COMMUN. MATH. SCI.
Vol. 22, No. 4, pp. 1099–1132

VARIATIONAL APPROACH TO SIMULTANEOUS FUSION AND DENOISING OF COLOR IMAGES WITH DIFFERENT SPATIAL RESOLUTION*

CIRO D'APICE†, PETER I. KOGUT‡, ROSANNA MANZO§, AND CLAUDIA PIPINO¶

Abstract. We propose a new variational model in Sobolev-Orlicz spaces with non-standard growth conditions of the objective functional and discuss its applications to the simultaneous fusion and denoising of color images with different spatial resolution. The characteristic feature of the proposed model is that we deal with a constrained minimization problem that lives in variable Sobolev-Orlicz spaces where the variable exponent, which is associated with non-standard growth, is unknown a priori and it depends on a particular function that belongs to the domain of objective functional. In view of this, we discuss the consistency of the proposed model, give the scheme for its regularization, derive the corresponding optimality system, and propose an iterative algorithm for practical implementations.

Keywords. Inverse problem; image fusion; denoising, constrained minimization problems; approximation methods; Sobolev-Orlicz space.

AMS subject classifications. 90C90; 94A08.

1. Introduction

The synthesis of several source images of the same scene into a single image that would contain much more visual information (see, for instance, [6, 8, 38]) is an important issue appearing in various fields such as remote sensing, medical diagnosis, defect inspection, and military surveillance. Since the observed source images are inevitably corrupted by noise, they can be blurred, and arguably, are geometrically dissimilar. A very promising approach to image quality enhancement is to fuse several sources, with different degradations, together to extract as much useful information as possible.

A significant part of the existing fusion methods (the so-called pixel-level methods) is based on the estimation of the value for each point in the fused image through some feature selection rule [26]. In particular, several methods have been developed such as spatial domain fusion methods [37], transform domain fusion methods [33], variational methods based on fusing the gradient information [44], or their combinations [34]. In [29], the authors proposed a new variational model by fusing the first- and second-order gradient information from the source images. However, this approach has originally been aimed at the fusion of images without visible noise corruptions.

Regarding the fusion methods of the noisy source images, apparently, [39] was one of the first papers dedicated to this problem. The authors proposed a weighted variational method based on the total variation (TV) regularization and with some regularization parameter in the objective functional that trades off the fit to the noisy source images and the smoothness from TV. So, the TV regularization term was added to the proposed model to reduce the influence of the noise.

*Received: February 23, 2023; Accepted (in revised form): October 12, 2023. Communicated by Shi Jin.

†Dipartimento di Scienze Aziendali - Management and Innovation Systems, University of Salerno, 132, Via Giovanni Paolo II, Fisciano, SA, Italy (cdapice@unisa.it).

‡Department of Differential Equations, Oles Honchar Dnipro National University, Gagarin av., 72, 49010 Dnipro, Ukraine, EOS Data Analytics Ukraine, Gagarin av., 103a, Dnipro, Ukraine (p.kogut@i.ua, peter.kogut@eosda.com).

§Dipartimento di Scienze Politiche e della Comunicazione, University of Salerno, Via Giovanni Paolo II, 132, Fisciano (SA), Italy (rmanzo@unisa.it).

¶Dipartimento di Scienze Aziendali - Management and Innovation Systems, University of Salerno, 132, Via Giovanni Paolo II, Fisciano, SA, Italy (cpipino@unisa.it).

Another approach has been introduced in [32], where the authors considered fractional-order derivatives as regularization in the variational model for image fusion and denoising. Their goal was to obtain a fused image of high quality, preserving sharp edges while maintaining smoothness in homogeneous regions, even when the source images are corrupted by noise. To achieve this, the authors of [32] aim to match the fractional-order gradient of the fused image with a target fractional-order gradient, using either L^2-norm or L^1-norm. However, selecting the appropriate target fractional-order gradient is a challenging task, and the practical implementation of this approach becomes complicated as a result.

Recent papers [17, 19, 31] also deserve mention, where the authors address the contrast enhancement, multimodal image fusion, and denoising problem using different techniques, such as a Retinex-based variational model, a Siamese convolutional neural network, and quaternion-based dictionary learning with saturation-value Total Variation regularization.

In this paper, we consider a constrained minimization problem with a special objective functional. The main feature of this functional is the fact that it contains a spatially variable exponent characterizing the growth conditions of the objective functional and it can be seen as a replacement for the 1-norm in TV regularization. The idea of using a spatially varying exponent in a TV-like regularization method for image denoising dates back as early as 1997 [5] and was put into practice in 2006 [9]. Both papers as well as some subsequent articles try to tackle variants of the problem

$$J(u) = \mathcal{D}(u) + \lambda \int_\Omega |\nabla u(x)|^{p(\nabla u(x))}\, dx \longrightarrow \inf, \tag{1.1}$$

where the exponent depends directly on the image, e.g.,

$$p(\nabla u) = 1 + \frac{a^2}{a^2 + |\nabla G_\sigma * u|^2}.$$

Here, $(G_\sigma * v)(x)$ determines the convolution of function v with the 2-dimensional Gaussian filter kernel G_σ

$$G_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} \exp\left(-\frac{|x|^2}{2\sigma^2}\right), \quad x \in \mathbb{R}^2,\ \sigma > 0 \text{ is a fixed parameter}, \tag{1.2}$$

$$(\nabla G_\sigma * u)(x) = \int_{\mathbb{R}^2} \nabla G_\sigma(x-y)\widetilde{u}(y)\, dy, \quad \forall x \in \Omega, \tag{1.3}$$

$\widetilde{u}$ is zero-extension of u outside Ω, $|\xi|$ stands for the Euclidean norm of $\xi \in \mathbb{R}^2$ given by the rule $|\xi| = \sqrt{(\xi, \xi)}$,

It has been demonstrated that this model possesses some favorable properties, particularly when edge preservation and effective noise suppression are primary goals in image reconstruction.

Furthermore, this model has been introduced specifically to address the issue of staircasing [35], which refers to the regularizer's inclination towards piecewise constant functions. The appearance of the staircase effect is a notable drawback of the classical TV model. However, the non-convex model (1.1) did not gain significant attention for a long period due to its high numerical complexity and the absence of a rigorous mathematical substantiation of its consistency. Only partial solutions to this problem have been derived for a smoothed version of the integrand, using a weak notion of solution (see, for instance, [40]).

A recently developed alternative variant is the TV-like method [25], which computes the variable exponent p in an offline step and keeps it as a fixed parameter in the final optimization problem. This approach allows the exponent to vary based on spatial location, enabling users to locally select whether to preserve edges or smooth intensity variations. However, there are only two natural types of imaging problems where this approach can be applied:

- single-channel imaging where first the exponent is computed from the given data and then is applied as prior in the subsequent minimization problem;
- dual-channel imaging where the secondary channel provides the exponent map that is used for regularization of the primary channel.

Thus, this circumstance imposes significant limitations from a practical point of view, especially in the case of multi-spectral satellite noisy images, where different channels can differ drastically (for instance, red and infrared channels).

The main purpose of this paper is to describe a robust approach for the simultaneous fusion and denoising of non-smooth multispectral images defined on grids with different resolution using for that the energy functional with nonstandard growth. We use the L^1-norm of the noise in the minimization function and a special form of anisotropic diffusion tensor for the regularization term. By following this approach, we aim to increase the noise robustness of the proposed model albeit it makes such variational problem completely non-smooth, non-convex, and, hence, significantly more difficult from a minimization point of view.

The main characteristic feature of the proposed model is that we consider the energy functional with nonstandard growth for each spectral channel separately. Moreover, the edge information for the fusion of two images with different resolution is mainly accumulated in the variable exponents $p_1(x), p_2(x), \dots, p_m(x)$. The second point that should be emphasized is the fact that we do not predefine the variable exponents $p_i(x)$ a priori using for that the original noisy images, but instead, we associate these characteristics with each feasible solution.

In summary, the main contributions of our paper can be enumerated as follows:

- The variational statement for the simultaneous fusion and denoising of multispectral images with different spatial resolution in the form of minimization problem in Sobolev-Orlicz spaces with non-standard growth conditions of the objective functional;
- Rigorous substantiation of the well posedness of the variational problem with non-standard growth functional;
- The proof of existence result in the proposed variational problem;
- The iterative algorithm for numerical implementations;
- Derivation of the first order necessary conditions for the approximating problem;
- Numerical experiments to study the performance of the new approach.

The remainder of the paper is organized as follows: In Section 2 we provide a formal statement of the fusion and denoising problem and describe its main properties. Section 3 focuses on the well-posedness of the proposed model and the solvability issues. In Section 4 we discuss the possible ways for the relaxation of the minimization problem. Specifically, we introduce a family of minimization problems with fictitious control and show that each of these problems is solvable and their solutions converge to the solution of the original problem in an appropriate topology. In Section 5 we introduce an iterative algorithm for the approximate solution of the fusion problem and discuss its convergence properties. The derivation of optimality conditions for the approximating problem and their rigorous substantiation are provided in Section 7. Since the introduced iterative algorithm leads to the so-called weak solutions, possible ways for the relaxation of the main minimization problem are discussed in Section 7. Finally, for illustration, we give in Section 8 some results of numerical experiments with standard test images.

2. Preliminaries and main contribution

Let $\Omega\subset\mathbb{R}^2$ be a bounded connected open set with a sufficiently smooth boundary $\partial\Omega$ and nonzero Lebesgue measure. In the majority of cases, Ω can be interpreted as a rectangular domain. Let G_H and G_L be two sample grids on Ω such that $G_H=\widehat{G}_H\cap\Omega$ and $G_H=\widehat{G}_H\cap\Omega$, where

$$\widehat{G}_H=\left\{(x_i,y_j)\,\middle|\,\begin{array}{l}x_1=x_H,\ x_i=x_1+\Delta_{H,x}(i-1),\ i=1,\dots,N_x,\\ y_1=y_H,\ y_j=y_1+\Delta_{H,y}(j-1),\ j=1,\dots,N_y,\end{array}\right\},$$
$$\widehat{G}_L=\left\{(x_i,y_j)\,\middle|\,\begin{array}{l}x_1=x_L,\ x_i=x_1+\Delta_{L,x}(i-1),\ i=1,\dots,M_x,\\ y_1=y_L,\ y_j=y_1+\Delta_{L,y}(j-1),\ j=1,\dots,M_y,\end{array}\right\},$$

with some fixed points (x_H,y_H) and (x_L,y_L). Hereinafter, it is assumed that $N_x>>M_x$ and $N_y>>M_y$.

