COMMUN. MATH. SCI.
Vol. 22, No. 3, pp. 777–788


# NON-UNIQUENESS OF TRANSONIC SHOCK SOLUTIONS TO EULER-POISSON SYSTEM WITH VARYING BACKGROUND CHARGES*

BEN DUAN†, YUANYUAN XING‡, AND HAORAN ZHENG§

**Abstract.** The Euler-Poisson equations with varying background charges in finitely long flat nozzles are investigated, for which two and only two transonic shock solutions are constructed. In [T. Luo and Z.P. Xin, Commun. Math. Sci., 10:419–462, 2012], Luo and Xin established the well-posedness of steady Euler-Poisson equations for the constant background charge. Motivated by their pioneering work and combined with the special physical character of semiconductor devices, we propose the transonic shock problem in which the density of the background charge is a piecewise constant function and its discontinuity is determined only by shock fronts. The existence and non-uniqueness of transonic shock solutions are obtained via the method of shock matching.

**Keywords.** Euler-Poisson equations; transonic shock; non-uniqueness; varying background charges.

**AMS subject classifications.** 35R35; 35L65; 35L67; 35Q81; 76H05.

## 1. Introduction

The charge transport governed by self-generated electric field in macroscopic scale can be described by the Euler-Poisson equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b, \end{cases} \tag{1.1}$$

where $u$, $\rho$, and $p$ represent the macroscopic particle velocity, density, and pressure, respectively. The electric field $E$ is generally generated by the Coulomb force of particles. And the function $b > 0$ denotes the density of positive background charge. The pressure $p$ satisfies

$$p(0) = 0,\ p'(\rho) > 0,\ p''(\rho) > 0,\ \text{for } \rho > 0,\ p(+\infty) = +\infty.$$

In gas dynamics, $c = \sqrt{p'(\rho)}$ represents the local sound speed. The flow is supersonic if $u > c$, and subsonic if $u < c$. Based on the special physical character of semiconductor devices, the density of the background charge is prescribed as

$$b = \begin{cases} b_1, & \text{if } u > c, \\ b_2, & \text{if } u < c, \end{cases} \tag{1.2}$$

where $b_1$ and $b_2$ are all positive constants.

The Euler-Poisson system under consideration consists of the isentropic Euler equations for the particle densities and momentum coupled with the Poisson equation through a source term modeling the electrostatic force. This system describes the

*Received: November 30, 2022; Accepted (in revised form): September 01, 2023. Communicated by Feimin Huang.

†School of Mathematics, Jilin University, Changchun, Jilin, China (bduan@jlu.edu.cn).

‡Corresponding author. School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, China (yxing_math@163.com).

§School of Mathematics, Jilin University, Changchun, Jilin, China (hrzheng22@mails.jlu.edu.cn).

dynamic behaviour of many important physical flows, including the propagation of electrons in submicron semiconductor devices (see [22]), the biological transport of ions for channel proteins (see [5]), the motion of plasmas and so on.

The well-posedness of steady Euler-Poisson equations is a hot topic in fluid dynamics. Many purely supersonic and subsonic solutions have been constructed for one-dimensional and multi-dimensional Euler-Poisson equations, one may refer to [2, 3, 10, 14, 24] and references cited therein. For transonic problems, based on a large number of experimental results, Courant and Friedrichs put forward a description involving shock fronts phenomenon in [9]: Given an appropriately large receiver pressure $p_e$ in a de Laval nozzle, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the widening part of the nozzle, a shock front intervenes, and the flow is compressed and slowed down to subsonic state. The position and the strength of the shock front are automatically adjusted so that the end pressure becomes $p_e$.

One can notice that if the right-hand side of the second equation in (1.1) is zero and the Poisson equation disappears, then the system (1.1) becomes Euler system. So far, there have been many significant studies on transonic shock problems for Euler equations, see [4, 6, 15, 18, 27–29]. Compared with Euler equations, the mathematical research of transonic shocks for Euler-Poisson equations is not extensive enough. In [1], Ascher, Markowich and Pietra considered a linear pressure function $p(\rho)=k\rho$ and a special boundary condition $\rho(0)=\rho(L)=\bar{\rho}$. They proved the existence of transonic shock solutions to Euler-Poisson equations with $0<b<\rho_s$, where $\bar{\rho}$ and $\rho_s$ are the densities on subsonic and sonic states respectively. In [25], Rosini presented a phase plane analysis for the steady Euler-Poisson system in terms of three different situations $0<b<\rho_s$, $b=\rho_s$ and $b>\rho_s$. Gamba constructed transonic shock solutions which may contain boundary layers via the vanishing viscosity method in [13], for which the solutions possibly admit more than one transonic shocks. For more general pressure functions and boundary conditions, Luo and Xin established a thorough well-posedness theory of isentropic Euler-Poisson equations in [19], where the existence and uniqueness of the transonic shock solution are proved for $0<b<\rho_s$, and the non-uniqueness is obtained for $b>\rho_s$. Their results revealed that the value of $b$ plays a key role on the well-posedness of transonic shock solutions. In [11], Duan and Zhang extended the non-uniqueness results in [19] to the non-isentropic Euler-Poisson equations with the same assumption $b>\rho_s$. In addition, for the Euler-Poisson system with relaxation terms in [7, 16, 17], it has been proved that the infinitely many transonic shock solutions exist for any $L^\infty$ background charge $b$ within the subsonic or supersonic regions but sufficiently close to the sonic line, or the transonic regions dominated by either the subsonic or supersonic regions. For other significant works, one may refer to [8, 12, 20, 21, 23, 26].

