolev embedding theory, we refer to [16].

The proof of following propositions for weighted Sobolev embedding theory, could be found in [8, Appendix C and D].

Proposition 2.1. *Let $k\in\mathbb{N}$ and $0<\delta<1$. Then $W^k_\delta(\Omega)\hookrightarrow W^{k,q}$, for $1\leq q<\frac{2}{1+\delta}$. In particular, $W^1_\delta(\Omega)\hookrightarrow L^p(\Omega)$, for $1\leq p<\frac{2}{\delta}$, and $W^{1/2}_\delta(\partial\Omega)\hookrightarrow L^q(\partial\Omega)$, for $1\leq q<\frac{2}{1+\delta}$.*

From the Proposition 2.1, we could deduce the following useful corollary.

Corollary 2.1. *Let $\delta_\omega=\max\{0,2-\frac{\pi}{\omega}\}\in[0,1)$ and $\delta\in(\delta_\omega,1)$. Then*

$$W^2_\delta(\Omega)\hookrightarrow H^s(\Omega),\quad \text{and}\quad W^{5/2}_\delta(\partial\Omega)\hookrightarrow H^{s+\frac{1}{2}}(\partial\Omega),\quad \text{when } 1<s<\min\{2,\frac{\pi}{\omega}\}.$$

Proof. For $1<p<\frac{2}{1+\delta}$, we employ Proposition 2.1 to deduce $W^2_\delta(\Omega)\hookrightarrow W^{2,p}(\Omega)$. Then we choose $s=3-\frac{2}{p}$. Clearly, $1<s<\min\{2,\frac{\pi}{\omega}\}$. Then we use the embedding $W^{2,p}(\Omega)\hookrightarrow H^{3-\frac{2}{p}}(\Omega)$ to deduce $W^2_\delta(\Omega)\hookrightarrow H^{3-\frac{2}{p}}(\Omega)=H^s(\Omega)$.

The above analysis also implies $W^3_\delta(\Omega)\hookrightarrow H^{s+1}(\Omega)$. For any $f\in W^{5/2}_\delta(\partial\Omega)$, there exists a $F\in W^3_\delta(\Omega)$ such that $\|F\|_{W^3_\delta(\Omega)}\lesssim\|f\|_{W^{5/2}_\delta(\partial\Omega)}$. Then $F\in H^{s+1}(\Omega)$, and $f\in H^{s+\frac{1}{2}}(\partial\Omega)$ satisfy

$$\|f\|_{H^{s+\frac{1}{2}}(\partial\Omega)}\lesssim\|F\|_{H^{s+1}(\Omega)}\lesssim\|F\|_{W^3_\delta(\Omega)}\lesssim\|f\|_{W^{5/2}_\delta(\partial\Omega)}.$$

□

Proposition 2.2. *Let $0<\delta<1$. Then for each $q\in[0,\infty)$,*

$$\|d^\delta f\|_{L^q(\Omega)}\lesssim\|f\|_{W^1_\delta}$$

for all $f\in W^1_\delta(\Omega)$.

Proposition 2.3. *Let $0<\delta<1$ and $\kappa\in(0,1)$. Suppose that $f\in W^{1/2}_\delta(\Sigma)$ and that $g\in H^{1/2+\kappa}(\Sigma)$. Then $fg\in W^{1/2}_\delta(\Sigma)$ and*

$$\|fg\|_{W^{1/2}_\delta}\lesssim\|f\|_{W^{1/2}_\delta}\|g\|_{1/2+\kappa}.$$

2.2. Weak formulation and basic estimates. Suppose that $\zeta=\zeta_0+\eta$ and that $\mathcal{A}$, $\mathcal{N}$ are in terms of η. We refer the velocity to v, pressure to q, temperature to ϑ and surface function to ξ in order to distinguish from (u,p,θ,η). That's because in our analysis, (v,q,ϑ,ξ) represent not only (u,p,θ,η), but also represent the temporal derivatives of (u,p,θ,η). We assume that (v,q,ϑ,ξ) satisfies

$$\begin{cases} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(v,q) - g\alpha\vartheta\nabla_{\mathcal{A}}\Phi_2 = F^1,\ \operatorname{div}_{\mathcal{A}} v = F^2,\ -k\Delta_{\mathcal{A}}\vartheta = F^3 & \text{in } \Omega, \\ S_{\mathcal{A}}(v,q)\mathcal{N} = g\xi\mathcal{N} - \sigma\partial_1\left(\frac{\partial_1\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}} + F^4\right)\mathcal{N} + F^5 & \text{on } \Sigma, \\ \kappa\nabla_{\mathcal{A}}\vartheta\cdot\mathcal{N} = F^6 & \text{on } \Sigma, \\ (S_{\mathcal{A}}(v,q)\nu - \beta v)\cdot\tau = F^7,\ v\cdot\nu = 0, \vartheta = 0 & \text{on } \Sigma_s, \\ \partial_t\xi = v\cdot\mathcal{N} + F^8 & \text{on } \Sigma, \\ k\partial_t\xi = \mp\sigma\left(\frac{\partial_1\xi}{(1+|\partial_1\zeta_0|^2)^{3/2}} + F^4\right) - \kappa F^9 & \text{at } \pm\ell. \end{cases} \tag{2.1}$$

LEMMA 2.1. *Assume that (v,q,ϑ,ξ) are sufficiently smooth and satisfy (2.1). Suppose that $\phi\in\mathcal{H}^1(t)$, and that $\psi\in\mathcal{W}(t)$. Then*

$$k(\vartheta,\phi)_{\mathcal{H}^1} = (F^3,\phi)_{\mathcal{H}^0} + \int_{-\ell}^{\ell} F^6\phi, \tag{2.2}$$

and

$$\begin{aligned} &((v,\psi)) - (q,\operatorname{div}_{\mathcal{A}}\psi)_{\mathcal{H}^0} + (\xi,\psi\cdot\mathcal{N})_{1,\Sigma} + [v\cdot\mathcal{N},\psi\cdot\mathcal{N}]_\ell - g\alpha(\vartheta\nabla_{\mathcal{A}}\Phi_2,\psi)_{\mathcal{H}^0} \\ =&(F^1,\psi)_{\mathcal{H}^0} - \int_{-\ell}^{\ell}[\sigma F^4(\psi\cdot\mathcal{N}) + F^5\cdot\psi] - \int_{\Sigma_s} F^7(\psi\cdot\tau)J - [F^8+F^9,\psi\cdot\mathcal{N}]_\ell. \end{aligned} \tag{2.3}$$

Proof. The proof is very standard. Multiplying ϕJ on both sides of the third equation of (2.1), we might integrate over Ω, and integrate by parts to reveal (2.2). Then (2.3) is derived by the same method, so we omit the details here. ∎

This lemma implies the following evolution of energy formula for (v,q,ϑ,ξ).

THEOREM 2.1. *Suppose that $\zeta=\zeta_0+\eta$ and that $\mathcal{A}$, $\mathcal{N}$ are determined in terms of η. Suppose that (v,q,ϑ,ξ) satisfy (2.1). Then*

$$\int_\Omega k|\nabla_{\mathcal{A}}\vartheta|^2 J = \int_\Omega F^3\vartheta J + \int_{-\ell}^{\ell} F^6\vartheta, \tag{2.4}$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-\ell}^{\ell}\frac{g}{2}|\xi|^2 + \frac{\sigma}{2}\frac{|\partial_1\xi|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right) + \int_\Omega\frac{\mu}{2}|\mathbb{D}_{\mathcal{A}}v|^2 J + \beta\int_{\Sigma_s}(v\cdot\tau)^2 + [v\cdot\mathcal{N}]_\ell^2 \\ =&g\alpha\int_\Omega\vartheta\nabla_{\mathcal{A}}\Phi_2\cdot v + \int_\Omega(F^1\cdot v + qF^2)J - \int_{\Sigma_s}F^7(v\cdot\tau)J - [F^8+F^9,v\cdot\mathcal{N}]_\ell \\ &-\int_{-\ell}^{\ell}\left[\sigma F^4(v\cdot\mathcal{N}) + F^5\cdot v - g\xi F^8 - \sigma\frac{\partial_1\xi\partial_1F^8}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right]. \end{aligned} \tag{2.5}$$

Proof. We choose the test functions $\phi=\vartheta$, $\psi=v$ in Lemma 2.1 to deduce that

$$k(\vartheta,\phi)_{\mathcal{H}^1} = (F^3,\phi)_{\mathcal{H}^0} + \int_{-\ell}^{\ell} F^6\phi, \tag{2.6}$$

and

$$\begin{aligned}&((v,v))-(q,\mathrm{div}_{\mathcal{A}}v)_{\mathcal{H}^0}+(\xi,v\cdot\mathcal{N})_{1,\Sigma}+[v\cdot\mathcal{N}v]_\ell^2-g\alpha(\vartheta\nabla_{\mathcal{A}}\Phi_2,v)_{\mathcal{H}^0}\\&=(F^1,v)_{\mathcal{H}^0}-\int_{-\ell}^{\ell}[\sigma F^4(v\cdot\mathcal{N})+F^5\cdot v]-\int_{\Sigma_s}F^7(v\cdot\tau)J-[F^8+F^9,v\cdot\mathcal{N}]_\ell.\end{aligned}\tag{2.7}$$

Equation (2.6) is exactly (2.4). Then we compute

$$\begin{aligned}(\xi,v\cdot\mathcal{N})_{1,\Sigma}&=(\xi,\partial_t\xi-F^8)_{1,\Sigma}\\&=\frac{d}{dt}\left(\int_{-\ell}^{\ell}\frac{g}{2}|\xi|^2+\frac{\sigma}{2}\frac{|\partial_1\xi|^2}{(1+|\partial_1\zeta_0|^2)^{3/2}}\right)-\int_{-\ell}^{\ell}g\xi F^8+\sigma\frac{\partial_1\xi\partial_1F^8}{(1+|\partial_1\zeta_0|^2)^{3/2}},\end{aligned}$$

then plug this into (2.7) to deduce (2.5). □

3. Elliptic estimates

In order to solve the Equation (1.11), we need some elliptic estimates. The cornerstone of these elliptic estimates is the part near the corner points.

3.1. Analysis for Poisson equations near the corner points. First, we introduce the notion of cone. Let (r,ρ) be the polar coordinates for $\mathbb{R}^2$.

$$K_\omega=\{x\in\mathbb{R}^2:r>0\text{ and }\rho\in(-\pi/2,-\pi/2+\omega)\}$$

denotes the cones with open angle $\omega\in(0,\pi)$. The lower and upper boundaries of K_ω are

$$\Gamma_-=\{x\in\mathbb{R}^2:r>0\text{ and }\rho=-\pi/2\}\text{ and }\Gamma_+=\{x\in\mathbb{R}^2:r>0\text{ and }\rho=-\pi/2+\omega\}$$

respectively.

Now, we consider the $\mathfrak{A}$-equation. We first give the proof of ϑ through the Poisson equations

$$\begin{cases}-k\Delta_{\mathfrak{A}}\vartheta=G^3 & \text{in}\quad K_\omega,\\ k\nabla_{\mathfrak{A}}\vartheta\cdot(\mathfrak{A}\nu)=G^5 & \text{on}\quad \Gamma_+,\\ \vartheta=0 & \text{on}\quad\Gamma_-,\end{cases}\tag{3.1}$$

where the differential operators $\nabla_{\mathfrak{A}}$ and $\Delta_{\mathfrak{A}}$ are defined in the same way as $\nabla_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}$, $S_{\mathfrak{A}}$ is defined in the same way as $S_{\mathcal{A}}$ and $\nabla\Phi_2=(\mathfrak{A}_{21},\mathfrak{A}_{22})$. Clearly, when $\mathfrak{A}=I_{2\times2}$, (3.1) becomes

$$\begin{cases}-k\Delta\vartheta=G^3 & \text{in}\quad K_\omega,\\ k\nabla\vartheta\cdot\nu=G^5 & \text{on}\quad \Gamma_+,\\ \vartheta=0 & \text{on}\quad\Gamma_-.\end{cases}\tag{3.2}$$

THEOREM 3.1. *Assume that the forcing terms in (3.1) satisfy $G^3\in W^0_\delta(K_\omega)$ and $G^5\in W^{1/2}_\delta(\Gamma_+)$. Suppose that $\vartheta\in H^1(K_\omega)$ satisfies*

$$\int_{K_\omega}k\nabla_{\mathfrak{A}}\vartheta\cdot\nabla_{\mathfrak{A}}\phi=\int_{K_\omega}G^3\phi+\int_{\Gamma_+}G^5\phi,$$

for all $\phi \in \{\phi \in H^1(K_\omega) : \phi|_{\Gamma_-} = 0\}$. Furthermore, assume that ϑ and all the forcing terms G^i are supported in $\bar{K}_\omega \cap B_1(0)$, where $B_1(0)$ is a unit disk centered at 0 with radius 1. Then $\nabla^2\vartheta \in W^0_\delta(K_\omega)$. Moreover,

$$\|\nabla^2\vartheta\|^2_{W^0_\delta(K_\omega)} \lesssim \|G^3\|^2_{W^0_\delta(K_\omega)} + \|G^5\|^2_{W^{1/2}_\delta(\Gamma_+)}. \tag{3.3}$$

Proof. The key point for proof is the utilization of [10, Theorem 8.2.1], which gives the condition in terms of eigenvalue of associated operator pencil (see [10] for the terminology) for solving the corresponding elliptic systems. The assumptions on $\mathfrak{A}$ guarantee that the two Poisson equations in (3.1) and (3.2) generate the same operator pencil. The eigenvalues of operator pencil for (3.2) could be easily derived (for instance, refer to [10]) as $\{\frac{n\pi}{\omega} : n \in \mathbb{Z}\setminus\{0\}\}$, which are not contained in

$$\{\lambda \in \mathbb{C} : 0 \le \Re\lambda \le 1-\delta\}.$$

Thus we could use [10, Theorem 8.2.1], and then argue as [11, Theorem 6.4.6] to derive that $\nabla^2\vartheta \in W^0_\delta(K_\omega)$ satisfying

$$\|\nabla^2\vartheta\|^2_{W^0_\delta} \lesssim \|G^3\|^2_{W^0_\delta} + \|G^5\|^2_{W^{1/2}_\delta}. \tag{3.4}$$

From the assumption on ϑ and the Poincáre inequality together with Sobolev embedding theory in [8, Appendix C], we also have

$$\|\vartheta\|^2_{H^1(K_\omega)} \lesssim \|G^3\|^2_{W^0_\delta} + \|G^5\|^2_{W^{1/2}_\delta}. \tag{3.5}$$

□

3.2. $\mathcal{A}$-Poisson equation. Now, we consider the Poisson equation

$$\begin{cases} -k\Delta\vartheta = G^3 & \text{in} \quad \Omega, \\ k\nabla\vartheta\cdot\nu = G^6 & \text{on} \quad \Sigma, \\ \vartheta = 0 & \text{on} \quad \Sigma_s, \end{cases} \tag{3.6}$$

where ν is the unit outward normal and unit tangent of $\partial\Omega$.