Let $S:G_H\to\mathbb{R}^m$ and $M:G_L\to\mathbb{R}^m$, $m\ge 1$, be a couple of multispectral images, containing the same scene albeit they are defined on grids with different resolution. The principle point is that the image with low resolution $M:G_L\to\mathbb{R}^m$ contains some extra objects that are invisible or absent in the image $S:G_H\to\mathbb{R}^m$. It is assumed that:

(i) Each of the given images $S:G_H\to\mathbb{R}^m$ and $M:G_L\to\mathbb{R}^m$ can be corrupted by some additive Gaussian noise with zero mean.

(ii) All spectral channels of the image $M=[M_1,M_2,\dots,M_m]$ have similar spectral characteristics to the corresponding channels of the image $S=[S_1,S_2,\dots,S_m]$, respectively;

(iii) The images $M:G_L\to\mathbb{R}^m$ and $S:G_H\to\mathbb{R}^m$ are rigidly co-registered. This means that the image M, after arguably some affine transformation and the image S after the resampling to the grid with low resolution G_L, could be successfully matched except for the zone where there are new objects.
In practice, the co-registration procedure is usually applied not to the original images directly, but rather to their spectral energies $Y_M:G_L\to\mathbb{R}$ and $Y_S:G_H\to\mathbb{R}$, where the last ones should be previously resampled to the grid of the low resolution G_L. Here,

$$Y_M(z):=\alpha_1M_1(z)+\alpha_2M_2(z)+\dots+\alpha_mM_m(z),\quad \forall z=(x,y)\in G_L,$$
$$Y_S(z):=\alpha_1S_1(z)+\alpha_2S_2(z)+\dots+\alpha_mS_m(z),\quad \forall z=(x,y)\in G_H$$

with appropriate weight coefficients α_i, $i=1,\dots,m$.

The main purpose of this paper is to present a robust approach for the simultaneous denoising and fusion of non-smooth multi-spectral images defined on grids with different resolution. With that in mind, we use a special form of anisotropic diffusion tensor for the regularization term and the L^1-norms for the fidelity terms. Namely, we deal with the following family of optimization problems:

$$\begin{aligned}J_i(v)=&\int_\Omega|\nabla v(x)|^{\mathfrak{F}(v(x))}\,dx+\lambda\int_\Omega|\nabla v(x)-\nabla S_i(x)|\,dx\\ &+\mu\int_\Omega|T_Sv(x)-S_i(x)|\,dx+\frac{1-\mu}{2}T_M\Big([(G_\sigma*v)(\cdot)-M_i(\cdot)]^2\Big)\quad\longrightarrow\inf,\end{aligned}\tag{2.1}$$

subject to the constraints

$$v \in W^{1,\mathfrak{F}(v(\cdot))}(\Omega), \quad 1 \leq \gamma_{i,0} \leq v(x) \leq \gamma_{i,1} \quad \text{a.e. in } \Omega, \tag{2.2}$$

where $i=1,\dots,m$, $S_i \in L^1(\Omega)$ and $M_i \in L^1(\Omega)$ are a particular spectral channel of the original noisy images $S=[S_1,S_2,\dots,S_m]^T \in L^1(\Omega;\mathbb{R}^m)$ and $M=[M_1,M_2,\dots,M_m]^T \in L^1(\Omega;\mathbb{R}^m)$, respectively, $\lambda>0$ and $\mu \in (0,1)$ are the tuning parameters, $W^{1,\mathfrak{F}(v(\cdot))}(\Omega)$ stands for the so-called Sobolev-Orlicz space associated with a feasible solution v, $T_S \in \mathcal{L}(L^1(\Omega))$ and $T_M \in \mathcal{L}(L^1(\Omega),\mathbb{R})$ are bounded linear operators with unbounded inverse,

$$\mathfrak{F}(v(x)) = 1 + g\left(\left|\left(\nabla G_\sigma * v\right)(x)\right|\right) \quad \text{in } \Omega, \tag{2.3}$$

and $g:[0,\infty) \to (0,\infty)$ is a continuous monotone decreasing function such that $g(0)=1$ and $g(t)>0$ for all $t>0$ with $\lim_{t\to+\infty} g(t)=0$.

In particular, if we set $p(x) := 1 + g(|(\nabla G_\sigma * v)(x)|)$, where the edge-stopping function $g(s)$ is taken in the form of the Cauchy law

$$g(t) = \frac{1}{1+(t/a)^2} \quad \text{with an appropriate } a>0, \tag{2.4}$$

it implies that $p(x) \approx 1$ in places in Ω where edges or discontinuities are present in the image $v(x)$, and $p(x) \approx 2$ in places where $v(x)$ is smooth or contains homogeneous features.

We define the parameters $\gamma_{i,0}$, $\gamma_{i,1}$, and the operator $T_M \in \mathcal{L}(L^1(\Omega),\mathbb{R})$, as follows:

$$\gamma_{i,0} = \min_{(x,y)\in G_H} S_i(x,y), \quad \gamma_{i,1} = \max_{(x,y)\in G_H} S_i(x,y), \quad T_M = \sum_{(x_i,y_j)\in S_L} \delta_{(x_i,y_j)}, \tag{2.5}$$

where $\delta_{(x_i,y_j)}$ is the Dirac's delta at the point (x_i,y_j) of the sample grid G_L.

It is worth emphasizing that, in contrast to the quadratic data-fitting term in the well-known model, introduced by Rudin et al. [36], we take the fidelity terms in L^1-norm just to increase the noise robustness of the model (2.1) albeit it makes such variational problem completely non-smooth and, hence, significantly more difficult from a minimization point of view.

Thus, the problem of simultaneous fusion and denoising of multi-spectral images with different spatial resolution consists in the generation of a new multi-spectral image $I^0 = \left[I_1^0, I_2^0, \ddots, I_m^0\right]^t : G_H \to \mathbb{R}^m$, which would be well defined on the entire grid G_H, such that

$$J_i(I_i^0) = \inf_{v\in\Xi_i} J_i(v), \quad \forall i=1,\dots,m, \tag{2.6}$$

where

$$\Xi_i = \left\{ u \in W^{1,\mathfrak{F}(v(\cdot))}(\Omega) \ : \ 1 \leq \gamma_{i,0} \leq u(x) \leq \gamma_{i,1} \quad \text{a.e. in } \Omega \right\} \tag{2.7}$$

stands for the set of feasible solutions to the minimization problem (2.1).

So, the main characteristic feature of the model (2.1) is the energy functional with nonstandard growth where the main information for the simultaneous fusion and denoising of images S and M is accumulated in the variable exponents $[\mathfrak{F}(v_1(x)),\dots,\mathfrak{F}(v_m(x))]$.

However, in contrast to [1, 9, 10, 28], we do not predefine the variable exponents $p(x)$ a priori using for that the original noisy images S or/and M, but instead we associate this characteristic with each feasible solution. As a result, we admit that each feasible solution to this problem lies in the corresponding individual functional space. Formally it means that we look for the true image $I^0 = \left[I_1^0, I_2^0, \ddots, I_m^0\right]^t$ such that

$$I^0 \in W^{1,\mathfrak{F}(I_1^0(\cdot))}(\Omega) \times W^{1,\mathfrak{F}(I_2^0(\cdot))}(\Omega) \times \cdots \times W^{1,\mathfrak{F}(I_m^0(\cdot))}(\Omega).$$

As follows from the definition of Sobolev-Orlicz space $W^{1,\mathfrak{F}(I_i^0(\cdot))}(\Omega)$, its regularity is completely determined by the exponent $\mathfrak{F}(I_i^0(\cdot))$ which depends on i-th spectral channel of the true image I^0 and, hence, is unknown a priori. Moreover, the exponents $\left\{\mathfrak{F}(I_1^0(\cdot)), \mathfrak{F}(I_2^0(\cdot)), \ldots, \mathfrak{F}(I_m^0(\cdot))\right\}$ may significantly differ from channel to channel. In particular, some pixels, which are the local minimum points in the red channel, become local maximum points in the near-infrared channel and vice versa. Moreover, the different feasible solutions $v \neq u$ to the above problem live in different functional spaces: we have $v \in W^{1,\mathfrak{F}(v(\cdot))}(\Omega)$ whereas $u \in W^{1,\mathfrak{F}(u(\cdot))}(\Omega)$. As a consequence, any minimizing sequence to this problem is a sequence living in the scale of variable spaces. As a result, notions such as convergence concept, compactness, density, and others should be specified for the case of variable Sobolev-Orlicz spaces.

Thus, although in the literature there are many approaches to the study of variational problems in abstract functional spaces, the above mentioned circumstances make the problem (2.1) rather challenging (see [9, 10] and [7, 12, 13, 20] for recent studies in this field).

3. Existence result

Our main intention in this section is to show that the constrained minimization problem (2.1)–(2.2) is consistent and admits at least one solution. Because of the specific form of the energy functional $J_i(v)$, the structure and main topological properties of the set of feasible solutions to the minimization problem (2.1)–(2.2) are challenging issues. So, the study of these issues is the main subject of this section (we can refer to [5, 7, 9, 14, 15] for some specific details that can appear in this case).

We begin with the following key assumptions:

(A1) The true intensities I_i^0 of all spectral channels for the retrieved image $I^0 = \left[I_1^0, \ldots, I_m^0\right]^t$ are subjected to the constraints $\gamma_{i,0} \leq I_i^0(x) \leq \gamma_{i,1}$ a.e. in Ω, where the thresholds $\gamma_{i,0}$ and $\gamma_{i,1}$ are defined in (2.5).

