COMMUN. MATH. SCI.
Vol. 22, No. 3, pp. 789–816

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF THE FULL EULER SYSTEM WITH DAMPING AND RADIATIVE EFFECTS IN $\mathbb{R}^3$*

SHIJIN DENG†, WENJUN WANG‡, FENG XIE§, AND XIONGFENG YANG¶

Abstract. In this paper, we study the global existence and the large-time behavior of solutions to the Cauchy problem of the full Euler system with damping and radiative effects around some constant equilibrium states. It is well-known that the solutions may blow up in finite time without the additional damping and radiative effects, and the global existence of the solutions obtained in this paper shows that these two effects together prevent the formation of the singularity when the initial perturbation is small. Combining the Green's function method and energy estimates, we consider the pointwise structures of the solutions to obtain a precise description of the system. The construction of the Green's function includes three steps: singularity removal, long wave-short wave decomposition and weighted energy estimate. Finally, we achieve the pointwise estimates of the solutions in the small perturbation framework by Duhamel's principle, the pointwise structure of the Green's function established for the linearized equations and bounded estimates for higher order derivatives of the solutions together.

Keywords. Full Euler system with damping and radiative effects; Global existence; Pointwise estimates; Green's function.

AMS subject classifications. 35L60; 35M31; 35Q35; 76N10.

1. Introduction

In this paper, we are concerned with the Cauchy problem for the following full Euler system with damping and radiative effects in $\mathbb{R}^3$:

$$\begin{cases} \partial_t \rho + div(\rho \vec{u}) = 0, \\ \partial_t(\rho\vec{u}) + div(\rho u \otimes u) + \nabla P = -\rho\vec{u}, \\ \partial_t(\rho\mathbf{E}) + div(\rho\mathbf{E}\vec{u} + P\vec{u}) = -div\vec{q}, \\ -\nabla div\vec{q} + \vec{q} + \nabla\left(\theta^4\right) = 0, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho,\vec{u},\theta)(\vec{x},0) = (\rho_0,\vec{u}_0,\theta_0)(\vec{x}). \tag{1.2}$$

Here, $\rho > 0$ and $\vec{u} = (u_1,u_2,u_3)$ denote the density and velocity respectively, $\theta > 0$ is the absolute temperature, $\vec{q} = (q_1,q_2,q_3)$ is the radiative heat flux, $P = P(\rho,\theta)$ denotes the pressure, $\mathbf{E} \equiv \frac{1}{2}|\vec{u}|^2 + \mathbf{e}$ is the total energy with $\mathbf{e}$ to be the internal energy. The equations of state and the internal energy for the (polytropic) fluid are given by

$$P = R\rho\theta, \qquad \mathbf{e} = C_v\theta, \tag{1.3}$$

where $R > 0$ is the universal gas constant and $C_v > 0$ is the specific heat at constant volume.

*Received: March 18, 2023; Accepted (in revised form): September 01, 2023. Communicated by Feimin Huang.
†School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai, China (matdengs@sjtu.edu.cn).
‡College of Science, University of Shanghai for Science and Technology, Shanghai, China (wwj001373@hotmail.com).
§School of Mathematical Sciences, CMA-Shanghai, and MOE-LSC, Shanghai Jiao Tong University, Shanghai, China (tzxief@sjtu.edu.cn).
¶School of Mathematical Sciences, CMA-Shanghai, and MOE-LSC, Shanghai Jiao Tong University, Shanghai, China (xf-yang@sjtu.edu.cn).

Before we proceed, it is necessary to give some comments about background of system (1.1), and review some known results. When the high temperature (more than 10000K) fluid is considered, the radiative effects should be included in the hydrodynamic equations. The most important effect of radiation process is the transport of energy, and it is the reason that the radiative flux is added into the energy equation. In addition, if the friction effect among molecules is also taken into consideration, the momentum equations should be supplemented with damping terms. These two factors are very important for the motion of fluid. Each of them has been extensively investigated in the study of hydrodynamics. However, there are very few results about the combination of these two effects in the literature. For the case where only radiation is considered, the system (1.1) without damping effects has been widely studied for the 1-D spatial variable [4–7, 9, 10, 16, 18, 20, 21, 25]. In these works, the global existence of smooth solutions, the nonlinear stability of elementary wave patterns, the large-time behavior of solutions and the hydrodynamic limit processes have been established. However, there is a center manifold around the equilibria which may lead to nonlinear instability for the multi-D spatial variable cases. The related mathematical analysis becomes extremely hard, especially for the global existence of smooth solutions. In other words, the radiative flux can not produce enough "good" properties to guarantee the global-in-time existence of smooth solutions, which is different from 1-D case. For the case where only the damping effect is included, the damping Euler system without radiative effect is considered in [1, 3, 17, 19, 22–24] and references therein. In [1, 3, 22, 23], the isentropic Euler system has been studied and the variation of entropy is not considered in the motion of fluid. In [17, 19, 24], the authors considered the full Euler equations with the damping effect only in the momentum equations and no dissipative mechanism in the energy equation. A center manifold arises again which may lead to nonlinear instability. However, different from the purely radiative model, the nonlinear structure of the model considered in [17, 19, 24] has a cancellation effect which yields the existence of the global solution. Here, we consider the non-isentropic Euler equations with the damping effect in the momentum equations and the radiative effect in the energy equation at the same time. The damping effect together with the radiative effect gives the nonlinear stability of the solution for the model (1.1). By the spectrum analysis, it shows that the structure of the damped system (1.1) is totally different from the model without damping effect. It is noted that the non-decaying component becomes to decay exponentially in time after the damping effect is added. Compared with the system without damping effect, there are even no Calderon-Zygmund operators; there is an extra exponentially decaying structure in time and this damping effect damps the slowly decaying structure in space. To show the intrinsic differences, we compare these two models in the linear level. The two systems, with and without damping effects, have similar behaviors in singularities. However, the main wave structure of the damped model (1.1) is like the heat kernel, while that of the model without damping effect is a combination of Navier-Stokes waves and the heat kernel. In the nonlinear level, the damping effect helps to make the system stable while the model without damping effect may need other mechanisms to cancel the instablity caused by the center manifold. If one makes a comparison between the model (1.1) and the model without the radiative effect in [17, 19, 24] from the spectrum analysis, one will find that they have almost the same wave behaviors except that the latter model contains a center manifold leading to instability. In the nonlinear level, the model in [17, 19, 24] has a special structure to ensure the existence of the global smooth solution, with some components of the solution remain just bounded.

To capture those differences between the model with and without damping effect, especially to describe the wave structures of this model, we aim to obtain a pointwise space-time structure of the solution in this paper. For this purpose, one could adopt the Green's function method. There are series of works about the construction of the Green's function for equations with dissipative mechanisms; see [8, 11–15] and the references therein.

In this paper, we focus on the small perturbed solution to (1.1)-(1.3) around the constant state $(\rho^*,\vec{u}^*,\theta^*,\vec{q}^*)$. Without loss of generality, we choose the constant state $(\rho^*,\vec{u}^*,\theta^*,\vec{q}^*)=(1,\vec{0},1,\vec{0})$ and denote $(\sigma,\vec{u},\Theta,\vec{q})\equiv(\rho-1,\vec{u},\theta-1,\vec{q})$. Then the corresponding linearized system can be written as follows.

$$\begin{cases}\sigma_t+div\vec{u}=-div(\sigma\vec{u}),\\ \vec{u}_t+R\nabla\sigma+R\nabla\Theta+\vec{u}=-\vec{u}\cdot\nabla\vec{u}-\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma,\\ \Theta_t+\frac{R}{C_v}div\vec{u}+\frac{div\vec{q}}{C_v}=-\vec{u}\cdot\nabla\Theta-\frac{R}{C_v}\Theta div\vec{u}+\frac{\vec{u}\cdot\vec{u}}{C_v}+\frac{\sigma div\vec{q}}{C_v(\sigma+1)},\\ -\nabla div\vec{q}+\vec{q}+4\nabla\Theta=-4\Theta\left(\Theta^2+3\Theta+3\right)\nabla\Theta,\end{cases}\tag{1.4}$$

with the initial data:

$$(\sigma,\vec{u},\Theta)(\vec{x},0)=(\rho_0-1,\vec{u}_0,\theta_0-1)(\vec{x})\equiv(\sigma_0,\vec{u}_0,\Theta_0)(\vec{x}).$$

To construct the Green's function for (1.4), we follow the construction method in [2]. The whole construction contains three steps: a separation of the singular-regular parts by introducing approximated spectra, the long wave-short wave decomposition in a finite Mach region and the weighted energy estimate outside the Mach region. The advantage of this construction is a precise structure of the Green's function and a clear separation of the singular-regular parts after which the sharp structure of the regular part could be obtained through the weighted energy method outside the finite Mach region. Once we have the pointwise structure of the Green's function, we could combine Duhamel's principle and bounded estimates from nonlinear energy estimates to yield a pointwise estimate for the solution. From this pointwise structure of the solution, it is quite obvious that the damping effect is very strong which does not only stabilize the nonlinear problem, but also changes the wave behaviors.

Our main result is as follows:

THEOREM 1.1. *There exist positive constants C_* and $0<\epsilon\ll1$ such that if the initial functions $(\rho_0,\vec{u}_0,\theta_0)$ satisfy*

$$\left\|(\rho_0,\vec{u}_0,\theta_0)(\cdot)-(1,\vec{0},1)\right\|_{H^6(\mathbb{R}^3)}\le C_*\epsilon,\tag{1.5}$$

then there exists a global unique solution $(\rho,\vec{u},\theta,\vec{q})$ to the Cauchy problem (1.1)-(1.3) in the following sense

$$\rho-1,\vec{u},\theta-1,\vec{q}\in C^0(0,\infty;H^6(\mathbb{R}^3))\cap C^1(0,\infty;H^5(\mathbb{R}^3)),$$

and the solution satisfies

$$\left\|(\rho,\vec{u},\theta,\vec{q})(\cdot,t)-(1,\vec{0},1,\vec{0})\right\|_{H^6(\mathbb{R}^3)}\le C\epsilon,$$

for a positive constant C.

Furthermore, assuming that the initial functions $(\rho_0,\vec{u}_0,\theta_0)$ also satisfy

$$|\partial_{\vec{x}}^\alpha(\rho_0-1,\vec{u}_0,\theta_0-1)(\vec{x})|\le C_*\epsilon e^{-|\vec{x}|}\ \text{for}\ 0\le|\alpha|\le3,\tag{1.6}$$

then for $t>0$ the solution to the Cauchy problem (1.1)-(1.3) has the following decaying structure

$$\left|\partial_{\vec{x}}^{\alpha}\left((\rho,\vec{u},\theta,\vec{q})(\vec{x},t)-(1,\vec{0},1,\vec{0})\right)\right|\leq C\epsilon\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{(3-|\alpha|)/2}}+e^{-(|\vec{x}|+t)/C}\right) \tag{1.7}$$

for $0\leq|\alpha|\leq 2$ and a positive constant C. Here, we denote the multi-index $\alpha\equiv(\alpha_1,\alpha_2,\alpha_3)$ with $|\alpha|=\sum_{j=1}^{3}\alpha_j$ and $\partial_{\vec{x}}^{\alpha}\equiv\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$.

Remark 1.1. It is well-known that for equations with dissipative mechanisms, the derivatives of the solution would decay faster in time than the solution itself. However, in the estimate (1.7), the derivatives even decay slower. The reason is that since (1.1) is a quasi-linear system, the representation of the solution or its derivative by Duhamel's principle always contains higher order derivatives for which we do not have pointwise space-time assumptions and could only use the bounded information from energy estimates. It leads to a loss of decaying rates for high order derivatives. The estimate in (1.7) could be improved if higher order regularity conditions are posed on the initial data in (1.6).

Remark 1.2. To guarantee the global existence of the solution, one only needs the H^4 regularity requirement for the initial data. Here, the H^6 requirement in (1.5) is for the later use of pointwise estimates.

Remark 1.3. In the construction of the Green's function and the justification of pointwise ansatz assumption, we decouple the equation for $\vec{q}$ and the equations for $\rho,\vec{u},\theta$ as follows: from the last equation of (1.1), one could obtain an elliptic equation for $div\vec{q}$:

$$-\Delta div\vec{q}+div\vec{q}+\Delta\left(\theta^4\right)=0$$

which could be solved directly:

$$div\vec{q}(\vec{x},t)=\mathbf{F}^{-1}\left[\frac{|\vec{\xi}|^2}{1+|\vec{\xi}|^2}\right]*_{\vec{x}}\left(\theta^4-1\right)(\vec{x},t)=\Big(\delta(\vec{x})-\mathbb{Y}(\vec{x})\Big)*_{\vec{x}}\left(\theta^4-1\right)(\vec{x},t), \tag{1.8}$$

with the 3-D Dirac-delta function $\delta(\vec{x})$ and the 3-D Yukawa potential $\mathbb{Y}(\vec{x})\equiv-\frac{e^{-|\vec{x}|}}{4\pi|\vec{x}|}$. One could substitute (1.8) into the fifth equation of (1.1) and after that the first five equations constitute a decoupled system for $\rho,\vec{u},\theta$. After one solves this decoupled system, one could estimate $div\vec{q}$ by combining the estimate for θ and (1.8). Thus the estimate for $\vec{q}$ follows directly from the last equation of (1.1) since one has that

$$\vec{q}=\nabla div\vec{q}-\nabla\left(\theta^4\right)=-\mathbb{Y}(\vec{x})*_{\vec{x}}\nabla\left(\theta^4\right)(\vec{x},t)=-\nabla\mathbb{Y}(\vec{x})*_{\vec{x}}\left(\theta^4-1\right)(\vec{x},t). \tag{1.9}$$

In a certain sense, from the above identity, one can find that both the decay structure and function space which $\vec{q}$ belongs to should be the same as the ones for θ under the assumptions of Theorem 1.1.

The rest of the paper is arranged as follows: In Section 2, we introduce the Green's function for the linearized system (1.4) and give some useful lemmas for the inverse Fourier transformation. In Section 3, we obtain the pointwise structure of the Green's function in space-time variables through the singularity removal, the long wave-short wave decomposition and the weighted energy method. In Section 4, we obtain the nonlinear stability of the solution by energy method. It provides the bounds of the high order derivatives of the solution. Finally, the pointwise estimate of the solution is obtained by using the solution representation from Duhamel's principle and the pointwise structure of the Green's function.

2. Preliminaries: The Green's function in Fourier variables

In this section, we introduce the Green's function for Cauchy problem of the linearized system (1.4) and some lemmas as a preparation for the study of pointwise structure of the Green's function in next section.

We denote the solution for the linear part of the system (1.4) to be $(V,\vec{U},W,\vec{Q})$ and the linear system is

$$\begin{cases} V_t + div\vec{U} = 0, \\ \vec{U}_t + R\nabla V + R\nabla W = -\vec{U}, \\ W_t + \frac{R}{C_v} div\vec{U} + \frac{div\vec{Q}}{C_v} = 0, \\ -\nabla div\vec{Q} + \vec{Q} + 4\nabla W = 0. \end{cases} \tag{2.1}$$

2.1. The Green's function in Fourier variables. Introduce the Fourier transformation of a function $g(\vec{x}) \in L^1(\mathbb{R}^3)$ or $g(\vec{x}) \in \mathcal{S}(\mathbb{R}^3)$: $\mathbf{F}[g](\vec{\xi}) \equiv \int_{\mathbb{R}^3} e^{-i\vec{\xi}\cdot\vec{x}} g(\vec{x}) d\vec{x}$. Here, $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space on $\mathbb{R}^3$. For $g(\vec{x}) \in \mathcal{S}'(\mathbb{R}^3)$, the Fourier transformation $\mathbf{F}[g]$ is defined by: $\forall \phi \in \mathcal{S}(\mathbb{R}^3)$, $< \mathbf{F}[g], \phi > = < g, \mathbf{F}[\phi] >$.