First, we study the weak solution of (3.6).

DEFINITION 3.1. *Assume that $G^3 \in W^0_\delta(\Omega)$ and $G^6 \in W^{1/2}_\delta(\Sigma)$ for some $0 < \delta < 1$. We say $\vartheta \in H^1(\Omega)$ is a weak solution of (3.6) if*

$$\int_\Omega k\nabla\vartheta\cdot\nabla\phi = \int_\Omega G^3\phi + \int_\Sigma G^6\phi \tag{3.7}$$

holds for any $\phi \in {}_0H^1(\Omega)$. Clearly, from Proposition 2.1, the integrations on the right-hand side of (3.7) are well-defined.

Now we sketch the proof of existence and uniqueness of weak solutions of elliptic Equation (3.6). First, (3.7) allows us to use Riesz representation theorem to obtain the existence of ϑ such that $\|\nabla\vartheta\|^2_0 \lesssim \|G^3\|^2_{W^0_\delta} + \|G^6\|^2_{W^{1/2}_\delta}$. Then the Poincáre inequality implies $\|\vartheta\|^2_1 \lesssim \|G^3\|^2_{W^0_\delta} + \|G^6\|^2_{W^{1/2}_\delta}$.

The next theorem shows that, regularity of weak solution of the Equation (3.6) could be improved to second-order.

THEOREM 3.2. *Assume that* $G^3 \in W^0_\delta(\Omega)$ *and* $G^6 \in W^{1/2}_\delta(\Sigma)$ *for some* $0<\delta<1$. *Then there exists a unique* ϑ *solving* (3.6) *such that* $\vartheta \in W^2_\delta(\Omega)$. *Moreover,*

$$\|\vartheta\|^2_{W^2_\delta(\Omega)} \lesssim \|G^3\|^2_{W^0_\delta} + \|G^6\|^2_{W^{1/2}_\delta}. \tag{3.8}$$

Proof. The idea of this proof is very standard. We divided Ω into three parts. One is away from the corners. Thus, we might use the standard elliptic theory (for instance, [1, Theorem 10.5]) to obtain the results. The other two parts are near corners. Then we might use Theorem 3.1 to derive the conclusion. The proof is in a similar way as that of [8, Theorem 5.6]. So we omit the details. ∎

Suppose that η and $\mathcal{A}$, $\mathcal{N}$, etc. are given. We consider the equation

$$\begin{cases} -k\Delta_{\mathcal{A}}\vartheta = G^3 & \text{in } \Omega, \\ k\nabla_{\mathcal{A}}\vartheta\cdot\mathcal{N} = G^6 & \text{on } \Sigma, \\ \vartheta = 0 & \text{on } \Sigma_s. \end{cases} \tag{3.9}$$

THEOREM 3.3. *Let* $\delta \in (\delta_\omega, 1)$. *Suppose that* $\|\eta\|_{W^{5/2}_\delta} < \gamma_0$ *where* $\gamma_0 << 1$. *Assume that* $G^3 \in W^0_\delta(\Omega)$ *and* $G^6 \in W^{1/2}_\delta(\Sigma)$. *Then there exists a unique* $\vartheta \in W^2_\delta(\Omega)$ *solving* (3.9). *Moreover,*

$$\|\vartheta\|^2_{W^2_\delta} \lesssim \|G^3\|^2_{W^0_\delta} + \|G^6\|^2_{W^{1/2}_\delta}. \tag{3.10}$$

Proof. We rewrite (3.9) as the perturbation form:

$$\begin{cases} -k\Delta\vartheta = G^3 - k\,\mathrm{div}_{I-\mathcal{A}}\nabla_{\mathcal{A}}\vartheta - k\,\mathrm{div}\nabla_{I-\mathcal{A}}\vartheta & \text{in } \Omega, \\ k\nabla\vartheta\cdot\mathcal{N}_0 = G^6 + k\nabla_{I-\mathcal{A}}\vartheta\cdot\mathcal{N} + k\nabla\vartheta\cdot(\mathcal{N}_0-\mathcal{N}) & \text{on } \Sigma, \\ \vartheta = 0 & \text{on } \Sigma_s. \end{cases} \tag{3.11}$$

We now employ fixed point theory to solve (3.9). Suppose that $\theta \in W^2_\delta(\Omega)$. Then we define the operator $T_\eta : W^2_\delta(\Omega) \to W^2_\delta(\Omega)$ via $\theta \mapsto \vartheta = T_\eta\theta$, where ϑ and θ satisfy

$$\begin{cases} -k\Delta\vartheta = G^3 - k\,\mathrm{div}_{I-\mathcal{A}}\nabla_{\mathcal{A}}\theta - k\,\mathrm{div}\nabla_{I-\mathcal{A}}\theta & \text{in } \Omega, \\ k\nabla\vartheta\cdot\mathcal{N}_0 = G^6 + k\nabla_{I-\mathcal{A}}\theta\cdot\mathcal{N} + k\nabla\theta\cdot(\mathcal{N}_0-\mathcal{N}) & \text{on } \Sigma, \\ \vartheta = 0 & \text{on } \Sigma_s. \end{cases} \tag{3.12}$$

In order to use Theorem 3.2, we need to estimate the right side of (3.12). We first choose r, $s>1$ satisfying $\frac{2}{q}+\frac{1}{2}=\frac{s}{2}$ to estimate

$$\begin{aligned} &\|\mathrm{div}_{I-\mathcal{A}}\nabla_{\mathcal{A}}\theta\|^2_{W^0_\delta} + \|\mathrm{div}\nabla_{I-\mathcal{A}}\theta\|^2_{W^0_\delta} \\ &\lesssim \|I-\mathcal{A}\|^2_{L^r}\|\nabla\mathcal{A}\|^2_{L^{2/(2-s)}}\|d^\delta\nabla\theta\|^2_{L^r} + \|I-\mathcal{A}\|^2_{L^{2/(2-s)}}\|d^\delta\nabla\theta\|^2_{L^{2/(1-s)}} \\ &\quad + \|I-\mathcal{A}\|^2_{L^\infty}(1+\|\mathcal{A}\|^2_{L^\infty})\|\theta\|^2_{W^2_\delta} \\ &\lesssim \|\eta\|^2_{W^{5/2}_\delta}\|\theta\|^2_{W^2_\delta}. \end{aligned} \tag{3.13}$$

Similarly, we use the trace theory to see that

$$\begin{aligned} &\|\nabla_{I-\mathcal{A}}\theta\cdot\mathcal{N}\|^2_{W^{1/2}_\delta(\Sigma)} + \|\nabla\theta\cdot(\mathcal{N}_0-\mathcal{N})\|^2_{W^{1/2}_\delta(\Sigma)} \\ &\lesssim \|\nabla_{I-\mathcal{A}}\theta\cdot(\mathcal{N}_0-\partial_1\bar{\eta}e_1)\|^2_{W^1_\delta} + \|\nabla\theta\cdot\partial_1\bar{\eta}e_1\|^2_{W^1_\delta} \end{aligned}$$

$$\begin{aligned}&\lesssim \big(\|\nabla(I-\mathcal{A})\|_{L^{2/(2-s)}}^2\|\mathcal{N}_0-\partial_1\bar{\eta}e_1\|_{L^\infty}^2+\|I-\mathcal{A}\|_{L^\infty}^2\|\nabla(\mathcal{N}_0-\partial_1\bar{\eta}e_1)\|_{L^{2/(2-s)}}^2\big)\\&\qquad\times\|d^\delta\nabla\theta\|_{L^{2/(s-1)}}+\big(\|I-\mathcal{A}\|_{L^\infty}^2\|\mathcal{N}_0-\partial_1\bar{\eta}e_1\|_{L^\infty}^2+\|\partial_1\bar{\eta}\|_{L^\infty}^2\big)\|\theta\|_{W_\delta^2}^2\\&\quad+\|\partial_1\bar{\eta}\|_{L^{2/(2-s)}}^2\|d^\delta\nabla\theta\|_{L^{2/(s-1)}}\\&\lesssim\|\eta\|_{W_\delta^{5/2}}^2\|\theta\|_{W_\delta^2}^2.\end{aligned}\tag{3.14}$$

We now use Theorem 3.2 to (3.12) to obtain that

$$\|\vartheta_1-\vartheta_2\|_{W_\delta^2}^2\lesssim\|\eta\|_{W_\delta^{5/2}}^2\|\theta_1-\theta_2\|_{W_\delta^2}^2,\tag{3.15}$$

for the mapping $T_\eta\theta_j=\vartheta_j$, $j=1,2$. Then we choose γ_0 sufficiently small such that

$$\|\vartheta_1-\vartheta_2|_{W_\delta^2}^2\le\frac{1}{4}\|\theta_1-\theta_2\|_{W_\delta^2}^2,\tag{3.16}$$

which yields T_η is strictly contractive. Thus, we use Banach's fixed point theory to deduce that (3.11) has a unique solution $\vartheta\in W_\delta^2$. ◻

Finally, we consider the equation

$$\begin{cases}-k\Delta_{\mathcal{A}}\vartheta=G^3 & \text{in }\Omega,\\ k\nabla_{\mathcal{A}}\vartheta\cdot\mathcal{N}+\vartheta=G^6 & \text{on }\Sigma,\\ \vartheta=0 & \text{on }\Sigma_s.\end{cases}\tag{3.17}$$

THEOREM 3.4. *Let $\delta\in(\delta_\omega,1)$. Suppose that $\|\eta\|_{W_\delta^{5/2}}<\gamma_0$ where γ_0 is the same as Theorem 3.3. Assume that $G^3\in W_\delta^0(\Omega)$ and $G^6\in W_\delta^{1/2}(\Sigma)$. Then there exists a unique $\vartheta\in W_\delta^2(\Omega)$ solving (3.17). Moreover,*

$$\|\vartheta\|_{W_\delta^2}^2\lesssim\|G^3\|_{W_\delta^0}^2+\|G^6\|_{W_\delta^{1/2}}^2.\tag{3.18}$$

Proof. For simplicity, we define the trace operator $R:W_\delta^2(\Omega)\to W_\delta^0(\Omega)\times W_\delta^{1/2}(\Sigma)\times W_\delta^{3/2}(\Sigma_s)$ via

$$R\vartheta=(0,0,\vartheta|_{\Sigma_s}).$$

Note that R is compact, since the embedding $W_\delta^{3/2}(\Sigma)\hookrightarrow W_\delta^{1/2}(\Sigma)$ is compact. So the operator $T_\eta+R$ is Fredholm, which means the dimensions of kernel and co-kernel of $T_\eta+R$ are both finite. However, for $\vartheta\in$ Ker$T_\eta+R$, we multiply the first equation in (3.17) by ϑJ, and integrate by parts over Ω to see that

$$k\int_\Omega|\nabla_{\mathcal{A}}\vartheta|^2J+\int_\Sigma|\vartheta|^2=0,\tag{3.19}$$

which implies $\vartheta=0$. So $T_\eta+R$ is injective. Then Fredholm alternative tells us that $T_\eta+R$ is also surjective. Hence $T_\eta+R$ is an isomorphism. Thus (3.17) is uniquely solvable and the estimate (3.18) holds. ◻

4. A priori estimates

Now, we employ the energy formulation in Theorem 2.1 and elliptic estimates to derive a priori estimates. The key point is that we could be able to estimate the interacting terms appearing on the right-hand side of (2.4) and (2.5). So, we first need to confirm the forcing terms F^i in (2.4) and (2.5) in the Appendix.

4.1. Estimates for interaction. Now, we could estimate the interaction terms on the right-hand side of (2.4) and (2.5). Thanks to [8, Section 6], some of them have been done. So, we only need to estimate the rest. And we only give the estimates for the twice temporally differentiated case. The corresponding estimates for once temporally differentiated problem and the problem without temporal differentiation are similar and much easier to handle.

In the remaining sections, we write $d = dist(\cdot, N)$, where $N = \{(-\ell, \zeta_0(-\ell)), (\ell, \zeta_0(\ell))\}$ is the set of corner points of $\partial\Omega$. In the subsequent estimates, the following lemma is useful. The proof is trivial, so we omit it.

LEMMA 4.1. *Suppose that $d = dist(\cdot, N)$ and that $0 < \delta < 1$. Then $d^{-\delta} \in L^r(\Omega)$ for $2 < r < \frac{2}{\delta}$.*

PROPOSITION 4.1.

$$\int_\Omega F^1 \cdot \psi J \lesssim \|\psi\|_1 (\sqrt{\mathcal{E}} + \mathcal{E}) \sqrt{\mathcal{D}}$$

for all $\psi \in H^1(\Omega)$.