(A2) There exist a couple of vector-valued functions $\widetilde{S} \in W^{1,1}(\Omega;\mathbb{R}^m)$ and $\widetilde{M} \in C(\overline{\Omega};\mathbb{R}^m)$ such that the grids G_H and G_L are the sets of Lebesgue point of $\widetilde{S}$ and $\widetilde{M}$, respectively, and

$$\widetilde{S}\big|_{G_H} = S, \quad \widetilde{M}\big|_{G_L} = M. \tag{3.1}$$

REMARK 3.1. Let us mention that in the case of digital images, the only accessible information is a sampled and quantized version of $I : \Omega \to \mathbb{R}^m$, i.e., $I(x_i, y_j)$, where $\{(x_i, y_j) \in \Omega\}$ is a set of discrete points and for each spectral channel $k = 1, \ldots, m$, $I_k(x_i, y_j)$ belongs in fact to a discrete set of values, $0, 1, \ldots, 255$ in most cases. Due to Shannon's theory, it is plausible to assume that I_k is recoverable at any point $(x, y) \in \Omega$ from the samples $I_k(x_i, y_j)$. So, in view of assumption (A2), we may assume that the images S and M are known in a continuous domain Ω and, therefore, the objective functional (2.1) should be interpreted as follows

$$J_i(v)=\int_\Omega |\nabla v(x)|^{\mathfrak{F}(v(x))}\,dx+\lambda\int_\Omega |\nabla v(x)-\nabla \widetilde{S}_i(x)|\,dx \\ +\mu\int_\Omega |T_S v(x)-\widetilde{S}_i(x)|\,dx+\frac{1-\mu}{2}T_M\left(|(G_\sigma * v)(\cdot)-\widetilde{M}_i(\cdot)|^2\right).$$

However, in practice, such reconstruction is not a trivial problem.

We say that a function $I^0=\left[I_1^0,\dots,I_m^0\right]^t:\Omega\to\mathbb{R}^m$ is the result of simultaneous fusion and denoising of the noise contaminated images $S:G_H\to\mathbb{R}^m$ and $M:G_L\to\mathbb{R}^m$ if for given regularization parameters $\lambda>0$, $\mu\in(0,1)$, and a given linear blur operator $T_S\in\mathcal{L}(L^1(\Omega))$, each spectral component I_i^0 is the solution of the corresponding constrained minimization problem (2.6)–(2.7), i.e., for each $i=1,\dots,m$,

$$I_i^0\in\Xi_i \quad\text{and}\quad J_i\left(I_i^0\right)=\inf_{v\in\Xi_i} J_i(v).$$

Hereinafter, we associate with each spectral channel v_i of an arbitrary image $v=[v_1,v_2,\dots,v_m]^t:\Omega\to\mathbb{R}^m$ the so-called texture index $p_i:\Omega\to\mathbb{R}$ following the rule

$$p_i(x):=\mathfrak{F}(v_i(x))=1+g(|(\nabla G_\sigma * v_i)(x)|),\ \forall x\in\Omega,\ \forall i=1,\dots,m, \tag{3.2}$$

where $g:[0,\infty)\to(0,\infty)$ is the edge-stopping function that we take in the form of the Cauchy law $g(t)=\frac{1}{1+(t/a)^2}$.

As follows from representation (3.2) and smoothness of the Gaussian filter kernel G_σ, we have the following estimates

$$|(\nabla G_\sigma * v)(x)|\le\int_\Omega |\nabla G_\sigma(x-y)||v(y)|\,dy \\ \le\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\|v\|_{L^1(\Omega)}\le\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}|\Omega|\gamma_{1,i},\quad \forall x\in\Omega,$$

$$\mathfrak{F}(v(x))=1+\frac{a^2}{a^2+(|(\nabla G_\sigma * v)(x)|)^2} \\ \ge 1+\frac{a^2}{a^2+\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\|v\|^2_{L^1(\Omega)}}\ge 1+\delta,\quad \forall x\in\Omega,$$

$$\mathfrak{F}(v(x))\le 2\quad\text{in }\Omega,$$

where

$$\delta=\frac{a^2}{a^2+\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega|^2\max_{1\le i\le m}\gamma_{1,i}^2}\ll 1, \tag{3.3}$$

$$\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}=\max_{\substack{z=x-y\\ x\in\overline{\Omega},y\in\overline{\Omega}}}\Big[|G_\sigma(z)|+|\nabla G_\sigma(z)|\Big]\frac{e^{-1}}{2\pi\sigma^2}\left[1+\frac{1}{\sigma^2}\operatorname{diam}\Omega\right]. \tag{3.4}$$

Hence,

$$\alpha\le\mathfrak{F}(v(x))\le\beta\quad\text{in }\Omega,\text{ where }\alpha:=1+\delta\ \text{ and }\ \beta:=2.$$

The following results play a crucial role in the sequel (for the proof, we refer to [15]).

LEMMA 3.1. *Let $\{v_k\}_{k\in\mathbb{N}}\subset L^\infty(\Omega)$ be a sequence of measurable non-negative functions such that $\gamma_{i,0}\le v_k(x)\le\gamma_{i,1}$ a.e. in Ω and $v_k(x)\to v(x)$ weakly-$*$ in $L^\infty(\Omega)$ for some $v\in L^\infty(\Omega)$, and each element of this sequence is extended by zero outside of Ω. Let $\{p_k=1+g(|(\nabla G_\sigma * v_k)|)\}_{k\in\mathbb{N}}$ be the corresponding sequence of texture indices. Then*

$$p_k(\cdot)\to p(\cdot)=1+g(|(\nabla G_\sigma * v)(\cdot)|) \quad \textit{uniformly in } \overline{\Omega} \textit{ as } k\to\infty,$$
$$\alpha:=1+\delta\le p_k(x)\le\beta:=2,\quad \forall x\in\Omega,\ \forall k\in\mathbb{N}. \tag{3.5}$$

PROPOSITION 3.1. *Let $\{p_k=1+g(|(\nabla G_\sigma * v_k)|)\}_{k\in\mathbb{N}}$ be a sequence of texture indices such that*

$$p_k(\cdot)\to p(\cdot)=1+g(|(\nabla G_\sigma * v)(\cdot)|) \quad \textit{uniformly in } \overline{\Omega} \textit{ as } k\to\infty$$

and conditions (3.5) hold true. If a bounded sequence $\left\{f_k\in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ converges weakly in $L^{1+\delta}(\Omega)$ to f, then $f\in L^{p(\cdot)}(\Omega)$, $f_k\rightharpoonup f$ in variable $L^{p_k(\cdot)}(\Omega)$, and

$$\liminf_{k\to\infty}\int_\Omega |f_k(x)|^{p_k(x)}\,dx\ge\int_\Omega |f(x)|^{p(x)}\,dx. \tag{3.6}$$

Following, in some technical aspects, recent studies [12, 13, 15, 20], we can give the following existence result.

THEOREM 3.1. *For each $i=1,\dots,m$ and given $\mu\in(0,1)$, $\lambda>0$, $S\in L^1(\Omega;\mathbb{R}^m)$, $M: G_L\to\mathbb{R}^m$, and $T_S\in\mathcal{L}(L^1(\Omega))$, the minimization problem (2.6)–(2.7) admits at least one solution $I_i^0\in\Xi_i$.*

Proof. Since $\Xi_i\ne\emptyset$ and $0\le J_i(v)<+\infty$ for all $v\in\Xi_i$, it follows that there exists a non-negative value $\zeta\ge0$ such that $\zeta=\inf_{v\in\Xi_i}J_i(v)$. Let $\{v_k\}_{k\in\mathbb{N}}\subset\Xi_i$ be a minimizing sequence to the problem (2.6)–(2.7), i.e.

$$v_k\in\Xi_i,\ \forall k\in\mathbb{N},\quad \textit{and}\quad \lim_{k\to\infty}J_i(v_k)=\zeta.$$

So, without loss of generality, we can suppose that $J_i(v_k)\le\zeta+1$ for all $k\in\mathbb{N}$.

Utilizing the fact that $v_k\in\Xi_i$, $\forall k\in\mathbb{N}$ and, therefore, $v_k(x)\le\gamma_{1,i}$ for almost all $x\in\Omega$, we see that

$$\|v_k\|_{L^1(\Omega)}\le\gamma_{1,i}|\Omega|,\quad \forall k\in\mathbb{N}.$$

Then setting $p_k(x)=1+g(|(\nabla G_\sigma * v_k)(x)|)$ in Ω and arguing as in Lemma 3.1, it can be shown that $p_k\in C^{0,1}(\overline{\Omega})$ and

$$\alpha:=1+\delta\le p_k(x)\le\beta:=2,\quad \forall x\in\Omega,\quad \forall k\in\mathbb{N}, \tag{3.7}$$

where δ is defined by the rule (3.3). From this, we deduce that

$$\int_\Omega |v_k(x)|^\alpha\,dx\le\int_\Omega\gamma_{1,i}^\alpha\,dx\le\gamma_{1,i}^\alpha|\Omega|,\quad \forall k\in\mathbb{N},$$
$$\int_\Omega |\nabla v_k(x)|^{p_k(x)}\,dx\le\zeta+1,\quad \forall k\in\mathbb{N}, \tag{3.8}$$

with $\alpha=1+\delta$.

Taking this fact into account, we infer from (3.8), (3.7), and (2.2) that

$$\|v_k\|_{W^{1,\alpha}(\Omega)} = \left(\int_\Omega\left[|v_k(x)|^\alpha+|\nabla v_k(x)|^\alpha\right]dx\right)^{1/\alpha}$$

$$\leq \quad (1+|\Omega|)^{1/\alpha}\left(\int_{\Omega}\left[|v_k(x)|^{p_k(x)}+|\nabla v_k(x)|^{p_k(x)}\right]dx+2\right)^{1/\alpha}$$
$$\overset{by\ (3.8)}{\leq} (1+|\Omega|)^{1/\alpha}\left(\gamma_{1,i}^2|\Omega|+\zeta+3\right)^{1/\alpha}$$

uniformly with respect to $k\in\mathbb{N}$. Therefore, there exists a subsequence of $\{v_k\}_{k\in\mathbb{N}}$, still denoted by the same index, and a function $I_i^0\in W^{1,\alpha}(\Omega)$ such that

$$\begin{aligned}&v_k\to I_i^0 \text{ strongly in } L^q(\Omega)\ for\ all\ q\in[1,\alpha^*),\\&v_k\rightharpoonup I_i^0 \text{ weakly in } W^{1,\alpha}(\Omega)\ as\ k\to\infty,\end{aligned}\tag{3.9}$$

where, by Sobolev embedding theorem, $\alpha^*=\frac{2\alpha}{2-\alpha}=\frac{2+2\delta}{1-\delta}>2+\delta$.