In this paper, we investigate the transonic shock problem in which the density of the background charge $b$ is a piecewise constant function and the discontinuity of $b$ is determined only by shock fronts. Since the shock is a free boundary connecting supersonic and subsonic regions, the function $b$ actually depends on transonic shock solutions. For the Euler-Poisson equations with varying background charges, the solution trajectories may have more plentiful phenomena than the ones of constant background charge. Our goal is to establish the existence and non-uniqueness of transonic shock solutions under a class of physical boundary conditions.

The steady Euler-Poisson equations are investigated in an interval $0\le x\le L$ of the form

$$\begin{cases} (\rho u)_x = 0, \\ (p(\rho)+\rho u^2)_x = \rho E, \\ E_x = \rho - b, \end{cases} \tag{1.3}$$

subject to the boundary conditions

$$(\rho,u,E)(0) = (\rho_0,u_0,E_0), \ \rho(L) = \rho_e. \tag{1.4}$$

The first equation in (1.3) shows that $\rho u(x) = J$ for some constant $J$. Thus the boundary value problem (1.3)-(1.4) can be reduced to

$$\begin{cases} \left(p(\rho)+\frac{J^2}{\rho}\right)_x = \rho E, \\ E_x = \rho - b, \\ (\rho,E)(0) = (\rho_0,E_0), \\ \rho(L) = \rho_e. \end{cases} \tag{1.5}$$

The equation $p'(\rho)\rho^2 = J^2$ admits a unique solution $\rho = \rho_s$, which is the density on sonic state. Then the flow is supersonic (respectively subsonic) if

$$p'(\rho) < J^2/\rho^2, i.e. \rho < \rho_s \text{ (respectively } p'(\rho) > J^2/\rho^2, i.e. \rho > \rho_s).$$

The goal of this work is to construct the non-uniqueness of transonic shock solutions to (1.5) in one dimensional flat nozzles, and to elucidate various analytical features especially including the monotonicity property between the shock location and the density at the exit of nozzles. We concentrate on the case where $0 < b < \rho_s$ and the electric field $E$ is negative in subsonic state.

The rest of this paper is organized as follows. In Section 2, the main theorem and the detailed behavior of transonic shock solutions to Euler-Poisson Equations (1.3) are elucidated via the method of shock matching. In Section 3, the existence result of transonic shocks is established by a monotonicity argument. Furthermore, we prove that there exist two and only two transonic shock solutions to the boundary value problem (1.5).

## 2. Existence of transonic shock solutions

For the boundary value problem (1.5), we assume that $\rho_0 < \rho_s$ and $\rho_e > \rho_s$. This implies that flows are supersonic at $x = 0$ and subsonic at $x = L$. The definition of transonic shock solutions to (1.5) is given as follows.

Definition 2.1. *The piecewise smooth function*

$$(\rho,E) = \begin{cases} (\rho_-,E_-)(x), \ 0 \le x < x^*, \\ (\rho_+,E_+)(x;x^*), \ x^* < x \le L, \end{cases} \tag{2.1}$$

*is a transonic shock solution to the boundary value problem* (1.5) *provided*

(i) $(\rho,E)$ *is separated by a shock discontinuity located at* $x^* \in (0,L)$;

(ii) $(\rho_-,E_-)$ *and* $(\rho_+,E_+)$ *satisfy* (1.5) *on the intervals* $(0,x^*)$ *and* $(x^*,L)$, *respectively;*

(iii) *the Rankine-Hugoniot conditions hold across the shock at* $x = x^*$,

$$\begin{cases} \left(p(\rho_-) + \frac{J^2}{\rho_-}\right)(x^*) = \left(p(\rho_+) + \frac{J^2}{\rho_+}\right)(x^*), \\ E_-(x^*) = E_+(x^*); \end{cases} \tag{2.2}$$

(iv) *the Lax's entropy condition holds at* $x = x^*$,

$$\rho_+(x^*) > \rho_s > \rho_-(x^*). \tag{2.3}$$

Solutions to (1.3) can be analyzed visually in $(\rho, E)$-phase plane via using the shock matching method. Any trajectory in $(\rho, E)$-plane satisfies

$$\mathrm{d}\left(\frac{1}{2}E^2 - H(\rho)\right) = 0, \quad \text{where} \quad H'(\rho) = \frac{\rho - b}{\rho}\left(p'(\rho) - \frac{J^2}{\rho^2}\right). \tag{2.4}$$

The trajectory passing through $(\rho_0, E_0)$ is obtained by

$$\frac{1}{2}E^2 - \int_{\rho_0}^{\rho} H'(\tau)\mathrm{d}\tau = \frac{1}{2}E_0^2. \tag{2.5}$$

The critical trajectory is defined as the trajectory passing through $(\rho_s, 0)$ of the form

$$\frac{1}{2}E^2 - \int_{\rho_s}^{\rho} H'(\tau)\mathrm{d}\tau = 0.$$

There are two branches of the critical trajectory, a supersonic branch $0 < b < \rho_s$, and a subsonic branch $b > \rho_s$. Hereafter, we focus on the case where $E_0 < 0$ and $(\rho_0, E_0)$ is in the supersonic critical trajectory for simplicity. When $(\rho_1, E_1)$ and $(\rho_2, E_2)$ are on the same trajectory, we define $l((\rho_1, E_1); (\rho_2, E_2))$ as the length of the variable $x$ for the trajectory of (1.3) which connects $(\rho_1, E_1)$ and $(\rho_2, E_2)$. If $E$ does not change sign on the trajectory connecting these two states, then

$$l((\rho_1, E_1); (\rho_2, E_2)) = \int_{\rho_1}^{\rho_2} \frac{h(\rho)}{\rho E(\rho)}\mathrm{d}\rho, \tag{2.6}$$

where $h(\rho) = p'(\rho) - \frac{J^2}{\rho^2}$.