Applying Fourier transformation to the first five equations in (2.1), one has that

$$\begin{cases} \mathbf{F}[V]_t + i\vec{\xi} \cdot \mathbf{F}[\vec{U}] = 0, \\ \mathbf{F}[\vec{U}]_t + Ri\vec{\xi} \cdot \mathbf{F}[V] + Ri\vec{\xi} \cdot \mathbf{F}[W] = -\mathbf{F}[\vec{U}], \\ \mathbf{F}[W]_t + \frac{R}{C_v} i\vec{\xi} \cdot \mathbf{F}[\vec{U}] + \frac{1}{C_v} \mathbf{F}[div\vec{Q}] = 0. \end{cases} \tag{2.2}$$

The last equation of (2.1) yields that $-\Delta div\vec{Q} + div\vec{Q} + 4\Delta W = 0$ and thus

$$\left(|\vec{\xi}|^2 + 1\right) \mathbf{F}[div\vec{Q}] = 4|\vec{\xi}|^2 \mathbf{F}[W]$$

and one could substitute it into the last equation in (2.2) to get:

$$\begin{cases} \mathbf{F}[V]_t + i\vec{\xi} \cdot \mathbf{F}[\vec{U}] = 0, \\ \mathbf{F}[\vec{U}]_t + Ri\vec{\xi} \cdot \mathbf{F}[V] + Ri\vec{\xi} \cdot \mathbf{F}[W] = -\mathbf{F}[\vec{U}], \\ \mathbf{F}[W]_t + \frac{R}{C_v} i\vec{\xi} \cdot \mathbf{F}[\vec{U}] + \frac{4|\vec{\xi}|^2}{C_v\left(1+|\vec{\xi}|^2\right)} \mathbf{F}[W] = 0. \end{cases} \tag{2.3}$$

Denote $\mathbb{G}(\vec{x},t)$ to be the Green's function of the system (1.4) and then it satisfies

$$\begin{cases} \mathbf{F}[\mathbb{G}]_t = A\mathbf{F}[\mathbb{G}], \\ \mathbf{F}[\mathbb{G}](\vec{\xi},0) = I_5, \end{cases} \tag{2.4}$$

where

$$A \equiv -\begin{pmatrix} 0 & i\vec{\xi} & 0 \\ Ri\vec{\xi}^T & I_3 & Ri\vec{\xi}^T \\ 0 & \frac{R}{C_v} i\vec{\xi} & \frac{4|\vec{\xi}|^2}{C_v\left(1+|\vec{\xi}|^2\right)} \end{pmatrix},$$

and I_n is the $n \times n$ identity matrix.

The general solution of the ODE system (2.4) could be written as

$$f(\xi,t)=\sum_{j=1}^{5}a_j e^{\lambda_j t}r_j, \tag{2.5}$$

where $\lambda_j, r_j (j=1,\cdots,5)$ are the eigenvalues and the corresponding right eigenvectors of the matrix A and the coefficients a_j are determined by initial function $f(\xi,0)$ of the ODE:

$$(a_1,a_2,a_3,a_4,a_5)^T \equiv (r_1,r_2,r_3,r_4,r_5)^{-1} f(\xi,0). \tag{2.6}$$

By a straightforward computation, one has

$$\lambda_1=\lambda_2=-1,$$

and $\lambda_j (j=3,4,5)$ are the three roots of the following characteristic equation:

$$C_v\left(1+|\vec{\xi}|^2\right)\lambda^3+\left(C_v+(4+C_v)|\vec{\xi}|^2\right)\lambda^2+|\vec{\xi}|^2\left(4+R(C_v+R)\left(1+|\vec{\xi}|^2\right)\right)\lambda+4R|\vec{\xi}|^4=0. \tag{2.7}$$

Moreover, the corresponding eigenvectors have the following forms.

$$r_1\equiv\begin{pmatrix}0\\-\xi_2/\xi_1\\1\\0\\0\end{pmatrix},\quad r_2\equiv\begin{pmatrix}0\\-\xi_3/\xi_1\\0\\1\\0\end{pmatrix},\quad r_j\equiv\begin{pmatrix}|\vec{\xi}|^2\\ i\xi_1\lambda_j\\ i\xi_2\lambda_j\\ i\xi_3\lambda_j\\ \frac{R|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)\lambda_j}{C_v\left(1+|\vec{\xi}|^2\right)\lambda_j+4|\vec{\xi}|^2}\end{pmatrix} \text{ for } j=3,4,5. \tag{2.8}$$

Now we substitute the initial data $\mathbf{F}[\mathbb{G}](\xi,0)=I_5$ into (2.6) and give the explicit formula of the Green's function $\mathbf{F}[\mathbb{G}](\vec{\xi},t)$ in Fourier variable:

$$\begin{aligned}\mathbf{F}[\mathbb{G}]\equiv{}& e^{-t}\begin{pmatrix}0&0&0&0&0\\0&1-\frac{\xi_1^2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&1-\frac{\xi_2^2}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&1-\frac{\xi_3^2}{|\vec{\xi}|^2}&0\\0&0&0&0&0\end{pmatrix}-\frac{\left(C_v\lambda_3\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_3 t}}{C_v(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_3\beta_3\\ &-\frac{\left(C_v\lambda_4\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_4 t}}{C_v(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_4\beta_4-\frac{\left(C_v\lambda_5\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_5 t}}{C_v(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_5\beta_5\\ \equiv{}&\mathbf{F}[\mathbb{G}_D]+\mathbf{F}[\mathbb{G}_H].\end{aligned} \tag{2.9}$$

Here, one denotes that for $j=3,4,5$,

$$\beta_j\equiv\left(\frac{R|\vec{\xi}|^2}{\lambda_j}\ \ i\xi_1\ \ i\xi_2\ \ i\xi_3\ \ \frac{RC_v|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}{C_v\lambda_j\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2}\right),$$

$$\mathbb{G}_D(\vec{x},t)\equiv\mathbf{F}^{-1}\left[e^{-t}\begin{pmatrix}0&0&0&0&0\\0&1-\frac{\xi_1^2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&1-\frac{\xi_2^2}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&1-\frac{\xi_3^2}{|\vec{\xi}|^2}&0\\0&0&0&0&0\end{pmatrix}\right]$$

$$+\mathbf{F}^{-1}\left[-\frac{\left(C_v\lambda_3\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_3 t}}{C_v(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_3\beta_3\right], \tag{2.10}$$

$$\mathbb{G}_H(\vec{x},t)\equiv\mathbf{F}^{-1}\left[-\frac{\left(C_v\lambda_4\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_4 t}}{C_v(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_4\beta_4\right]$$

$$+\mathbf{F}^{-1}\left[-\frac{\left(C_v\lambda_5\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)e^{\lambda_5 t}}{C_v(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)|\vec{\xi}|^2\left(1+|\vec{\xi}|^2\right)}r_5\beta_5\right]. \tag{2.11}$$

REMARK 2.1. In (2.9), the Green's function $\mathbb{G}(\vec{x},t)$ is decomposed into two parts: the damping wave $\mathbb{G}_D(\vec{x},t)$ and the heat diffusion wave like part $\mathbb{G}_H(\vec{x},t)$.

Roughly speaking, the damping wave $\mathbb{G}_D(\vec{x},t)$ decays exponentially fast in both space and time variables. It seems that it contains a Calderon-Zygmund operator. Due to the exponentially fast decaying rate in time variable, it does not play an important role in low frequency. In high frequency, we will use some approximations to separate this operator from the fast decaying remainders, which will be carried out in next section.

The decay structure of low frequency part of $\mathbb{G}_H(\vec{x},t)$ is similar to that of the heat kernel. It is the reason we name it "heat diffusion wave like part". We will also show this property in next section.

2.2. Asymptotic behaviors of spectra. Next, we study the basic properties of the eigenvalues λ_j, the roots of characteristic Equation (2.7), as a preparation for the inverse transformation:

LEMMA 2.1. *The eigenfunctions $\lambda_j (j=3,4,5)$ satisfy $Re(\lambda_j)\le 0$ for $\vec{\xi}\in\mathbb{R}^3$, and there exist constants $\kappa_0,\kappa_1>0$, such that for $|\vec{\xi}|>\kappa_0$,*

$$Re(\lambda_j)\le-\kappa_1. \tag{2.12}$$

Furthermore, when $|\vec{\xi}|$ is near 0, the eigenfunctions $\lambda_3(\vec{\xi}),\lambda_4(\vec{\xi})$ and $\lambda_5(\vec{\xi})$ are analytic with respect to $|\vec{\xi}|^2$, and have the following asymptotical expansions:

$$\begin{cases}\lambda_3(\vec{\xi})=-1+\frac{R(C_v+R)}{C_v}|\vec{\xi}|^2+O\left(|\vec{\xi}|^4\right),\\ \lambda_{4,5}(\vec{\xi})=-\frac{4+RC_v+R^2\pm\sqrt{(4+RC_v+R^2)^2-16RC_v}}{2C_v}|\vec{\xi}|^2+O\left(|\vec{\xi}|^4\right).\end{cases} \tag{2.13}$$

When $|\vec{\xi}|\to\infty$, the asymptotic behaviors are listed as follows:

$$\begin{cases}\lambda_3(\vec{\xi})=-\frac{4}{C_v+R}+O\left(|\vec{\xi}|^{-2}\right),\\ \lambda_{4,5}(\vec{\xi})=\pm i\sqrt{R\left(1+\frac{R}{C_v}\right)}|\vec{\xi}|-\frac{C_v^2+RC_v+4R}{2C_v(C_v+R)}\\ \mp i\sqrt{R\left(1+\frac{R}{C_v}\right)}\left(C_v^4+2RC_v^3+R^2C_v^2-8RC_v^2-8R^2C_v+64RC_v+16R^2\right)|\vec{\xi}|^{-1}+O\left(|\vec{\xi}|^{-2}\right).\end{cases} \tag{2.14}$$

Proof. The property (2.12) could be verified by the Routh-Hurwitz criteria: for the given polynomial, $P(\lambda)=\lambda^3+p_1\lambda^2+p_2\lambda+p_3$, all the roots of the polynomial $P(\lambda)=0$

have negative real parts if and only if $p_1>0$, $p_3>0$ and $p_1p_2>p_3$. Since $\lambda_j(j=3,4,5)$ are the roots of (2.7), one has that

$$p_1=\frac{C_v+(4+C_v)|\vec{\xi}|^2}{C_v(1+|\vec{\xi}|^2)},\quad p_2=\frac{|\vec{\xi}|^2\left(4+R(C_v+R)(1+|\vec{\xi}|^2)\right)}{C_v(1+|\vec{\xi}|^2)},\quad p_3=\frac{4R|\vec{\xi}|^4}{C_v(1+|\vec{\xi}|^2)},$$

and for $|\vec{\xi}|\neq 0$, denote $P(|\vec{\xi}|)\equiv C_v(4+C_vR+R^2)+2(C_v^2R+C_vR^2+2R^2+2C_v+8)|\vec{\xi}|^2+R(C_vR+C_v^2+4R)|\vec{\xi}|^4$ and one has that

$$p_1>0,\quad p_3>0,$$

$$p_1p_2-p_3=\frac{P(|\vec{\xi}|)|\vec{\xi}|^2}{\left(C_v(1+|\vec{\xi}|^2)\right)^2}>0.$$

Thus, the real parts of $\lambda_j(j=3,4,5)$ are negative when $|\vec{\xi}|\neq 0$.

The asymptotics (2.13) and (2.14) could be obtained by straightforward computations and we omit the details. □

2.3. A useful lemma. Denote

$$\mathbf{D}_\delta\equiv\{\vec{\xi}\in\mathbb{C}^n\big|\,|Im(\xi_k)|<\delta \text{ for } k=1,\cdots,n\}. \tag{2.15}$$

The following lemma holds:

Lemma 2.2 (Lemma 2.1, [2]). *Suppose that a function $f\in L^1(\mathbb{R}^n)$ and its Fourier transformation $\mathbf{F}[f](\vec{\xi})$, $\vec{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)$ is analytic in $\mathbf{D}_{\nu_0}$, and satisfies*

$$\left|\mathbf{F}[f](\vec{\xi})\right|<\frac{E_1}{\left(1+|\vec{\xi}|\right)^{n+1}} \text{ for } |Im(\xi_j)|<\nu_0 \text{ and } j=1,\cdots,n.$$

Then the function $f(\vec{x})$ satisfies

$$|f(\vec{x})|\leq E_1e^{-\nu_0|\vec{x}|/C},$$

for any positive constant $C>1$.

3. Pointwise structure of the Green's function

In this section, we obtain the pointwise structure of the Green's function $\mathbb{G}(\vec{x},t)$ defined by (2.4) in space-time variables via inverse Fourier transformation. The whole construction includes a decomposition of the singular and the regular parts, the calculations of the truncation errors, the direct inverse of the singular part, and a region separation for estimates of the regular part; and the regular part of the Green's function $\mathbb{G}(\vec{x},t)$ satisfies

Theorem 3.1 (Pointwise structure of regular part of $\mathbb{G}(\vec{x},t)$). *There exists a positive constant C such that for $0\leq|\alpha|\leq 3$,*

$$|\partial_{\vec{x}}^\alpha(\mathbb{G}(\vec{x},t)-\mathbb{G}_S^*(\vec{x},t))|\leq C\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{(3+|\alpha|)/2}}+e^{-(|\vec{x}|+t)/C}\right).$$

In the above theorem, the function $\mathbb{G}_S^*(\vec{x},t)$ denotes the singular part of the Green's function in the sense that one could take derivatives up to order 3 of the remainder. The description about $\mathbb{G}_S^*(\vec{x},t)$ is too tedious and is given by (A.9) in the Appendix.

The existence of the singularity in $\mathbb{G}(\vec{x},t)$ could be captured after an asymptotic expansion of $\mathbf{F}[\mathbb{G}]$ for $\vec{\xi}$ goes to infinity: The property (2.14) yields that when $|\vec{\xi}|\to\infty$,

$$\mathbf{F}[\mathbb{G}_D]=e^{-t}\begin{pmatrix}0&0&0&0&0\\0&1-\frac{\xi_1^2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_2}{|\vec{\xi}|^2}&1-\frac{\xi_2^2}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&0\\0&-\frac{\xi_1\xi_3}{|\vec{\xi}|^2}&-\frac{\xi_2\xi_3}{|\vec{\xi}|^2}&1-\frac{\xi_3^2}{|\vec{\xi}|^2}&0\\0&0&0&0&0\end{pmatrix}$$
$$+e^{-\frac{4t}{C_v+R}}\begin{pmatrix}\frac{R}{C_v+R}&0\;0\;0&-\frac{C_v}{C_v+R}\\0&0\;0\;0&0\\0&0\;0\;0&0\\0&0\;0\;0&0\\-\frac{R}{C_v+R}&0\;0\;0&\frac{C_v}{C_v+R}\end{pmatrix}+O\left(\frac{1}{|\vec{\xi}|}\right),$$

and

$$\mathbf{F}[\mathbb{G}_H]=e^{-\frac{\left(Cv^2+RC_v+4R\right)t}{2C_v(C_v+R)}}$$
$$\cdot\begin{pmatrix}\frac{C_v\cos\tilde{C}|\vec{\xi}|t}{C_v+R}&-\frac{\sqrt{C_v}\sin\tilde{C}|\vec{\xi}|t}{\sqrt{R(C_v+R)}}\cdot\frac{i\vec{\xi}}{|\vec{\xi}|}&\frac{C_v\cos\tilde{C}|\vec{\xi}|t}{C_v+R}\\-\frac{\sqrt{RC_v}\sin\tilde{C}|\vec{\xi}|t}{\sqrt{C_v+R}}\cdot\frac{i\vec{\xi}^T}{|\vec{\xi}|}&\cos\tilde{C}|\vec{\xi}|t\cdot\frac{\vec{\xi}^T\vec{\xi}}{|\vec{\xi}|^2}&-\frac{\sqrt{RC_v}\sin\tilde{C}|\vec{\xi}|t}{\sqrt{C_v+R}}\cdot\frac{i\vec{\xi}^T}{|\vec{\xi}|}\\\frac{R\cos\tilde{C}|\vec{\xi}|t}{C_v+R}&-\frac{\sqrt{R}\sin\tilde{C}|\vec{\xi}|t}{\sqrt{C_v(C_v+R)}}\frac{i\vec{\xi}}{|\vec{\xi}|}&\frac{R\cos\tilde{C}|\vec{\xi}|t}{C_v+R}\end{pmatrix}+O\left(\frac{1}{|\vec{\xi}|}\right),$$

with $\tilde{C}\equiv\sqrt{\frac{R}{C_v}(C_v+R)}$. The above expansions suggest the existence of the Dirac-delta function and other singular structures in the Green's function.