Moreover, passing to a subsequence if necessary, we have (see Proposition 3.1 and Lemma 3.1):

$$\begin{aligned}&v_k(x)\to I_i^0(x) \text{ a.e. in } \Omega\\&v_k\rightharpoonup I_i^0 \text{ weakly in } L^{p_k(\cdot)}(\Omega),\\&\nabla v_k\rightharpoonup \nabla I_i^0 \text{ weakly in } L^{p_k(\cdot)}(\Omega;\mathbb{R}^N),\\&p_k(\cdot)\to p_i^0(\cdot)=1+g\left(\left|\left(\nabla G_\sigma * I_i^0\right)(\cdot)\right|\right) \text{ uniformly in } \overline{\Omega}\ as\ k\to\infty,\end{aligned}\tag{3.10}$$

where $I_i^0\in W^{1,p^0(\cdot)}(\Omega)$.

Since $\gamma_{0,i}\leq v_k(x)\leq\gamma_{1,i}$ a.a. in Ω for all $k\in\mathbb{N}$, it follows from (3.10) that the limit function I_i^0 is also subjected to the same restriction. Thus, I_i^0 is a feasible solution to the minimization problem (2.6)–(2.7).

Let us show that I_i^0 is a minimizer of this problem. With that in mind, we note that due to the obvious inequality

$$|T_S(v_k(x))-\widetilde{S}_i(x)|\leq\left(\|T_S\|_{\mathcal{L}(L^1(\Omega))}\gamma_{1,i}+|\widetilde{S}_i(x)|\right),$$

we have: the sequence $\left\{T_S(v_k(x))-\widetilde{S}_i(x)\right\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, equi-integrable in Ω, and because of (3.10), it strongly converges in $L^1(\Omega)$ to $T_S\left(I_i^0\right)-\widetilde{S}_i$. Hence,

$$\liminf_{k\to\infty}\int_\Omega|T_S(v_k(x))-\widetilde{S}_i(x)|\,dx=\int_\Omega\left|T_S\left(I_i^0(x)\right)-\widetilde{S}_i(x)\right|dx.\tag{3.11}$$

In view of the piecewise convergence (3.10), we have a similar relation for the last term in (2.1)

$$\liminf_{k\to\infty}T_M\left(|(G_\sigma*v_k)(\cdot)-\widetilde{M}_i(\cdot)|^2\right)=T_M\left(|\left(G_\sigma*I_i^0\right)(\cdot)-\widetilde{M}_i(\cdot)|^2\right).\tag{3.12}$$

It remains to notice that due to the properties (3.8), (3.9), the sequence $\left\{|\nabla v_k|\in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is bounded and weakly convergent to $|\nabla I_i^0|$ in $L^\alpha(\Omega)$. Hence, by Proposition 3.1, the following lower semicontinuous properties

$$\liminf_{k\to\infty}\int_\Omega|\nabla v_k(x)|^{p_k(x)}\,dx\geq\int_\Omega|\nabla I_i^0(x)|^{p_i^0(x)}\,dx,\tag{3.13}$$

$$\liminf_{k\to\infty}\int_\Omega|\nabla v_k(x)-\nabla\widetilde{S}_i(x)|\,dx\geq\int_\Omega|\nabla I_i^0(x)-\nabla\widetilde{S}_i(x)|\,dx\tag{3.14}$$

hold.

As a result, utilizing relations (3.11)–(3.14), we finally obtain

$$\zeta = \inf_{v\in\Xi_i} J_i(v) = \lim_{k\to\infty} J_i(v_k) = \liminf_{k\to\infty} J_i(v_k) \geq J_i(I_i^0).$$

Thus, I_i^0 is a minimizer to the problem (2.6)–(2.7), whereas its uniqueness remains an open question. □

4. On relaxation of the minimization problem (2.6)–(2.7)

It is clear that because of the specific choice of the exponent

$$\mathfrak{F}(v(x)) = 1 + g(|(\nabla G_\sigma * v)(x)|) \quad \text{in } \Omega,$$

constrained minimization problem (2.6)–(2.7) is not trivial in its practical implementation. Moreover, in this case, the objective functional $J_i(v)$ is not convex. Even if we represent the minimization problem (2.6)–(2.7) in the form

$$\text{Find } \ (I_i^0, p_i^0) \in \Lambda_i \ \text{ such that } \quad F_i\left(I_i^0, p_i^0\right) = \inf_{(v,p)\in\Lambda_i} F_i(v,p), \tag{4.1}$$

where

$$\begin{aligned} F_i(v,p) = &\int_\Omega |\nabla v(x)|^{p(x)}\,dx + \lambda \int_\Omega |\nabla v(x) - \nabla \widetilde{S}_i(x)|\,dx \\ &+ \mu \int_\Omega |T_S v(x) - \widetilde{S}_i(x)|\,dx + \frac{1-\mu}{2} T_M\left(|(G_\sigma * v)(\cdot) - \widetilde{M}_i(\cdot)|^2\right), \end{aligned} \tag{4.2}$$

$$\Lambda_i = \left\{ (u,p) \in W^{1,\mathfrak{F}(v(\cdot))}(\Omega) \times C^{0,1}(\overline{\Omega}) \;\middle|\; \begin{array}{l} 1 \leq \gamma_{i,0} \leq u(x) \leq \gamma_{i,1} \quad \text{a.e. in } \Omega \\ p(x) = 1 + g(|(\nabla G_\sigma * u)(x)|) \ \text{ in } \Omega \end{array} \right\} \tag{4.3}$$

the main difficulty in its study comes from the state constraints

$$p(x) = 1 + g(|(\nabla G_\sigma * v)(x)|) \tag{4.4}$$

with the non-convex right-hand side. This motivates us to pass to some relaxation scheme of variational problem (4.1)–(4.3). It will be shown in the sequel that using this approach, the non-convexity can be negligible in practice and that reliable solutions can be computed using a variety of different optimization algorithms.

As the main step of this procedure, we propose to consider the function $p(\cdot) := \mathfrak{F}(v(\cdot))$ as a fictitious control subjected to some special constraints and interpret the fulfillment of equality $\mathfrak{F}(v(x)) = 1 + g(|(\nabla G_\sigma * v)(x)|)$ with some accuracy in Ω. To do so, we notice that if $v \in \Xi_i$ is a feasible solution to the problem (2.6)–(2.7) then $\mathfrak{F}(v(\cdot))$ is subjected to the two-side inequality (3.7) with $\delta \in (0,1)$ given by (3.3). Keeping this in mind and following in some aspects the standard penalty method [41, Chapter 2] (see also [21–24, 27]), we consider the following family of approximating problems:

$$\begin{aligned} \text{Minimize } J_{i,\varepsilon}(v,p) = &\int_\Omega |\nabla v(x)|^{p(x)}\,dx + \lambda \int_\Omega |\nabla v(x) - \nabla \widetilde{S}_i(x)|\,dx \\ &+ \mu \int_\Omega |T_S v(x) - \widetilde{S}_i(x)|\,dx + \frac{1-\mu}{2} T_M\left(|(G_\sigma * v)(\cdot) - \widetilde{M}_i(\cdot)|^2\right) \\ &+ \frac{1}{\varepsilon} \int_\Omega |p(x) - 1 - g(|(\nabla G_\sigma * v)(x)|)|^2\,dx \end{aligned} \tag{4.5}$$

subject to the constraints $(v,p)\in\Xi_{i,\varepsilon}$, where

$$\Xi_{i,\varepsilon}=\left\{(v,p)\;\middle|\;\begin{array}{c} v\in W^{1,\alpha}(\Omega),\ p\in\mathfrak{S}_{ad},\ J_{i,\varepsilon}(v,p)<+\infty,\\ 0\le\gamma_{i,0}\le v(x)\le\gamma_{i,1}\ a.e.\ in\ \Omega,\end{array}\right\} \tag{4.6}$$

$$\mathfrak{S}_{ad}=\left\{h\in C(\Omega)\;\middle|\;\begin{array}{c} |h(x)-h(y)|\le C|x-y|,\ \forall x,y\in\Omega,\\ 1<\alpha\le h(\cdot)\le\beta\ in\ \overline{\Omega}.\end{array}\right\} \tag{4.7}$$

Here, $\alpha=1+\delta$, $\delta>0$ is given by (3.3), $\beta=2$, and

$$C:=\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_{1,i}^2|\Omega|C_G}{a^2} \tag{4.8}$$

with a positive constant C_G coming from the inequality

$$\int_\Omega|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)|\,dz\le C_G|x-y|,\quad\forall x,y\in\Omega.$$

To justify the choice (4.8) for the constant C, we make use of the following observation. If we assume for a moment that $p(x)=1+g\left(|(\nabla G_\sigma*v)|\right)$ for some $v\in\Xi_i$, then the following chain of estimates holds true

$$\begin{aligned}|p(x)-p(y)|&\le a^2\left|\frac{|(\nabla G_\sigma*v)(x)|^2-|(\nabla G_\sigma*v)(y)|^2}{(a^2+|(\nabla G_\sigma*v)(x)|^2)(a^2+|(\nabla G_\sigma*v)(y)|^2)}\right|\\ &\le\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\|v\|_{L^1(\Omega)}}{a^2}\Big||(\nabla G_\sigma*v)(x)|-|(\nabla G_\sigma*v)(y)|\Big|\\ &\le\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega|}{a^2}\int_\Omega|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)|\,dz,\\ &\qquad\forall x,y\in\Omega\ \text{ with }\gamma_1=\|v\|_{L^\infty(\Omega)}\le\gamma_{i,1}.\end{aligned}$$

Then taking into account the smoothness of the function $\nabla G_\sigma(\cdot)$, we deduce: there exists a positive constant $C_G>0$ independent of k such that

$$|p(x)-p(y)|\le\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_{1,i}^2|\Omega|C_G}{a^2}|x-y|,\ \forall x,y\in\Omega.$$

Hereinafter, we assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers which converges to 0. So, when we write $\varepsilon>0$, we consider only the elements of this sequence.