The existence result of transonic shock solutions is given in the following theorem.

THEOREM 2.1. *Let* $J$, $L$, $0 < b_1 < b_2 < \rho_s$ *be the given positive constants and* $(\rho_0, E_0)$ *be the supersonic boundary data with* $u_0 = J/\rho_0$, *then there exists an interval* $I = (\underline{\rho}, \overline{\rho})$. *For any density at the exit of nozzles* $\rho_e \in I$, *the boundary value problem* (1.5) *has exactly two transonic shock solutions on* $(0, L)$.

The flow pattern of the solution to (1.5) is described in Figure 2.1. Given $\rho_0$, $E_0$ and $\rho_e$, there exist two kinds of flow patterns with shocks located at $x_1^*$ and $x_2^*$, respectively. From $(\rho_0, E_0)$ to $B$ or $C$, the flow is supersonic on the interval $(0, x_1^*)$ or $(0, x_2^*)$, with the density of the background charge being $b_1$. Across the shock, the trajectory jumps from $B$ to $B'$ or from $C$ to $C'$. Then, the phase plane changes to the one with respect to the background charge $b_2$. The flow becomes subsonic and travels to the state $\rho = \rho_e$.

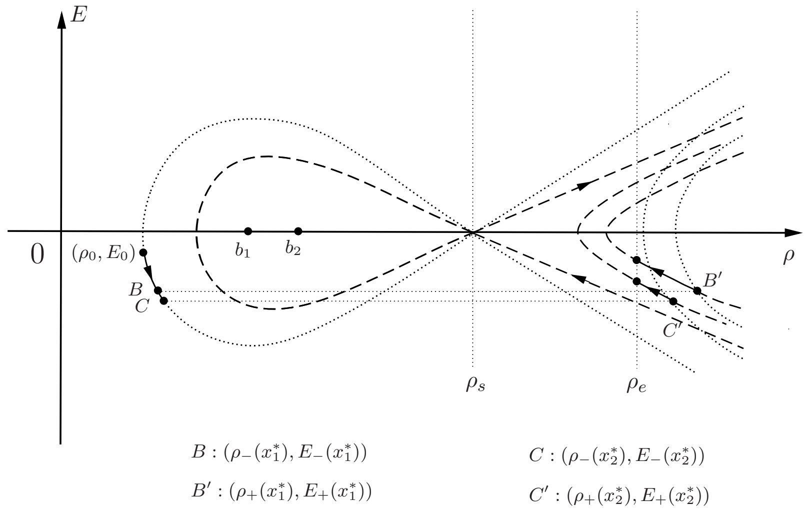


FIG. 2.1. *Flow pattern of transonic shock solutions.*

## 3. Proof of the main theorem

In this section, the monotonicity between the shock location and the density at the exit of the nozzle is established. Furthermore, we prove the existence and non-uniqueness of transonic shock solutions to the boundary value problem (1.5). Lemma 3.1 and Lemma 3.2 are the key ingredients to our proof.

LEMMA 3.1. *For the given constants $J$, $L$, $b_1$, $b_2$ and any supersonic boundary data $(\rho_0,E_0)$, suppose that*

$$L=x^*+l((\rho_+(x^*),E_+(x^*));(\rho_+(L;x^*),E_+(L;x^*))), \tag{3.1}$$

*where $(\rho_+(L;x^*),E_+(L;x^*))$ is the state at the exit of the nozzle. If $E$ does not change sign along the trajectory from $(\rho_+(x^*),E_+(x^*))$ to $(\rho_+(L;x^*),E_+(L;x^*))$, then*

$$\begin{aligned}\frac{\mathrm{d}\rho_+(L;x^*)}{\mathrm{d}x^*}=&\frac{\rho_+(L;x^*)E_+(L;x^*)}{h(\rho_+(L;x^*))}\Bigg(\frac{\rho_-(x^*)}{\rho_+(x^*)}-1\\&+\left(\frac{\rho_-(x^*)}{\rho_+(x^*)}\cdot b_2-b_1\right)E_-(x^*)\int_{\rho_+(x^*)}^{\rho_+(L;x^*)}\frac{h(\tau)}{\tau E_+^3(\tau,x^*)}\mathrm{d}\tau\Bigg),\end{aligned} \tag{3.2}$$

*provided $E_+(\tau,x^*)$ satisfies*

$$\frac{1}{2}E_+^2(\tau,x^*)-\int_{\rho_+(x^*)}^{\tau}\frac{(t-b_2)h(t)}{t}\mathrm{d}t=\frac{1}{2}E_+^2(x^*) \tag{3.3}$$