Proof. We first use Hölder inequality, Sobolev embedding theory, and trace theory to estimate

$$\begin{aligned}
&\left| \int_\Omega 2g\alpha \partial_t \theta \nabla_{\partial_t \mathcal{A}} \Phi_2 \cdot \psi J + 2g\alpha\theta \nabla_{\partial_t \mathcal{A}} \partial_t \Phi_2 \cdot \psi J \right| \\
&\lesssim \int_\Omega |\partial_t \theta| (|\partial_t \bar\eta| + |\nabla \partial_t \bar\eta|)(1 + |\nabla \bar\eta|)|\psi| + |\theta| (|\partial_t \bar\eta| + |\nabla \partial_t \bar\eta|)^2 \psi \\
&\lesssim \|\partial_t \theta\|_{L^3} (\|\partial_t \bar\eta\|_{L^3} + \|\nabla \partial_t \bar\eta\|_{L^3}) \|\psi\|_{L^3} + \|\partial_t \theta\|_{L^4} (\|\partial_t \bar\eta\|_{L^4} + \|\nabla \partial_t \bar\eta\|_{L^4}) \|\nabla \bar\eta\|_{L^4} \|\psi\|_{L^4} \\
&\qquad + \|\theta\|_{L^4} (\|\partial_t \bar\eta\|_{L^2} + \|\nabla \partial_t \bar\eta\|_{L^2}) \|\psi\|_{L^4} \\
&\lesssim \|\partial_t \theta\|_1 \|\partial_t \eta\|_{3/2} (1 + \|\eta\|_{3/2}) \|\psi\|_1 + \|\theta\|_1 \|\partial_t \eta\|_{3/2}^2 \|\psi\|_1 \lesssim \|\psi\|_1 (\sqrt{\mathcal{E}} + \mathcal{E}) \sqrt{\mathcal{D}}. \qquad (4.1)
\end{aligned}$$

We then use Hölder inequality, usual Sobolev embedding theory, Corollary 2.1, and trace theory to estimate

$$\begin{aligned}
\left| \int_\Omega 2g\alpha \partial_t \theta \nabla_{\mathcal{A}} \partial_t \Phi_2 \cdot \psi J \right| &\lesssim \|\mathcal{A}\|_{L^\infty} \|\partial_t \theta\|_{L^3} \|\nabla \partial_t \bar\eta\|_{L^3} \|\psi\|_{L^3} \\
&\lesssim (1 + \|\eta\|_{W^{5/2}_\delta}) \|\partial_t \theta\|_1 \|\partial_t \eta\|_{3/2} \|\psi\|_1 \lesssim \|\psi\|_1 (\sqrt{\mathcal{E}} + \mathcal{E}) \sqrt{\mathcal{D}}. \quad (4.2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left| \int_\Omega g\alpha\theta \nabla_{\partial_t^2 \mathcal{A}} \Phi_2 \cdot \psi J + g\alpha\theta \nabla_{\mathcal{A}} \partial_t^2 \Phi_2 \cdot \psi J \right| \\
&\lesssim \int_\Omega |\theta| [(|\nabla \partial_t^2 \bar\eta| + |\partial_t^2 \bar\eta|)(1 + |\nabla \bar\eta| + |\bar\eta|) + |\nabla \partial_t \bar\eta|^2 + |\partial_t \bar\eta|^2] \psi \\
&\lesssim \|\theta\|_1 \|\partial_t^2 \eta\|_{3/2} (1 + \|\eta\|_{3/2}) \|\psi\|_1 + \|\theta\|_1 \|\partial_t \eta\|_{3/2}^2 \|\psi\|_1 \lesssim \|\psi\|_1 (\sqrt{\mathcal{E}} + \mathcal{E}) \sqrt{\mathcal{D}}. \qquad (4.3)
\end{aligned}$$

Then we use the same method to estimate the convection terms in F^1.

$$\begin{aligned}
&\left| \int_\Omega \partial_t^2 \cdot \nabla_{\mathcal{A}} u \cdot \psi J + 2\partial_t u \cdot \nabla_{\mathcal{A}} \partial_t u \cdot \psi J + u \cdot \nabla_{\mathcal{A}} \partial_t u \cdot \psi J \right| \\
&\lesssim \|\mathcal{A}\|_{L^\infty} \int_\Omega (|\partial_t^2 u| |\nabla u| + |\partial_t u| |\nabla \partial_t u| + |u| |\nabla \partial_t^2 u|) |\psi|
\end{aligned}$$

$$\lesssim \|\mathcal{A}\|_{L^\infty}\left(\|\partial_t^2 u\|_{L^4}\|\nabla u\|_{L^2}+\|\partial_t u\|_{L^4}\|\nabla\partial_t u\|_{L^2}+\|u\|_{L^4}\|\nabla\partial_t^2 u\|_{L^2}\right)\|\psi\|_{L^4}$$
$$\lesssim (1+\|\eta\|_{W_\delta^{5/2}})(\|\partial_t^2 u\|_1\|u\|_1+\|\partial_t u\|_1^2)\|\psi\|_1\lesssim \|\psi\|_1(\sqrt{\mathcal{E}}+\mathcal{E})\sqrt{\mathcal{D}}. \tag{4.4}$$

We now turn to estimate

$$\begin{aligned}&\left|\int_\Omega 2(\partial_t u\cdot\nabla_{\partial_t\mathcal{A}})u\cdot\psi J+2(u\cdot\nabla_{\partial_t\mathcal{A}})\partial_t u\cdot\psi J\right|\\&\lesssim \int_\Omega (|\partial_t u||\nabla u|+|u||\nabla\partial_t u|)(|\partial_t\bar\eta|+|\nabla\partial_t\bar\eta|)|\psi|\\&\lesssim \|\partial_t u\|_1\|u\|_1\|\partial_t\eta\|_{3/2}\|\psi\|_1\lesssim \|\psi\|_1\mathcal{E}\sqrt{\mathcal{D}},\end{aligned} \tag{4.5}$$

and finally we estimate

$$\begin{aligned}\left|\int_\Omega (u\cdot\nabla_{\partial_t^2\mathcal{A}})u\cdot\psi J\right|&\lesssim \int_\Omega |u|(|\nabla\partial_t^2\bar\eta|+|\partial_t^2\bar\eta|+|\nabla\partial_t\bar\eta|^2+|\partial_t\bar\eta|^2)|\nabla u||\psi|\\&\lesssim \|u\|_1^2(\|\partial_t^2\eta\|_{3/2}+\|\partial_t\eta\|_{3/2}^2)\|\psi\|_1\lesssim \|\psi\|_1\mathcal{E}\sqrt{\mathcal{D}}.\end{aligned} \tag{4.6}$$

Then combining all the above estimates together with [8, Proposition 6.2] completes the proof. ∎

Proposition 4.2.

$$\int_\Omega F^3\phi J\lesssim \|\phi\|_1(\sqrt{\mathcal{E}}+\mathcal{E})\sqrt{\mathcal{D}}$$

for all $\phi\in\mathcal{H}^1$.

Proof. We first estimate

$$\begin{aligned}\left|\int_\Omega (2\mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\mathcal{A}}\partial_t\theta)\phi J\right|&\lesssim \int_\Omega |\partial_t\nabla\bar\eta||\nabla\partial_t^2\theta||\phi||\nabla\partial_t\bar\eta||\nabla^2\bar\eta||\nabla\partial_t\theta||\phi|\\&\quad+\int_\Omega|\nabla\partial_t\bar\eta||\nabla\bar\eta||\nabla^2\partial_t\theta||\phi|\\&:=I+II.\end{aligned}$$

For I, we choose $2<r<\frac{2}{\delta}$, and $q>1$ such that $\frac{2}{q}+\frac{1}{r}=\frac{1}{2}$. Then we employ Hölder inequality, Lemma 4.1, Propositions 2.1, 2.2 for the weighted Sobolev inequality, usual Sobolev embedding inequality and usual trace theory to deduce that

$$\begin{aligned}I&\le \|\nabla\partial_t\bar\eta\|_{L^q}\|d^{-\delta}\|_{L^r}\|d^\delta\nabla^2\partial_t\theta\|_{L^2}\|\phi\|_{L^q}+\|\nabla\partial_t\bar\eta\|_{L^p}\|\nabla^2\bar\eta\|_{L^2}\|\nabla\partial_t\theta\|_{L^r}\|\phi\|_{L^p}\\&\lesssim \|\partial_t\eta\|_{3/2}\|\partial_t\theta\|_{W_\delta^2}\|\phi\|_1+\|\partial_t\eta\|_{3/2}\|\eta\|_{3/2}\|\partial_t\theta\|_{W_\delta^2}\|\phi\|_1\lesssim \|\phi\|_1(\sqrt{\mathcal{E}}+\mathcal{E})\sqrt{\mathcal{D}}.\end{aligned} \tag{4.7}$$

For II, we choose $2<r<\frac{2}{\delta}$, and $p>1$ such that $\frac{3}{p}+\frac{1}{r}=\frac{1}{2}$. Then we employ Hölder inequality, Lemma 4.1, Sobolev embedding inequality and usual trace theory to deduce that

$$\begin{aligned}II&\le \|\nabla\partial_t\bar\eta\|_{L^p}\|\nabla\bar\eta\|_{L^p}\|d^{-\delta}\|_{L^r}\|d^\delta\nabla^2\partial_t\theta\|_{L^2}\|\phi\|_{L^p}\\&\lesssim \|\partial_t\eta\|_{3/2}\|\eta\|_{3/2}\|\partial_t\theta\|_{W_\delta^2}\|\phi\|_1\lesssim \|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}.\end{aligned} \tag{4.8}$$

Then we could use the same method to estimate the other terms as follows. In the following three estimates, we choose $2<r<\frac{2}{\delta}$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{r}=\frac{1}{2}$ and $\frac{2}{q}+\frac{1}{r}=\frac{1}{2}$.

$$\left|\int_\Omega (\mathrm{div}_{\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\partial_t\theta)\phi J\right|$$

$$\lesssim \int_\Omega |\nabla^2\bar\eta||\nabla\partial_t\theta||\phi| + |\nabla\partial_t\bar\eta||\nabla^2\partial_t\theta||\phi|$$
$$\lesssim \|\nabla^2\partial_t\bar\eta|\|_{L^2}\|\nabla\partial_t\theta\|_{L^r}\|\phi\|_{L^p} + \|\nabla\partial_t\bar\eta\|_{L^q}\|d^\delta\|_{L^r}\|d^\delta\nabla^2\partial_t\theta\|_{L^2}\|\phi\|_{L^q}$$
$$\lesssim \|\partial_t\eta\|_{3/2}\|\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.9}$$

We now turn to the term

$$\left|\int_\Omega (\mathrm{div}_{\partial_t^2\mathcal{A}}\nabla_{\mathcal{A}}\theta)\phi J\right| \lesssim \int_\Omega |\nabla\partial_t^2\bar\eta||\nabla^2\theta||\phi| + |\nabla\partial_t^2\bar\eta||\nabla^2\bar\eta||\nabla\theta|$$
$$\lesssim \|\nabla\partial_t^2\eta\|_{L^q}\|d^{-\delta}\|_{L^r}\|d^\delta\nabla^2\theta\|_{L^2}\|\phi\|_{L^q} + \|\nabla\partial_t^2\bar\eta\|_{L^q}\|\nabla^2\bar\eta\|_{L^2}\|\nabla\theta\|_{L^r}\|\phi\|_{L^q}$$
$$\lesssim \|\partial_t^2\theta\|_{3/2}\|\theta\|_{W^2_\delta}\|\phi\|_1 + \|\partial_t\eta\|_{3/2}^2\|\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1(\sqrt{\mathcal{E}}+\mathcal{E})\sqrt{\mathcal{D}}. \tag{4.10}$$

With the same tools, we have

$$\left|\int_\Omega (\mathrm{div}_{\mathcal{A}}\nabla_{\partial_t^2\mathcal{A}}\theta)\phi J\right| \lesssim \int_\Omega |\nabla\partial_t^2\bar\eta||\nabla^2\theta||\phi| + |\nabla^2\partial_t^2\bar\eta|$$
$$\lesssim \|\nabla\partial_t^2\bar\eta\|_{L^q}\|d^{-\delta}\|_{L^r}\|d^\delta\nabla^2\theta\|_{L^2}\|\phi\|_{L^q} + \|\nabla^2\partial_t^2\bar\eta|_{L^2}\|\nabla\theta\|_{L^r}\|\phi\|_{L^p}$$
$$\lesssim \|\partial_t^2\eta\|_{3/2}\|\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.11}$$

Now, we choose $2<r<\frac{2}{\delta}$, and $p,q>1$ such that $\frac{2}{p}+\frac{1}{r}=\frac{1}{2}$ and $\frac{3}{q}+\frac{1}{r}=\frac{1}{2}$ to estimate

$$\left|\int_\Omega (\mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\theta)\right| \lesssim \int_\Omega |\nabla\partial_t\bar\eta|^2|\nabla^2\theta||\phi| + |\nabla\partial_t\bar\eta||\nabla^2\partial_t\bar\eta||\nabla\theta||\phi|$$
$$\lesssim \|\nabla\partial_t\bar\eta\|_{L^q}^2\|d^{-\delta}\|_{L^r}\|d^\delta\nabla^2\theta\|_{L^2}\|\phi\|_{L^q} + \|\nabla\partial_t\bar\eta\|_{L^p}\|\nabla^2\partial_t\bar\eta\|_{L^2}\|\nabla\theta\|_{L^r}\|\phi\|_{L^p}$$
$$\lesssim \|\partial_t\eta\|_{3/2}^2\|\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1\mathcal{E}\sqrt{\mathcal{D}}. \tag{4.12}$$

□

Proposition 4.3.

$$\int_{-\ell}^{\ell} F^6\phi \lesssim \|\phi\|_1(\sqrt{\mathcal{E}}+\mathcal{E})\sqrt{\mathcal{D}}$$

for all $\phi\in\mathcal{H}^1(\Omega)$.