Definition 4.1. *We say that a pair (v,p) is quasi-feasible to minimization problem (4.1)–(4.3) if $(v,p)\in\Xi_{i,\varepsilon}$ for some $\varepsilon>0$ small enough. We also say that $(u^0_{i,\varepsilon},p^0_{i,\varepsilon})\in W^{1,p^0_\varepsilon(\cdot)}(\Omega)\times C^{0,1}(\overline{\Omega})$ is a quasi-optimal solution to the problem (4.1)–(4.3) if*

$$(u^0_{i,\varepsilon},p^0_{i,\varepsilon})\in\Xi_{i,\varepsilon}\quad\textit{and}\quad J_{i,\varepsilon}(u^0_{i,\varepsilon},p^0_{i,\varepsilon})=\inf_{(v,p)\in\Xi_{i,\varepsilon}}J_{i,\varepsilon}(v,p).$$

Remark 4.1. It is clear that condition $p\in\mathfrak{S}_{ad}$ together with the fact that $\mathfrak{S}_{ad}$ is a compact subset in $C(\overline{\Omega})$ implies: every cluster point of a sequence $\{p_k\}_{k\in\mathbb{N}}\subset\mathfrak{S}_{ad}$ with respect to the uniform topology is a regular exponent, i.e. it is an exponent satisfying the log-Hölder continuity condition [43]. In this case, the set $C_0^\infty(\mathbb{R}^2)$ is dense in $W^{1,p(\cdot)}(\Omega)$ [11] and this fact plays a crucial role in the study of minimization problem (4.5).

The principle point in the statement of the approximated problem (4.5) is the fact that we pass from the state constrained optimization problem (4.1) with the variable exponent $p(x)=\mathfrak{F}(v(x))$ strongly depending on the function of interest v to its approximation where we eliminate the equality constraint $p(x)=\mathfrak{F}(v(x))$ for the state $v(x)$ and the exponent $p(x)$ and allow such pairs to run freely in their respective sets of feasibility.

We begin with the following existence result.

THEOREM 4.1. *For each $i=1,\dots,M$, every positive value $\varepsilon>0$, and given $\mu>0$, $\lambda>0$, $\widetilde{S}_i,\widetilde{M}_i\in L^1(\Omega)$, and $T_S\in\mathcal{L}(L^1(\Omega))$, the minimization problem (4.5) has at least one solution.*

Proof. Since the set $\Xi_{i,\varepsilon}$ is nonempty, we can assert the existence of a minimizing sequence $\{(u_k,p_k)\}_{k\in\mathbb{N}}\subset\Xi_{i,\varepsilon}$. Then arguing as in the proof of Theorem 3.1, we deduce the boundedness of the sequence $\{u_k\}_{k\in\mathbb{N}}$ in $W^{1,p_k(\cdot)}(\Omega)$ and, hence, the existence of a subsequence, still denoted in the same way, such that $u_k\rightharpoonup u_\varepsilon^0$ in $W^{1,\alpha}(\Omega)$ and in variable $W^{1,p_k(\cdot)}(\Omega)$. As for the sequence $\{p_k\}_{k\in\mathbb{N}}$, we see that

$$\{p_k(\cdot)\}\subset\mathfrak{S}=\left\{h\in C^{0,1}(\Omega)\;\middle|\;\begin{array}{c}|h(x)-h(y)|\le C|x-y|,\ \forall x,y,\in\Omega,\\ 1<\alpha\le h(\cdot)\le\beta \text{ in } \overline{\Omega},\end{array}\right\}$$

and $\max_{x\in\overline{\Omega}}|p_k(x)|\le\beta$. Since each element of the sequence $\{p_k\}_{k\in\mathbb{N}}$ has the same modulus of continuity, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli theorem the sequence $\{p_k\}_{k\in\mathbb{N}}$ is relatively compact with respect to the norm topology of $C(\overline{\Omega})$. Since the set $\mathfrak{S}$ is closed with respect to the uniform convergence, it follows that

$$p_k(\cdot)\to p_\varepsilon^0(\cdot)\ \text{ uniformly in } \overline{\Omega}\ \text{ as }\ k\to\infty\quad\text{and, therefore,}\quad p_\varepsilon^0\in\mathfrak{S}_{ad}.$$

Thus, we can suppose that for a given minimizing sequence there exists a subsequence of $\{(u_k,p_k)\}_{k\in\mathbb{N}}$ in $W^{1,p_k(\cdot)}(\Omega)\times C^{0,1}(\Omega)$, still denoted in the same way, and a pair $\left(u_\varepsilon^0,p_\varepsilon^0\right)$ such that $p_k\to p_\varepsilon^0$ in $C(\overline{\Omega})$, $u_k\rightharpoonup u_\varepsilon^0$ in $W^{1,\alpha}(\Omega)$ and in variable $W^{1,p_k(\cdot)}(\Omega)$. Then, by the Sobolev embedding theorem, we deduce that $u_k\to u_\varepsilon^0$ strongly in $L^q(\Omega)$ for all $q\in[1,\frac{2\alpha}{2-\alpha})$, and, therefore, we can suppose that $u_k(x)\to u_\varepsilon^0(x)$ almost everywhere in Ω as $k\to\infty$. As a result, we have

$$\begin{aligned}
&\gamma_{0,i}\le u_\varepsilon^0(x)\le\gamma_{1,i}\quad\text{and}\quad\alpha\le p_\varepsilon^0(x)\le\beta \text{ a.a. in } \Omega,\\
&\lim_{k\to\infty}\int_\Omega|T_S(u_k(x))-\widetilde{S}_i(x)|\,dx=\int_\Omega|T_i(u_\varepsilon^0(x))-\widetilde{S}_i(x)|\,dx,\\
&\lim_{k\to\infty}T_M\left(|(G_\sigma*u_k)(\cdot)-\widetilde{M}_i(\cdot)|^2\right)=T_M\left(|\left(G_\sigma*u_\varepsilon^0\right)(\cdot)-\widetilde{M}_i(\cdot)|^2\right),\\
&\liminf_{k\to\infty}\int_\Omega|\nabla u_k(x)|^{p_k(x)}\,dx\overset{by\ (3.6)}{\ge}\int_\Omega|\nabla u_\varepsilon^0(x)|^{p_\varepsilon^0(x)}\,dx.
\end{aligned}$$

Thus, $\left(u_\varepsilon^0,p_\varepsilon^0\right)\in\Xi_{i,\varepsilon}$. It remains to notice that

$$\left|p_k-1-g\left(|(\nabla G_\sigma*u_k)(x)|\right)\right|^2\to\left|p_\varepsilon^0-1-g\left(\left|\left(\nabla G_\sigma*u_\varepsilon^0\right)(x)\right|\right)\right|^2\ \ in\ \ C(\overline{\Omega}),$$

and the Lebesgue dominated convergence theorem implies

$$\lim_{k\to\infty}\int_\Omega \left|p_k-1-g\left(\left|\left(\nabla G_\sigma*u_k\right)(x)\right|\right)\right|^2dx=\int_\Omega\left|p^0_\varepsilon-1-g\left(\left|\left(\nabla G_\sigma*u^0_\varepsilon\right)(x)\right|\right)\right|^2dx.$$

Utilizing the above mentioned properties, we finally obtain

$$J_{i,\varepsilon}(u^0_\varepsilon,p^0_\varepsilon)\le\liminf_{k\to\infty}J_{i,\varepsilon}(u_k,p_k)=\inf_{(v,p)\in\Xi_i}J_{i,\varepsilon}(v,p).$$

Thus, $\left(u^0_\varepsilon,p^0_\varepsilon\right)\in\Xi_{i,\varepsilon}$ is an optimal pair to the problem (4.5). ∎

Taking this existence result into account, we pass to the study of approximation properties of the problems (4.5). Namely, we establish the convergence of minima of (4.5) to minima of (4.1)–(4.3) as ε tends to zero. In other words, we show that some optimal solutions to (4.1)–(4.3) can be approximated by the quasi-optimal solutions of this problem.

THEOREM 4.2. *Let $\left\{(u^0_\varepsilon,p^0_\varepsilon)\in\Xi_{i,\varepsilon}\right\}_{\varepsilon>0}$ be a sequence of minimizers to the problem (4.5). Then there exists a subsequence of $\left\{(u^0_\varepsilon,p^0_\varepsilon)\right\}_{\varepsilon>0}$, still denoted by the same index ε, such that*

$$p^0_\varepsilon\to p^0\quad in\ C(\overline{\Omega})\ as\ \varepsilon\to0, \tag{4.9}$$
$$u^0_\varepsilon\rightharpoonup u^0\quad in\ W^{1,\alpha}(\Omega)\ as\ \varepsilon\to0, \tag{4.10}$$
$$u^0_\varepsilon\rightharpoonup u^0\quad in\quad W^{1,p^0_\varepsilon(\cdot)}(\Omega),\ u^0\in W^{1,p^0(\cdot)}(\Omega), \tag{4.11}$$
$$p^0(x)=1+g\left(\left|\left(\nabla G_\sigma*u^0\right)(x)\right|\right)\qquad in\quad \Omega, \tag{4.12}$$
$$J_i(u^0)=\inf_{v\in\Xi_i}J_i(v)=\lim_{\varepsilon\to0}\inf_{(u,p)\in\Xi_{i,\varepsilon}}J_{i,\varepsilon}(v,p)=\lim_{\varepsilon\to0}J_{i,\varepsilon}(u^0_\varepsilon,p^0_\varepsilon), \tag{4.13}$$

and $u^0\in\Xi_i$.