*for $\tau\in[\rho_+(x^*),\rho_+(L;x^*)]$.*

*Proof.* It follows from (2.6) that

$$L=x^*+\int_{\rho_+(x^*)}^{\rho_+(L;x^*)}\frac{h(\tau)}{\tau E_+(\tau,x^*)}\mathrm{d}\tau, \tag{3.4}$$

where

$$x^* = l((\rho_0, E_0), (\rho_-(x^*), E_-(x^*))) = \int_{\rho_0}^{\rho_-(x^*)} \frac{h(\tau)}{\tau E_-(\tau)} \mathrm{d}\tau. \tag{3.5}$$

A direct computation implies

$$\frac{\mathrm{d}\rho_-(x^*)}{\mathrm{d}x^*} = \frac{\rho_-(x^*) E_-(x^*)}{h(\rho_-(x^*))}. \tag{3.6}$$

Differentiating (3.4) with respect to $x^*$ yields

$$\begin{aligned} 0 = 1 &+ \frac{h(\rho_+(L;x^*))}{\rho_+(L;x^*) E_+(L;x^*)} \cdot \frac{\mathrm{d}\rho_+(L;x^*)}{\mathrm{d}x^*} - \frac{h(\rho_+(x^*))}{\rho_+(x^*) E_+(x^*)} \cdot \frac{\mathrm{d}\rho_+(x^*)}{\mathrm{d}x^*} \\ &- \int_{\rho_+(x^*)}^{\rho_+(L;x^*)} \frac{h(\tau)}{\tau E_+^2(\tau, x^*)} \cdot \frac{\partial E_+(\tau, x^*)}{\partial x^*} \mathrm{d}\tau. \end{aligned} \tag{3.7}$$

Using the first equation in the Rankine-Hugoniot conditions (2.2), one can as well deduce

$$\left(p'(\rho_-(x^*)) - \frac{J^2}{\rho_-(x^*)^2}\right) \frac{\mathrm{d}\rho_-(x^*)}{\mathrm{d}x^*} = \left(p(\rho_+(x^*)) - \frac{J^2}{\rho_+(x^*)^2}\right) \frac{\mathrm{d}\rho_+(x^*)}{\mathrm{d}x^*}. \tag{3.8}$$

Substituting (3.6) into (3.8) yields

$$\frac{\mathrm{d}\rho_+(x^*)}{\mathrm{d}x^*} = \frac{h(\rho_-(x^*))}{h(\rho_+(x^*))} \cdot \frac{\mathrm{d}\rho_-(x^*)}{\mathrm{d}x^*} = \frac{\rho_-(x^*) E_-(x^*)}{h(\rho_+(x^*))}. \tag{3.9}$$

Employing the definition of $E_+(\tau, x^*)$ in (3.3), it holds that

$$E_+(\tau, x^*) \frac{\partial E_+(\tau, x^*)}{\partial x^*} = E_+(x^*) \frac{\mathrm{d}E_+(x^*)}{\mathrm{d}x^*} - \frac{(\rho_+(x^*) - b_2) h(\rho_+(x^*))}{\rho_+(x^*)} \cdot \frac{\mathrm{d}\rho_+(x^*)}{\mathrm{d}x^*}. \tag{3.10}$$

Similarly, returning to the Equation (2.5) we have

$$\frac{1}{2} E_-^2(x^*) - \int_{\rho_0}^{\rho_-(x^*)} \frac{(t - b_1) h(t)}{t} \mathrm{d}t = \frac{1}{2} E_0^2 \tag{3.11}$$

and so deduce

$$E_-(x^*) \frac{\mathrm{d}E_-(x^*)}{\mathrm{d}x^*} = \frac{(\rho_-(x^*) - b_1) h(\rho_-(x^*))}{\rho_-(x^*)} \cdot \frac{\mathrm{d}\rho_-(x^*)}{\mathrm{d}x^*}, \tag{3.12}$$

provided $E_-(x^*) = E_+(x^*)$. Hence

$$\begin{aligned} E_+(\tau, x^*) \frac{\partial E_+(\tau, x^*)}{\partial x^*} &= \frac{(\rho_-(x^*) - b_1) h(\rho_-(x^*))}{\rho_-(x^*)} \cdot \frac{\mathrm{d}\rho_-(x^*)}{\mathrm{d}x^*} \\ &\quad - \frac{(\rho_+(x^*) - b_2) h(\rho_+(x^*))}{\rho_+(x^*)} \cdot \frac{\mathrm{d}\rho_+(x^*)}{\mathrm{d}x^*} \\ &= \left(\frac{\rho_-(x^*)}{\rho_+(x^*)} \cdot b_2 - b_1\right) E_-(x^*). \end{aligned} \tag{3.13}$$

This, together with (3.7) and (3.9), yields (3.2). The proof is completed. □

In the following, we define

$$\mathcal{B}=\{\bar{x}:\rho_-(\bar{x})<\frac{b_1}{b_2}\rho_s\},$$

where $\bar{x}$ is a shock location determined later. Then the following lemma is obtained.