Proof. We choose q,p,r such that $1<q<\frac{2}{1+\delta}$, $\frac{2}{p}+\frac{1}{q}=1$ and $\frac{3}{r}+\frac{1}{q}=1$. First, we employ the weighted Sobolev inequality in Proposition 2.1, usual Sobolev embedding theory and trace theory, Hölder inequality to estimate

$$\left|\int_{-\ell}^{\ell} (\nabla_{\partial_t\mathcal{A}}\partial_t\theta\cdot\mathcal{N})\phi\right| \lesssim \|\partial_t\mathcal{A}\|_{L^p(\Sigma)}\|\nabla\partial_t\theta\|_{L^q(\Sigma)}\|\mathcal{N}\|_{L^\infty}\|\phi\|_{L^p(\Sigma)}$$
$$\lesssim \|\partial_t\bar\eta\|_{H^{3/2}(\Sigma)}\|\partial_t\theta\|_{W^{3/2}_\delta(\Sigma)}\|\phi\|_{H^{1/2}(\Sigma)}$$
$$\lesssim \|\partial_t\eta\|_{3/2}\|\partial_t\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.13}$$

The same tools allow us to estimate

$$\left|\int_{-\ell}^{\ell} (\nabla_{\partial_t^2\mathcal{A}}\theta\cdot\mathcal{N})\phi\right| \lesssim \|\partial_t^2\mathcal{A}\|_{L^p(\Sigma)}\|\nabla\theta\|_{L^q(\Sigma)}\|\mathcal{N}\|_{L^\infty}\|\phi\|_{L^p(\Sigma)}$$
$$\lesssim \|\partial_t^2\eta\|_{3/2}\|\theta\|_{W^2_\delta}\|\phi\|_1 \lesssim \|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.14}$$

Then it is much easier to estimate

$$\left|\int_{-\ell}^{\ell}(\nabla_{\partial_t\mathcal{A}}\theta\cdot\partial_t\mathcal{N})\phi\right|\lesssim\|\partial_t\mathcal{A}\|_{L^r(\Sigma)}\|\nabla\theta\|_{L^q(\Sigma)}\|\partial_t\mathcal{N}\|_{L^r(\Sigma)}\|\phi\|_{L^r(\Sigma)}$$
$$\lesssim\|\partial_t\eta\|_{3/2}^2\|\theta\|_{W^2_\delta}\|\phi\|_1\lesssim\|\phi\|_1\mathcal{E}\sqrt{\mathcal{D}}. \tag{4.15}$$

In the following, we have

$$\left|\int_{-\ell}^{\ell}(\nabla_{\mathcal{A}}\partial_t\theta\cdot\partial_t\mathcal{N})\phi\right|\lesssim\|\mathcal{A}\|_{L^\infty}\|\nabla\partial_t\theta\|_{L^q(\Sigma)}\|\partial_t\partial_1\eta\|_{L^p(\Sigma)}\|\phi\|_{L^p(\Sigma)}$$
$$\lesssim\|\partial_t\eta\|_{3/2}\|\partial_t\theta\|_{W^2_\delta}\|\phi\|_1\lesssim\|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.16}$$

The term of two time derivatives on the normal direction is estimated by

$$\left|\int_{-\ell}^{\ell}(\nabla_{\mathcal{A}}\theta\cdot\partial_t^2\mathcal{N})\phi\right|\lesssim\|\mathcal{A}\|_{L^\infty}\|\nabla\partial_t^2\theta\|_{L^p(\Sigma)}\|\partial_1\eta\|_{L^q(\Sigma)}\|\phi\|_{L^p(\Sigma)}$$
$$\lesssim\|\partial_t^2\eta\|_{3/2}\|\theta\|_{W^2_\delta}\|\phi\|_1\lesssim\|\phi\|_1\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}. \tag{4.17}$$

□

4.2. Estimates for elliptic terms. We now estimate for the right-hand side of the elliptic estimate in Theorem 3.3. We only give the estimates for the time differentiated case. The case without temporal differentiation, may be handled in the same way and is much easier.

PROPOSITION 4.4.

$$\|F^1\|_{W^0_\delta}^2\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.$$

Proof. First, we use Hölder inequality and the usual Sobolev inequality to derive that

$$\begin{aligned}
&\|g\alpha\theta\nabla_{\partial_t\mathcal{A}}\Phi_2\|_{W^0_\delta}^2+\|g\alpha\theta\nabla_{\mathcal{A}}\partial_t\Phi_2\|_{W^0_\delta}^2\\
&\lesssim\|g\alpha\nabla_{\partial_t\mathcal{A}}\Phi_2\|_{L^2}^2+\|g\alpha\theta\nabla_{\mathcal{A}}\partial_t\Phi_2\|_{L^2}^2\\
&\lesssim\|\theta\|_{L^4}^2\|\partial_t\mathcal{A}\|_{L^4}^2\|\nabla\Phi_2\|_{L^\infty}^2+\|\mathcal{A}\|_{L^\infty}^2\|\theta\|_{L^4}^2\|\nabla\partial_t\Phi_2\|_{L^4}^2\\
&\lesssim\|\theta\|_1^2\|\partial_t\eta\|_{3/2}^2\lesssim\mathcal{E}\mathcal{D}.
\end{aligned}$$

Then we choose q,p such that $2<q<\frac{2}{\delta}$ and $\frac{2}{p}+\frac{1}{q}=\frac{1}{2}$. The Hölder inequality, Proposition 2.1 and Corollary 2.1 for weighted Sobolev inequality and usual Sobolev inequality and trace theory reveal that

$$\begin{aligned}
&\|\partial_t u\cdot\nabla_{\mathcal{A}}u\|_{W^0_\delta}^2+\|u\cdot\nabla_{\mathcal{A}}\partial_t u\|_{W^0_\delta}^2+\|u\cdot\nabla_{\partial_t\mathcal{A}}u\|_{W^0_\delta}^2\\
&\lesssim\|\mathcal{A}\|_{L^\infty}^2\|\partial_t u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\|\mathcal{A}\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\nabla\partial_t u\|_{L^2}^2+\|u\|_{L^p}^2\|\partial_t\mathcal{A}\|_{L^p}^2\|\nabla u\|_{L^q}^2\\
&\lesssim\|\partial_t u\|_s^2\|u\|_1^2+\|u\|_s^2\|\partial_t u\|_1^2+\|u\|_1^2\|\partial_t\eta\|_{3/2}^2\|u\|_{W^2_\delta}^2\\
&\lesssim\|\partial_t u\|_{W^2_\delta}^2\|u\|_1^2+\|u\|_{W^2_\delta}^2\|\partial_t u\|_1^2+\|u\|_1^2\|\partial_t\eta\|_{3/2}^2\|u\|_{W^2_\delta}^2\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.
\end{aligned}$$

Then combining the estimates in [8, Proposition 7.1], we have the conclusion. □

PROPOSITION 4.5.

$$\|F^3\|_{W^2_\delta}^2\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.$$

Proof. First, for $1<s<\min\{2,\frac{\pi}{\omega}\}$, we choose p such that $\frac{2}{p}+\frac{2-s}{2}=\frac{1}{2}$. Then we employ Hölder inequality, Proposition 2.1 and Corollary 2.1 for weighted Sobolev inequality and usual Sobolev inequality and trace theory to deduce that

$$\begin{aligned}\|\mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\mathcal{A}}\theta\|^2_{W^0_\delta} &\lesssim \|d^\delta\nabla^2\theta\|^2_{L^2}\|\mathcal{A}\|^2_{L^\infty}\|\partial_t\mathcal{A}\|^2_{L^\infty}+\|\partial_t\mathcal{A}\|^2_{L^p}\|\nabla\mathcal{A}\|^2_{L^{\frac{2}{2-s}}}\|\nabla\theta\|^2_{L^p}\\ &\lesssim \|\partial_t\eta\|^2_{s+1/2}\|\theta\|^2_{W^2_\delta}+\|\partial_t\eta\|^2_{3/2}\|\eta\|^2_{s+1/2}\|\theta\|^2_{W^2_\delta}\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathrm{div}_{\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\theta\|^2_{W^0_\delta} &\lesssim \|\mathcal{A}\|^2_{L^\infty}(\|\partial_t\mathcal{A}\|^2_{L^\infty}\|d^\delta\nabla^2\theta\|^2_{L^2}+\|\nabla\partial_t\mathcal{A}\|^2_{L^{\frac{2}{2-s}}}\|d^\delta\nabla\theta\|_{L^{\frac{2}{s-1}}})\\ &\lesssim \|\partial_t\eta\|^2_{s+1/2}\|\theta\|^2_{W^2_\delta}\lesssim\mathcal{E}\mathcal{D}.\end{aligned}$$

Then we choose q,r such that $2<q<\frac{2}{\delta}$ and $\frac{2}{r}+\frac{1}{q}=\frac{1}{2}$. The Hölder inequality, Proposition 2.1 and Corollary 2.1 for weighted Sobolev inequality and usual Sobolev inequality and trace theory reveal that

$$\begin{aligned}&\|\partial_t u\cdot\nabla_{\mathcal{A}}\theta\|^2_{W^0_\delta}+\|u\cdot\nabla_{\mathcal{A}}\partial_t\theta\|^2_{W^0_\delta}+\|u\cdot\nabla_{\partial_t\mathcal{A}}\theta\|^2_{W^0_\delta}\\ &\lesssim\|\partial_t u\|^2_{W^2_\delta}\|\theta\|^2_1+\|u\|^2_{W^2_\delta}\|\partial_t\theta\|^2_1+\|u\|^2_1\|\partial_t\eta\|^2_{3/2}\|\theta\|^2_{W^2_\delta}\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.\end{aligned}$$

□

PROPOSITION 4.6.

$$\|F^6\|^2_{W^{1/2}_\delta}\lesssim(\mathcal{E}+\mathcal{E}^2)\mathcal{D}.$$

Proof. For $1<s<\min\{2,\frac{\pi}{\omega}\}$, we employ Proposition 2.3, Corollary 2.1 and trace theory to derive that

$$\begin{aligned}\|\nabla_{\partial_t\mathcal{A}}\theta\cdot\mathcal{N}\|^2_{W^{1/2}_\delta(\Sigma)} &\lesssim\|\nabla\theta\|^2_{W^{1/2}_\delta(\Sigma)}\|\partial_t\mathcal{A}\|_{s-\frac{1}{2}}\|\mathcal{N}\|_{s-\frac{1}{2}}\\ &\lesssim\|\theta\|^2_{W^2_\delta(\Omega)}\|\partial_t\eta\|^2_{W^{5/2}_\delta}(1+\|\eta\|^2_{W^{5/2}_\delta})\lesssim\mathcal{D}(\mathcal{E}^2+\mathcal{E}).\end{aligned}$$

Similarly,

$$\begin{aligned}\|\nabla_{\mathcal{A}}\theta\cdot\partial_t\mathcal{N}\|^2_{W^{1/2}_\delta(\Sigma)} &\lesssim\|\nabla\theta\|^2_{W^{1/2}_\delta(\Sigma)}\|\mathcal{A}\|_{s-\frac{1}{2}}\|\partial_1\partial_t\eta\|_{s-\frac{1}{2}}\\ &\lesssim\|\theta\|^2_{W^2_\delta(\Omega)}\|\eta\|^2_{W^{5/2}_\delta}(1+\|\partial_t\eta\|^2_{W^{5/2}_\delta})\lesssim\mathcal{D}(\mathcal{E}^2+\mathcal{E}).\end{aligned}$$

□

Finally, we present the following theorem.

THEOREM 4.1. *Let $\omega\in(0,\pi)$ be the angle generated by ζ_0 at the corners, $\delta_\omega\in\max\{0,2-\frac{\pi}{\omega}\}$ and $\delta\in(\delta_\omega,1)$. Suppose that $\|\eta\|_{W^{5/2}_\delta}<\gamma_0$, where γ_0 is given as Theorem 3.3. Then we have*

$$\sum_{j=0}^{1}\|\partial_t^j u\|^2_{W^2_\delta}+\|\partial_t^j p\|^2_{\mathring{W}^1_\delta}+\|\partial_t^j\theta\|^2_{W^2_\delta}+\|\partial_t^j\eta\|^2_{W^{5/2}_\delta}\lesssim\mathcal{D}_{\shortparallel}+\mathcal{D}(\mathcal{E}^2+\mathcal{E}),\tag{4.18}$$

where, the lower order of dissipation is $\mathcal{D}_{\shortparallel}=\sum_{j=0}^2\left(\|\partial_t^j u\|^2_1+\|\partial_t^j u\|^2_{H^0(\Sigma_s)}+[\partial_t^j u\cdot\mathcal{N}]^2_\ell\right)+\sum_{j=0}^2\left(\|\partial_t^j p\|^2_0+\|\partial_t^j\theta\|^2_1+\|\partial_t^j\eta\|^2_{3/2}\right)+\|\partial_t^3\eta\|^2_{1/2}$. This theorem is based on Propositions 4.4–4.6, and is proved in a similar way as in [8]. So we omit the details here.

4.3. A priori estimates. Now we present a priori estimates, which are the most important part in this section. First, we will develop a decay estimate for some lower-order terms. Then we will present a higher-order bound for energy and dissipation.

THEOREM 4.2. *There exists a universal constant $\gamma_0 > 0$ such that if*

$$\sup_{0\le t\le T}\mathcal{E}(t)+\int_0^T\mathcal{D}(t)\,\mathrm{d}t\le\gamma_0,$$

then there exists a universal constant $\lambda > 0$ such that

$$\sup_{0\le t\le T}e^{\lambda t}\Big(\mathcal{E}_{\shortparallel}(t)+\|u(t)\|_1^2+\|u(t)\cdot\tau\|_{L^2(\Sigma_s)}^2+[u(t)\cdot\mathcal{N}(t)]_\ell^2+\|p(t)\|_0^2+\|\theta(t)\|_1^2\Big)\lesssim\mathcal{E}_{\shortparallel}(0). \tag{4.19}$$

Then we present a higher-order bound for energy and dissipation.

THEOREM 4.3. *For δ being the same as in Theorem 3.3, there exists a γ_0 such that if*

$$\sup_{0\le t\le T}\mathcal{E}(t)+\int_0^T\mathcal{D}(t)\,\mathrm{d}t\le\gamma_0,$$

then

$$\sup_{0\le t\le T}\mathcal{E}(t)+\int_0^T\mathcal{D}(t)\,\mathrm{d}t\lesssim\mathcal{E}(0). \tag{4.20}$$

Theorems 4.2 and 4.3 are proved in the similar way as in [8], based on Propositions 4.1– 4.6 and Theorem 4.1. So we omit the details here.