Proof. Let $u^*\in\Xi_i$ be an arbitrary feasible solution to the original problem (2.6)–(2.7). We set $p^*=\mathfrak{F}(u^*(\cdot))$ in Ω. Then $u^*\in W^{1,\alpha}(\Omega)$, $p^*\in\mathfrak{S}_{ad}$, $J_{i,\varepsilon}(u^*,p^*)=J_i(u^*)<+\infty$, and, as a consequence, $(u^*,p^*)\in\Xi_{i,\varepsilon}$ for each $\varepsilon>0$.

Since $J_{i,\varepsilon}(u^0_\varepsilon,p^0_\varepsilon)\le J_{i,\varepsilon}(u^*,p^*)=J_i(u^*)=:C^*$, it follows from (4.5) that

$$\sup_{\varepsilon>0}\int_\Omega|\nabla u^0_\varepsilon(x)|^{p^0_\varepsilon(x)}dx\le C^*, \tag{4.14}$$
$$\int_\Omega\left|p^0_\varepsilon(x)-1-g\left(\left|\left(\nabla G_\sigma*u^0_\varepsilon\right)(x)\right|\right)\right|^2dx\le\varepsilon C^*,\quad\forall\varepsilon>0. \tag{4.15}$$

Since $\left\{p^0_\varepsilon\in C^{0,1}(\overline{\Omega})\right\}$ is a bounded sequence in $C(\overline{\Omega})$ with the same modulus of continuity, it follows, by Arzelà–Ascoli theorem, that this sequence is relatively compact with respect to the norm topology of $C(\overline{\Omega})$. Without loss of generality, we can suppose that there exists a function $p^0\in C(\overline{\Omega})$ such that assertion (4.9) is valid. Moreover, as follows from definition of the set $\mathfrak{S}_{ad}$, the limit function p^0 is subjected to the pointwise constraints

$$\alpha:=1+\delta\le p^0(x)\le\beta:=2,\quad\forall x\in\Omega. \tag{4.16}$$

Arguing similarly, we can infer from (4.14) and the two-side inequality

$$0\le\gamma_{0,i}\le u^0_\varepsilon(x)\le\gamma_{1,i}\quad\text{a.a. in }\Omega,\ \forall\varepsilon>0 \tag{4.17}$$

that the sequence $\{u_\varepsilon^0\}$ is relatively compact with respect to the weak topology of $W^{1,\alpha}(\Omega)$. Indeed, taking into account (4.17) and observing that

$$\sup_{\varepsilon>0}\int_\Omega |u_\varepsilon^0(x)|^{p_\varepsilon^0(x)}\,dx \overset{by\ (4.17)}{\leq} +\infty,$$

we see that $u_\varepsilon^0 \in W^{1,p_\varepsilon^0(\cdot)}(\Omega)$ for all $\varepsilon>0$ and the sequence $\{u_\varepsilon^0\}$ is bounded in variable space $W^{1,p_\varepsilon^0(\cdot)}(\Omega)$. Hence, this sequence is bounded in $W^{1,\alpha}(\Omega)$. Therefore, in view of completeness of $W^{1,\alpha}(\Omega)$, there exists a function $u^0\in W^{1,\alpha}(\Omega)$ such that, up to a subsequence, property (4.10) holds true. As a result, Proposition 3.1 and Sobolev embedding theorem lead us to the conclusion:

$$\begin{aligned}
&u_\varepsilon^0 \rightharpoonup u^0 \text{ in } W^{1,p_\varepsilon^0(\cdot)}(\Omega),\ u^0\in W^{1,p^0(\cdot)}(\Omega),\\
&u_\varepsilon^0 \to u^0 \text{ strongly in } L^q(\Omega) \text{ for all } q\in[1,\alpha^*),
\end{aligned}\tag{4.18}$$

where $\alpha^*=\frac{2\alpha}{2-\alpha}$. So, we can suppose that $u_\varepsilon^0(x)\to u^0(x)$ a.e. in Ω. Then passing to the limit in (4.17) as $\varepsilon\to 0$, we see that the limit function u^0 is also subjected to the point-wise constraints

$$0\le \gamma_{0,i}\le u^0(x)\le \gamma_{1,i} \quad \text{a.a. in } \Omega. \tag{4.19}$$

Moreover, utilizing the estimate (4.15) and properties (4.9)–(4.10), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \left|p_\varepsilon^0(x)-1-g\left(\left|\left(\nabla G_\sigma * u_\varepsilon^0\right)(x)\right|\right)\right|^2 dx\\
&=\int_\Omega \left|p^0(x)-1-g\left(\left|\left(\nabla G_\sigma * u^0\right)(x)\right|\right)\right|^2 dx = 0.
\end{aligned}$$

Hence, $p^0(x)=1+g\left(\left|\left(\nabla G_\sigma * u^0\right)(x)\right|\right)$ in Ω. Thus, $u^0\in W^{1,\mathfrak{F}(u^0(\cdot))}(\Omega)$. Combining this fact with (4.19), we see that the limit function u^0 is a feasible solution to the minimization problem (2.6)–(2.7).

Let us show that this function is optimal to the problem (2.6)–(2.7). Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega |T_S(u_\varepsilon^0(x))-\widetilde{S}_i(x)|\,dx \overset{by\ (4.18)}{=} \int_\Omega |T_S(u^0(x))-\widetilde{S}_i(x)|\,dx,\\
&\lim_{\varepsilon\to 0} T_M\left(\left|\left(G_\sigma * u_\varepsilon^0\right)(\cdot)-\widetilde{M}_i(\cdot)\right|^2\right) \overset{by\ (4.18)}{=} T_M\left(\left|\left(G_\sigma * u^0\right)(\cdot)-\widetilde{M}_i(\cdot)\right|^2\right),
\end{aligned}$$

it follows from Proposition 3.1 that

$$\liminf_{\varepsilon\to 0} J_{i,\varepsilon}(u_\varepsilon^0,p_\varepsilon^0)\ge J_i(u^0). \tag{4.20}$$

Then, assuming the converse—namely, there is a function $\widehat{u}\in\Xi_i$ such that $J_i(\widehat{u})<J_i(u^0)$, we get:

$$\begin{aligned}
&(\widehat{u},\widehat{p})\in\Xi_{i,\varepsilon} \quad \forall\varepsilon>0 \ \text{ with } \ \widehat{p}:=\mathfrak{F}(\widehat{u}(\cdot)),\\
&J_i(\widehat{u})\equiv J_{i,\varepsilon}(\widehat{u},\widehat{p})\ge \inf_{(v,p)\in\Xi_{i,\varepsilon}} J_{i,\varepsilon}(v,p)=J_{i,\varepsilon}(u_\varepsilon^0,p_\varepsilon^0).
\end{aligned}$$

Hence,

$$J_i(\widehat{u})\ge \limsup_{\varepsilon\to 0} J_{i,\varepsilon}(u_\varepsilon^0,p_\varepsilon^0)\ge \liminf_{\varepsilon\to 0} J_{i,\varepsilon}(u_\varepsilon^0,p_\varepsilon^0) \overset{by\ (4.20)}{\geq} J_i(u^0), \tag{4.21}$$

and we come into contradiction with the initial assumptions. Thus, u^0 is a solution of the original problem (2.6)–(2.7). In order to establish the equality (4.13), it is enough, instead of $(\widehat{u},\widehat{p})$, to take (u^0,p^0) in (4.21). ∎

Since Theorem 4.2 does not answer whether the entire set of solutions to the problem (2.6)–(2.7) can be attained in such a way, the following result sheds some light on this matter.

Corollary 4.1. *Let $u^0\in\Xi_i$ be a minimizer to optimization problem (2.6)–(2.7) such that there is a closed neighborhood $\mathcal{U}(u^0)$ of u^0 in the norm topology of $L^\alpha(\Omega)$ satisfying*

$$J_i(u^0)<J_i(v)\quad \forall v\in\Xi_i\cap\mathcal{U}(u^0). \tag{4.22}$$

Then there exists a sequence of local minima $\left\{(u^0_\varepsilon,p^0_\varepsilon)\right\}_{\varepsilon>0}$ of problems (4.5) such that

$$(u^0_\varepsilon,p^0_\varepsilon)\to(u^0,\mathfrak{F}(u^0(\cdot)))\quad \textit{in the sense of Theorem 4.2.}$$

Proof. By the strict local optimality of u^0, we have that it is the unique solution of the problem

$$\min_{v\in\Xi_i,v\in\mathcal{U}(u^0)} J_i(v). \tag{4.23}$$

For every $\varepsilon>0$ let us consider the following optimization problems

$$\min_{(v,p)\in\Xi_{i,\varepsilon},v\in\mathcal{U}(u^0)} J_{i,\varepsilon}(v,p). \tag{4.24}$$

Since the set $\left\{(v,p)\in\Xi_{i,\varepsilon},v\in\mathcal{U}(u^0)\right\}$ is nonempty, it follows that the problem (4.24) has at least one solution $(u^0_\varepsilon,p^0_\varepsilon)$ for every $\varepsilon>0$. Now, arguing as in the proof of Theorem 4.2, we deduce that $(u^0_\varepsilon,p^0_\varepsilon)\to(\tilde{u}^0,\tilde{p}^0)$ in the sense of convergences (4.9)–(4.13), and $\tilde{u}^0$ is a solution of (4.23). Since u^0 is the unique solution of (4.23), we infer that $u^0=\tilde{u}^0$ and, therefore, $(u^0_\varepsilon,p^0_\varepsilon)\to(u^0,\mathfrak{F}(u^0(\cdot)))$ in the sense of Theorem 4.2. This implies the existence of $\varepsilon^0>0$ such that u^0_ε belongs to the interior of $\mathcal{U}(u^0)$ for every $\varepsilon\le\varepsilon^0$. Consequently, $(u^0_\varepsilon,p^0_\varepsilon)$ is a local minimum of (4.5) for every $\varepsilon\le\varepsilon^0$. This concludes the proof. ∎