LEMMA 3.2. *Let $J$, $L$, $0<b_1<b_2<\rho_s$ be the given positive constants, then there exists a non-empty parameter set $\mathcal{B}$. For any supersonic boundary data $(\rho_0,E_0)$, if $E<0$ along the trajectory from $(\rho_+(x^*),E_+(x^*))$ to $(\rho_+(L;x^*),E_+(L;x^*))$, then there exists a unique shock location $\bar{x}\in\mathcal{B}$ and an $\check{x}$ satisfying $E_+(L,\check{x})=0$ such that*

$$\lim_{x^*\to\check{x}}\frac{\partial\rho_+(L;x^*)}{\partial x^*}=-\infty,\quad \begin{cases}\frac{\partial\rho_+(L;x^*)}{\partial x^*}<0, & \text{for } \check{x}<x^*<\bar{x},\\ \frac{\partial\rho_+(L;x^*)}{\partial x^*}>0, & \text{for } \bar{x}<x^*<L\end{cases} \tag{3.14}$$

*and thus*

$$\rho_+(L;\bar{x})=\min_{\check{x}<x^*<L}\rho_+(L;x^*). \tag{3.15}$$

*Proof.* Since $E<0$ and $E$ does not change sign along the trajectory from $(\rho_+(x^*),E_+(x^*))$ to $(\rho_+(L;x^*),E_+(L;x^*))$, it follows from Lemma 3.1 that

$$\frac{\partial\rho_+(L;x^*)}{\partial x^*}=\frac{\rho_+(L;x^*)E_+(L;x^*)}{h(\rho_+(L;x^*))}Q(L;x^*), \tag{3.16}$$

provided

$$Q(L;x^*)=\frac{\rho_-(x^*)}{\rho_+(x^*)}-1+\left(\frac{\rho_-(x^*)}{\rho_+(x^*)}\cdot b_2-b_1\right)E_-(x^*)\int_{\rho_+(x^*)}^{\rho_+(L;x^*)}\frac{h(\tau)}{\tau E_+^3(\tau,x^*)}\mathrm{d}\tau. \tag{3.17}$$

In order to determine the sign of (3.16), we first analyze the term $Q(L;x^*)$. According to the subsonic trajectory, there exists a unique shock location $\check{x}$ such that

$$\frac{1}{2}E_+^2(\check{x})-H(\rho_+(\check{x}))=-H(\rho_+(L;\check{x})), \tag{3.18}$$

that is

$$\lim_{x^*\to\check{x}}E_+(L,x^*)=0. \tag{3.19}$$

In addition

$$\int_{\rho_+(\check{x})}^{\rho_+(L;\check{x})}\frac{h(\tau)}{\tau E_+^3(\tau,\check{x})}\mathrm{d}\tau=+\infty. \tag{3.20}$$

To verify this, note that $E_+(\tau,\check{x})$ satisfies

$$\frac{1}{2}E_+^2(\tau,\check{x})-H(\tau)=\frac{1}{2}E_+^2(\check{x})-H(\rho_+(\check{x})) \tag{3.21}$$

and $-\infty<E_+(\tau,\check{x})<0$. It is derived that

$$\frac{\partial E_+^2(\tau,\check{x})}{\partial\tau}=2H'(\tau)=\frac{2(\tau-b_2)}{\tau}\left(p'(\tau)-\frac{J^2}{\tau^2}\right) \tag{3.22}$$

for $\rho_+(L;\check{x}) \le \tau \le \rho(\check{x})$ and $\tau > b_2$. Therefore, there exist positive constants $C_1$ and $C_2$ such that

$$C_1 \le \frac{\partial E_+^2(\tau,\check{x})}{\partial \tau} \le C_2. \tag{3.23}$$

Since $E_+^2(\tau,\check{x}) = 0$ at $\tau = \rho_+(L;\check{x})$, then

$$E_+^2(\tau,\check{x}) = O(|\tau - \rho_+(L;\check{x})|) \tag{3.24}$$

for small $|\tau - \rho_+(L;\check{x})|$, which confirms (3.20) and thus

$$Q(L;\check{x}) = +\infty. \tag{3.25}$$

Besides, plugging $x^* = L$ into the Equation (3.16) implies that

$$Q(L;L) = \frac{\rho_-(L;L)}{\rho_+(L;L)} - 1 < 0. \tag{3.26}$$

In view of the fact that

$$\operatorname{sgn}\left(\frac{\partial \rho_+(L;x^*)}{\partial x^*}\right) = -\operatorname{sgn}(Q(L;x^*)) \tag{3.27}$$

for $x^* \in [\check{x}, L]$, $\frac{\partial \rho_+(L;x^*)}{\partial x^*}$ changes the sign in this interval. Therefore, there exists some $\bar{x} \in \mathcal{B}$ between $\check{x}$ and $L$ such that

$$\frac{\partial \rho_+(L;\bar{x})}{\partial \bar{x}} = Q(L;\bar{x}) = 0. \tag{3.28}$$

Differentiating $Q(L;\bar{x})$ with respect to $\bar{x}$ gives

$$\begin{aligned}
\frac{\partial Q(L;\bar{x})}{\partial \bar{x}} &= \frac{\rho_-(\bar{x})E_-(\bar{x})}{\rho_+(\bar{x})h(\rho_-(\bar{x}))} - \frac{\rho_-^2(\bar{x})E_-(\bar{x})}{\rho_+^2(\bar{x})h(\rho_+(\bar{x}))} \\
&\quad + \left(\frac{\rho_-(\bar{x})E_-(\bar{x})}{\rho_+(\bar{x})h(\rho_-(\bar{x}))} - \frac{\rho_-^2(\bar{x})E_-(\bar{x})}{\rho_+^2(\bar{x})h(\rho_+(\bar{x}))}\right) \cdot b_2 E_-(\bar{x}) \int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})} \frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})} \mathrm{d}\tau \\
&\quad - \left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})} \cdot b_2 - b_1\right) \cdot \frac{\rho_-(\bar{x})}{\rho_+(\bar{x})E_-(\bar{x})} \\
&\quad + \left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})} \cdot b_2 - b_1\right) \cdot (\rho_-(\bar{x}) - b_1) \int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})} \frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})} \mathrm{d}\tau \\
&\quad - \left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})} \cdot b_2 - b_1\right)^2 \cdot E_-^2(\bar{x}) \int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})} \frac{3h(\tau)}{\tau E_+^5(\tau,\bar{x})} \mathrm{d}\tau.
\end{aligned} \tag{3.29}$$