5. Linear problem

5.1. Construction of initial data. Before we study the well-posedness of (1.11), we first consider the initial data and the initial energy $\mathcal{E}(0)$. Suppose that $\eta_0\in W_\delta^{5/2}(\Sigma)$, $\partial_t\eta(0)\in H^{3/2}(\Sigma)$, $\partial_t^2\eta(0)\in H^1(\Sigma)$, and that

$$\mathfrak{E}_0(\eta):=\|\eta_0\|_{W_\delta^{5/2}(\Sigma)}^2+\|\partial_t\eta(0)\|_{H^{3/2}(\Sigma)}^2+\sum_{j=0}^2\|\partial_t^j\eta(0)\|_{H^1(\Sigma)}^2\le\gamma_0$$

where $\gamma_0 > 0$ is small enough to satisfy the conditions in Theorem 3.3. We now construct the initial data $u(t=0)=u_0$, $p(t=0)=p_0$ and $\theta(t=0)=\theta_0$. When $t=0$, we consider the elliptic equation

$$\begin{cases}\operatorname{div}_{\mathcal{A}(0)}S_{\mathcal{A}(0)}(p_0,u_0)-g\alpha\theta_0\nabla_{\mathcal{A}(0)}\Phi_2(0)=-u_0\nabla_{\mathcal{A}(0)}u_0 & \text{in}\quad\Omega,\\ \operatorname{div}_{\mathcal{A}(0)}u_0=0 & \text{in}\quad\Omega,\\ -k\operatorname{div}_{\mathcal{A}(0)}\nabla_{\mathcal{A}(0)}\theta_0=-u_0\nabla_{\mathcal{A}(0)}\theta_0 & \text{in}\quad\Omega,\\ u_0\cdot\mathcal{N}(0)=\partial_t\eta(0),\quad \mu\mathbb{D}_{\mathcal{A}(0)}u_0\mathcal{N}(0)\cdot\mathcal{T}(0)=0 & \text{on}\quad\Sigma,\\ u_0\cdot\nu=0,\quad \mu\mathbb{D}_{\mathcal{A}(0)}u_0\nu\cdot\tau-\beta u_0\cdot\tau=0 & \text{on}\quad\Sigma_s,\\ k\nabla_{\mathcal{A}(0)}\theta_0\cdot\mathcal{N}(0)=0 & \text{on}\quad\Sigma,\\ \theta_0=0 & \text{on}\quad\Sigma_s.\end{cases} \tag{5.1}$$

First, we consider the linear equation

$$\begin{cases} \operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0^{(0)}, u_0^{(0)}) - g\alpha\theta_0^{(0)} \nabla_{\mathcal{A}(0)} \Phi_2(0) = 0 & \text{in} \quad \Omega, \\ \operatorname{div}_{\mathcal{A}(0)} u_0^{(0)} = 0 & \text{in} \quad \Omega, \\ -k \operatorname{div}_{\mathcal{A}(0)} \nabla_{\mathcal{A}(0)} \theta_0^{(0)} = 0 & \text{in} \quad \Omega, \\ u_0^{(0)} \cdot \mathcal{N}(0) = \partial_t \eta(0), \quad \mu \mathbb{D}_{\mathcal{A}(0)} u_0^{(0)} \mathcal{N}(0) \cdot \mathcal{T}(0) = 0, \quad k \nabla_{\mathcal{A}(0)} \theta_0^{(0)} \cdot \mathcal{N}(0) = 0 & \text{on} \quad \Sigma, \\ u_0^{(0)} \cdot \nu = 0, \quad \mu \mathbb{D}_{\mathcal{A}(0)} u_0^{(0)} \nu \cdot \tau - \beta u_0^{(0)} \cdot \tau = 0, \quad \theta_0^{(0)} = 0 & \text{on} \quad \Sigma_s. \end{cases} \tag{5.2}$$

We employ Theorem 3.3 for Equation (5.2) to deduce that there exists a unique $(u_0^{(0)}, p_0^{(0)}, \theta_0^{(0)}) \in W_\delta^2 \times \mathring{W}_\delta^1 \times W_\delta^2$, and

$$\|u_0^{(0)}\|_{W_\delta^2}^2 + \|p_0^{(0)}\|_{\mathring{W}_\delta^1}^2 + \|\theta_0^{(0)}\|_{W_\delta^2}^2 \lesssim \|\partial_t \eta(0)\|_{W_\delta^{3/2}}^2 \lesssim \|\partial_t \eta(0)\|_{3/2}^2. \tag{5.3}$$

Actually, $\theta_0^{(0)} = 0$. For simplicity, we rewrite the linear Equation (5.2) as

$$L(u_0^{(0)}, p_0^{(0)}, \theta_0^{(0)}) = (0, 0, 0, \partial_t \eta(0), 0, 0, 0, 0, 0),$$

with the linear operator $L : \mathfrak{S} \to \mathfrak{N}$ defined as

$$\begin{aligned} &L(u, p, \theta) = (\operatorname{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(p, u) - g\alpha\theta \nabla_{\mathcal{A}(0)} \Phi_2(0), \ \operatorname{div}_{\mathcal{A}(0)} u, \ -k\Delta_{\mathcal{A}(0)} \theta, \\ &u \cdot \mathcal{N}(0), \ \mu \mathbb{D}_{\mathcal{A}(0)} u \mathcal{N}(0) \cdot \mathcal{T}(0), \ u \cdot \nu, \ \mu \mathbb{D}_{\mathcal{A}(0)} u \nu \cdot \tau - \beta u \cdot \tau, \\ &k \nabla_{\mathcal{A}(0)} \theta \cdot \mathcal{N}(0), \ \theta|_{\Sigma_s}), \end{aligned}$$

where the space

$$\mathfrak{S} = W_\delta^2(\Omega) \times \mathring{W}_\delta^1(\Omega) \times W_\delta^2(\Omega)$$

and

$$\begin{aligned} \mathfrak{N} = W_\delta^0(\Omega) \times W_\delta^1(\Omega) \times W_\delta^0(\Omega) \times W_\delta^{3/2}(\Sigma) \times W_\delta^{1/2}(\Sigma) \times W_\delta^{3/2}(\Sigma_s) \times W_\delta^{1/2}(\Sigma_s) \\ \times W_\delta^{1/2}(\Sigma) \times W_\delta^{3/2}(\Sigma_s). \end{aligned}$$

Then we define the nonlinear operator $N : \mathfrak{S} \to \mathfrak{N}$ as

$$N(u, p, \theta) = (u \cdot \nabla_{\mathcal{A}(0)} u, 0, u \cdot \nabla_{\mathcal{A}(0)} \theta, 0, 0, 0, 0, 0, 0).$$

Then nonlinear Equation (5.1) might be rewritten as

$$L(u_0, p_0, \theta_0) + N(u_0, p_0, \theta_0) = (0, 0, 0, \partial_t \eta(0), 0, 0, 0, 0, 0) = L(u_0^{(0)}, p_0^{(0)}, \theta_0^{(0)}).$$

From the solvable equation of (5.2), L has a bounded inverse L^{-1}. Thus

$$(u_0, p_0, \theta_0) = (u_0^{(0)}, p_0^{(0)}, \theta_0^{(0)}) - L^{-1} N(u_0, p_0, \theta_0).$$

Since $N(0, 0, 0) = 0$ and $\|N(u, p, \theta) - N(v, q, \vartheta)\| \lesssim (\|u\|_{W_\delta^2} + \|v\|_{W_\delta^2} + \|\theta\|_{W_\delta^2})(\|u - v\|_{W_\delta^2} + \|\theta - \vartheta\|_{W_\delta^2})$, we then could use a standard argument (for instance, see [15, Proposition 1.38]) to deduce the existence and uniqueness of $(u_0, p_0, \theta_0) \in W_\delta^2 \times \mathring{W}_\delta^1 \times W_\delta^2$, and

$$\|u_0\|_{W_\delta^2}^2 + \|p_0\|_{\mathring{W}_\delta^1}^2 + \|\theta_0\|_{W_\delta^2}^2 \lesssim \|\partial_t \eta(0)\|_{W_\delta^{3/2}}^2 \lesssim \|\partial_t \eta(0)\|_{3/2}^2. \tag{5.4}$$

Clearly, from the embedding $W^2_\delta(\Omega)\hookrightarrow H^1(\Omega)$ and the boundary condition, $u_0\in\mathcal{V}(0)$.

Then we construct $\partial_t u(0)$ and $\partial_t p(0)$. In order to preserve the divergence-free condition, we construct $D_t u(0)$ instead of $\partial_t u(0)$, where $D_t u$ is defined via

$$D_t u = \partial_t u - (\partial_t(K\nabla\Phi))(K\nabla\Phi)u. \tag{5.5}$$

The advantage of D_t is that it preserves the $\mathrm{div}_{\mathcal{A}}$ free condition. Now we temporally differentiate the Equation (1.11), then take $t=0$,

$$\begin{cases} \mathrm{div}_{\mathcal{A}(0)} S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0)) - g\alpha\partial_t\theta(0)\nabla_{\mathcal{A}(0)}\Phi_2(0) + (D_t u(0))\cdot\nabla_{\mathcal{A}(0)}u_0 & \\ \qquad + u_0\cdot\nabla_{\mathcal{A}(0)}(D_t u(0)) = \tilde{F}(0) & \text{in } \Omega,\\ \mathrm{div}_{\mathcal{A}(0)} D_t u(0) = 0 & \text{in } \Omega,\\ -k\Delta_{\mathcal{A}(0)}\partial_t\theta(0) + (D_t u(0))\cdot\nabla_{\mathcal{A}(0)}\theta_0 + u_0\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0) = \tilde{F}^3(0) & \text{in } \Omega,\\ S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0))\mathcal{N}(0) = g\partial_t\eta(0)\mathcal{N}(0) - \sigma\partial_1\left(\dfrac{\partial_1\partial_t\eta(0)}{(1+|\partial_1\zeta_0|)^{3/2}}\right)\mathcal{N}(0) & \\ \qquad + \partial_t F^4(0)\mathcal{N}(0) + \tilde{F}^4(0) & \text{on } \Sigma,\\ (S_{\mathcal{A}(0)}(\partial_t p(0), D_t u(0))\nu - \beta D_t u(0))\cdot\tau = \tilde{F}^5, \quad D_t u(0)\cdot\nu = 0, \quad \partial_t\theta(0) = 0 & \text{on } \Sigma_s,\\ D_t u(0)\cdot\mathcal{N}(0) = \partial_t^2\eta(0), \quad k\nabla_{\mathcal{A}(0)}\partial_t\theta(0)\cdot\mathcal{N}(0) = \tilde{F}^6 & \text{on } \Sigma,\\ \kappa\partial_t^2\eta(\pm\ell,0) + \kappa\partial_t\hat{\mathscr{W}}(\partial_t\eta(\pm\ell))(0) = \mp\sigma\left(\dfrac{\partial_1\partial_t\eta(0)}{(1+|\zeta_0|^2)^{3/2}} + \partial_t F^3(0)\right)(\pm\ell), & \end{cases} \tag{5.6}$$

where

$$\begin{aligned} \tilde{F}^1(0) &= -\mathrm{div}_{\partial_t\mathcal{A}(0)} S_{\mathcal{A}(0)}(p_0,u_0) + \mu\,\mathrm{div}_{\mathcal{A}(0)}\mathbb{D}_{\partial_t\mathcal{A}(0)}u_0 + \mu\,\mathrm{div}_{\mathcal{A}(0)}\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0)\\ &\qquad - g\alpha\theta_0(\nabla_{\partial_t\mathcal{A}(0)}\Phi_2(0) + \nabla_{\mathcal{A}(0)}\partial_t\Phi_2(0)) - (R(0)u_0)\cdot\nabla_{\mathcal{A}(0)}u_0 - u_0\cdot\nabla_{\partial_t\mathcal{A}(0)}u_0,\\ \tilde{F}^3(0) &= k\,\mathrm{div}_{\partial_t\mathcal{A}(0)}\nabla_{\mathcal{A}(0)}\theta_0 + k\,\mathrm{div}_{\mathcal{A}(0)}\nabla_{\partial_t\mathcal{A}(0)}\theta_0 - (R(0)u_0)\cdot\nabla_{\mathcal{A}(0)}\theta_0 - u_0\cdot\nabla_{\partial_t\mathcal{A}(0)}\theta_0\\ \partial_t F^4(0) &= \partial_z\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\partial_1\partial_t\eta(0),\\ \tilde{F}^4(0) &= \mu\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0)\mathcal{N}(0) + \mu\mathbb{D}_{\partial_t\mathcal{A}(0)}u_0\mathcal{N}(0)\\ &\qquad + \left[g\eta_0 - \sigma\partial_1\left(\frac{\partial_1\eta_0}{(+|\partial_1\zeta_0|^2)^{3/2}} + \mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\right)\right]\partial_t\mathcal{N}(0),\\ \tilde{F}^5(0) &= \mu\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0)\nu\cdot\tau + \mu\mathbb{D}_{\partial_t\mathcal{A}(0)}u_0\nu\cdot\tau + \beta R(0)u_0\cdot\tau,\\ \tilde{F}^6(0) &= -k\nabla_{\partial_t\mathcal{A}(0)}\theta_0\cdot\mathcal{N}(0) - k\nabla_{\mathcal{A}(0)}\theta_0\cdot\partial_t\mathcal{N}(0). \end{aligned}$$