3.1. Approximated spectra and singularity removal. In this subsection, we aim to remove the singularity at the beginning and make a preparation for the treatment of the remaining regular part as a whole. First, one introduces the following approximated spectra:

$$\begin{cases}\lambda_3^*(\vec{\xi})\equiv a_{3,0}+\frac{a_{3,-2}}{1+|\vec{\xi}|^2}+\frac{a_{3,-4}^*}{\left(1+|\vec{\xi}|^2\right)^2}+\frac{a_{3,-6}^*}{\left(1+|\vec{\xi}|^2\right)^3}+\frac{a_{3,-8}^*}{\left(1+|\vec{\xi}|^2\right)^4}-\frac{J_0^*}{\left(1+|\vec{\xi}|^2\right)^5},\\ \lambda_4^*(\vec{\xi})\equiv\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|\left(1+\frac{a_{-1}}{1+|\vec{\xi}|^2}+\frac{a_{-3}^*}{\left(1+|\vec{\xi}|^2\right)^2}+\frac{a_{-5}^*}{\left(1+|\vec{\xi}|^2\right)^3}+\frac{a_{-7}^*}{\left(1+|\vec{\xi}|^2\right)^4}+\frac{J_{11}^*}{\left(1+|\vec{\xi}|^2\right)^5}\right)\\ \qquad+a_0+\frac{a_{-2}}{1+|\vec{\xi}|^2}+\frac{a_{-4}^*}{\left(1+|\vec{\xi}|^2\right)^2}+\frac{a_{-6}^*}{\left(1+|\vec{\xi}|^2\right)^3}+\frac{a_{-8}^*}{\left(1+|\vec{\xi}|^2\right)^4}-\frac{J_{12}^*}{\left(1+|\vec{\xi}|^2\right)^5},\\ \lambda_5^*(\vec{\xi})\equiv-\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|\left(1+\frac{a_{-1}}{1+|\vec{\xi}|^2}+\frac{a_{-3}^*}{\left(1+|\vec{\xi}|^2\right)^2}+\frac{a_{-5}^*}{\left(1+|\vec{\xi}|^2\right)^3}+\frac{a_{-7}^*}{\left(1+|\vec{\xi}|^2\right)^4}+\frac{J_{11}^*}{\left(1+|\vec{\xi}|^2\right)^5}\right)\\ \qquad+a_0+\frac{a_{-2}}{1+|\vec{\xi}|^2}+\frac{a_{-4}^*}{\left(1+|\vec{\xi}|^2\right)^2}+\frac{a_{-6}^*}{\left(1+|\vec{\xi}|^2\right)^3}+\frac{a_{-8}^*}{\left(1+|\vec{\xi}|^2\right)^4}-\frac{J_{12}^*}{\left(1+|\vec{\xi}|^2\right)^5},\end{cases}\tag{3.1}$$

with constants $a_{3,-j}^*(j=0,2,4,6,8)$ and $a_{-j}^*(j=3,4,\cdots,8)$ given by

$$\begin{cases}a_{3,-4}^*=a_{3,-2}+a_{3,-4},\\a_{3,-6}^*=a_{3,-2}+2a_{3,-4}+a_{3,-6},\\a_{3,-8}^*=a_{3,-2}+3a_{3,-4}+3a_{3,-6}+a_{3,-8},\end{cases}\tag{3.2}$$

$$\begin{cases} a^*_{-3} = a_{-1} + a_{-3}, \\ a^*_{-5} = a_{-1} + 2a_{-3} + a_{-5}, \\ a^*_{-7} = a_{-1} + 3a_{-3} + 3a_{-5} + a_{-7}, \\ a^*_{-4} = a_{-2} + a_{-4}, \\ a^*_{-6} = a_{-2} + 2a_{-4} + a_{-6}, \\ a^*_{-8} = a_{-2} + 3a_{-4} + 3a_{-6} + a_{-8}. \end{cases} \tag{3.3}$$

Here, $a_{3,-j}(j=0,2,4,6,8)$ and $a_{-j}(j=0,1,2,\cdots,8)$ are constants in the expansions of $\lambda_j(\vec{\xi})$ in the Appendix A.1. The constants J^*_0, J^*_{11} and J^*_{12} are chosen to be positive and large enough to ensure that for $j,k=3,4,5$ and $j \neq k$,

$$\sup_{\vec{\xi}\in\mathbf{D}_{1/2}} Re(\lambda^*_j(\vec{\xi})) < -\nu_0 < 0, \tag{3.4}$$

and

$$\inf_{\vec{\xi}\in\mathbf{D}_{1/2}} |\lambda^*_j(\vec{\xi}) - \lambda^*_k(\vec{\xi})| > 0. \tag{3.5}$$

These conditions (3.4)-(3.5) can be satisfied by choosing suitable constants since

$$a_{3,0} \equiv -\frac{4}{C_v + R} < 0, \qquad a_0 \equiv -\frac{C_v^2 + C_v R + 4R}{2C_v(C_v + R)} < 0.$$

The approximated spectrum λ^*_3 defined by (3.1) with constants chosen above is up to degree 10 approximation of λ_3 and similarly $\lambda^*_4(\lambda^*_5)$ is up to degree 9 approximation of $\lambda_4(\lambda_5)$ for $|\vec{\xi}| \to \infty$, i.e.

$$\sup_{\vec{\xi}\in\mathbf{D}_{1/2}} |\vec{\xi}|^{10}(\lambda^*_3(\vec{\xi}) - \lambda_3(\vec{\xi})) < \infty, \quad \sup_{\vec{\xi}\in\mathbf{D}_{1/2}} |\vec{\xi}|^{9}(\lambda^*_{4,5}(\vec{\xi}) - \lambda_{4,5}(\vec{\xi})) < \infty, \tag{3.6}$$

where $\mathbf{D}_{1/2}$ is defined in (2.15) with $\delta = 1/2$.

Furthermore, define the **singular support functions** $\mathbb{G}^*_D$ and $\mathbb{G}^*_H$ as follows:

$$\begin{cases} \mathbb{G}^*_D(\vec{x},t) \equiv \mathbf{F}^{-1}\left[e^{-t}\begin{pmatrix} 0 & 0 & 0 \\ 0 & I_3 - p(\vec{\xi})\vec{\xi}^T\vec{\xi} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{p(\vec{\xi})\big(C_v\lambda^*_3(1+|\vec{\xi}|^2)+4|\vec{\xi}|^2\big)e^{\lambda^*_3 t}}{C_v(\lambda^*_3-\lambda^*_4)(\lambda^*_3-\lambda^*_5)\big(1+|\vec{\xi}|^2\big)} r^*_3\beta^*_3 \right], \\ \mathbb{G}^*_H(\vec{x},t) \\ \quad \equiv \mathbf{F}^{-1}\left[-\frac{p(\vec{\xi})\big(C_v\lambda^*_4(1+|\vec{\xi}|^2)+4|\vec{\xi}|^2\big)e^{\lambda^*_4 t}}{C_v(\lambda^*_4-\lambda^*_3)(\lambda^*_4-\lambda^*_5)\big(1+|\vec{\xi}|^2\big)} r^*_4\beta^*_4 - \frac{p(\vec{\xi})\big(C_v\lambda^*_5(1+|\vec{\xi}|^2)+4|\vec{\xi}|^2\big)e^{\lambda^*_5 t}}{C_v(\lambda^*_5-\lambda^*_3)(\lambda^*_5-\lambda^*_4)\big(1+|\vec{\xi}|^2\big)} r^*_5\beta^*_5 \right], \end{cases} \tag{3.7}$$

with

$$r^*_k \equiv \begin{pmatrix} |\vec{\xi}|^2 \\ i\xi_1\lambda^*_k \\ i\xi_2\lambda^*_3 \\ i\xi_3\lambda^*_3 \\ \frac{R|\vec{\xi}|^2(1+|\vec{\xi}|^2)\lambda^*_k}{C_v(1+|\vec{\xi}|^2)\lambda^*_k+4|\vec{\xi}|^2} \end{pmatrix}, \qquad \beta^*_k \equiv \begin{pmatrix} \frac{R|\vec{\xi}|^2}{\lambda^*_k} & i\xi_1 & i\xi_2 & i\xi_3 & \frac{RC_v|\vec{\xi}|^2(1+|\vec{\xi}|^2)}{C_v\lambda^*_k(1+|\vec{\xi}|^2)+4|\vec{\xi}|^2} \end{pmatrix},$$

and the polynomial $p(\vec{\xi})$, a high order approximation of $|\vec{\xi}|^{-2}$ for $|\vec{\xi}|\to\infty$:

$$p(\vec{\xi})\equiv\sum_{j=1}^{5}\left(1+|\vec{\xi}|^2\right)^{-j}=|\vec{\xi}|^{-2}+O(|\vec{\xi}|^{-12}) \text{ for } |\vec{\xi}|\to\infty.$$

The singular support functions (3.7) and the degree 10(9) approximation (3.6) result in the following lemma:

Lemma 3.1. *The remainder of the Green's functions $\mathbb{G}_D,\mathbb{G}_H$ and singular support functions $\mathbb{G}_D^*,\mathbb{G}_H^*$ satisfy that for $\vec{\xi}\in\mathbb{R}^3$,*

$$|\mathbf{F}[\mathbb{G}_D^*]-\mathbf{F}[\mathbb{G}_D]|=O(1)(1+|\vec{\xi}|)^{-10},\quad |\mathbf{F}[\mathbb{G}_H^*]-\mathbf{F}[\mathbb{G}_H]|=O(1)(1+|\vec{\xi}|)^{-9}. \tag{3.8}$$

The conclusion (3.8) follows directly after one substitutes the degree 10(9) approximation into the Definition (3.7) of the singular support functions. Lemma 3.1 results in that

$$\|(\mathbb{G}_D^*+\mathbb{G}_H^*-\mathbb{G}_D-\mathbb{G}_H)(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}=O(1),$$

which reveals that $\mathbb{G}_D^*+\mathbb{G}_H^*$ contains all the singular structures of $\mathbb{G}_D+\mathbb{G}_H$. It leads to a further study of the singular part $\mathbb{G}_D^*+\mathbb{G}_H^*$. For more details about the singular structure, refer to Subsection A.2 in the Appendix and we only list the result here:

Lemma 3.2. *There exists a positive constant C such that*

$$|\mathbb{G}_D^*(\vec{x},t)+\mathbb{G}_H^*(\vec{x},t)-\mathbb{G}_S^*(\vec{x},t)|\le Ce^{-(|\vec{x}|+t)/C},$$

with $\mathbb{G}_S^*(\vec{x},t)$ given by (A.9).

3.2. Estimates of regular part: proof of Theorem 3.1. Now, one denotes

$$\mathbb{G}^*\equiv\mathbb{G}_D^*+\mathbb{G}_H^*,\qquad \mathbb{G}_R\equiv\mathbb{G}-\mathbb{G}^*, \tag{3.9}$$

and Lemma 3.1 suggests that $\mathbb{G}_R$ should be regular enough. We derive the equations satisfied by $\mathbb{G}_R$ as a preparation for the study of the detailed estimates.

The function $\mathbb{G}_R$ satisfies the following system:

$$\begin{cases}(\partial_t I_5-A)\mathbb{G}_R=-(\partial_t I_5-A)(\mathbb{G}_D^*+\mathbb{G}_H^*)\equiv\mathbb{E}_1(\vec{x},t),\\ \mathbb{G}_R(\vec{x},0)\equiv\mathbb{E}_2(\vec{x})\end{cases} \tag{3.10}$$

where A is defined in (2.4) and the Fourier transformations of the truncation errors $\mathbb{E}_1(\vec{x},t)$ and $\mathbb{E}_2(\vec{x})$ satisfy

$$\begin{aligned}\mathbf{F}[\mathbb{E}_1]=&-\left(1-p(\xi)|\xi|^2\right)\begin{pmatrix}0 & i\vec{\xi} & 0\\ 0 & 0 & 0\\ 0 & i\frac{R}{C_v}\xi & 0\end{pmatrix}e^{-t}+\frac{p(\vec{\xi})\tilde{P}(\lambda_3^*)}{C_v\left(1+|\vec{\xi}|^2\right)(\lambda_3^*-\lambda_4^*)(\lambda_3^*-\lambda_5^*)}\begin{pmatrix}0\\ i\vec{\xi}^T\\ 0\end{pmatrix}\beta_3^*e^{\lambda_3^*t}\\ &+\frac{p(\vec{\xi})\tilde{P}(\lambda_4^*)}{C_v\left(1+|\vec{\xi}|^2\right)(\lambda_4^*-\lambda_3^*)(\lambda_4^*-\lambda_5^*)}\begin{pmatrix}0\\ i\vec{\xi}^T\\ 0\end{pmatrix}\beta_4^*e^{\lambda_4^*t}\\ &+\frac{p(\vec{\xi})\tilde{P}(\lambda_5^*)}{C_v\left(1+|\vec{\xi}|^2\right)(\lambda_5^*-\lambda_3^*)(\lambda_5^*-\lambda_4^*)}\begin{pmatrix}0\\ i\vec{\xi}^T\\ 0\end{pmatrix}\beta_5^*e^{\lambda_5^*t},\end{aligned} \tag{3.11}$$

and

$$\mathbf{F}[\mathbb{E}_2]=\begin{pmatrix}1&0&0\\0&I_3&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0\\0&I_3-p(\vec{\xi})\vec{\xi}^T\vec{\xi}&0\\0&0&0\end{pmatrix}-p(\vec{\xi})\begin{pmatrix}e_2^{1,1}&0&0\\0&\vec{\xi}^T\vec{\xi}&0\\0&0&e_2^{5,5}\end{pmatrix}, \tag{3.12}$$

with

$$\begin{aligned}\tilde{P}(\lambda_j^*)\equiv&C_v\left(1+|\vec{\xi}|^2\right)(\lambda_j^*)^3+\left(C_v+(4+C_v)|\vec{\xi}|^2\right)(\lambda_j^*)^2\\&+|\vec{\xi}|^2\left(4+R(C_v+R)\left(1+|\vec{\xi}|^2\right)\right)\lambda_j^*+4R|\vec{\xi}|^4,\quad j=3,4,5,\\e_2^{1,1}\equiv&-\frac{4R|\vec{\xi}|^6}{C_v(1+|\vec{\xi}|^2)\lambda_3^*\lambda_4^*\lambda_5^*},\\e_2^{5,5}\equiv&\frac{4C_vR^2|\vec{\xi}|^6(1+|\vec{\xi}|^2)^2}{\left(C_v(1+|\vec{\xi}|^2)\lambda_3^*+4|\vec{\xi}|^2\right)\left(C_v(1+|\vec{\xi}|^2)\lambda_4^*+4|\vec{\xi}|^2\right)\left(C_v(1+|\vec{\xi}|^2)\lambda_5^*+4|\vec{\xi}|^2\right)}.\end{aligned}$$

Substituting the Definition (3.1) of approximated spectra $\lambda_k^*(k=3,4,5)$ in (3.11) and (3.12), one has the following lemma:

LEMMA 3.3. *The Fourier transformations $\mathbf{F}[\mathbb{E}_1](\vec{\xi},t)$ and $\mathbf{F}[\mathbb{E}_2](\vec{\xi})$ of truncation error functions are analytic in $\mathbf{D}_{1/2}$, and there exists a positive constant C such that the following estimates*

$$\left|\mathbf{F}[\mathbb{E}_1](\vec{\xi},t)\right|<\frac{Ce^{-t/C}}{(1+|\vec{\xi}|)^9},\qquad\left|\mathbf{F}[\mathbb{E}_2](\vec{\xi})\right|<\frac{C}{(1+|\vec{\xi}|)^9}$$

hold true for $\vec{\xi}\in\mathbf{D}_{1/2}$.

This lemma, together with Lemma 2.2, yields the following estimates for truncation errors:

LEMMA 3.4. *There exists $C_0>0$, such that*

$$|\mathbb{E}_1(\vec{x},t)|\le C_0e^{-(|\vec{x}|+t)/C_0},\qquad|\mathbb{E}_2(\vec{x})|\le C_0e^{-|\vec{x}|/C_0},$$

$$\int_{\mathbb{R}^3}e^{|\vec{x}|/C_0}\left|\partial_{\vec{x}}^{\alpha}\mathbb{E}_1(\vec{x},t)\right|^2d\vec{x}\le C_0e^{-t/C_0}\ \textit{for}\ |\alpha|<9-\frac{3}{2}, \tag{3.13}$$

$$\int_{\mathbb{R}^3}e^{|\vec{x}|/C_0}\left|\partial_{\vec{x}}^{\alpha}\mathbb{E}_2(\vec{x})\right|^2d\vec{x}\le C_0\ \textit{for}\ |\alpha|<9-\frac{3}{2}. \tag{3.14}$$

From Lemma 3.4, one finds that truncation errors are regular enough and also decay exponentially fast in space-time variables. It allows us to apply weighted energy method later for the pointwise structure of the regular part $\mathbb{G}_R$ outside the cone.