Owing to (3.16) and (3.28), we have

$$Q(L;\bar{x}) = \frac{\rho_-(\bar{x})}{\rho_+(\bar{x})} - 1 + \left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})} \cdot b_2 - b_1\right) E_-(\bar{x}) \int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})} \frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})} \mathrm{d}\tau = 0. \tag{3.30}$$

Substituting (3.30) into (3.29) yields

$$\frac{\partial Q(L;\bar{x})}{\partial \bar{x}} = \frac{\rho_+(\bar{x})(b_2 - b_1)}{\rho_-(\bar{x})b_2 - \rho_+(\bar{x})b_1} \cdot \left(\frac{\rho_-(\bar{x})E_-(\bar{x})}{\rho_+(\bar{x})h(\rho_-(\bar{x}))} - \frac{\rho_-^2(\bar{x})E_-(\bar{x})}{\rho_+^2(\bar{x})h(\rho_+(\bar{x}))}\right)$$

$$
\begin{aligned}
&-\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)\cdot\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})E_-(\bar{x})}\\
&+\frac{(\rho_-(\bar{x})-b_1)(\rho_+(\bar{x})-\rho_-(\bar{x}))}{\rho_+(\bar{x})E_-(\bar{x})}\\
&-\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)^2\cdot E_-^2(\bar{x})\int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})}\frac{3h(\tau)}{\tau E_+^5(\tau,\bar{x})}\mathrm{d}\tau.
\end{aligned}\tag{3.31}
$$

Based on (3.30), a direct computation gives

$$
\begin{aligned}
&\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}b_2-b_1\right)^2\cdot E_-^2(\bar{x})\int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})}\frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})}\cdot\frac{1}{E_-^2(\bar{x})}\mathrm{d}\tau\\
=&\left(1-\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\right)\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}b_2-b_1\right)\frac{1}{E_-(\bar{x})}.
\end{aligned}\tag{3.32}
$$

Then we rewrite (3.31) as

$$
\frac{\partial Q(L;\bar{x})}{\partial\bar{x}}=\sum_{i=1}^{5}I_i,\tag{3.33}
$$

provided

$$
\begin{aligned}
I_1&=\frac{\rho_+(\bar{x})(b_2-b_1)}{\rho_-(\bar{x})b_2-\rho_+(\bar{x})b_1}\cdot\left(\frac{\rho_-(\bar{x})E_-(\bar{x})}{\rho_+(\bar{x})h(\rho_-(\bar{x}))}-\frac{\rho_-^2(\bar{x})E_-(\bar{x})}{\rho_+^2(\bar{x})h(\rho_+(\bar{x}))}\right),\\
I_2&=-\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)\cdot\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})E_-(\bar{x})}+\frac{(\rho_-(\bar{x})-b_1)(\rho_+(\bar{x})-\rho_-(\bar{x}))}{\rho_+(\bar{x})E_-(\bar{x})},\\
I_3&=-\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)^2\cdot E_-^2(\bar{x})\int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})}\frac{3h(\tau)}{\tau E_+^5(\tau,\bar{x})}\mathrm{d}\tau,\\
I_4&=\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}b_2-b_1\right)^2\cdot E_-^2(\bar{x})\int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})}\frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})}\frac{1}{E_-^2(\bar{x})}\mathrm{d}\tau,\\
I_5&=-\left(1-\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\right)\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}b_2-b_1\right)\frac{1}{E_-(\bar{x})}.
\end{aligned}
$$

Combining $I_3$ with $I_4$ together yields

$$
I_3+I_4=\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}b_2-b_1\right)^2E_-^2(\bar{x})\int_{\rho_+(\bar{x})}^{\rho_+(L;\bar{x})}\frac{h(\tau)}{\tau E_+^3(\tau,\bar{x})}\Big(\frac{1}{E_-^2(\bar{x})}-\frac{3}{E_+^2(\tau,\bar{x})}\Big)\mathrm{d}\tau<0,\tag{3.34}
$$

where the inequalities $E_-^2(\bar{x})>E_+^2(\tau,\bar{x})$ and $\rho_+(\bar{x})>\rho_+(L)$ have been used. Therefore,

$$
\frac{\partial Q(L;\bar{x})}{\partial\bar{x}}<I_1+I_2+I_5.\tag{3.35}
$$

(3.35) can be transformed into

$$
\frac{\partial Q(L;\bar{x})}{\partial\bar{x}}<-\frac{1}{\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)E_-(\bar{x})}(J_1+J_2),\tag{3.36}
$$

provided

$$J_1=(b_2-b_1)\left(\frac{\rho_-^2(\bar{x})E_-^2(\bar{x})}{\rho_+^2(\bar{x})h(\rho_+(\bar{x}))}-\frac{\rho_-(\bar{x})E_-^2(\bar{x})}{\rho_+(\bar{x})h(\rho_-(\bar{x}))}\right), \tag{3.37}$$

$$J_2=-\frac{b_2-b_1}{b_2}\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)(\rho_-(\bar{x})-b_1)+\frac{\rho_-(\bar{x})-b_1+b_2}{b_2}\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)^2. \tag{3.38}$$