Then the pressureless weak formulation could be rewritten, by utilizing the last equation of (5.6), as

$$B[(D_t u(0),\partial_t\theta(0)),(w,\phi)] = L[(w,\phi)], \tag{5.7}$$

where $w\in\mathcal{V}(0)$ and $\phi\in\mathcal{H}^1(0)$, and

$$\begin{aligned} &B[(D_t u(0),\partial_t\theta(0)),(w,\phi)]\\ &:= ((D_t u(0),w)) + (D_t u(0)\cdot\mathcal{N}(0), w\cdot\mathcal{N}(0))_{1,\Sigma} + k\int_\Omega \nabla_{\mathcal{A}(0)}\partial_t\theta^{(0)}(0)\cdot\nabla_{\mathcal{A}(0)}\phi J(0)\\ &\quad + \int_\Omega [-g\alpha\partial_t\theta(0)\nabla_{\mathcal{A}(0)}\Phi_2(0) + (D_t u(0))\cdot\nabla_{\mathcal{A}(0)}u_0 + u_0\cdot\nabla_{\mathcal{A}(0)}(D_t u(0))]\cdot wJ(0) \end{aligned}$$

$$+\int_{\Omega}[(D_tu(0))\cdot\nabla_{\mathcal{A}(0)}\theta_0+u_0\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0)]\phi J(0), \tag{5.8}$$

and

$$\begin{aligned}L[(w,\phi)]:=&(\partial_t^2\eta(0),w\cdot\mathcal{N}(0))_{1,\Sigma}-(\partial_t\eta(0),w\cdot\mathcal{N}(0))_{1,\Sigma}-k\int_{-\ell}^{\ell}\nabla_{\mathcal{A}(0)}\theta_0\cdot\partial_t\mathcal{N}(0)\phi\\&-\int_{-\ell}^{\ell}\left[g\eta_0-\sigma\partial_1\left(\frac{\partial_1\eta_0}{(+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\right)\right]\partial_t\mathcal{N}(0)\cdot w\\&-\int_{-\ell}^{\ell}\partial_z\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\partial_1\partial_t\eta(0)\partial_1(w\cdot\mathcal{N}(0))-\int_{\Sigma_s}\beta(R(0)u_0\cdot\tau)(w\cdot\tau)J(0)\\&-\int_{\Omega}\Big(\operatorname{div}_{\partial_t\mathcal{A}(0)}S_{\mathcal{A}(0)}(p_0,u_0)\cdot w+\frac{\mu}{2}\mathbb{D}_{\partial_t\mathcal{A}(0)}u_0:\mathbb{D}_{\mathcal{A}(0)}w\\&+\frac{\mu}{2}\mathbb{D}_{\mathcal{A}(0)}(R(0)u_0):\mathbb{D}_{\mathcal{A}(0)}w\Big)J(0)+k\int_{\Omega}\operatorname{div}_{\partial_t\mathcal{A}(0)}\nabla_{\mathcal{A}(0)}\theta_0\phi\\&-[\partial_t^2\eta(0),w\cdot\mathcal{N}(0)]_\ell-[\partial_t\hat{\mathscr{W}}(\partial_t\eta)(0),w\cdot\mathcal{N}(0)]_\ell J(0).\end{aligned} \tag{5.9}$$

If we denote $(D_tu(0),\partial_t\theta(0))$ be a new unknown, and (w,ϕ) be a new test function, then $B[\cdot,\cdot]:(\mathcal{V}(0)\times\mathcal{H}^1(0))\times(\mathcal{V}(0)\times\mathcal{H}^1(0))\to\mathbb{R}$ is a bilinear mapping satisfying

$$B[(v,\vartheta),(w,\phi)]\lesssim(\|v\|_{\mathcal{W}}+\|\vartheta\|_{\mathcal{H}^1})(\|w\|_{\mathcal{W}}+\|\phi\|_{\mathcal{H}^1}),$$

and $L[\cdot]:\mathcal{V}(0)\times\mathcal{H}^1(0)\to\mathbb{R}$ is a bounded linear functional on $\mathcal{V}(0)\times\mathcal{H}^1(0)$. Now we show the bilinear form $B[\cdot,\cdot]$ is coercive. We utilize Hölder inequality and Sobolev inequalities to deduce that

$$\begin{aligned}&k\int_{\Omega}\nabla_{\mathcal{A}(0)}\partial_t\theta(0)\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0)J(0)\\&\quad+\int_{\Omega}[(D_tu(0))\cdot\nabla_{\mathcal{A}(0)}\theta_0+u_0\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0)]\partial_t\theta(0)J(0)\\&\gtrsim\|\partial_t\theta(0)\|_{\mathcal{H}^1}^2-\|D_tu(0)\|_{L^4}\|\nabla\theta_0\|_{L^2}\|\partial_t\theta(0)\|_{L^4}+\|u_0\|_{L^4}\|\nabla\partial_t\theta(0)\|_{L^2}\|\partial_t\theta(0)\|_{L^4}\\&\gtrsim\|\partial_t\theta(0)\|_{\mathcal{H}^1}^2-\|D_tu(0)\|_1\|\theta_0\|_{W_\delta^2}\|\partial_t\theta_0\|_1-\|u_0\|_{W_\delta^2}\|\partial_t\theta(0)\|_1^2.\end{aligned} \tag{5.10}$$

Then Cauchy-Schwarz inequality, (5.4) and smallness of γ_0 imply

$$\begin{aligned}&k\textstyle\int_\Omega\nabla_{\mathcal{A}(0)}\partial_t\theta(0)\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0)J(0)+\int_\Omega[(D_tu(0))\cdot\nabla_{\mathcal{A}(0)}\theta_0+u_0\cdot\nabla_{\mathcal{A}(0)}\partial_t\theta(0)]\partial_t\theta(0)J(0)\\&\gtrsim\|\partial_t\theta(0)\|_{\mathcal{H}^1}^2-\|\partial_t\eta(0)\|_{3/2}^2\|D_tu(0)\|_1^2.\end{aligned} \tag{5.11}$$

Similarly,

$$\begin{aligned}&((D_tu(0),D_tu(0)))+(D_tu(0)\cdot\mathcal{N}(0),D_tu(0)\cdot\mathcal{N}(0))_{1,\Sigma}\\&\quad+\int_\Omega[-g\alpha\partial_t\theta(0)\nabla_{\mathcal{A}(0)}\Phi_2(0)+(D_tu(0))\cdot\nabla_{\mathcal{A}(0)}u_0+u_0\cdot\nabla_{\mathcal{A}(0)}(D_tu(0))]\cdot D_tu(0)J(0)\\&\gtrsim\|D_tu(0)\|_{\mathcal{W}}^2-\|\partial_t\theta(0)\|_1^2.\end{aligned} \tag{5.12}$$

Then we plug (5.11) into (5.12) to deduce that

$$B[(D_tu(0),\partial_t\theta(0)),(D_tu(0),\partial_t\theta(0))]\gtrsim\|D_tu(0)\|_{\mathcal{W}}^2. \tag{5.13}$$

Similarly, plugging (5.12) into (5.11) reveals

$$B[(D_t u(0),\partial_t\theta(0)),(D_t u(0),\partial_t\theta(0))]\gtrsim \|\partial_t\theta(0)\|_{\mathcal{H}^1}^2. \tag{5.14}$$

Combining (5.13) and (5.14) imply

$$B[(D_t u(0),\partial_t\theta(0)),(D_t u(0),\partial_t\theta(0))]\gtrsim \|D_t u(0)\|_{\mathcal{W}}^2+\|\partial_t\theta(0)\|_{\mathcal{H}^1}^2. \tag{5.15}$$

Thus the bilinear form $B[\cdot,\cdot]$ is coercive. Then Lax-Millgram theorem guarantees that there exists a unique pair $(D_t u(0),\partial_t\theta(0))\in\mathcal{V}(0)\times\mathcal{H}^1(0)$, such that

$$\|D_t u(0)\|_1^2+\|\partial_t\theta(0)\|_1^2\lesssim \|\eta_0\|_{W_\delta^{5/2}}^2+\|\partial_t\eta(0)\|_{3/2}^2+\|\partial_t^2\eta(0)\|_1^2. \tag{5.16}$$

Now from [8, Theorem 4.6], we may recover $\partial_t p(0)\in \mathring{H}^0(\Omega)$ such that

$$\|\partial_t p(0)\|_0^2\lesssim \|\eta_0\|_{W_\delta^{5/2}}^2+\|\partial_t\eta(0)\|_{3/2}^2+\|\partial_t^2\eta(0)\|_1^2. \tag{5.17}$$

In the construction of initial data above, η_0, $\partial_t\eta(0)$, and $\partial_t^2\eta(0)$ need to satisfy some compatibility conditions. At the corner points $x_1=\pm\ell$,

$$\kappa\partial_t\eta(0)+\kappa\hat{\mathscr{W}}(\partial_t\eta(0))=\mp\sigma\left(\frac{\partial_1\eta_0}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\right) \tag{5.18}$$

and

$$\kappa\partial_t^2\eta(0)+\kappa\hat{\mathscr{W}}'(\partial_t\eta(0))\partial_t^2\eta(0)=\mp\sigma\left(\frac{\partial_1\partial_t\eta(0)}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\partial_z\mathcal{R}(\partial_1\zeta_0,\partial_1\eta_0)\partial_1\partial_t\eta(0)\right). \tag{5.19}$$

5.2. Linear problem for Poisson system. Suppose that η is given and that $\mathcal{A}$, J, $\mathcal{N}$, etc. are determined in terms of η. Before turning to an analysis of the linear problem, we define various quantities in terms of η:

$$\mathfrak{D}(\eta):=\sum_{j=0}^{1}\|\partial_t^j\eta\|_{L^2W_\delta^{5/2}}^2+\sum_{j=0}^{2}\|\partial_t^j\eta\|_{L^2H^{3/2}}^2+\|\partial_t^3\eta\|_{L^2W_\delta^{1/2}}^2+\sum_{j=1}^{3}\|[\partial_t^j\eta]_\ell\|_{L^2([0,T])}^2,$$

$$\mathfrak{E}(\eta):=\|\eta\|_{L^\infty W_\delta^{5/2}}^2+\|\partial_t\eta\|_{L^\infty H^{3/2}}^2+\sum_{j=0}^{2}\|\partial_t^j\eta\|_{L^\infty H^1}^2,\quad \mathfrak{K}(\eta):=\mathfrak{D}(\eta)+\mathfrak{E}(\eta),$$

$$\mathfrak{E}_0=\mathfrak{E}_0(\eta):=\|\eta_0\|_{W_\delta^{5/2}}^2+\|\partial_t\eta(0)\|_{H^{3/2}}^2+\sum_{j=0}^{2}\|\partial_t^j\eta(0)\|_{H^1}^2. \tag{5.20}$$

Throughout this section, we always assume that $\mathfrak{K}(\eta)\le\gamma_0$ and $\gamma_0>0$ is sufficiently small.

For the purpose of constructing solutions to the nonlinear system, we need to consider the following modified linear problem

$$\begin{cases}-k\Delta_{\mathcal{A}}\theta=F^3 & \text{in}\quad\Omega,\\ k\nabla_{\mathcal{A}}\theta\cdot\mathcal{N}+\theta=F^6 & \text{on}\quad\Sigma,\\ \theta=0 & \text{on}\quad\Sigma_s.\end{cases} \tag{5.21}$$

To analyze (5.21), we need to consider two notions of solution: weak and strong.

Definition 5.1. *Suppose that $F^3+F^6\in(\mathcal{H}^1_T)^*$. θ is called a weak solution of* (5.21)*, provided that $\theta\in L^2([0,T];{}_0H^1(\Omega))$ and satisfies*

$$k\int_0^T(\nabla_{\mathcal{A}}\theta,\nabla_{\mathcal{A}}\phi)_{\mathcal{H}^0}=\int_0^T(F^3,\phi)_{\mathcal{H}^0}+\int_0^T\int_{-\ell}^{\ell}(F^6-\theta)\phi \tag{5.22}$$

for each $\phi\in\mathcal{H}^1_T$.

In the following, we will see that weak solutions will arise as a byproduct of the construction of strong solutions to (5.21). Hence, we now ignore the existence of weak solutions and record a uniqueness result based on some integral equalities and bounds satisfied by weak solutions.

Proposition 5.1. *Weak solutions to* (5.21) *are unique.*

Proof. If θ^1 and θ^2 are both weak solutions to (5.21), then $\theta=\theta^1-\theta^2$ is a weak solution of (5.21) with $F^3=F^6=0$. Using the test function $\theta\chi_{[0,t]}\in\mathcal{H}^1_T$, where $\chi_{[0,t]}$ is a temporal indicator function, we have that

$$\int_0^t\|\theta\|^2_{\mathcal{H}^1}+\int_0^t\|\theta\|^2_{H^0(\Sigma)}=0, \tag{5.23}$$

which implies $\theta=0$. ◻

We now give our definition of strong solutions.

Definition 5.2. *Suppose that the forcing functions satisfy*

$$\begin{aligned}&F^3\in L^2([0,T];W^{3/2}_\delta(\Omega)),\ F^6\in L^2([0,T];W^{1/2}_\delta(\Sigma)),\\&\partial_t(F^3+F^6)\in L^2([0,T];(\mathcal{H}^1)^*).\end{aligned} \tag{5.24}$$

If there exists a θ satisfying (5.21) *in the strong sense of*

$$\theta\in L^2([0,T];W^2_\delta(\Omega)),\quad \partial_t^j\theta\in L^2([0,T];{}_0H^1(\Omega)), \tag{5.25}$$

for $j=0,1$, we call it a strong solution.

The proof of the following lemma is in the similar way as in [14].

Lemma 5.1. *Suppose that the right-hand side of the following are finite. Then $\theta\in C^0([0,T];{}_0H^1(\Omega))$ and $u\in C^0([0,T];{}_0H^1(\Omega))$, satisfying*

$$\|\theta\|^2_{L^\infty H^1}\lesssim\|\theta_0\|^2_{W^2_\delta}+\|\theta\|^2_{L^2H^1}+\|\partial_t\theta\|^2_{L^2H^1}.$$

Now we state our main theorem for the strong solutions.