3.2.1. Long wave-short wave decomposition and wave structure in a finite Mach region. First, we study the regular part $\mathbb{G}_R$ defined by (3.9) in a finite Mach region. Introduce the classical long wave-short wave decomposition

$$\begin{cases}f(\vec{x},t)=f^L(\vec{x},t)+f^S(\vec{x},t),\\\mathbf{F}[f^L]=\Lambda\left(\frac{|\vec{\xi}|}{\epsilon_0}\right)\mathbf{F}[f],\\\mathbf{F}[f^S]=\left(1-\Lambda\left(\frac{|\vec{\xi}|}{\epsilon_0}\right)\right)\mathbf{F}[f],\end{cases}$$

with the parameter $\epsilon_0 \ll 1$, where

$$\Lambda(r) \equiv H(1-|r|) \equiv \begin{cases} 1, & 0 \le |r| < 1, \\ 0, & |r| \ge 1. \end{cases}$$

Long wave component in a finite Mach region. Although it seems that $\mathbf{F}[\mathbb{G}_D]$ contains a Calderon-Zygmund operator which slows down the decaying rate in $\vec{x}$ variable, the exponentially fast decaying rate in t variable here makes a compensation and allows us to ignore this effect. Since $\mathbf{F}[\mathbb{G}_D](\vec{\xi},t)$ is bounded for $|\vec{\xi}| \ll 1$ and due to the spectrum gap stated in Lemma 2.1, there exists a positive constant C such that

$$\left|\mathbb{G}_D^L(\vec{x},t)\right| \le Ce^{-t/C}. \tag{3.15}$$

The long wave component of $\mathbb{G}_H$ behaves like the heat kernel: A straightforward computation after substituting (2.13) into (2.11) yields that when $|\vec{\xi}| \to 0$, one has that

$$\mathbf{F}[\mathbb{G}_H] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(|\vec{\xi}|). \tag{3.16}$$

Moreover, when $|\vec{\xi}|$ is small, $\mathbf{F}[\mathbb{G}_H]$ is analytic with respect to $\vec{\xi}$ from Lemma 2.1. The estimate (3.16) together with Lemma 2.1 and contour integral yields the wave structure of the long wave component of the function $\mathbb{G}_H(\vec{x},t)$:

LEMMA 3.5. *For $|\vec{x}| < C_1 t$ with C_1 being any positive constant, there exists a positive constant C such that the long wave component $\mathbb{G}_H^L(\vec{x},t)$ satisfies*

$$\left|\partial_{\vec{x}}^{\alpha} \mathbb{G}_H^L(\vec{x},t)\right| \le C(1+t)^{-(3+|\alpha|)/2} e^{-\frac{|\vec{x}|^2}{Ct}}.$$

Here, $\alpha \equiv (\alpha_1,\alpha_2,\alpha_3)$, $\partial_{\vec{x}}^{\alpha} \equiv \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $|\alpha| \equiv \alpha_1+\alpha_2+\alpha_3$.

The proof is similar to Appendix A in [2] and we omit the details.

The estimate (3.15) and Lemma 3.5 give us the long wave structure of the Green's function $\mathbb{G}$. On the other aspect, the long wave component $\mathbb{G}^{*L}$ of the singular support function $\mathbb{G}^*$ decays very fast due to the spectrum gap (3.4); meanwhile (3.5) ensures that there is no pole in $\mathbf{F}[\mathbb{G}^*]$. These two facts give that

$$\left|\mathbb{G}^{*L}(\vec{x},t)\right| \le Ce^{-t/C}, \tag{3.17}$$

for a positive constant C. Finally, one has the long wave structure of $\mathbb{G}_R$. For any positive constant C_1, when $|\vec{x}| < C_1 t$, there exists a positive constant C such that

$$\left|\partial_{\vec{x}}^{\alpha}\mathbb{G}_R^L(\vec{x},t)\right| = \left|\partial_{\vec{x}}^{\alpha}\left(\mathbb{G}^L - \mathbb{G}^{*L}\right)(\vec{x},t)\right| \le C\left((1+t)^{-(3+|\alpha|)/2} e^{-\frac{|\vec{x}|^2}{Ct}} + e^{-(|\vec{x}|+t)/C}\right). \tag{3.18}$$

Short wave component. The regularity stated in Lemma 3.1 combined with the spectrum gap stated in Lemma 2.1 and (3.4)-(3.5) leads to the following estimates for the short wave component of $\mathbb{G}_R$. For $0 \le |\alpha| < 6$, one has that

$$\begin{aligned} \left|\partial_{\vec{x}}^{\alpha}\mathbb{G}_R^S(\vec{x},t)\right| &= \left|\int_{\mathbb{R}^3} e^{i\vec{x}\cdot\vec{\xi}}\left(1-\Lambda(|\vec{\xi}|/\epsilon_0)\right)|\vec{\xi}|^{|\alpha|}\left(\mathbf{F}[\mathbb{G}_D]+\mathbf{F}[\mathbb{G}_H]-\mathbf{F}[\mathbb{G}_D^*]-\mathbf{F}[\mathbb{G}_H^*]\right)d\vec{\xi}\right| \\ &\le Ce^{-t/C}. \end{aligned} \tag{3.19}$$

3.2.2. Weighted energy method outside a finite Mach region. Outside the finite Mach region, one applies the weighted energy method to obtain the exponentially fast decaying structure of $\mathbb{G}_R$. Denote

$$V^R \equiv \left(1\ 0\ 0\ 0\ 0\right)\mathbb{G}_R, \quad \vec{U}^R \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbb{G}_R, \quad W^R \equiv \left(0\ 0\ 0\ 0\ 1\right)\mathbb{G}_R,$$

$$\mathbb{E}_1^1 \equiv \left(1\ 0\ 0\right)\mathbb{E}_1, \quad \vec{\mathbb{E}}_1^2 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbb{E}_1, \quad \mathbb{E}_1^3 \equiv \left(0\ 0\ 1\right)\mathbb{E}_1.$$

Define $\vec{Q}^R$ by the equation $-\nabla div\vec{Q}^R + \vec{Q}^R + 4\nabla W^R = 0$. Then the functions $(V^R, \vec{U}^R, W^R, \vec{Q}^R)$ satisfy:

$$\begin{cases} V_t^R + div\vec{U}^R = \mathbb{E}_1^1, \\ \vec{U}_t^R + R\nabla V^R + R\nabla W^R + \vec{U}^R = \vec{\mathbb{E}}_1^2, \\ W_t^R + \frac{R}{C_v} div\vec{U}^R + \frac{div\vec{Q}^R}{C_v} = \mathbb{E}_1^3, \\ -\nabla div\vec{Q}^R + \vec{Q}^R + 4\nabla W^R = 0, \\ (V^R, \vec{U}^R, W^R)^T(\vec{x}, 0) = \mathbb{E}_2(\vec{x}), \end{cases} \tag{3.20}$$

and one has the following estimates for the solution $(V^R, \vec{U}^R, W^R)$:

LEMMA 3.6. *For any positive constant C, when $|\vec{x}| > Ct$ and $0 \le |\alpha| \le 5$, there exist positive constants C_0 and C_1 such that the solution $(V^R, \vec{U}^R, W^R)$ of (3.20) satisfies*

$$\left|\partial_{\vec{x}}^\alpha V^R\right| + \left|\partial_{\vec{x}}^\alpha \vec{U}^R\right| + \left|\partial_{\vec{x}}^\alpha W^R\right| \le C_1 e^{-(|\vec{x}|+t)/C_0}.$$

Proof. One chooses the weighted function $\mathbf{W}(\vec{x}, t)$ as follows:

$$\mathbf{W}(\vec{x}, t) \equiv e^{(|\vec{x}| - at)/M}$$

with the constant a to be determined later and $M \gg 1$, and then

$$\mathbf{W}_t = -\frac{a}{M}\mathbf{W}, \qquad \nabla\mathbf{W} = \frac{\vec{x}}{M|\vec{x}|}\mathbf{W}. \tag{3.21}$$

Integrate the inner product of $\mathbf{W}(RV^R, \vec{U}^R, C_v W^R)$ and the first five equations in (3.20) with respect to $\vec{x}$ in $\mathbb{R}^3$ to yield that

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} \Bigg(R\mathbf{W}V^R\left(V_t^R + div\vec{U}^R - \mathbb{E}_1^1\right) + \mathbf{W}\vec{U}^R \cdot \left(\vec{U}_t^R + R\nabla V^R + R\nabla W^R + \vec{U}^R - \vec{\mathbb{E}}_1^2\right) \\ &+ C_v \mathbf{W} W^R \left(W_t^R + \frac{R}{C_v} div\vec{U}^R + \frac{div\vec{Q}^R}{C_v} - \mathbb{E}_1^3 \right)\Bigg) d\vec{x} \\ = &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \mathbf{W}\left(R\left(V^R\right)^2 + \vec{U}^R \cdot \vec{U}^R + C_v\left(W^R\right)^2\right) d\vec{x} \\ &+ \frac{a}{2M}\int_{\mathbb{R}^3} \mathbf{W}\left(R\left(V^R\right)^2 + \vec{U}^R \cdot \vec{U}^R + C_v\left(W^R\right)^2\right) d\vec{x} + \int_{\mathbb{R}^3} \mathbf{W}\left(W^R\right)^2 d\vec{x} \end{aligned}$$

$$
\begin{aligned}
&-\frac{R}{M}\int_{\mathbb{R}^3}\mathbf{W}\left(V^R+W^R\right)\vec{U}^R\cdot\frac{\vec{x}}{|\vec{x}|}d\vec{x}+\int_{\mathbb{R}^3}\mathbf{W}W^R div\vec{Q}^R d\vec{x}\\
&-\int_{\mathbb{R}^3}\mathbf{W}\left(V^R\mathbb{E}_1^1+\vec{U}^R\cdot\vec{\mathbb{E}}_1^2+W^R\mathbb{E}_1^3\right)d\vec{x}. \qquad (3.22)
\end{aligned}
$$

Here, the term $\int_{\mathbb{R}^3}\mathbf{W}W^R div\vec{Q}^R d\vec{x}$ could be estimated by using the final equation in (3.20):

$$
\begin{aligned}
\int_{\mathbb{R}^3}\mathbf{W}W^R div\vec{Q}^R d\vec{x}=&-\frac{1}{M}\int_{\mathbb{R}^3}\mathbf{W}W^R\frac{x}{|\vec{x}|}\cdot\vec{Q}^R d\vec{x}-\frac{1}{4}\int_{\mathbb{R}^3}\mathbf{W}\left(\nabla div\vec{Q}^R-\vec{Q}^R\right)\cdot\vec{Q}^R d\vec{x}\\
=&-\frac{1}{M}\int_{\mathbb{R}^3}\mathbf{W}W^R\frac{x}{|\vec{x}|}\cdot\vec{Q}^R d\vec{x}+\frac{1}{4}\int_{\mathbb{R}^3}\mathbf{W}\left(\left(div\vec{Q}^R\right)^2+\vec{Q}^R\cdot\vec{Q}^R\right)d\vec{x}\\
&+\frac{1}{4M}\int_{\mathbb{R}^3}\mathbf{W}\left(div\vec{Q}^R\right)\frac{\vec{x}}{|\vec{x}|}\cdot\vec{Q}^R d\vec{x}.
\end{aligned}
$$

Substitute it into (3.22) and choose $a>0$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\mathbf{W}\left(R\left(V^R\right)^2+\vec{U}^R\cdot\vec{U}^R+C_v\left(W^R\right)^2\right)d\vec{x}\\
&\quad+\frac{a}{2M}\int_{\mathbb{R}^3}\mathbf{W}\left(R\left(V^R\right)^2+\vec{U}^R\cdot\vec{U}^R+C_v\left(W^R\right)^2\right)d\vec{x}\\
\le& C\int_{\mathbb{R}^3}\mathbf{W}\left(\left|\mathbb{E}_1^1\right|^2+\left|\vec{\mathbb{E}}_1^2\right|^2+\left|\mathbb{E}_1^3\right|^2\right)d\vec{x}.
\end{aligned}
$$

It, together with Lemma 3.4, results in that there exists a positive constant $C_*>0$ such that

$$
\int_{\mathbb{R}^3}\mathbf{W}\left(\left(V^R\right)^2+\vec{U}^R\cdot\vec{U}^R+\left(W^R\right)^2\right)d\vec{x}\le C_*e^{-t/C_*}.
$$

Similarly, one could obtain the high order estimates:

$$
\int_{\mathbb{R}^3}\mathbf{W}\left(\left(\partial_{\vec{x}}^{\alpha}V^R\right)^2+\partial_{\vec{x}}^{\alpha}\vec{U}^R\cdot\partial_{\vec{x}}^{\alpha}\vec{U}^R+\left(\partial_{\vec{x}}^{\alpha}W^R\right)^2\right)d\vec{x}\le C_*e^{-t/C_*}\ \text{for } 0<|\alpha|\le 7.
$$

By Sobolev inequality, it follows that there exist $C_0,C_1>0$ such that for $0\le|\alpha|\le 5$,

$$
\sup_{(\vec{x},t)\in\mathbb{R}^3\times\mathbb{R}^+}e^{(|\vec{x}|-at)/2M}\left(\left|\partial_{\vec{x}}^{\alpha}V^R\right|+\left|\partial_{\vec{x}}^{\alpha}\vec{U}^R\right|+\left|\partial_{\vec{x}}^{\alpha}W^R\right|\right)\le C_0e^{-t/C_0},
$$

and thus when $|\vec{x}|>2at$ and $0\le|\alpha|\le 5$,

$$
\left|\partial_{\vec{x}}^{\alpha}V^R\right|+\left|\partial_{\vec{x}}^{\alpha}\vec{U}^R\right|+\left|\partial_{\vec{x}}^{\alpha}W^R\right|\le C_1e^{-(|\vec{x}|+t)/C_0}.
$$

□

Finally, we conclude the pointwise structure of the Green's function in space and time variables from (3.9), Lemma 3.2, (3.18), (3.19) and Lemma 3.6.

4. Nonlinear stability: proof of the main Theorem

The solution of the nonlinear equations could be represented by the Green's function $\mathbb{G}(\vec{x},t)$ and Duhamel's principle:

$$\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)=\int_{\mathbb{R}^3}\mathbb{G}(\vec{x}-\vec{y},t)\begin{pmatrix}\sigma_0\\ \vec{u}_0\\ \Theta_0\end{pmatrix}(\vec{y})d\vec{y}$$
$$+\int_0^t\int_{\mathbb{R}^3}\mathbb{G}(\vec{x}-\vec{y},t-s)\begin{pmatrix}-div(\sigma\vec{u})\\ -\vec{u}\cdot\nabla\vec{u}-\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\\ -\vec{u}\cdot\nabla\Theta-\frac{R}{C_v}\Theta div\vec{u}+\frac{\vec{u}\cdot\vec{u}}{C_v}+\frac{\sigma div\vec{q}}{C_v(\sigma+1)}\end{pmatrix}(\vec{y},s)d\vec{y}ds. \quad (4.1)$$

Since the system (1.4) is quasi-linear, the representation (4.1) contains derivatives and one needs to obtain a priori estimates for the highest order derivatives from energy method before we investigate the wave structure by (4.1) and the pointwise estimate of the Green's function.