5. Proximal alternating minimization algorithm and its modification

In this section, we discuss an algorithm that will attempt to numerically compute the solutions to the state constrained minimization problem (4.1)–(4.3). As follows from Theorem 4.2, some optimal solutions to (4.1)–(4.3) can be obtained as cluster points of the quasi-optimal solutions to this problem. From a practical point of view, it means that we can focus on the mathematical model of approximating problem (4.5)–(4.6), with $\varepsilon>0$ small enough, which models the solution that we are after. For a concise presentation, we cast problem (4.5)–(4.6) in the form

$$(v^*,p^*)\in \operatorname*{Argmin}_{(v,p)\in\Xi_{i,\varepsilon}} J_{i,\varepsilon}(v,p). \tag{5.1}$$

Since the objective functional $J_{i,\varepsilon}(v,p)$ is neither convex in the joint variables (v,p) nor bi-convex (i.e., convex in each of the variables v and p), an abstract algorithm for finding solution of (5.1) is the proximal alternating minimization algorithm [2]. Given the initial pair $(v_0,p_0)\in\Lambda_i\subset\Xi_{i,\varepsilon}$, where

$$v_0(x)=\widetilde{M}_i(x)\quad\text{and}\quad p_0(x)=1+g\left(\left|(\nabla G_\sigma*v_0)(x)\right|\right)\quad\text{in }\Omega \tag{5.2}$$

and the step sizes $\tau_k^u, \tau_k^q > 0$, the next iterations can be computed by the update scheme

$$(v_k, p_k) \longrightarrow (v_{k+1}, p_k) \longrightarrow (v_{k+1}, p_{k+1}), \tag{5.3}$$

$$v_{k+1} \in \operatorname*{Argmin}_{\substack{u \in W^{1,\alpha}(\Omega) \\ \gamma_{i,0} \le u(x) \le \gamma_{i,1}}} \left\{ \frac{1}{2\tau_k^u} \|u - v_k\|^2_{L^2(\Omega)} + J_{i,\varepsilon}(u, p_k) \right\}, \tag{5.4}$$

$$p_{k+1} \in \operatorname*{Argmin}_{q \in \mathfrak{S}_{ad}} \left\{ \frac{1}{2\tau_k^q} \|q - p_k\|^2_{L^2(\Omega)} + J_{i,\varepsilon}(v_{k+1}, q) \right\}. \tag{5.5}$$

It is well known that under reasonably mild conditions on the regularity of $J_{i,\varepsilon}$ (which are obviously satisfied in our case, see [2] for the details), the proximal alternating minimization algorithm monotonously decreases the objective functional and its iterates converge to a critical point of $J_{i,\varepsilon}$. However, as it was mentioned in [2], very few general results ensure that the sequence $\{(v_k, p_k)\}_{k \in \mathbb{N}}$ converges to a global minimizer of (4.1)–(4.3), even for strictly convex functions. Meanwhile, exploiting the fact that minimization problem (5.5) with ε small enough admits a unique minimizer p_{k+1} at each step of iteration, we see that

$$p_{k+1}(x) = 1 + g\left(\left|(\nabla G_\sigma * v_{k+1})(x)\right|\right) \quad \text{in } \Omega. \tag{5.6}$$

It means that due to the equality (5.6), we can alleviate this approach. Indeed, in view of the representation (5.6), we can specify the above mentioned iteration procedure as follows

$$(v_k, p_k) \longrightarrow (v_{k+1}, p_{k+1}), \tag{5.7}$$

$$v_{k+1} \in \operatorname*{Argmin}_{\substack{u \in W^{1,\alpha}(\Omega) \\ \gamma_{i,0} \le u(x) \le \gamma_{i,1}}} \left\{ \frac{1}{2\tau_k^u} \|u - v_k\|^2_{L^2(\Omega)} + J_{i,\varepsilon}(u, p_k) \right\}, \tag{5.8}$$

$$p_{k+1}(x) = 1 + g\left(\left|(\nabla G_\sigma * v_{k+1})(x)\right|\right) \quad \text{in } \Omega, \tag{5.9}$$

provided the parameter $\varepsilon > 0$ is chosen small enough. However, as follows from the structure of the penalized objective functional $J_{i,\varepsilon}$, we still deal with a non-convex optimization problem in (5.8).

In view of this, the main idea we are going to push forward in this section is to represent the iteration procedure (5.7)–(5.9) as follows

$$(v_k, p_k) \longrightarrow (v_{k+1}, p_{k+1}), \tag{5.10}$$

$$v_{k+1} \in \operatorname*{Argmin}_{u \in \mathcal{B}_{i,p_k(\cdot)}} \left\{ \frac{1}{2\tau_k^u} \|u - v_k\|^2_{L^2(\Omega)} + F_i(u, p_k) \right\}, \tag{5.11}$$

$$p_{k+1}(x) = 1 + g\left(\left|(\nabla G_\sigma * v_{k+1})(x)\right|\right) \quad \text{in } \Omega, \tag{5.12}$$

where the cost functional F_i is defined in (4.2) and

$$\mathcal{B}_{i,p(\cdot)} = \left\{ v \in W^{1,p(\cdot)}(\Omega) : 1 \le \gamma_{i,0} \le v(x) \le \gamma_{i,1} \quad a.e.\ in \quad \Omega \right\}.$$

The main benefit of this modification is to pass to convex optimization problems at each step of iteration. Then arguing as in the proof of Theorem 3.1 and using convexity arguments, it can be shown that, for each $p_k(\cdot) \in \mathfrak{S}_{ad}$, there exists a unique element $v_{k+1} \in \mathcal{B}_{i,p_k(\cdot)}$ such that $v_{k+1} = \operatorname*{Argmin}_{u \in \mathcal{B}_{i,p_k(\cdot)}} \left\{ \frac{1}{2\tau_k^u} \|u - v_k\|^2_{L^2(\Omega)} + F_i(u, p_k) \right\}$. This fact reflects the principal difference between optimization problems (5.11) and (4.1), where the problem (5.11) can be viewed as a minimization of the growth energy functional (4.2) with the frozen exponent $p_k(x)$. Thus, the sequence $\{v_k\}_{k\in\mathbb{N}}$ can be defined in a unique way. Moreover, the iteration procedure (5.7)–(5.9) possesses the following property.

PROPOSITION 5.1. *For any sequence of stepsizes $\{\tau_k^u\}_{k\in\mathbb{N}}$, $\{\tau_k^q\}_{k\in\mathbb{N}}$ such that $\tau_k^u,\tau_k^q \in (r_-,+\infty)$ for all $k\in\mathbb{N}$ with some positive r_-, the numerical sequence $\{F_i(v_k,p_k)\}_{k\in\mathbb{N}}$ does not increase and the estimates*

$$F_i(v_{k+1},p_{k+1}) + \frac{1}{2\tau_k^q}\|p_{k+1}-p_k\|^2_{L^2(\Omega)} + \frac{1}{2\tau_k^u}\|v_{k+1}-v_k\|^2_{L^2(\Omega)} \le F_i(v_k,p_k), \quad \forall k\in\mathbb{N}, \tag{5.13}$$

$$\sum_{k=1}^{\infty}\left[\|v_k-v_{k-1}\|^2_{L^2(\Omega)} + \|p_k-p_{k-1}\|^2_{L^2(\Omega)}\right] < +\infty \tag{5.14}$$

hold.

Proof. To begin with, we notice that the equality (5.12) can be rewritten in an equivalent form as follows

$$p_{k+1} \in \operatorname*{Argmin}_{q\in\mathfrak{S}_{ad}}\left\{\frac{1}{2\tau_k^q}\|q-p_k\|^2_{L^2(\Omega)} + F_i(v_{k+1},q)\right\} \tag{5.15}$$

provided the stepsize τ_k^q is greater than a fixed positive parameter which can be chosen arbitrarily large. In this case the algorithm (5.7)–(5.9) is very close to a coordinate descent method. Then

$$\begin{aligned} F_i(v_{k+1},p_k) + \frac{1}{2\tau_k^u}\|v_{k+1}-v_k\|^2_{L^2(\Omega)} &\overset{\text{by (5.11)}}{\le} F_i(v_k,p_k),\\ F_i(v_{k+1},p_{k+1}) + \frac{1}{2\tau_k^q}\|p_{k+1}-p_k\|^2_{L^2(\Omega)} &\overset{\text{by (5.15)}}{\le} F_i(v_{k+1},p_k). \end{aligned}$$

Hence, an elementary induction

$$\begin{aligned} &F_i(v_{k+1},p_{k+1}) + \frac{1}{2\tau_k^q}\|p_{k+1}-p_k\|^2_{L^2(\Omega)} + \frac{1}{2\tau_k^u}\|v_{k+1}-v_k\|^2_{L^2(\Omega)}\\ &\le F_i(v_{k+1},p_k) + \frac{1}{2\tau_k^u}\|v_{k+1}-v_k\|^2_{L^2(\Omega)} \le F_i(v_k,p_k), \quad \forall k\in\mathbb{N} \end{aligned}$$

ensures that estimate (5.13) is valid.

As for estimate (5.14), it is enough to observe that $F_i(v,p)\ge 0$ for all feasible pairs (v,p). Hence, (5.14) immediately follows from (5.13). As a consequence, we have,

$$\lim_{k\to\infty}\|v_k-v_{k-1}\|_{L^2(\Omega)} = \lim_{k\to\infty}\|p_k-p_{k-1}\|_{L^2(\Omega)} = 0. \tag{5.16}$$

∎

We say that a function $\widehat{u}_i$ is a weak solution to the original problem (2.6)–(2.7) if

$$\begin{aligned} &\widehat{u}_i = \operatorname*{Argmin}_{v\in\mathcal{B}_{\widehat{p}_i}} F_i(v,\widehat{p}_i(\cdot)),\ \widehat{u}_i\in\mathcal{B}_{i,\widehat{p}_i(\cdot)},\\ &\widehat{p}_i(x) = 1 + g(|(\nabla G_\sigma * \widehat{u}_i)(x)|),\ \forall x\in\Omega. \end{aligned} \tag{5.17}$$

REMARK 5.1. The relation between a weak solution and a solution to the problem (2.6)–(2.7) is rather intricate. Since the uniqueness of solutions to (2.6)–(2.7) is a questionable option, it follows that, in principle, these definitions can describe the different functions in Ξ_i. As immediately follows from (5.17), a weak solution is a merely feasible one to the original problem. However, if the problem (4.1) admits a unique solution $(u_i^0,p_i^0)\in\Lambda_i$, then (5.17) implies that the function u_i^0 can be considered as a weak solution.