Theorem 5.1. *Suppose that the forcing terms F^3 and F^6 satisfy the condition* (5.24)*, that the initial data are the same as in Section 5.1. Suppose that $\mathfrak{K}(\eta)\le\gamma_0$ is smaller than γ_0 in Theorem 3.3. Then there exists a unique strong solution θ solving* (5.21) *such that θ satisfies* (5.25)*. The solution obeys the estimate*

$$\begin{aligned}&\sum_{j=0}^1\|\partial_t^j\theta\|^2_{L^2H^1}+\|\theta\|_{L^\infty H^1}+\|\theta\|^2_{L^2W^2_\delta}\\&\lesssim\mathfrak{E}_0+\|(F^3+F^6)(0)\|^2_{(\mathcal{H}^1)^*}+\mathfrak{E}(\eta)\big(\|F^3\|^2_{L^2W^0_\delta}+\|F^6\|^2_{L^2W^{1/2}_\delta}\big)\\&\qquad+(1+\mathfrak{E}(\eta))(\|\partial_t(F^3+F^6)\|^2_{(\mathcal{H}^1_T)^*}).\end{aligned} \tag{5.26}$$

Moreover, $\partial_t\theta$ satisfies

$$\begin{cases} -k\Delta_{\mathcal{A}}\partial_t\theta = \partial_t F^3 + G^3 & in \quad \Omega, \\ k\nabla_{\mathcal{A}}\partial_t\theta\cdot\mathcal{N} + \partial_t\theta = \partial_t F^6 + G^6 & on \quad \Sigma, \\ \partial_t\theta = 0 & on \quad \Sigma_s, \end{cases} \tag{5.27}$$

in the weak sense of (5.22)*, where G^3 is defined by*

$$G^3 = k\,\mathrm{div}_{\partial_t\mathcal{A}}(-R\nabla_{\mathcal{A}}\theta + \nabla_{\partial_t\mathcal{A}}\theta), \tag{5.28}$$

and G^6 by

$$G^6 = -k\nabla_{\partial_t\mathcal{A}}\theta\cdot\mathcal{N} - k\nabla_{\mathcal{A}}\theta\cdot\partial_t\mathcal{N}. \tag{5.29}$$

More precisely, (5.27) *holds in the weak sense of*

$$\begin{aligned} \int_\Omega \nabla_{\mathcal{A}}\partial_t\theta\cdot\nabla_{\mathcal{A}}\phi J = \int_\Omega [\partial_t F^3 + F^3\partial_t JK]\phi J + \int_\Sigma (\partial_t F^6 - \partial_t^j\theta)\phi \\ - \int_\Omega [\nabla_{\partial_t\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi + \nabla_{\mathcal{A}}\theta\cdot\nabla_{\partial_t\mathcal{A}}\phi + \nabla_{\mathcal{A}}\theta\cdot\nabla_{\partial_t\mathcal{A}}\phi\partial_t JK]J. \end{aligned} \tag{5.30}$$

Proof. We only need to prove the results concerning the temperature θ. Then due to the proof of [19, Theorem 4.13], we may make some refinement to obtain our results for u here. Since the equations for θ are a Poisson system, we expect to use the elliptic analysis to obtain the existence and uniqueness of θ for a.e. $t\in[0,T]$. Nevertheless for solving the equations for u, we are necessarily to gain the control of one temporal regularity for θ. That is the reason why we still use the Galerkin method.

In order to utilize the Galerkin method (for instance, see [4]), we must first construct a countable basis of $H^2(\Omega)\cap\mathcal{H}^1(t)$ for each $t\in[0,T]$. For each $t\in[0,T]$, the space $H^2(\Omega)\cap\mathcal{H}^1(t)$ is separable, so the existence of a countable basis is not an issue.

Since $H^2(\Omega)\cap{}_0H^1(\Omega)$ is separable, it possess a countable basis $\{w^j\}_{j=1}^\infty$. Note that this basis is not time–dependent. Since $H^2(\Omega)\cap\mathcal{H}^1(t)$ is time-dependent, we define $\phi^j(t) = K(t)w^j$. Then it is easy to show that $\{\phi^j(t)\}_{j=1}^\infty$ is a countable basis of $H^2(\Omega)\cap\mathcal{H}^1(t)$ for each $t\in[0,T]$. Moreover, we could express $\partial_t\phi^j(t)$ in terms of $\phi^j(t)$ as

$$\partial_t\phi^j(t) = \partial_t K(t)w^j = \partial_t K(t)J(t)K(t)w^j = \partial_t K(t)J(t)\phi^j(t).$$

For any integer $m\geq 1$, we define the finite dimensional space

$$\mathcal{H}^1_m = \mathrm{span}\{\phi^1(t),\phi^2(t),\dots,\phi^m(t)\}\subseteq H^2(\Omega)\cap\mathcal{H}^1(t). \tag{5.31}$$

Then we define an approximate solution

$$\theta^m(t) := d_j^m(t)\phi^j(t), \text{ with } d_j^m:[0,T]\to\mathbb{R} \text{ for } j=1,\dots,m, \tag{5.32}$$

where as usual we use the Einstein convention of summation of the repeated index j. We want to choose the coefficients $d_j^m\in C^0([0,T])$, so that

$$(\nabla_{\mathcal{A}}\theta^m,\nabla_{\mathcal{A}}\phi)_{\mathcal{H}^0} = (F^3,\phi)_{\mathcal{H}^0} + \int_\Sigma F^6\phi \tag{5.33}$$

for each $\phi\in\mathcal{H}^1_m(t)$. Then we plug (5.32) into (5.33) to deduce the equation for d_j^m,

$$d_j^m\left[(\nabla_{\mathcal{A}}\phi^j,\nabla_{\mathcal{A}}\phi^k)_{\mathcal{H}^0} + (\phi^j,\phi^k)_{H^0(\Sigma)}\right] = (F^3,\phi^k)_{\mathcal{H}^0} + \int_\Sigma F^6\phi^k. \tag{5.34}$$

From the definition of $\mathcal{H}^1$, the matrix with entry $(\nabla_{\mathcal{A}}\phi^j,\nabla_{\mathcal{A}}\phi^k)_{\mathcal{H}^0}+(\phi^j,\phi^k)_{H^0(\Sigma)}$ is invertible. Since the coefficients of linear system (5.33) are in $C^0([0,T])$, and the forcing terms are in $C^0([0,T])$, we find that $d_j^m\in C^0([0,T])$. Then from the assumptions for the forcing terms, we could temporally differentiate (5.33) to find that $d_j^m\in C^{0,1}([0,T])$ actually.

By our construction, $\theta^m\in\mathcal{H}^1_m$. Then we take the test function $\phi=\theta^m$ in (5.33) to derive that

$$\|\theta^m\|^2_{\mathcal{H}^1}+\|\theta^m\|_{H^0(\Sigma)}=\int_\Omega F^3\theta^m J+\int_\Sigma F^6\phi. \tag{5.35}$$

Then we choose p,q,r such that $1<q<\frac{2}{1+\delta}$ with $0<\delta<1$ and $\frac{1}{p}+\frac{1}{q}=1$, $2<r<\frac{2}{\delta}$ and $r'=\frac{2r}{r-2}$. Then we employ Hölder inequality, Lemma 4.1, Proposition 2.1, Sobolev inequality and usual trace theory to deduce that

$$\begin{aligned}\|\theta^m\|^2_{\mathcal{H}^1}&\lesssim\|F^3\|_{W^0_\delta}\|d^{-\delta}\|_{L^r}\|\theta^m\|_{L^{r'}}+\|F^6\|_{L^q}\|\theta^m\|_{L^p}\\&\lesssim\|F^3\|_{W^0_\delta}\|\theta^m\|_1+\|F^6\|_{W^{1/2}_\delta}\|\theta^m\|_1.\end{aligned} \tag{5.36}$$

Then together with Cauchy inequality we have

$$\|\theta^m\|^2_1\lesssim\|F^3\|^2_{W^0_\delta}+\|F^6\|^2_{W^{1/2}_\delta}. \tag{5.37}$$

Then integrating from 0 to T, we have that

$$\|\theta^m\|^2_{L^2H^1}\lesssim\|F^3\|^2_{L^2W^0_\delta}+\|F^6\|^2_{L^2W^{1/2}_\delta}. \tag{5.38}$$

Suppose that $\phi=a_j^m\phi^j$ with $a_j^m\in C^{0,1}([0,T])$. It is easy to verify that $\partial_t\phi\in\mathcal{H}^1_m$. We take this ϕ in (5.33), then temporally differentiate (5.33), and then subtract (5.33) with the test function ϕ replaced by $\partial_t\phi$. This eliminates the terms for $\partial_t\phi$ and leaves us the equality

$$\begin{aligned}(\nabla_{\mathcal{A}}\partial_t\theta^m,\nabla_{\mathcal{A}}\phi)_{\mathcal{H}^0}=&\int_\Omega[\partial_tF^3+F^3\partial_tJK]\phi J+\int_\Sigma\partial_tF^6\phi-\int_\Omega\nabla_{\partial_t\mathcal{A}}\theta^m\cdot\nabla_{\mathcal{A}}\phi J\\&-\int_\Omega\nabla_{\mathcal{A}}\theta^m\cdot\nabla_{\partial_t\mathcal{A}}\phi J-\int_\Omega\nabla_{\mathcal{A}}\theta^m\cdot\nabla_{\mathcal{A}}\phi\partial_tJKJ.\end{aligned} \tag{5.39}$$

Then, we choose the test function $\phi=\partial_t\theta^m$ and utilize the Hölder inequality to find that

$$\begin{aligned}\|\partial_t\theta^m(t)\|^2_{\mathcal{H}^1}\lesssim&\|\partial_t\mathcal{A}\|_{L^\infty}\|\theta^m\|_1\|\partial_t\theta^m\|_1+\|\partial_tJ\|_{L^\infty}\|\theta^m\|_1\|\partial_t\theta^m\|_1\\&+\|\partial_t(F^3+F^6)\|_{(\mathcal{H}^1)^*}\|\partial_t\theta^m\|_1+\|\partial_tJ\|_1\|F^3\|_{W^0_\delta}\|\partial_t\theta^m\|_1\\\lesssim&\|\partial_t\eta\|_{W^{5/2}_\delta}\|\theta^m\|_1\|\partial_t\theta^m\|_1+\|\partial_t(F^3+F^6)\|_{(\mathcal{H}^1)^*}\|\partial_t\theta^m\|_1\\&+\|\partial_t\eta\|_{3/2}\|F^3\|_{W^0_\delta}\|\partial_t\theta^m\|_1.\end{aligned} \tag{5.40}$$

Then after an integration from 0 to T, Cauchy inequality, Lemma 5.1 and the smallness of γ_0 imply

$$\begin{aligned}\|\partial_t\theta^m\|^2_{L^2H^1}\lesssim&\|\partial_t\eta\|_{L^2W^{5/2}_\delta}\|\theta^m\|^2_{L^\infty H^1}+\|\partial_t\eta\|^2_{L^\infty H^{3/2}}+\|F^3\|^2_{L^2W^0_\delta}+\|\partial_t(F^3+F^6)\|^2_{L^2(\mathcal{H}^1)^*}\\\lesssim&\mathfrak{E}_0+(1+\mathfrak{K}(\eta))(\|F^3\|^2_{L^2W^0_\delta}+\|F^6\|^2_{L^2W^{1/2}_\delta})+(1+\mathfrak{K}(\eta))\|\partial_t(F^3+F^6)\|^2_{L^2(\mathcal{H}^1)^*}.\end{aligned} \tag{5.41}$$

From the energy estimates for θ^m and $\partial_t\theta^m$, we know that the sequences $\{\theta^m\}$ and $\{\partial_t\theta^m\}$ are uniformly bounded in ${L^2}_0H^1$. Then up to an extraction of a subsequence, we have that

$$\theta^m \rightharpoonup \theta \text{ weakly in } {L^2}_0H^1, \quad \partial_t\theta^m \rightharpoonup \partial_t\theta \text{ weakly in } {L^2}_0H^1.$$

By lower semicontinuity, the energy estimates imply that

$$\|\theta\|^2_{L^2H^1} + \|\partial_t\theta\|^2_{L^2H^1}$$

is bounded. Then we pass to the limit for (5.35) with almost $t\in[0,T]$,

$$k(\nabla_{\mathcal{A}}\theta,\nabla_{\mathcal{A}}\phi)_{\mathcal{H}^0} = \int_\Omega F^3\phi J + \int_\Sigma F^6\phi, \tag{5.42}$$

which means θ is the weak solution for the elliptic equation

$$-k\Delta_{\mathcal{A}}\theta = F^3 \text{ in } \Omega, \quad k\nabla_{\mathcal{A}}\theta\cdot\mathcal{N} + \theta = F^6 \text{ on } \Sigma, \quad \theta = 0 \text{ on } \Sigma_s. \tag{5.43}$$

The elliptic estimates arguments similar to those in Theorem 3.3 imply that the elliptic Equation (5.43) admits a unique strong solution with

$$\|\theta(t)\|^2_{W^2_\delta} \lesssim \|F^3(t)\|^2_{W^0_\delta} + \|F^6(t)\|^2_{W^{1/2}_\delta}. \tag{5.44}$$

After integration from 0 to T,

$$\|\theta\|^2_{L^2W^2_\delta} \lesssim \|F^3\|^2_{L^2W^0_\delta} + \|F^6\|^2_{L^2W^{1/2}_\delta}. \tag{5.45}$$

Then we pass to the limit for (5.39) with almost $t\in[0,T]$,

$$\begin{aligned}(\nabla_{\mathcal{A}}\partial_t\theta,\nabla_{\mathcal{A}}\phi)_{\mathcal{H}^0} + (\partial_t\theta,\phi)_{H^0(\Sigma)} &= \int_\Omega [\partial_tF^3 + F^3\partial_tJK]\phi J + \int_\Sigma \partial_tF^6\phi - \int_\Omega \nabla_{\partial_t\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi J \\ &\quad - \int_\Omega \nabla_{\mathcal{A}}\theta\cdot\nabla_{\partial_t\mathcal{A}}\phi J - \int_\Omega \nabla_{\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi\partial_tJKJ. \end{aligned}\tag{5.46}$$