4.1. A priori estimate from energy method. We pose the following a priori assumption:

$$N(T)\equiv\sup_{0<t<T}\left\{\|(\sigma,\vec{u},\Theta)(\cdot,t)\|_{H^l(\mathbb{R}^3)}\right\}\le\delta_0,\quad 0<\delta_0\ll1, l\ge4. \quad (4.2)$$

It, combined with Sobolev inequality, yields that

$$\sum_{0\le|\alpha|\le l-2}\sup_{0<t<T,\ \vec{x}\in\mathbb{R}^3}\left|\partial_{\vec{x}}^{\alpha}(\sigma,\vec{u},\Theta)(\vec{x},t)\right|\le C\delta_0. \quad (4.3)$$

Estimate A. Integrate the inner product of $(R\sigma,\vec{u},C_v\Theta)$ and the first five equations of (1.4) over $\mathbb{R}^3$ to yield:

$$\begin{aligned}0=&\int_{\mathbb{R}^3}(R\sigma\sigma_t+R\sigma div\vec{u}+R\sigma div(\sigma\vec{u})\\&+\vec{u}\cdot\vec{u}_t+R\vec{u}\cdot\nabla\sigma+R\vec{u}\cdot\nabla\Theta+\vec{u}\cdot\vec{u}+\vec{u}\cdot(\vec{u}\cdot\nabla\vec{u})+R\vec{u}\cdot\left(\frac{\Theta-\sigma}{\sigma+1}\right)\nabla\sigma\\&+C_v\Theta\Theta_t+R\Theta div\vec{u}+\frac{\Theta div\vec{q}}{\sigma+1}+C_v\Theta\vec{u}\cdot\nabla\Theta+R\Theta^2 div\vec{u}-\Theta\vec{u}\cdot\vec{u}\Big)d\vec{x}\\=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(R\sigma^2+\vec{u}\cdot\vec{u}+C_v\Theta^2\right)d\vec{x}+\int_{\mathbb{R}^3}(\vec{u}\cdot\vec{u}+\Theta div\vec{q})d\vec{x}+\int_{\mathbb{R}^3}\mathbf{N}_1(\vec{x},t)d\vec{x},\end{aligned} \quad (4.4)$$

with

$$\begin{aligned}\mathbf{N}_1\equiv&R\sigma div(\sigma\vec{u})+\vec{u}\cdot(\vec{u}\cdot\nabla\vec{u})+R\vec{u}\cdot\left(\frac{\Theta-\sigma}{\sigma+1}\right)\nabla\sigma\\&+C_v\Theta\vec{u}\cdot\nabla\Theta+R\Theta^2 div\vec{u}-\Theta\vec{u}\cdot\vec{u}-\Theta\frac{\sigma}{\sigma+1}div\vec{q}.\end{aligned}$$

To treat the term $\int_{\mathbb{R}^3}\Theta div\vec{q}d\vec{x}$ in (4.4), one uses the last equation of (1.4):

$$\begin{aligned}0=&\int_{\mathbb{R}^3}\vec{q}\cdot\left(-\nabla div\vec{q}+\vec{q}+4\nabla\Theta+4\Theta\left(\Theta^2+3\Theta+3\right)\nabla\Theta\right)d\vec{x}\\=&\int_{\mathbb{R}^3}\left((div\vec{q})^2+\vec{q}\cdot\vec{q}+4\vec{q}\cdot\nabla\Theta+4\vec{q}\cdot\Theta\left(\Theta^2+3\Theta+3\right)\nabla\Theta\right)d\vec{x}.\end{aligned} \quad (4.5)$$

The nonlinear term $\int_{\mathbb{R}^3}\mathbf{N}_1(\vec{x},t)d\vec{x}$ could be estimated as follows:

$$\left|\int_{\mathbb{R}^3}\mathbf{N}_1(\vec{x},t)d\vec{x}\right|\le C\delta_0\left(\int_{\mathbb{R}^3}\nabla\sigma\cdot\nabla\sigma d\vec{x}+\int_{\mathbb{R}^3}\nabla\Theta\cdot\nabla\Theta d\vec{x}\right.$$
$$\left.+\int_{\mathbb{R}^3}\vec{u}\cdot\vec{u}d\vec{x}+\sum_{j=1}^{3}\int_{\mathbb{R}^3}\nabla u_j\cdot\nabla u_j d\vec{x}+\int_{\mathbb{R}^3}\vec{q}\cdot\vec{q}d\vec{x}\right). \quad (4.6)$$

Combining (4.4)+$\frac{1}{4}$(4.5) and (4.6), one has that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(R\sigma^2+\vec{u}\cdot\vec{u}+C_v\Theta^2\right)d\vec{x}+\frac{1}{8}\int_{\mathbb{R}^3}\left(4\vec{u}\cdot\vec{u}+\vec{q}\cdot\vec{q}+(div\vec{q})^2\right)d\vec{x}$$
$$\le C\delta_0\left(\int_{\mathbb{R}^3}\nabla\sigma\cdot\nabla\sigma d\vec{x}+\int_{\mathbb{R}^3}\nabla\Theta\cdot\nabla\Theta d\vec{x}+\sum_{j=1}^{3}\int_{\mathbb{R}^3}\nabla u_j\cdot\nabla u_j d\vec{x}\right). \quad (4.7)$$

Estimate B. Differentiate (1.4) with respect to $x_k(k=1,2,3)$ and integrate its inner product with $(R\sigma_{x_k},\vec{u}_{x_k},C_v\Theta_{x_k})$ over $\mathbb{R}^3$ and consider the sum with respect to k to result in:

$$\begin{aligned}0=&\sum_{k=1}^{3}\int_{\mathbb{R}^3}\Big(R\sigma_{x_k}\sigma_{x_kt}+R\sigma_{x_k}div\vec{u}_{x_k}+R\sigma_{x_k}div(\sigma\vec{u})_{x_k}\\&+\vec{u}_{x_k}\cdot\vec{u}_{x_kt}+R\vec{u}_{x_k}\cdot\nabla\sigma_{x_k}+R\vec{u}_{x_k}\cdot\nabla\Theta_{x_k}+\vec{u}_{x_k}\cdot\vec{u}_{x_k}\\&+\vec{u}_{x_k}\cdot(\vec{u}\cdot\nabla\vec{u})_{x_k}+R\vec{u}_{x_k}\cdot\left(\left(\frac{\Theta-\sigma}{\sigma+1}\right)\nabla\sigma\right)_{x_k}\\&+C_v\Theta_{x_k}\Theta_{x_kt}+R\Theta_{x_k}div\vec{u}_{x_k}+\Theta_{x_k}div\vec{q}_{x_k}+C_v\Theta_{x_k}(\vec{u}\cdot\nabla\Theta)_{x_k}\\&+R\Theta_{x_k}(\Theta div\vec{u})_{x_k}-\Theta_{x_k}(\vec{u}\cdot\vec{u})_{x_k}+\Theta_{x_k}\left(\frac{\sigma div\vec{q}}{C_v(\sigma+1)}\right)_{x_k}\Big)d\vec{x}\\=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(R\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+C_v\nabla\Theta\cdot\nabla\Theta\right)d\vec{x}\\&+\int_{\mathbb{R}^3}\left(\sum_{k=1}^{3}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla div\vec{q}\right)d\vec{x}+\int_{\mathbb{R}^3}\mathbf{N}_2(\vec{x},t)d\vec{x},\end{aligned} \quad (4.8)$$

with

$$\begin{aligned}\mathbf{N}_2\equiv&R\nabla\sigma\cdot\nabla div(\sigma\vec{u})+\sum_{k=1}^{3}\vec{u}_{x_k}\cdot(\vec{u}\cdot\nabla\vec{u})_{x_k}+\sum_{k=1}^{3}R\vec{u}_{x_k}\cdot\left(\left(\frac{\Theta-\sigma}{\sigma+1}\right)\nabla\sigma\right)_{x_k}\\&+C_v\Theta_{x_k}(\vec{u}\cdot\nabla\Theta)_{x_k}+R\Theta_{x_k}(\Theta div\vec{u})_{x_k}-\Theta_{x_k}(\vec{u}\cdot\vec{u})_{x_k}-\Theta_{x_k}\left(\frac{\sigma}{\sigma+1}div\vec{q}\right)_{x_k}.\end{aligned}$$

Similar to (4.5) and (4.6), one has the following estimates:

$$\begin{aligned}0=&\int_{\mathbb{R}^3}\vec{q}_{x_k}\cdot\left(-\nabla div\vec{q}_{x_k}+\vec{q}_{x_k}+4\nabla\Theta_{x_k}+4\left(\Theta(\Theta^2+3\Theta+3)\nabla\Theta\right)_{x_k}\right)d\vec{x}\\=&\int_{\mathbb{R}^3}\left((div\vec{q}_{x_k})^2+\vec{q}_{x_k}\cdot\vec{q}_{x_k}+4\vec{q}_{x_k}\cdot\nabla\Theta_{x_k}+4\vec{q}_{x_k}\cdot\left(\Theta(\Theta^2+3\Theta+3)\nabla\Theta\right)_{x_k}\right)d\vec{x},\end{aligned} \quad (4.9)$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\mathbf{N}_2(\vec{x},t)d\vec{x}\right.\\&\quad\left.+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{R\sigma\nabla\sigma\cdot\nabla\sigma}{\sigma+1}+\sum_{k=1}^{3}\frac{\Theta-\sigma}{\Theta+1}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\frac{C_v(\Theta^2+2\Theta-\sigma)}{(\Theta+1)^2}\nabla\Theta\cdot\nabla\Theta\right)d\vec{x}\right|\\&\le C\delta_0\left(\int_{\mathbb{R}^3}\nabla\sigma\cdot\nabla\sigma d\vec{x}+\sum_{j=1}^{3}\int_{\mathbb{R}^3}\nabla u_j\cdot\nabla u_j d\vec{x}+\int_{\mathbb{R}^3}\nabla\Theta\cdot\nabla\Theta d\vec{x}\right.\\&\quad\left.+\int_{\mathbb{R}^3}(div\vec{q})^2d\vec{x}+\int_{\mathbb{R}^3}\nabla div\vec{q}\cdot\nabla div\vec{q}d\vec{x}\right).\end{aligned}\tag{4.10}$$

The estimates (4.8), (4.9) and (4.10) result in

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{R}{\sigma+1}\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\frac{\sigma+1}{\Theta+1}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\frac{C_v(\sigma+1)}{(\Theta+1)^2}\nabla\Theta\cdot\nabla\Theta\right)d\vec{x}\\&\quad+\frac{1}{8}\int_{\mathbb{R}^3}\left(\sum_{k=1}^{3}(4\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\vec{q}_{x_k}\cdot\vec{q}_{x_k})+\nabla div\vec{q}\cdot\nabla div\vec{q}\right)d\vec{x}\\&\le C\delta_0\left(\int_{\mathbb{R}^3}\nabla\sigma\cdot\nabla\sigma d\vec{x}+\int_{\mathbb{R}^3}(div\vec{q})^2d\vec{x}+\int_{\mathbb{R}^3}\vec{q}\cdot\vec{q}d\vec{x}\right).\end{aligned}\tag{4.11}$$

Here, we use the last equation in (1.4) and a priori Assumption (4.2) to obtain:

$$\left|\int_{\mathbb{R}^3}\nabla\Theta\cdot\nabla\Theta d\vec{x}\right|\le C\left(\int_{\mathbb{R}^3}\nabla div\vec{q}\cdot\nabla div\vec{q}d\vec{x}+\int_{\mathbb{R}^3}\vec{q}\cdot\vec{q}d\vec{x}\right).\tag{4.12}$$

It, together with (4.7) yields

$$\begin{aligned}&\tfrac{d}{dt}\int_{\mathbb{R}^3}\left(R\sigma^2+\vec{u}\cdot\vec{u}+C_v\Theta^2+\tfrac{R}{\sigma+1}\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\tfrac{\sigma+1}{\Theta+1}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\tfrac{C_v(\sigma+1)}{(\Theta+1)^2}\nabla\Theta\cdot\nabla\Theta\right)d\vec{x}\\&\quad+\frac{1}{8}\int_{\mathbb{R}^3}\left(8\vec{u}\cdot\vec{u}+\vec{q}\cdot\vec{q}+(div\vec{q})^2+\sum_{k=1}^{3}(4\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\vec{q}_{x_k}\cdot\vec{q}_{x_k})+\nabla div\vec{q}\cdot\nabla div\vec{q}\right)d\vec{x}\\&\le C\delta_0\int_{\mathbb{R}^3}\nabla\sigma\cdot\nabla\sigma d\vec{x},\end{aligned}\tag{4.13}$$

and thus

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\sigma^2+\vec{u}\cdot\vec{u}+\Theta^2+\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta\right)(\vec{x},t)d\vec{x}\\&\quad+C\int_0^t\int_{\mathbb{R}^3}\left(\vec{u}\cdot\vec{u}+\vec{q}\cdot\vec{q}+(div\vec{q})^2+\sum_{k=1}^{3}(\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\vec{q}_{x_k}\cdot\vec{q}_{x_k})+\nabla div\vec{q}\cdot\nabla div\vec{q}\right)(\vec{x},s)d\vec{x}ds\\&\le C\int_{\mathbb{R}^3}\left(\sigma^2+\vec{u}\cdot\vec{u}+\Theta^2+\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta\right)(\vec{x},0)d\vec{x}\\&\quad+C\delta_0\int_0^t\int_{\mathbb{R}^3}(\nabla\sigma\cdot\nabla\sigma)(\vec{x},s)d\vec{x}ds.\end{aligned}\tag{4.14}$$

Estimate C. Differentiate the first and the fifth equations of (1.4) with respect to t and the second to the fourth equations with respect to $\vec{x}$ to yield that:

$$\begin{cases} \sigma_{tt}-R\Delta\sigma-R\Delta\Theta+\sigma_t=-div(\sigma\vec{u})-div(\sigma\vec{u})_t+div(\vec{u}\cdot\nabla\vec{u})+div\left(\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\right),\\ \Theta_{tt}-\frac{R^2}{C_v}\Delta\sigma-\frac{R^2}{C_v}\Delta\Theta+\frac{1}{C_v}div\vec{q}_t\\ =-(\vec{u}\cdot\nabla\Theta)_t-\frac{R}{C_v}(\Theta div\vec{u})_t+\left(\frac{\vec{u}\cdot\vec{u}}{C_v}\right)_t+\left(\frac{\sigma div\vec{q}}{C_v(\sigma+1)}\right)_t+div(\vec{u}\cdot\nabla\vec{u})+div\left(\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\right). \end{cases} \tag{4.15}$$

Integrate the inner product of $(R\sigma_t, C_v\Theta_t)$ and (4.15) over $(0,t)\times\mathbb{R}^3$ to result in:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}\left(R\sigma_t^2+C_v\Theta_t^2+R^2\nabla\sigma\cdot\nabla\sigma+R^2\nabla\Theta\cdot\nabla\Theta+R^2\nabla\sigma\cdot\nabla\Theta\right)(\vec{x},s)d\vec{x}\Big|_{s=0}^{t}\\ &\quad+\int_0^t\int_{\mathbb{R}^3}\left(R\sigma_t^2+\frac{1}{4}\left((div\vec{q}_t)^2+\vec{q}_t\cdot\vec{q}_t\right)\right)(\vec{x},s)d\vec{x}ds\\ &=\int_0^t\int_{\mathbb{R}^3}\mathbf{N}_3(\vec{x},s)d\vec{x}ds\\ &\equiv\int_0^t\int_{\mathbb{R}^3}R\sigma_t\left(-div(\sigma\vec{u})-div(\sigma\vec{u})_t+div(\vec{u}\cdot\nabla\vec{u})+div\left(\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\right)\right)(\vec{x},s)d\vec{x}ds\\ &\quad+\int_0^t\int_{\mathbb{R}^3}\Theta_t\left(-C_v(\vec{u}\cdot\nabla\Theta)_t-R(\Theta div\vec{u})_t+(\vec{u}\cdot\vec{u})_t+\left(\frac{\sigma div\vec{q}}{\sigma+1}\right)_t\right.\\ &\quad\left.+C_v div(\vec{u}\cdot\nabla\vec{u})+C_v div\left(\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\right)\right)(\vec{x},s)d\vec{x}ds\\ &\quad-\int_0^t\int_{\mathbb{R}^3}\vec{q}_t\cdot\left(\Theta(\Theta^2+3\Theta+3)\nabla\Theta\right)_t(\vec{x},s)d\vec{x}ds. \end{aligned} \tag{4.16}$$

Here, we use the last equation of (1.4) to obtain:

$$\begin{aligned} \int_{\mathbb{R}^3}\Theta_t div\vec{q}_t d\vec{x}&=-\int_{\mathbb{R}^3}\nabla\Theta_t\cdot\vec{q}_t d\vec{x}\\ &=\frac{1}{4}\int_{\mathbb{R}^3}\left((div\vec{q}_t)^2+\vec{q}_t\cdot\vec{q}_t\right)d\vec{x}+\int_{\mathbb{R}^3}\vec{q}_t\cdot\left(\Theta(\Theta^2+3\Theta+3)\nabla\Theta\right)_t d\vec{x}. \end{aligned}$$