Indeed, $J_1>0$, owing to $h(\rho_-(\bar{x}))<0$ and $h(\rho_+(\bar{x}))>0$. On the other hand, it can be deduced from $\bar{x}\in\mathcal{B}$ that

$$\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1<0. \tag{3.39}$$

Consequently, if $\rho_-(\bar{x})-b_1\geq 0$, then

$$\begin{aligned}
J_2=&\frac{\rho_-(\bar{x})-b_1+b_2}{b_2}\left(\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)-\frac{(b_2-b_1)(\rho_-(\bar{x})-b_1)}{2(\rho_-(\bar{x})-b_1+b_2)}\right)^2\\
&-\frac{(b_2-b_1)^2(\rho_-(\bar{x})-b_1)^2}{4(\rho_-(\bar{x})-b_1+b_2)b_2}\\
\geq&\frac{\rho_-(\bar{x})-b_1+b_2}{b_2}\cdot\frac{(b_2-b_1)^2(\rho_-(\bar{x})-b_1)^2}{4(\rho_-(\bar{x})-b_1+b_2)^2}-\frac{(b_2-b_1)^2(\rho_-(\bar{x})-b_1)^2}{4(\rho_-(\bar{x})-b_1+b_2)b_2}\\
\geq&0.
\end{aligned}$$

Otherwise, if $\rho_-(\bar{x})-b_1<0$, then

$$\begin{aligned}
J_2=&\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)\left(\frac{\rho_-(\bar{x})-b_1+b_2}{b_2}\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)-\frac{b_2-b_1}{b_2}(\rho_-(\bar{x})-b_1)\right)\\
=&\frac{\rho_-(\bar{x})-b_1+b_2}{b_2}\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)\left(\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)-\frac{(b_2-b_1)(\rho_-(\bar{x})-b_1)}{\rho_-(\bar{x})+b_2-b_1}\right)\\
=&\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\left(\frac{\rho_-(\bar{x})}{\rho_+(\bar{x})}\cdot b_2-b_1\right)(b_2-b_1-\rho_+(\bar{x})+\rho_-(\bar{x}))\\
\geq&0.
\end{aligned}$$

Here the inequality $0<b_1<b_2<\rho_s<\rho_+(\bar{x})$ has been employed. Then

$$\frac{\partial Q(L;\bar{x})}{\partial\bar{x}}<0 \tag{3.40}$$

is valid. Therefore, $Q(L;x^*)$ only changes the sign once at $\bar{x}\in(\check{x},L)$. We conclude from (3.25), (3.26) and (3.40) that

$$\lim_{x^*\to\check{x}}Q(x^*)=+\infty,\quad \begin{cases}Q(L;x^*)>0, & \text{for}\quad \check{x}<x^*<\bar{x},\\ Q(L;\bar{x})=0,\\ Q(L;x^*)<0, & \text{for}\quad \bar{x}<x^*<L.\end{cases} \tag{3.41}$$

This confirms (3.14) and (3.15). ∎

The proof of Theorem 2.1: In Lemma 3.2, the monotonicity of $\rho_+(L;x^*)$ is obtained. Set

$$\underline{\rho}:=\rho(L;\bar{x}),\quad \overline{\rho}:=\min\{\rho(L;L),\rho(L;\check{x})\}\quad\text{and}\quad I=(\underline{\rho},\overline{\rho}]. \tag{3.42}$$

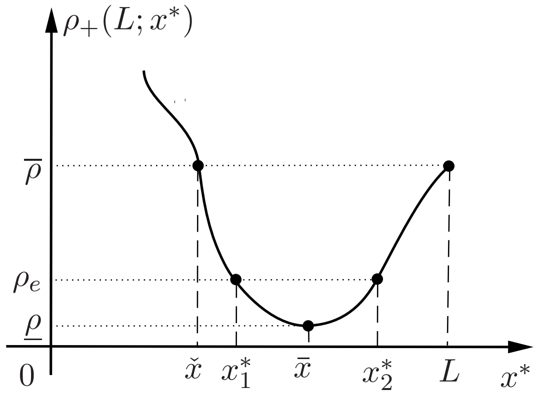


FIG. 3.1. *Monotonicity.*

As shown in Figure 3.1, for any $\rho_e \in I$, there exist exactly two shock locations $x_1^*$ and $x_2^*$ such that $\rho(L;x_1^*) = \rho(L;x_2^*) = \rho_e$. Therefore, the boundary value problem (1.5) admits precisely two transonic shock solutions which satisfy the Rankine-Hugoniot conditions (2.2) and the Lax's entropy condition (2.3). This completes the proof of Theorem 2.1.

**Acknowledgement.** The research of Ben Duan is supported by NSFC No. 12271205 and No. 12171498. The research of Haoran Zheng is partly supported by the National Key Research and Development Program of China 2020YFA0713602, NSFC No. 12171199 and No. 11971198, Jilin Provincial Department of science and technology 20210201015GX.