Then an integration by parts reveals that

$$\begin{aligned} &-\int_\Omega \nabla_{\partial_t\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi J - \int_\Omega \nabla_{\mathcal{A}}\theta\cdot\nabla_{\partial_t\mathcal{A}}\phi J - \int_\Omega \nabla_{\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi\partial_tJKJ \\ &= -\int_\Omega \nabla_{\partial_t\mathcal{A}}\theta\cdot\nabla_{\mathcal{A}}\phi J - \int_\Omega \nabla_{\mathcal{A}}\theta\cdot R^\top\nabla_{\mathcal{A}}\phi J \\ &= -\int_\Omega (\nabla_{\partial_t\mathcal{A}}\theta + R\nabla_{\mathcal{A}}\theta)\cdot\nabla_{\mathcal{A}}\phi J \\ &= (\operatorname{div}_{\mathcal{A}}(\nabla_{\partial_t\mathcal{A}}\theta + R\nabla_{\mathcal{A}}\theta),\phi) - \langle\nabla_{\partial_t\mathcal{A}}\theta\cdot\mathcal{N} + \nabla_{\mathcal{A}}\theta\cdot\partial_t\mathcal{N},\phi\rangle_{-1/2}. \end{aligned}\tag{5.47}$$

This completes the proof for θ. □

In order to state our higher regularity results for the problem (5.21), we must be able to define the forcing terms and initial data for the problem that results from temporally differentiating (5.21) one time. First, we define some mappings. Given F^4, v, q, $\tilde{\xi}$, we define the vector fields $\mathfrak{G}^1$, $\mathfrak{G}^3$ in Ω, $\mathfrak{G}^5$, $\mathfrak{G}^6$ on Σ and $\mathfrak{G}^7$ on Σ_s by

$$\begin{aligned}\mathfrak{G}^3(\vartheta) &= \operatorname{div}_{\mathcal{A}}(-R\nabla_{\mathcal{A}}\vartheta + \nabla_{\partial_t\mathcal{A}}\vartheta), \\ \mathfrak{G}^6(\vartheta) &= -\nabla_{\mathcal{A}}\vartheta\cdot\partial_t\mathcal{N} - \nabla_{\partial_t\mathcal{A}}\vartheta\cdot\mathcal{N},\end{aligned}\tag{5.48}$$

These mappings allow us to define the forcing terms as follows. We write $F^{3,0}=F^3$ and $F^{6,0}=F^6$. Then we write

$$F^{3,1}:=\partial_t F^3+\mathfrak{G}^3(\theta),\quad F^{6,1}:=\partial_t F^6+\mathfrak{G}^6(\theta). \tag{5.49}$$

When F^i are sufficiently regular for the following to make sense, we define the vectors

$$F^{3,2}:=\mathfrak{G}^3(\partial_t\theta)+\partial_t F^{3,1},\quad F^{6,2}:=\mathfrak{G}^6(\partial_t\theta)+\partial_t F^{6,1}. \tag{5.50}$$

In the following theorem, we present the higher order regularity of Equation (5.21), which is a direct corollary of Theorem 5.1.

THEOREM 5.2. *Suppose that $\mathfrak{K}(\eta)\le\alpha$ is sufficiently small, and that $\partial_t^j F^i$, $i=3,6$, $j=1,2$, satisfy the assumptions in (5.24). Let $\theta_0\in W_\delta^2(\Omega)$ and $\partial_t\theta(0)\in H^1(\Omega)$ be determined in terms of η_0, $\partial_t\eta(0)$ and $\partial_t^2\eta(0)$ as in Section 5.1. Then there exists $T_0>0$ such that for $0<T\le T_0$, then there exists a unique strong solution θ to (5.21) on $[0,T_0]$ such that $\partial_t^j\theta$ satisfies*

$$\begin{cases} -k\Delta_{\mathcal{A}}\partial_t^j\theta=F^{3,j} & \text{in}\quad \Omega,\\ k\nabla_{\mathcal{A}}\partial_t^j\theta\cdot\mathcal{N}+\partial_t^j\theta=F^{6,j} & \text{on}\quad \Sigma,\\ \partial_t^j\theta=0 & \text{on}\quad \Sigma_s,\end{cases} \tag{5.51}$$

in the strong sense with initial data $\partial_t^j\theta(0)$ for $j=0,1$ and in the weak sense for $j=2$. Moreover, the solution satisfies the estimate

$$\begin{aligned}\mathfrak{K}(\theta)\lesssim \mathfrak{E}_0(1&+\|F^3(0)\|_{W_\delta^0}^2+\|F^6(0)\|_{W_\delta^{1/2}}^2+\|\partial_t(F^3+F^6)(0)\|_{(\mathcal{H}^1)^*}^2)\\ &+(1+\mathfrak{K}(\eta))\sum_{j=0}^1\left(\|\partial_t^j F^3\|_{L^2W_\delta^0}^2+\|\partial_t^j F^6\|_{L^2W_\delta^{1/2}}^2\right.\\ &+(1+\mathfrak{E}(\eta))\|\partial_t^2(F^3+F^6)\|_{(\mathcal{H}^1)_T^*}^2,\end{aligned} \tag{5.52}$$

where $\mathfrak{K}(\theta)=\|\theta\|_{L^\infty W_\delta^2}^2+\|\partial_t\theta\|_{L^\infty H^1}^2+\|\theta\|_{L^2W_\delta^2}^2+\sum_{j=0}^2\|\partial_t^j\theta\|_{L^2H^1}^2$.

6. The full nonlinear equation

We finally turn to our main result for solutions global and decaying in time. The local well posedness for (1.10) is proved in the similar way as in [19] based on linear theory in Theorem 5.2. So we omit the details. We directly sketch the proof of our main results.

Proof. **(Proof of Theorem 1.1.)** We set

$$\begin{aligned}T^*:=\sup\Big\{&T>0:\ \text{For each choice of the initial data } (\eta_0,\partial_t\eta(0),\partial_t^2\eta(0)) \text{ satisfying the}\\ &\text{compatibility condition (5.18) and (5.19) such that there exists a universal small}\\ &\text{parameter } \gamma_0>0, \text{ and the initial energy satisfies } \mathcal{E}(0)\le\gamma_0. \text{ There exists a unique}\\ &\text{solution on } [0,T] \text{ satisfying}\\ &\sup_{0\le t\le T}\left[\mathcal{E}(t)+e^{\lambda t}\left(\|u(t)\|_1^2+\|u(t)\cdot\tau\|_{L^2(\Sigma_s)}^2+[u(t)\cdot\mathcal{N}(t)]_\ell^2+\|p(t)\|_0^2+\|\theta(t)\|_1^2\right)\right]\\ &+\int_0^T\mathcal{D}(t)\,\mathrm{d}t\le\mathcal{E}(0)\Big\}.\end{aligned} \tag{6.1}$$

Then with the local existence theory, the set on the right side of (6.1) is nonempty. Then a standard continuity argument coupling Theorem 4.2 and Theorem 4.3 implies $T^* = +\infty$. This completes the proof. ∎

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Appendix. Forcing terms. The forcing terms F^i in (2.1) are in terms of the temporal differentiation for velocity, pressure, temperature and surface functions. In particular,

(1) When $(v,q,\vartheta,\xi) = (u,p,\theta,\eta)$,

$$F^1 = -u\cdot\nabla_{\mathcal{A}}u,\ F^2 = 0,\ F^3 = -u\cdot\nabla_{\mathcal{A}}\theta,\ F^4 = \mathcal{R}(\partial_1\zeta_0,\partial_1\eta),$$

$$F^5 = F^6 = F^7 = F^8 = 0,\ F^9 = \kappa\hat{\mathscr{W}}(\partial_t\eta).$$

(2) When $(v,q,\vartheta,\xi) = (\partial_t u,\partial_t p,\partial_t\theta,\partial_t\eta)$,

$$\begin{aligned}F^1 = &-\mathrm{div}_{\partial_t\mathcal{A}} S_{\mathcal{A}}(u,p) + \mu\,\mathrm{div}_{\mathcal{A}}\mathbb{D}_{\partial_t\mathcal{A}}u + g\alpha\theta\nabla_{\partial_t\mathcal{A}}\Phi_2 + g\alpha\theta\nabla_{\mathcal{A}}\partial_t\Phi_2\\ &-\partial_t u\cdot\nabla_{\mathcal{A}}u - u\cdot\nabla_{\partial_t\mathcal{A}}u - u\cdot\nabla_{\mathcal{A}}\partial_t u,\end{aligned}$$

$$F^2 = -\mathrm{div}_{\partial_t\mathcal{A}}u,$$

$$F^3 = \mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\mathcal{A}}\theta + \mathrm{div}_{\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\theta - \partial_t u\cdot\nabla_{\mathcal{A}}\theta - u\cdot\nabla_{\partial_t\mathcal{A}}\theta - u\cdot\nabla_{\mathcal{A}}\partial_t\theta,$$

$$F^4 = \partial_t[\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)],$$

$$F^5 = g\eta\partial_t\mathcal{N} - \sigma\partial_1\left(\frac{\partial_1\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}} - \mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\right)\partial_t\mathcal{N} + \mu\mathbb{D}_{\partial_t\mathcal{A}}u\mathcal{N} - S_{\mathcal{A}}(p,u)\partial_t\mathcal{N},$$

$$F^6 = -\nabla_{\partial_t\mathcal{A}}\theta\cdot\mathcal{N} - \nabla_{\mathcal{A}}\theta\cdot\partial_t\mathcal{N},\ F^7 = -\mu\mathbb{D}_{\partial_t\mathcal{A}}u\nu\cdot\tau,$$

$$F^8 = u\cdot\partial_t\mathcal{N},\ F^9 = \hat{\mathscr{W}}'(\partial_t\eta)\partial_t^2\eta.$$

(3) When $(v,q,\vartheta,\xi) = (\partial_t^2 u,\partial_t^2 p,\partial_t\theta,\partial_t^2\eta)$,

$$\begin{aligned}F^1 = &-2\,\mathrm{div}_{\partial_t\mathcal{A}} S_{\mathcal{A}}(\partial_t u,\partial_t p) + 2\mu\,\mathrm{div}_{\mathcal{A}}\mathbb{D}_{\partial_t\mathcal{A}}\partial_t u + 2g\alpha\partial_t\theta\nabla_{\partial_t\mathcal{A}}\Phi_2 + 2g\alpha\partial_t\theta\nabla_{\mathcal{A}}\partial_t\Phi_2\\ &-\mathrm{div}_{\partial_t^2\mathcal{A}} S_{\mathcal{A}}(u,p) + 2\mu\,\mathrm{div}_{\partial_t\mathcal{A}}\mathbb{D}_{\partial_t\mathcal{A}}u + \mu\,\mathrm{div}_{\mathcal{A}}\mathbb{D}_{\partial_t^2\mathcal{A}}u + g\alpha\theta\nabla_{\partial_t^2\mathcal{A}}\Phi_2\\ &+2g\alpha\theta\nabla_{\partial_t\mathcal{A}}\partial_t\Phi_2 + g\alpha\theta\nabla_{\mathcal{A}}\partial_t^2\Phi_2 - \partial_t^2 u\cdot\nabla_{\mathcal{A}}u - 2\partial_t u\cdot\nabla_{\partial_t\mathcal{A}}u - 2\partial_t u\cdot\nabla_{\mathcal{A}}\partial_t u\\ &-u\cdot\nabla_{\partial_t^2\mathcal{A}}u - 2u\cdot\nabla_{\partial_t\mathcal{A}}\partial_t u - u\cdot\nabla_{\mathcal{A}}\partial_t^2 u,\end{aligned}$$

$$F^2 = -\mathrm{div}_{\partial_t^2\mathcal{A}}u - 2\,\mathrm{div}_{\partial_t\mathcal{A}}\partial_t u,$$

$$F^3 = 2\,\mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\mathcal{A}}\partial_t\theta + 2\,\mathrm{div}_{\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\partial_t\theta + 2\,\mathrm{div}_{\partial_t\mathcal{A}}\nabla_{\partial_t\mathcal{A}}\theta + \mathrm{div}_{\partial_t^2\mathcal{A}}\nabla_{\mathcal{A}}\theta + \mathrm{div}_{\mathcal{A}}\nabla_{\partial_t^2\mathcal{A}}\theta,$$

$$F^4 = \partial_t^2[\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)],$$

$$\begin{aligned}F^5 = &2\mu\mathbb{D}_{\partial_t\mathcal{A}}\partial_t u\mathcal{N} + \mu\mathbb{D}_{\partial_t^2\mathcal{A}}u\mathcal{N} + \mu\mathbb{D}_{\partial_t\mathcal{A}}u\partial_t\mathcal{N}\\ &+2\left[g\partial_t\eta - \sigma\partial_1\left(\frac{\partial_1\partial_t\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}} + \partial_t[\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)]\right) - S_{\mathcal{A}}(\partial_t p,\partial_t u)\right]\partial_t\mathcal{N}\end{aligned}$$

$$+2\left[g\eta-\sigma\partial_1\left(\frac{\partial_\eta}{(1+|\partial_1\zeta_0|^2)^{3/2}}+\mathcal{R}(\partial_1\zeta_0,\partial_1\eta)\right)-S_{\mathcal{A}}(p,u)\right]\partial_t^2\mathcal{N},$$

$$F^6=-2\nabla_{\partial_t\mathcal{A}}\partial_t\theta\cdot\mathcal{N}-2\nabla_{\mathcal{A}}\partial_t\theta\cdot\partial_t\mathcal{N}-\nabla_{\partial_t^2\mathcal{A}}\theta\cdot\mathcal{N}-2\nabla_{\partial_t\mathcal{A}}\theta\cdot\partial_t\mathcal{N}-\nabla_{\mathcal{A}}\theta\partial_t^2\mathcal{N},$$

$$F^7=2\mu\mathbb{D}_{\partial_t\mathcal{A}}\partial_t u\nu\cdot\tau+\mu\mathbb{D}_{\partial_t^2\mathcal{A}}u\nu\cdot\tau,$$

$$F^8=2\partial_t u\cdot\partial_t\mathcal{N}+u\cdot\partial_t^2\partial_t\mathcal{N},\ F^9=\hat{\mathscr{W}}'(\partial_t\eta)\partial_t^3\eta+\hat{\mathscr{W}}''(\partial_t\eta)(\partial_t^2\eta)^2.$$

A key feature for all of the F^8 terms is that they vanish at the point $x_1=\pm\ell$, since both u_1 and $\partial_t u_1$ vanish at $x_1=\pm\ell$. We usually use this feature throughout the paper.

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