The nonlinear estimate is similar to (4.10):

$$\begin{aligned} \left|\int_0^t\int_{\mathbb{R}^3}\mathbf{N}_3(\vec{x},s)d\vec{x}ds\right|\le& C\delta_0\int_{\mathbb{R}^3}\left(\sigma_t^2+\Theta_t^2+\nabla\sigma\cdot\nabla\sigma+\nabla\Theta\cdot\nabla\Theta+\nabla\sigma\cdot\nabla\Theta\right)(\vec{x},t)d\vec{x}\\ &+C\delta_0\int_{\mathbb{R}^3}\left(\sigma_t^2+\Theta_t^2+\nabla\sigma\cdot\nabla\sigma+\nabla\Theta\cdot\nabla\Theta+\nabla\sigma\cdot\nabla\Theta\right)(\vec{x},0)d\vec{x}\\ &+C\delta_0\int_0^t\int_{\mathbb{R}^3}\left(\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^{3}\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta+\vec{u}\cdot\vec{u}\right.\\ &\left.+(div\vec{q})^2+\vec{q}_t\cdot\vec{q}_t+(div\vec{q}_t)^2\right)(\vec{x},s)d\vec{x}ds. \end{aligned} \tag{4.17}$$

Here, we use the Equations (1.4) and a priori Assumption (4.2) to obtain

$$\sigma_t,\vec{u}_t,\Theta_t\le C\delta_0,$$

$$\left|\int_0^t\int_{\mathbb{R}^3}\sigma_t^2(\vec{x},s)d\vec{x}ds\right|\le C\int_0^t\int_{\mathbb{R}^3}\left(\nabla\sigma\cdot\nabla\sigma+(div\vec{u})^2\right)(\vec{x},s)d\vec{x}ds,$$

$$\left|\int_0^t\int_{\mathbb{R}^3}\Theta_t^2(\vec{x},s)d\vec{x}ds\right|\leq C\int_{\mathbb{R}^3}\left((div\vec{q})^2+(div\vec{u})^2\right)(\vec{x},s)d\vec{x}ds$$
$$+C\delta_0\int_0^t\int_{\mathbb{R}^3}(\nabla\Theta\cdot\nabla\Theta+\vec{u}\cdot\vec{u})(\vec{x},s)d\vec{x}ds.$$

Combining (4.16) and (4.17), one has that:

$$\begin{aligned}&\int_{\mathbb{R}^3}(\sigma_t^2+\Theta_t^2+\nabla\sigma\cdot\nabla\sigma+\nabla\Theta\cdot\nabla\Theta)(\vec{x},t)d\vec{x}+C\int_0^t\int_{\mathbb{R}^3}\left(\sigma_t^2+(div\vec{q}_t)^2+\vec{q}_t\cdot\vec{q}_t\right)(\vec{x},s)d\vec{x}ds\\&\leq C\int_{\mathbb{R}^3}(\sigma_t^2+\Theta_t^2+\nabla\sigma\cdot\nabla\sigma+\nabla\Theta\cdot\nabla\Theta)(\vec{x},0)d\vec{x}\\&\quad+C\delta_0\int_0^t\int_{\mathbb{R}^3}\left(\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^3\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta+\vec{u}\cdot\vec{u}+(div\vec{q})^2\right)(\vec{x},s)d\vec{x}ds.\end{aligned}\tag{4.18}$$

Next, integrate the product of σ and the first equation of (4.15) to result in

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}\sigma^2(\vec{x},s)d\vec{x}\Big|_{s=0}^t+\int_0^t\int_{\mathbb{R}^3}\left(-\sigma_t^2+\frac{R}{2}\nabla\sigma\cdot\nabla\sigma\right)(\vec{x},s)d\vec{x}ds\\&\leq\frac{R}{2}\int_0^t\int_{\mathbb{R}^3}(\nabla\Theta\cdot\nabla\Theta)(\vec{x},s)d\vec{x}ds+C\delta_0\int_0^t\int_{\mathbb{R}^3}\left(\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^3\vec{u}_{x_k}\cdot\vec{u}_{x_k}\right)(\vec{x},s)d\vec{x}ds.\end{aligned}\tag{4.19}$$

Now, choosing $0<\delta_0\ll\delta_1\ll1$ and considering (4.14)+(4.18)+δ_1(4.19) combined with (4.12), one has that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\sigma^2+\vec{u}\cdot\vec{u}+\Theta^2+\sigma_t^2+\Theta_t^2+\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^3\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta\right)(\vec{x},t)d\vec{x}\\&\quad+C\int_0^t\int_{\mathbb{R}^3}\Big(\vec{u}\cdot\vec{u}+\vec{q}\cdot\vec{q}+\sigma_t^2+\nabla\sigma\cdot\nabla\sigma+\vec{q}_t\cdot\vec{q}_t+(div\vec{q})^2+(div\vec{q}_t)^2\\&\quad+\sum_{k=1}^3(\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\vec{q}_{x_k}\cdot\vec{q}_{x_k})+\nabla div\vec{q}\cdot\nabla div\vec{q}\Big)(\vec{x},s)d\vec{x}ds\\&\leq C\int_{\mathbb{R}^3}\left(\sigma^2+\vec{u}\cdot\vec{u}+\Theta^2+\nabla\sigma\cdot\nabla\sigma+\sum_{k=1}^3\vec{u}_{x_k}\cdot\vec{u}_{x_k}+\nabla\Theta\cdot\nabla\Theta\right)(\vec{x},0)d\vec{x}.\end{aligned}\tag{4.20}$$

Thus, we finish the estimate for the solution in H^1 norm.

High order estimates. The estimates for high order derivatives could be obtained similarly. Here, we state the conclusion and omit the details:

$$\begin{aligned}&\|\sigma(\cdot,t)\|_{H^l(\mathbb{R}^3)}+\|\vec{u}(\cdot,t)\|_{H^l(\mathbb{R}^3)}+\|\Theta(\cdot,t)\|_{H^l(\mathbb{R}^3)}\\&\quad+C\int_0^t\left(\|\nabla\sigma(\cdot,t)\|_{H^{l-1}(\mathbb{R}^3)}+\|\vec{u}(\cdot,t)\|_{H^l(\mathbb{R}^3)}+\|\nabla\Theta(\cdot,t)\|_{H^{l-1}(\mathbb{R}^3)}\right)ds\\&\leq C\left(\|\sigma(\cdot,0)\|_{H^l(\mathbb{R}^3)}+\|\vec{u}(\cdot,0)\|_{H^l(\mathbb{R}^3)}+\|\Theta(\cdot,0)\|_{H^l(\mathbb{R}^3)}\right).\end{aligned}\tag{4.21}$$

Thus, we verify the a priori Assumption (4.2).

REMARK 4.1. The local existence can be done by using the standard iteration arguments and fixed point theorem. Then, by the standard continuity argument, the global existence theory and the bounded estimate for the solution in $H^6(\mathbb{R}^3)$ norm in Theorem 1.1 follows from (4.21) and (1.9).

4.2. Pointwise structure. We go back to the solution representation (4.1). When the initial data $(\sigma_0,\vec{u}_0,\Theta_0)(\vec{x})$ satisfy the initial condition (1.6), from Theorem 3.1, one has that when $0\le|\alpha|\le 2$, there exist constants $C_I>0$ and $C>1$ such that

$$\begin{aligned}&\left|\partial_{\vec{x}}^{\alpha}\int_{\mathbb{R}^3}\mathbb{G}(\vec{x}-\vec{y},t)\begin{pmatrix}\sigma_0\\ \vec{u}_0\\ \Theta_0\end{pmatrix}(\vec{y})d\vec{y}\right|\\ \le&\left|\int_{\mathbb{R}^3}\partial_{\vec{x}}^{\alpha}(\mathbb{G}-\mathbb{G}_S^*)(\vec{x}-\vec{y},t)\begin{pmatrix}\sigma_0\\ \vec{u}_0\\ \Theta_0\end{pmatrix}(\vec{y})d\vec{y}\right|+\left|\int_{\mathbb{R}^3}\mathbb{G}_S^*(\vec{x}-\vec{y},t)\partial_{\vec{y}}^{\alpha}\begin{pmatrix}\sigma_0\\ \vec{u}_0\\ \Theta_0\end{pmatrix}(\vec{y})d\vec{y}\right|\\ \le& C_I\epsilon\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{(3+|\alpha|)/2}}+e^{-(|\vec{x}|+t)/C}\right).\end{aligned}\tag{4.22}$$

Based on the linear estimate (4.22), one poses the following ansatz assumption: For $0\le|\alpha|\le 2$,

$$\left|\partial_{\vec{x}}^{\alpha}\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\le 2C_I\epsilon\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{(3-|\alpha|)/2}}+e^{-(|\vec{x}|+t)/C}\right).\tag{4.23}$$

REMARK 4.2. The ansatz Assumption (4.23) is quite different from the linear estimates (4.22) and seems unusual. In the linear level, the derivatives gain extra decaying rates in time while in nonlinear level the derivatives decay even slower than the solution itself. The reason is as follows: according to the solution representation (4.1), in the justification of the ansatz Assumption (4.23) for derivatives, the third and the fourth order derivatives are involved; however, they are not included in the Assumption (4.23) and one could only use bounded estimates from energy method for them. Therefore, the lack of pointwise space-time structures of the third and the fourth order derivatives in (4.23) results in the loss of decaying rate for the first and the second order derivatives in (4.23). One could improve the initial regularity to compensate this kind of loss.

The ansatz Assumption (4.23) yields that

$$\begin{aligned}\mathbf{N}(\vec{x},t)\equiv&\begin{pmatrix}-div(\sigma\vec{u})\\ -\vec{u}\cdot\nabla\vec{u}-\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\\ -\vec{u}\cdot\nabla\Theta-\frac{R}{C_v}\Theta div\vec{u}+\frac{\vec{u}\cdot\vec{u}}{C_v}+\frac{\sigma div\vec{q}}{C_v(\sigma+1)}\end{pmatrix}(\vec{x},t)\\ =&O(1)\epsilon^2\left(\frac{e^{-\frac{2|\vec{x}|^2}{Ct}}}{(1+t)^{5/2}}+e^{-2(|\vec{x}|+t)/C}\right),\end{aligned}$$

$$\begin{aligned}\partial_{\vec{x}}\mathbf{N}(\vec{x},t)=&\partial_{\vec{x}}\begin{pmatrix}-div(\sigma\vec{u})\\ -\vec{u}\cdot\nabla\vec{u}-\frac{R(\Theta-\sigma)}{\sigma+1}\nabla\sigma\\ -\vec{u}\cdot\nabla\Theta-\frac{R}{C_v}\Theta div\vec{u}+\frac{\vec{u}\cdot\vec{u}}{C_v}+\frac{\sigma div\vec{q}}{C_v(\sigma+1)}\end{pmatrix}(\vec{x},t)\\ =&O(1)\epsilon^2\left(\frac{e^{-\frac{2|\vec{x}|^2}{Ct}}}{(1+t)^{2}}+e^{-2(|\vec{x}|+t)/C}\right).\end{aligned}$$

Here, the estimate for $div\vec{q}$ could be obtained from (1.8) and (4.23):

$$div\vec{q}=O(1)\epsilon\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{3/2}}+e^{-(|\vec{x}|+t)/C}\right).$$

Thus, for $0<\epsilon\ll 1$, one has that

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^3}\mathbb{G}(\vec{x}-\vec{y},t-s)\mathbf{N}(\vec{y},s)d\vec{x}ds\right|\\
\le&\left|\int_0^t\int_{\mathbb{R}^3}(\mathbb{G}-\mathbb{G}_S^*)(\vec{x}-\vec{y},t-s)\mathbf{N}(\vec{y},s)d\vec{x}ds\right|+\left|\int_0^t\int_{\mathbb{R}^3}\mathbb{G}_S^*(\vec{x}-\vec{y},t-s)\mathbf{N}(\vec{y},s)d\vec{x}ds\right|\\
\le& C\epsilon^2\left(\int_0^t\int_{\mathbb{R}^3}\frac{e^{-\frac{|\vec{x}-\vec{y}|^2}{C(t-s)}}}{(1+t-s)^{3/2}}\frac{e^{-\frac{2|\vec{y}|^2}{Cs}}}{(1+s)^{5/2}}d\vec{y}ds+\int_0^t e^{-(t-s)/C}\frac{e^{-\frac{|\vec{x}|^2}{Cs}}}{(1+s)^2}ds\right)\\
\le& C\epsilon^2\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{3/2}}+e^{-(|\vec{x}|+t)/C}\right)\ll C_I\epsilon\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{3/2}}+e^{-(|\vec{x}|+t)/C}\right).
\end{aligned}$$

It, together with (4.22) verifies the ansatz Assumption (4.23) for $|\alpha|=0$.

The justification of (4.23) for $|\alpha|=1$ and $|\alpha|=2$ is similar to the one for $|\alpha|=0$. One uses the following nonlinear estimates:

$$\begin{aligned}
\partial_{\vec{x}}^2\mathbf{N}(\vec{x},t)=&O(1)\left(\left|\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\left|\partial_{\vec{x}}^3\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|+\left|\partial_{\vec{x}}\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\left|\partial_{\vec{x}}^2\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\right)\\
=&O(1)\epsilon^2\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{(1+t)^{3/2}}+e^{-(|\vec{x}|+t)/C}\right),
\end{aligned}$$

$$\begin{aligned}
\partial_{\vec{x}}^3\mathbf{N}(\vec{x},t)=&O(1)\left(\left|\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\left|\partial_{\vec{x}}^4\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\right.\\
&\left.+\left|\partial_{\vec{x}}\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|\left|\partial_{\vec{x}}^3\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|+\left|\partial_{\vec{x}}^2\begin{pmatrix}\sigma\\ \vec{u}\\ \Theta\end{pmatrix}(\vec{x},t)\right|^2\right)\\
=&O(1)\epsilon^2\left(\frac{e^{-\frac{|\vec{x}|^2}{Ct}}}{1+t}+e^{-(|\vec{x}|+t)/C}\right).
\end{aligned}$$

Here, for the third and the fourth order derivatives, we use the bounded estimates from energy method. It is also the reason for the loss of decaying rates for the first and the second order derivatives in (4.23).

Thus, we verify the ansatz Assumption (4.23) and finish the proof of Theorem 1.1.

Acknowledgement. S. Deng is supported by National Nature Science Foundation of China 11831011 and Shanghai Science and Technology Innovation Action Plan No. 21JC1403600. W. Wang is supported by National Nature Science Foundation of China 11871341, 12071152, and 12371222. F. Xie is supported by National Nature Science Foundation of China 12271359, 11831003, 12161141004 and Shanghai Science and Technology Innovation Action Plan No. 20JC1413000. X. Yang is supported by National Natural Science Foundation of China 12271356, Shanghai Science and Technology Innovation Action Plan No. 21JC1403600 and the Fundamental Research Funds for the Central Universities. The authors would also like to thank the referees for the valuable comments and suggestions.

Appendix. In the appendix, we list details of tedious computations omitted during the construction of the Green's function in Section 3.