REFERENCES


[1] U.M. Ascher, P.A. Markowich, P. Pietra, and C. Schmeiser, *A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model*, Math. Model. Meth. Appl. Sci., 1:347–376, 1991. 1

[2] M. Bae, B. Duan, J.J. Xiao, and C.J. Xie, *Structural stability of supersonic solutions to the Euler-Poisson system*, Arch. Ration. Mech. Anal., 239:679–731, 2021. 1

[3] M. Bae, B. Duan, and C.J. Xie, *Subsonic flow for the multidimensional Euler-Poisson system*, Arch. Ration. Mech. Anal., 220:155–191, 2016. 1

[4] M. Bae, G.Q. Chen, and M. Feldman, *Regularity of solutions to regular shock reflection for potential flow*, Invent. Math., 175:505–543, 2009. 1

[5] D.P. Chen, R.S. Eisenberg, J.W. Jerome, and C.W. Shu, *A hydrodynamic model of temperature change in open ionic channels*, Biophys. J., 69:2304–2322, 1995. 1

[6] S.X. Chen, *Stability of transonic shock fronts in two–dimensional Euler systems*, Trans. Am. Math. Soc., 357:287–308, 2005. 1

[7] L. Chen, M. Mei, G.J. Zhang, and K.J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary and transonic doping profile*, J. Differ. Equ., 269:8173–8211, 2020. 1

[8] L. Chen, M. Mei, and G.J. Zhang, *Radially symmetric spiral flows of the compressible Euler–Poisson system for semiconductors*, J. Differ. Equ., 373:359–388, 2023. 1

[9] R. Courant and K.O. Friendrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, New York, 1948. 1

[10] P. Degond and P.A. Markowich, *A steady state potential flow model for semiconductors*, Ann. Mat. Pura Appl., 165:87–98, 1993. 1

[11] B. Duan and N. Zhang, *Non–uniqueness of transonic shock solutions to non–isentropic Euler–Poisson system*, Commun. Math. Sci., 19:903–917, 2021. 1

[12] B. Duan, Z. Luo, and J.J. Xiao, *Transonic shock solutions to the Euler–Poisson system in quasi*

*one–dimensional nozzles*, Commun. Math. Sci., 14:1023–1047, 2016. 1
[13] I.M. Gamba, *Stationary transonic solutions of a one–dimensional hydrodynamic model for semiconductors*, Commun. Partial Differ. Equ., 17:553–577, 1992. 1
[14] Y. Guo and W. Strauss, *Stability of semiconductor states with insulating and contact boundary conditions*, Arch. Ration. Mech. Anal., 179:1–30, 2006. 1
[15] J. Li, Z.P. Xin, and H.C. Yin, *On transonic shocks in a nozzle with variable end pressures*, Commun. Math. Phys., 291:111–150, 2009. 1
[16] J. Li, M. Mei, G.J. Zhang, and K.J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary: (I) Subsonic doping profile*, SIAM J. Math. Anal., 49:4767–4811, 2017. 1
[17] J. Li, M. Mei, G.J. Zhang, and K.J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary: (II) Supersonic doping profile*, SIAM J. Math. Anal., 50:718–734, 2018. 1
[18] T.P. Liu, *Nonlinear stability and instability of transonic flows through a nozzle*, Commun. Math. Phys., 83:243–260, 1982. 1
[19] T. Luo and Z.P. Xin, *Transonic shock solutions for a system of Euler–Poisson equations*, Commun. Math. Sci., 10:419–462, 2012. 1
[20] T. Luo and J. Smoller, *Existence and non–linear stability of rotating star solutions of the compressible Euler–Poisson equations*, Arch. Ration. Mech. Anal., 191:447–496, 2009. 1
[21] T. Luo, J. Ranch, C.J. Xie, and Z.P. Xin, *Stability of transonic shock solutions for one–dimensional Euler–Poisson equations*, Arch. Ration. Mech. Anal., 202:787–827, 2011. 1
[22] P.A. Markowich, C.A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer Verlag, Vienna, 1990. 1
[23] L.S. Mai and K.J. Zhang, *Asymptotic stability of steady state solutions for the relativistic Euler–Poisson equations*, Discrete Contin. Dyn. Syst., 36:981–1004, 2016. 1
[24] Y.J. Peng and I. Violet, *Example of supersonic solutions to a steady state Euler–Poisson system*, Appl. Math. Lett., 19:1335–1340, 2006. 1
[25] M.D. Rosini, *A phase analysis of transonic solutions for the hydrodynamic semiconductor model*, Quart. Appl. Math., 63:251–268, 2005. 1
[26] H. Sun, M. Mei, and K.J. Zhang, *Sub–exponential convergence to steady–states for 1–D Euler–Poisson equations with time–dependent damping*, Commun. Math. Sci., 20:447–478, 2022. 1
[27] Z.P. Xin and H.C. Yin, *Transonic shock in a nozzle. I. Two–dimensional case*, Commun. Pure Appl. Math., 58:999–1050, 2005. 1
[28] Z.P. Xin, W. Yan, and H.C. Yin, *Transonic shock problem for the Euler system in a nozzle*, Arch. Ration. Mech. Anal., 194:1–47, 2009. 1
[29] Z.P. Xin and H.C. Yin, *The transonic shock in a nozzle, 2–D and 3–D complete Euler systems*, J. Differ. Equ., 245:1014–1085, 2008. 1