A.1. High order expansion of spectra. The following expansions of spectra for $|\vec{\xi}|\to\infty$ could be obtained after a direct computation:

$$\begin{cases}\lambda_3(\vec{\xi})=a_{3,0}+a_{3,-2}|\vec{\xi}|^{-2}+a_{3,-4}|\vec{\xi}|^{-4}+a_{3,-6}|\vec{\xi}|^{-6}+a_{3,-8}|\vec{\xi}|^{-8}+O(|\vec{\xi}|^{-10}),\\ \lambda_4(\vec{\xi})=\sqrt{\dfrac{R(C_v+R)}{C_v}}i|\vec{\xi}|\left(1+a_{-1}|\vec{\xi}|^{-2}+a_{-3}|\vec{\xi}|^{-4}+a_{-5}|\vec{\xi}|^{-6}+a_{-7}|\vec{\xi}|^{-8}\right)\\ \qquad+\left(a_0+a_{-2}|\vec{\xi}|^{-2}+a_{-4}|\vec{\xi}|^{-4}+a_{-6}|\vec{\xi}|^{-6}+a_{-8}|\vec{\xi}|^{-8}\right)+O(|\vec{\xi}|^{-9}),\\ \lambda_5(\vec{\xi})=-\sqrt{\dfrac{R(C_v+R)}{C_v}}i|\vec{\xi}|\left(1+a_{-1}|\vec{\xi}|^{-2}+a_{-3}|\vec{\xi}|^{-4}+a_{-5}|\vec{\xi}|^{-6}+a_{-7}|\vec{\xi}|^{-8}\right)\\ \qquad+\left(a_0+a_{-2}|\vec{\xi}|^{-2}+a_{-4}|\vec{\xi}|^{-4}+a_{-6}|\vec{\xi}|^{-6}+a_{-8}|\vec{\xi}|^{-8}\right)+O(|\vec{\xi}|^{-9}).\end{cases}$$

A.2. Singular support functions. Now we turn to study the structure of the singular support function $\mathbb{G}^*$. Substitute the explicit representations (3.1) of approximated spectra into $e^{\lambda_k^* t}(k=3,4,5)$ to yield

$$\begin{aligned}e^{\lambda_3^* t}&=e^{a_{3,0}t}e^{\frac{a_{3,-2}}{1+|\vec{\xi}|^2}t+\frac{a^*_{3,-4}}{(1+|\vec{\xi}|^2)^2}t+\frac{a^*_{3,-6}}{(1+|\vec{\xi}|^2)^3}t+\frac{a^*_{3,-8}}{(1+|\vec{\xi}|^2)^4}t-\frac{J_0^*}{(1+|\vec{\xi}|^2)^5}t}\\&=e^{a_{3,0}t}+te^{a_{3,0}t}\frac{a_{3,-2}}{1+|\vec{\xi}|^2}+te^{a_{3,0}t}\frac{a^*_{3,-4}+\frac12(a_{3,-2})^2t}{\left(1+|\vec{\xi}|^2\right)^2}+\mathbf{O}_1(\vec{\xi},t),\end{aligned}\tag{A.1}$$

$$\begin{aligned}e^{\lambda_4^* t}&=e^{\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\left(1+\frac{a_{-1}}{1+|\vec{\xi}|^2}+\frac{a^*_{-3}}{(1+|\vec{\xi}|^2)^2}+\frac{a^*_{-5}}{(1+|\vec{\xi}|^2)^3}+\frac{a^*_{-7}}{(1+|\vec{\xi}|^2)^4}+\frac{J^*_{11}}{(1+|\vec{\xi}|^2)^5}\right)}\\&\quad\cdot e^{a_0t}e^{\frac{a_{-2}}{1+|\vec{\xi}|^2}t+\frac{a^*_{-4}}{(1+|\vec{\xi}|^2)^2}t+\frac{a^*_{-6}}{(1+|\vec{\xi}|^2)^3}t+\frac{a^*_{-8}}{(1+|\vec{\xi}|^2)^4}t-\frac{J^*_{12}}{(1+|\vec{\xi}|^2)^5}t}\\&=e^{\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t+a_0t}\left(1+\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\cdot\frac{a_{-1}}{1+|\vec{\xi}|^2}\right.\\&\qquad\left.+\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\cdot\frac{a^*_{-3}+\frac12(a_{-1})^2\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t}{\left(1+|\vec{\xi}|^2\right)^2}\right)\\&\qquad\cdot\left(1+t\frac{a_{-2}}{1+|\vec{\xi}|^2}+t\frac{a^*_{-4}+\frac12(a_{-2})^2t}{\left(1+|\vec{\xi}|^2\right)^2}\right)+\mathbf{O}_2(\vec{\xi},t),\end{aligned}$$

and

$$\begin{aligned}e^{\lambda_5^* t}&=e^{-\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\left(1+\frac{a_{-1}}{1+|\vec{\xi}|^2}+\frac{a^*_{-3}}{(1+|\vec{\xi}|^2)^2}+\frac{a^*_{-5}}{(1+|\vec{\xi}|^2)^3}+\frac{a^*_{-7}}{(1+|\vec{\xi}|^2)^4}+\frac{J^*_{11}}{(1+|\vec{\xi}|^2)^5}\right)}\\&\quad\cdot e^{a_0t}e^{\frac{a_{-2}}{1+|\vec{\xi}|^2}t+\frac{a^*_{-4}}{(1+|\vec{\xi}|^2)^2}t+\frac{a^*_{-6}}{(1+|\vec{\xi}|^2)^3}t+\frac{a^*_{-8}}{(1+|\vec{\xi}|^2)^4}t-\frac{J^*_{12}}{(1+|\vec{\xi}|^2)^5}t}\\&=e^{-\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t+a_0t}\left(1-\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\cdot\frac{a_{-1}}{1+|\vec{\xi}|^2}\right.\end{aligned}$$

$$-\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t\cdot\frac{a_{-3}^*-\frac{1}{2}(a_{-1})^2\sqrt{\frac{R(C_v+R)}{C_v}}i|\vec{\xi}|t}{\left(1+|\vec{\xi}|^2\right)^2}\Bigg)$$

$$\cdot\left(1+t\frac{a_{-2}}{1+|\vec{\xi}|^2}+t\frac{a_{-4}^*+\frac{1}{2}(a_{-2})^2t}{\left(1+|\vec{\xi}|^2\right)^2}\right)+\mathbf{O}_3(\vec{\xi},t).$$

Here, the functions $\mathbf{O}_j(\vec{\xi},t)(j=1,2,3)$ are analytic in $\vec{\xi}\in\mathbf{D}_{1/2}$ and satisfy

$$\left|\mathbf{O}_j(\vec{\xi},t)\right|=O(1)\frac{e^{a_{3,0}t/2}}{\left(1+|\vec{\xi}|^2\right)^3}\qquad\text{for }\vec{\xi}\in\mathbf{D}_{1/2}.$$

One goes on to expand the coefficients and the matrices $r_k^*\beta_k^*$ in (3.7):

$$\begin{aligned}&-\frac{p(\vec{\xi})\left(C_v\lambda_3^*\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)}{C_v(\lambda_3^*-\lambda_4^*)(\lambda_3^*-\lambda_5^*)\left(1+|\vec{\xi}|^2\right)}r_3^*\beta_3^*=\begin{pmatrix}-\frac{R\left(4+C_va_{3,0}\right)}{C_va_{3,0}c^2}&0&-\frac{R}{c^2}\\0&0&0\\-\frac{R^2}{C_vc^2}&0&-\frac{R^2a_{3,0}}{(4+C_va_{3,0})c^2}\end{pmatrix}\\&+\frac{1}{1+|\vec{\xi}|^2}\begin{pmatrix}c_3^{1,1}&-\frac{4+C_va_{3,0}}{C_vc^2}i\vec{\xi}&c_3^{1,5}\\-\frac{R\left(4+C_va_{3,0}\right)}{C_vc^2}i\vec{\xi}^T&\frac{a_{3,0}\left(4+C_va_{3,0}\right)}{C_vc^2}\frac{i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2}&-\frac{Ra_{3,0}}{c^2}i\vec{\xi}^T\\c_3^{5,1}&-\frac{Ra_{3,0}}{C_vc^2}i\vec{\xi}&c_3^{5,5}\end{pmatrix}+\mathbf{O}_{M3}(\vec{\xi},t),\end{aligned}\tag{A.2}$$

$$\begin{aligned}&-\frac{p(\vec{\xi})\left(C_v\lambda_4^*\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)}{C_v(\lambda_4^*-\lambda_3^*)(\lambda_4^*-\lambda_5^*)\left(1+|\vec{\xi}|^2\right)}r_4^*\beta_4^*=\begin{pmatrix}\frac{R}{2c^2}&0&\frac{R}{2c^2}\\0&0&0\\\frac{R^2}{2C_vc^2}&0&\frac{R^2}{2C_vc^2}\end{pmatrix}\\&+\frac{1}{1+|\vec{\xi}|^2}\cdot\begin{pmatrix}c_4^{1,1}&\frac{4+C_va_{3,0}}{2C_vc^2}i\vec{\xi}+\frac{c_4^{1,2}i|\vec{\xi}|}{1+|\vec{\xi}|^2}i\vec{\xi}&c_4^{1,3}\\\frac{R(4+C_va_{3,0})}{2C_vc^2}i\vec{\xi}^T+\frac{c_4^{1,2}Ri|\vec{\xi}|}{1+|\vec{\xi}|^2}i\vec{\xi}^T&\frac{i\vec{\xi}^T\cdot i\vec{\xi}}{2}+\frac{c_4^{2,2,2}\cdot i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2}&\frac{Ra_{3,0}}{2c^2}i\vec{\xi}^T+\frac{c_4^{2,3}i|\vec{\xi}|\cdot i\vec{\xi}^T}{1+|\vec{\xi}|^2}\\\frac{Rc_4^{1,3}}{C_v}&\frac{Ra_{3,0}}{2C_vc^2}i\vec{\xi}+\frac{c_4^{2,3}i|\vec{\xi}|\cdot i\vec{\xi}}{C_v\left(1+|\vec{\xi}|^2\right)}&c_4^{3,3}\end{pmatrix}\\&+\frac{1}{1+|\vec{\xi}|^2}\cdot\begin{pmatrix}-\frac{R(4+C_va_{3,0}-C_va_0)}{2C_vc^3}i|\vec{\xi}|&\frac{i|\vec{\xi}|}{2c}i\vec{\xi}&-\frac{R\left(a_{3,0}+a_0\right)}{2c^3}i|\vec{\xi}|\\\frac{iR|\vec{\xi}|}{2c}i\vec{\xi}^T&\frac{c_4^{2,2,1}i|\vec{\xi}\cdot i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2}&\frac{Ri|\vec{\xi}|\cdot i\vec{\xi}^T}{2c}\\-\frac{R^2\left(a_{3,0}+a_0\right)}{2C_vc^3}i|\vec{\xi}|&\frac{Ri|\vec{\xi}|\cdot i\vec{\xi}}{2C_vc}&\frac{R^2\left(4-C_va_{3,0}+C_va_0\right)}{2(C_v)^2c^3}i|\vec{\xi}|\end{pmatrix}+\mathbf{O}_{M4}(\vec{\xi},t),\end{aligned}\tag{A.3}$$

and

$$\begin{aligned}&-\frac{p(\vec{\xi})\left(C_v\lambda_5^*\left(1+|\vec{\xi}|^2\right)+4|\vec{\xi}|^2\right)}{C_v(\lambda_5^*-\lambda_3^*)(\lambda_5^*-\lambda_3^*)\left(1+|\vec{\xi}|^2\right)}r_5^*\beta_5^*=\begin{pmatrix}\frac{R}{2c^2}&0&\frac{R}{2c^2}\\0&0&0\\\frac{R^2}{2C_vc^2}&0&\frac{R^2}{2C_vc^2}\end{pmatrix}\\&+\frac{1}{1+|\vec{\xi}|^2}\cdot\begin{pmatrix}c_4^{1,1}&\frac{4+C_va_{3,0}}{2C_vc^2}i\vec{\xi}+\frac{c_4^{1,2}i|\vec{\xi}|}{1+|\vec{\xi}|^2}i\vec{\xi}&c_4^{1,3}\\\frac{R(4+C_va_{3,0})}{2C_vc^2}i\vec{\xi}^T+\frac{c_4^{1,2}Ri|\vec{\xi}|}{1+|\vec{\xi}|^2}i\vec{\xi}^T&\frac{i\vec{\xi}^T\cdot i\vec{\xi}}{2}+\frac{c_4^{2,2,2}\cdot i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2}&\frac{Ra_{3,0}}{2c^2}i\vec{\xi}^T+\frac{c_4^{2,3}i|\vec{\xi}|\cdot i\vec{\xi}^T}{1+|\vec{\xi}|^2}\\\frac{Rc_4^{1,3}}{C_v}&\frac{Ra_{3,0}}{2C_vc^2}i\vec{\xi}+\frac{c_4^{2,3}i|\vec{\xi}|\cdot i\vec{\xi}}{C_v\left(1+|\vec{\xi}|^2\right)}&c_4^{3,3}\end{pmatrix}\end{aligned}$$

$$-\frac{1}{1+|\vec{\xi}|^2}\cdot\begin{pmatrix} -\frac{R(4+C_v a_{3,0}-C_v a_0)}{2C_v c^3}i|\vec{\xi}| & \frac{i|\vec{\xi}|}{2c}i\vec{\xi} & -\frac{R\left(a_{3,0}+a_0\right)}{2c^3}i|\vec{\xi}| \\ \frac{iR|\vec{\xi}|}{2c}i\vec{\xi}^T & \frac{c_4^{2,2,1}i|\vec{\xi}\cdot i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2} & \frac{Ri|\vec{\xi}|\cdot i\vec{\xi}^T}{2c} \\ -\frac{R^2\left(a_{3,0}+a_0\right)}{2C_v c^3}i|\vec{\xi}| & \frac{Ri|\vec{\xi}|\cdot i\vec{\xi}}{2C_v c} & \frac{R^2\left(4-C_v a_{3,0}+C_v a_0\right)}{2(C_v)^2c^3}i|\vec{\xi}| \end{pmatrix}+\mathbf{O}_{M5}(\vec{\xi},t), \tag{A.4}$$

where the functions $\mathbf{O}_{Mk}(\vec{\xi})(k=3,4,5)$ are analytic in $\vec{\xi}\in\mathbf{D}_{1/2}$ and satisfy

$$\left|\mathbf{O}_{Mk}(\vec{\xi})\right|=O(1)\frac{1}{\left(1+|\vec{\xi}|\right)^3}\ \text{for}\ \vec{\xi}\in\mathbf{D}_{1/2},$$

and

$$c\equiv\sqrt{R\left(1+\frac{R}{C_v}\right)}.$$

The expansion (A.1), together with (A.2) and the definition of $p(\vec{\xi})$ in Lemma 3.1, yields that

$$\mathbf{F}[\mathbb{G}_D^*]=e^{-t}\begin{pmatrix}0&0&0\\0&I_3-\frac{\vec{\xi}^T\vec{\xi}}{1+|\vec{\xi}|^2}&0\\0&0&0\end{pmatrix}+e^{a_{3,0}t}\begin{pmatrix}-\frac{R(4+C_v a_{3,0})}{C_v a_{3,0}c^2}&0&-\frac{R}{c^2}\\0&0&0\\-\frac{R^2}{C_v c^2}&0&-\frac{R^2 a_{3,0}}{(4+C_v a_{3,0})c^2}\end{pmatrix}$$
$$+\frac{e^{a_{3,0}t}}{1+|\vec{\xi}|^2}\begin{pmatrix}0&-\frac{4+C_v a_{3,0}}{C_v c^2}i\vec{\xi}&0\\-\frac{R(4+C_v a_{3,0})}{C_v c^2}i\vec{\xi}^T&\frac{a_{3,0}(4+C_v a_{3,0})}{C_v c^2}\frac{i\vec{\xi}^T\cdot i\vec{\xi}}{1+|\vec{\xi}|^2}&-\frac{Ra_{3,0}}{c^2}i\vec{\xi}^T\\0&-\frac{Ra_{3,0}}{C_v c^2}i\vec{\xi}&0\end{pmatrix}+\frac{\mathbb{C}_{M3}e^{a_{3,0}t}}{1+|\vec{\xi}|^2}+\mathbf{O}_{f3}(\vec{\xi},t), \tag{A.5}$$

with the entries of matrix $\mathbb{C}_{M3}$ being first order polynomial of t, and coefficients depending only on $a_{3,0},a_{3,-2},a_0$ and a_{-1}. Moreover, $\mathbf{O}_{f3}(\vec{\xi},t)$ is analytic in $\vec{\xi}\in\mathbf{D}_{1/2}$, and satisfies

$$\left|\mathbf{O}_{f3}(\vec{\xi},t)\right|=O(1)\frac{e^{-t/C}}{\left(1+|\vec{\xi}|\right)^3}\ \text{for}\ \vec{\xi}\in\mathbf{D}_{1/2}.$$

Thus, (A.5), together with Lemma 2.2, results in that

$$\Bigg|\mathbf{F}^{-1}\left[\mathbf{F}[\mathbb{G}_D^*]\right](\vec{x},t)-\begin{pmatrix}0&0&0\\0&I_3&0\\0&0&0\end{pmatrix}e^{-t}\delta(\vec{x})+\begin{pmatrix}0&0&0\\0&\partial_{\vec{x}}^T\mathbb{Y}(\vec{x})*_{\vec{x}}\partial_{\vec{x}}&0\\0&0&0\end{pmatrix}e^{-t}\mathbb{Y}(\vec{x})$$
$$-\begin{pmatrix}-\frac{R\left(4+C_v a_{3,0}\right)}{C_v a_{3,0}c^2}&0&-\frac{R}{c^2}\\0&0&0\\-\frac{R^2}{C_v c^2}&0&-\frac{Ra_{3,0}}{(4+C_v a_{3,0})c^2}\end{pmatrix}e^{a_{3,0}t}\delta(\vec{x})$$
$$-\begin{pmatrix}0&-\frac{4+C_v a_{3,0}}{C_v c^2}\partial_{\vec{x}}&0\\-\frac{R\left(4+C_v a_{3,0}\right)}{C_v c^2}\partial_{\vec{x}}^T&\frac{a_{3,0}\left(4+C_v a_{3,0}\right)}{C_v c^2}\partial_{\vec{x}}^T\mathbb{Y}(\vec{x})*_{\vec{x}}\partial_{\vec{x}}&-\frac{Ra_{3,0}}{c^2}\partial_{\vec{x}}^T\\0&-\frac{Ra_{3,0}}{C_v c^2}\partial_{\vec{x}}&0\end{pmatrix}e^{a_{3,0}t}\mathbb{Y}(\vec{x})-\mathbb{C}_{M3}e^{a_{3,0}t}\mathbb{Y}(\vec{x})\Bigg|$$

$$\leq Ce^{-(|\vec{x}|+t)/C}, \tag{A.6}$$

for a positive constant C. Here, one denotes $\partial_{\vec{x}}=(\partial_{x_1},\partial_{x_2},\partial_{x_3})$ and $\partial_{\vec{x}}^T=(\partial_{x_1},\partial_{x_2},\partial_{x_3})^T$ and $\mathbb{Y}(\vec{x})$ stands for the 3-dimensional Yukawa potential with unit mass (i.e. $\mathbb{Y}\equiv(1+\Delta)^{-1}\delta(\vec{x})$):

$$\mathbb{Y}(\vec{x})\equiv\mathbf{F}^{-1}\left[\frac{1}{1+|\vec{\xi}|^2}\right]=-\frac{e^{-|\vec{x}|}}{4\pi|\vec{x}|}. \tag{A.7}$$

Similarly, one has that

$$\begin{aligned}
\mathbf{F}[\mathbb{G}_H^*]=&\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}C_v&0&C_v\\0&0&0\\R&0&R\end{pmatrix}\left(\mathbf{C}(\vec{\xi},t)-a_{-1}c^2\mathbf{S}(\vec{\xi},t)\right)\\
&+\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}4+C_va_{3,0}-C_va_0&0&C_v(a_{3,0}+a_0)\\0&0&0\\R(a_{3,0}+a_0)&0&\frac{R(4-C_va_{3,0}+C_va_0)}{C_v}\end{pmatrix}\mathbf{S}(\vec{\xi},t)\\
&+e^{a_0t}\begin{pmatrix}0&i\vec{\xi}&0\\Ri\vec{\xi}^T&0&Ri\vec{\xi}^T\\0&\frac{R}{C_v}i\vec{\xi}&0\end{pmatrix}\mathbf{S}(\vec{\xi},t)+\frac{\mathbf{C}-a_{-1}c^2\mathbf{S}}{1+|\vec{\xi}|^2}\begin{pmatrix}0&0&0\\0&i\vec{\xi}^T\cdot i\vec{\xi}&0\\0&0&0\end{pmatrix}e^{a_0t}\\
&+\mathbb{C}_{Mc}^1e^{a_0t}\frac{\mathbf{C}(\vec{\xi},t)}{1+|\vec{\xi}|^2}+\mathbb{C}_{Ms}^1e^{a_0t}\frac{\mathbf{S}(\vec{\xi},t)}{1+|\vec{\xi}|^2}+\mathbb{C}_{Mc}^2e^{a_0t}\frac{i\vec{\xi}\mathbf{C}(\vec{\xi},t)}{1+|\vec{\xi}|^2}+\mathbb{C}_{Ms}^2e^{a_0t}\frac{i\vec{\xi}\mathbf{S}(\vec{\xi},t)}{1+|\vec{\xi}|^2}\\
&+\mathbb{C}_{Mc}^3e^{a_0t}\frac{i\vec{\xi}\cdot i|\vec{\xi}|\mathbf{C}(\vec{\xi},t)}{\left(1+|\vec{\xi}|^2\right)^2}+\mathbb{C}_{Ms}^3e^{a_0t}\frac{i\vec{\xi}\cdot i|\vec{\xi}|\mathbf{S}(\vec{\xi},t)}{1+|\vec{\xi}|^2}\\
&+\mathbb{C}_{Mc}^4e^{a_0t}\frac{i\vec{\xi}^Ti\vec{\xi}\mathbf{C}(\vec{\xi},t)}{\left(1+|\vec{\xi}|^2\right)^2}+\mathbb{C}_{Ms}^4e^{a_0t}\frac{i\vec{\xi}^Ti\vec{\xi}\mathbf{S}(\vec{\xi},t)}{1+|\vec{\xi}|^2}+\mathbf{O}_{f4}(\vec{\xi},t)
\end{aligned}$$

where $\mathbf{O}_{f4}(\vec{\xi},t)$ is analytic and also satisfies

$$\left|\mathbf{O}_{f4}(\vec{\xi},t)\right|=O(1)\frac{e^{-t/C}}{\left(1+|\vec{\xi}|\right)^3}$$

in $\vec{\xi}\in\mathbf{D}_{1/2}$, and then

$$\begin{aligned}
\Bigg|\mathbb{G}_H^*-&\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}C_v&0&C_v\\0&0&0\\R&0&R\end{pmatrix}\left(\mathbb{C}(\vec{x},t)-a_{-1}c^2\mathbb{S}(\vec{x},t)\right)\\
&-\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}4+C_va_{3,0}-C_va_0&0&C_v(a_{3,0}+a_0)\\0&0&0\\R(a_{3,0}+a_0)&0&\frac{R(4-C_va_{3,0}+C_va_0)}{C_v}\end{pmatrix}\mathbb{S}(\vec{x},t)\\
&-e^{a_0t}\begin{pmatrix}0&\partial_{\vec{x}}&0\\R\partial_{\vec{x}}^T&0&R\partial_{\vec{x}}^T\\0&\frac{R}{C_v}\partial_{\vec{x}}&0\end{pmatrix}\mathbb{S}(\vec{x},t)-e^{a_0t}\begin{pmatrix}0&0&0\\0&\partial_{\vec{x}}^T\left(\mathbb{C}-a_{-1}c^2\mathbb{S}\right)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})&0\\0&0&0\end{pmatrix}\\
&-\mathbb{C}_{M4}^1e^{a_0t}\mathbb{C}(\vec{x},t)*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}_{M5}^1e^{a_0t}\mathbb{S}(\vec{x},t)*_{\vec{x}}\mathbb{Y}(\vec{x})\\
&-\mathbb{C}_{M4}^2e^{a_0t}\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}_{M5}^2e^{a_0t}\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})\\
&-\mathbb{C}_{M4}^3e^{a_0t}\partial_t\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}_{M5}^3e^{a_0t}\partial_t\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})
\end{aligned}$$

$$
\begin{aligned}
&-\mathbb{C}^4_{M4}e^{a_0t}\partial_{\vec{x}}\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}^4_{M5}e^{a_0t}\partial_{\vec{x}}\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})\Big|\\
\le & Ce^{-(|\vec{x}|+t)/C}.
\end{aligned}
\tag{A.8}
$$

Here, one denotes

$$
\begin{cases}\mathbf{C}(\vec{\xi},t)\equiv\cos c|\vec{\xi}|t,\\ \mathbf{S}(\vec{\xi},t)\equiv\frac{\sin c|\vec{\xi}|t}{c|\vec{\xi}|},\end{cases}\quad\text{and}\quad\begin{cases}\mathbb{C}(\vec{x},t)\equiv\mathbf{F}^{-1}\left[\cos c|\vec{\xi}|t\right],\\ \mathbb{S}(\vec{x},t)\equiv\mathbf{F}^{-1}\left[\frac{\sin c|\vec{\xi}|t}{c|\vec{\xi}|}\right].\end{cases}
$$

The entries of matrices $\mathbb{C}^j_{M4}$ and $\mathbb{C}^j_{M5}(j=1,2,3,4)$ are also first order polynomial of t and with coefficients depending only on $a_{3,0},a_{3,-2},a_0$ and a_{-1}.

The estimates (A.6) and (A.8) conclude the singular structure in the Green's function $\mathbb{G}$: Denote

$$
\begin{aligned}
\mathbb{G}^*_S(\vec{x},t)\equiv&\begin{pmatrix}0&0&0\\0&I_3&0\\0&0&0\end{pmatrix}e^{-t}\delta(\vec{x})-\begin{pmatrix}0&0&0\\0&\partial^T_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\partial_{\vec{x}}&0\\0&0&0\end{pmatrix}e^{-t}\mathbb{Y}(\vec{x})\\
&+\begin{pmatrix}-\frac{R(4+C_va_{3,0})}{C_va_{3,0}c^2}&0&-\frac{R}{c^2}\\0&0&0\\-\frac{R^2}{C_vc^2}&0&-\frac{R^2a_{3,0}}{(4+C_va_{3,0})c^2}\end{pmatrix}e^{a_{3,0}t}\delta(\vec{x})\\
&+\begin{pmatrix}0&-\frac{4+C_va_{3,0}}{C_vc^2}\partial_{\vec{x}}&0\\-\frac{R(4+C_va_{3,0})}{C_vc^2}\partial^T_{\vec{x}}&\frac{a_{3,0}(4+C_va_{3,0})}{C_vc^2}\partial^T_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\partial_{\vec{x}}&-\frac{Ra_{3,0}}{c^2}\partial^T_{\vec{x}}\\0&-\frac{Ra_{3,0}}{C_vc^2}\partial_{\vec{x}}&0\end{pmatrix}e^{a_{3,0}t}\mathbb{Y}(\vec{x})+\mathbb{C}_{M3}e^{a_{3,0}t}\mathbb{Y}(\vec{x})\\
&+\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}C_v&0&C_v\\0&0&0\\R&0&R\end{pmatrix}\left(\mathbb{C}(\vec{x},t)-a_{-1}c^2\mathbb{S}(\vec{x},t)\right)\\
&-\frac{e^{a_0t}}{C_v+R}\begin{pmatrix}4+C_va_{3,0}-C_va_0&0&C_v(a_{3,0}+a_0)\\0&0&0\\R(a_{3,0}+a_0)&0&\frac{R(4-C_va_{3,0}+C_va_0)}{C_v}\end{pmatrix}\mathbb{S}(\vec{x},t)\\
&-e^{a_0t}\begin{pmatrix}0&\partial_{\vec{x}}&0\\R\partial^T_{\vec{x}}&0&R\partial^T_{\vec{x}}\\0&\frac{R}{C_v}\partial_{\vec{x}}&0\end{pmatrix}\mathbb{S}(\vec{x},t)-e^{a_0t}\begin{pmatrix}0&0&0\\0&\partial^T_{\vec{x}}\left(\mathbb{C}-a_{-1}c^2\mathbb{S}\right)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})&0\\0&0&0\end{pmatrix}\\
&-\mathbb{C}^1_{M4}e^{a_0t}\mathbb{C}(\vec{x},t)*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}^1_{M5}e^{a_0t}\mathbb{S}(\vec{x},t)*_{\vec{x}}\mathbb{Y}(\vec{x})\\
&-\mathbb{C}^2_{M4}e^{a_0t}\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}^2_{M5}e^{a_0t}\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})\\
&-\mathbb{C}^3_{M4}e^{a_0t}\partial_t\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}^3_{M5}e^{a_0t}\partial_t\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})\\
&-\mathbb{C}^4_{M4}e^{a_0t}\partial_{\vec{x}}\mathbb{C}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x})-\mathbb{C}^4_{M5}e^{a_0t}\partial_{\vec{x}}\mathbb{S}(\vec{x},t)*_{\vec{x}}\partial_{\vec{x}}\mathbb{Y}(\vec{x})*_{\vec{x}}\mathbb{Y}(\vec{x}),
\end{aligned}
\tag{A.9}
$$

and one has Lemma 3.2.

REFERENCES

[1] C. Dafermos, *A system of hyperbolic conservation laws with frictional damping*, Z. Angew. Math. Phys., 46:294–307, 1995. 1

[2] S. Deng and S.-H. Yu, *Green's function and pointwise convergence for compressible Navier-Stokes equations*, Quart. Appl. Math., 75(3):433–503, 2017. 1, 2.2, 3.2.1

[3] L. Hsiao and T.-P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Commun. Math. Phys., 143:599–605, 1992. 1

[4] S. Kawashima, Y. Nikkuni, and S. Nishibata, *The initial value problem for hyperbolic-elliptic coupled systems and applications to radiation hydrodynamics*, in H. Freistuhler (ed.), Analysis of Systems of Conservation Laws, Chapman and Hall/CRC, 99:87–127, 1997. 1

[5] S. Kawashima and S. Nishibata, *A singular limit for hyperbolic-elliptic coupled systems in radiation hydrodynamics*, Indiana Univ. Math. J., 50(1):567–589, 2001. 1

[6] S. Kawashima, Y. Nikkuni, and S. Nishibata, *Large-time behavior of solutions to hyperbolic-elliptic coupled systems*, Arch. Ration. Mech. Anal., 170(4):297–329, 2003. 1

[7] C. Lattanzio, C. Mascia, and D. Serre, *Shock waves for radiative hyperbolic-elliptic systems*, Indiana Univ. Math. J., 56(5):2601–2640, 2007. 1

[8] H.-L. Li, T. Yang, and M. Zhong, *Green's function and pointwise space-time behaviors of the Vlasov-Poisson-Boltzmann system*, Arch. Ration. Mech. Anal., 235(2):1011–1057, 2020. 1

[9] C.J. Lin, J.F. Coulombel, and T. Goudon, *Shock profiles for non-equilibrium radiating gases*, Phys. D, 218(1):83–94, 2006. 1

[10] C.J. Lin, *Asymptotic stability of rarefaction waves in radiative hydrodynamics*, Commun. Math. Sci., 9(1):207–223, 2001. 1

[11] Y.C. Lin, M.J. Lyu, H. Wang, and K.C. Wu, *Space-time behavior of the solution to the Boltzmann equation with soft potentials*, J. Differ. Equ., 322:180–236, 2022. 1

[12] Y.C. Lin, H. Wang, and K.C. Wu, *Spatial behavior of the solution to the linearized Boltzmann equation with hard potentials*, J. Math. Phys., 61(2):021504, 2020. 1

[13] T.P. Liu and W. Wang, *The pointwise estimates of diffusion wave for the Navier-Stokes systems in odd multi-dimensions*, Commun. Math. Phys., 196(1):145–173, 1998. 1

[14] T.P. Liu and S.H. Yu, *The Green's function and large-time behavior of solutions for the one-dimensional Boltzmann equation*, Commun. Pure Appl. Math., 57(12):1543–1608, 2004. 1

[15] T.P. Liu and Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws*, Mem. Amer. Math. Soc., 125(599):1997. 1

[16] T. Nguyen, R.G. Plaza, and K. Zumbrun, *Stability of radiative shock profiles for hyperbolic-elliptic coupled systems*, Phys. D, 239(1):428–453, 2010. 1

[17] P. Qu and Y.J. Wang, *Global classical solutions to partially dissipative hyperbolic systems violating the Kawashima condition*, J. Math. Pures Appl., 109(9):93–146, 2018. 1

[18] C. Rohde and F. Xie, *Decay rates to viscous contact waves for a 1D compressible radiation hydrodynamics model*, Math. Models Meth. Appl. Sci., 23(3):441–469, 2013. 1

[19] Z. Tan and G. Wu, *Large time behavior of solutions for compressible Euler equations with damping in* R^3, J. Differ. Equ., 252(2):1546–1561, 2012. 1

[20] J. Wang and F. Xie, *Singular limit to strong contact discontinuity for a 1D compressible radiation hydrodynamics model*, SIAM J. Math. Anal., 43(3):1189–1204, 2011. 1

[21] J. Wang and F. Xie, *Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system*, J. Differ. Equ., 251(4-5):1030–1055, 2011. 1

[22] W.-K. Wang and T. Yang, *The pointwise estimates of solutions for Euler equations with damping in multi-dimensions*, J. Differ. Equ., 173(2):410–450, 2011. 1

[23] W.-K. Wang and T. Yang, *Existence and stability of planar diffusion waves for 2-D Euler equations with damping*, J. Differ. Equ., 242(1):40–71, 2007. 1

[24] Z. Wu and X. Miao, *Large time behavior for the system of compressible adiabatic flow through porous media in* R^3, J. Math. Anal. Appl., 472(1):112–132, 2019. 1

[25] F. Xie, *Nonlinear stability of combination of viscous contact wave with rarefaction waves for a 1D radiation hydrodynamics model*, Discrete Contin. Dyn. Syst. Ser. B, 17(3):1075–1100, 2012. 1