Before proceeding further, we note that, for given $i=1,\dots,m$, the sequence of exponents $\{p_k\}_{k\in\mathbb{N}}$ is compact with respect to the strong topology of $C(\overline{\Omega})$. Our next goal is to establish the existence of a weak solution to the original problem (2.6)–(2.7) and show that it can be attained by the iterative algorithm (5.10)–(5.12). To do so, we begin with some technical results.

LEMMA 5.1. *For each $i=1,\dots,m$ and given $\mu\in(0,1)$, $\lambda>0$, $\widetilde{S}\in L^1(\Omega;\mathbb{R}^m)$, $\widetilde{M}:G_L\to\mathbb{R}^m$, and $T_S\in\mathcal{L}(L^1(\Omega))$, the sequence of minimizers $\left\{v_k\in W^{1,p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ of (5.11) is compact with respect to the weak topology of $W^{1,\alpha}(\Omega)$.*

Proof. Let us show that the sequence of minimizers $\{v_k\}_{k\in\mathbb{N}}$ of (5.11) is bounded in the following sense

$$\limsup_{k\to\infty}\int_\Omega |v_k(x)|^{p_k(x)}\,dx<+\infty.$$

Let $\widehat{u}\in C^1(\overline{\Omega})$ be an arbitrary function such that $\gamma_{0,i}\le\widehat{u}(x)\le\gamma_{1,i}$ in Ω. Since

$$F_i(v_k,p_k)\le F_i(\widehat{u},p_k)+\frac{1}{2\tau_{k-1}^u}\|\widehat{u}-v_{k-1}\|^2_{L^2(\Omega)},\quad\forall k=1,2,\dots \tag{5.18}$$

and

$$\begin{aligned}
&\int_\Omega|\nabla\widehat{u}(x)|^{p_k(x)}\,dx\le\int_\Omega\left(1+\|\widehat{u}\|_{C^1(\overline{\Omega})}\right)^{p_k(x)}dx\le|\Omega|\left(1+\|\widehat{u}\|_{C^1(\overline{\Omega})}\right)^2,\\
&\int_\Omega|\nabla\widehat{u}(x)-\nabla\widetilde{S}_i(x)|\,dx\le\int_\Omega\left[\|\widehat{u}\|_{C^1(\overline{\Omega})}+|\nabla\widetilde{S}_i(x)|\right]dx\le|\Omega|\|\widehat{u}\|_{C^1(\overline{\Omega})}+\|\widetilde{S}_i\|_{W^{1,1}(\Omega)},\\
&\int_\Omega|T_S\widehat{u}(x)-\widetilde{S}_i(x)|\,dx\le\int_\Omega\left[\|T_S\|_{\mathcal{L}(L^1(\Omega))}\gamma_{1,i}+|\widetilde{S}_i(x)|\right]dx\\
&\qquad\qquad\le|\Omega|\|T_S\|_{\mathcal{L}(L^1(\Omega))}\gamma_{1,i}+\|\widetilde{S}_i\|_{L^1(\Omega)},\\
&T_M\left(|(G_\sigma*\widehat{u}(\cdot))-\widetilde{M}_i(\cdot)|^2\right)\le\left(\|G_\sigma*\widehat{u}\|_{C(\overline{\Omega})}+\|\widetilde{M}_i\|_{C(\overline{\Omega})}\right)^2\\
&\qquad\qquad\le\left(\frac{1}{\left(\sqrt{2\pi}\sigma\right)^2}\|\widehat{u}\|_{C(\overline{\Omega})}+\|\widetilde{M}_i\|_{C(\overline{\Omega})}\right)^2,\\
&\|\widehat{u}-v_{k-1}\|^2_{L^2(\Omega)}\le4\gamma_{1,i}^2|\Omega|,
\end{aligned}$$

it follows that

$$\sup_{k\in\mathbb{N}}F_i(v_k,p_k)\le\sup_{k\in\mathbb{N}}\left[F_i(\widehat{u},p_k)+\frac{1}{2\tau_{k-1}^u}\|\widehat{u}-v_{k-1}\|^2_{L^2(\Omega)}\right]\le\widehat{C}$$

with some appropriate constant $\widehat{C}>0$.

From this and definition of the set $\mathcal{B}_{i,p_k(\cdot)}$, we deduce

$$\int_\Omega|v_k(x)|^\alpha\,dx\le\gamma_{1,i}^\alpha|\Omega|,\quad\forall k\in\mathbb{N}, \tag{5.19}$$

$$\int_\Omega |\nabla v_k(x)|^{p_k(x)}\,dx \le \widehat{C}, \quad \forall k\in\mathbb{N}. \tag{5.20}$$

Since (see [11, 16, 42] for the details)

$$\|f\|^{\alpha}_{L^{p(\cdot)}(\Omega)} - 1 \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^{\beta}_{L^{p(\cdot)}(\Omega)} + 1, \quad \forall f\in L^{p(\cdot)}(\Omega), \tag{5.21}$$

it follows that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,\alpha}(\Omega)$. So, its weak compactness is a direct consequence of the reflexivity of $W^{1,\alpha}(\Omega)$. □

We notice that boundedness of the sequence $\{v_k\}_{k\in\mathbb{N}}$ in $W^{1,\alpha}(\Omega)$ and compactness of the embedding $W^{1,\alpha}(\Omega)\hookrightarrow L^q(\Omega)$ for $q\in\left[1,\frac{2\alpha}{2-\alpha}\right)$ imply the existence of an element $u^*\in W^{1,\alpha}(\Omega)$ such that, up to a subsequence,

$$v_k(x)\to u^*(x) \text{ a.e. in } \Omega, \tag{5.22}$$

$$v_k\to u^* \quad in \quad L^q(\Omega), \text{ and } \nabla v_k \rightharpoonup \nabla u^* \text{ in } L^\alpha(\Omega;\mathbb{R}^2). \tag{5.23}$$

Then using (5.22) and passing to the limit in two-side inequality $\gamma_{0,i}\le v_k(x)\le\gamma_{1,i}$, we obtain

$$\gamma_{0,i}\le u^*(x)\le\gamma_{1,i} \qquad for\ a.a.\ \ x\in\Omega. \tag{5.24}$$

Utilizing this fact together with the pointwise convergence (5.22), by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{k\to\infty} p_k(x) = \lim_{k\to\infty}\mathfrak{F}(v_k(x)) &= 1 + \frac{a^2}{a^2 + \left(\lim_{k\to\infty}\left|(\nabla G_\sigma * v_k)(x)\right|\right)^2}\\ &= 1 + \frac{a^2}{a^2 + \left(\left|\left(\nabla G_\sigma * \lim_{k\to\infty} v_k\right)(x)\right|\right)^2} = \mathfrak{F}(u^*(x)), \quad \forall x\in\Omega.\end{aligned} \tag{5.25}$$

Since, by Arzelà–Ascoli theorem, the set $\left\{p_k = 1 + g\left(\left|(\nabla G_\sigma * v_k)(x)\right|\right)\right\}_{k\in\mathbb{N}}$ is compact with respect to the norm topology of $C(\overline{\Omega})$, it follows from (5.25) (see also the proof of Lemma 3.1) that

$$p_k\to p^* = \mathfrak{F}(u^*(x)) \ \text{ strongly in } C(\overline{\Omega}) \text{ as } k\to\infty, \text{ and } p^*\in\mathfrak{S}_{ad}. \tag{5.26}$$

Then properties (5.22)–(5.26) and Proposition 3.1 imply:

$$u^*\in\mathcal{B}_{i,p^*(\cdot)} = \left\{u\in W^{1,p^*(\cdot)}(\Omega)\ :\ 1\le\gamma_{i,0}\le u(x)\le\gamma_{i,1} \text{ a.e. in } \Omega\right\}. \tag{5.27}$$

Thus, the iterative procedure (5.7)–(5.9) has a cluster point $(u^*,p^*)\in\mathcal{B}_{i,p^*(\cdot)}\times\mathfrak{S}_{ad}$ with respect to the convergence (5.22)–(5.23), (5.26).

We are now in a position to state the main result of this section.

THEOREM 5.1. *Let $\mu\in(0,1)$, $\lambda>0$, $\widetilde{S}\in L^1(\Omega;\mathbb{R}^m)$, $\widetilde{M}:G_L\to\mathbb{R}^m$, and $T_S\in\mathcal{L}(L^1(\Omega))$ be given data. Let $\{\tau_k^u\}_{k\in\mathbb{N}}$ be a monotonically increasing sequence of positive stepsizes such that $\tau_k^u\to\infty$ as $k\to\infty$. Then, for each $i=1,\dots,m$, the sequence $\{(v_k,p_k)\}_{k\in\mathbb{N}}$, coming from the iteration procedure (5.7)–(5.9), possesses the following asymptotic properties:*

$$v_k(x)\to\widetilde{u}_i(x) \ a.e.\ in\ \Omega, \tag{5.28}$$

$$v_k \to \widetilde{u}_i \ \text{in}\ L^q(\Omega),\ \forall q \in \left[1, \frac{2\alpha}{2-\alpha}\right), \nabla v_k \rightharpoonup \nabla \widetilde{u}_i \ \text{in}\ L^\alpha(\Omega;\mathbb{R}^2), \tag{5.29}$$

$$p_k \to \widetilde{p}_i = \mathfrak{F}(\widetilde{u}_i) \ \text{strongly in}\ C(\overline{\Omega}) \ \text{as}\ k \to \infty, \tag{5.30}$$

where $\widetilde{u}_i$ is a weak solution to the problem (2.6)–(2.7), that is,

$$\widetilde{u}_i \in \mathcal{B}_{i,\widetilde{p}_i(\cdot)}, \quad \widetilde{u}_i = \operatorname*{Argmin}_{v \in \mathcal{B}_{i,\widetilde{p}_i(\cdot)}} F_i(v, \widetilde{p}_i),$$

and, in addition, the following variational property holds true

$$F_i(v_k, p_k) \geq F_i(v_{k+1}, p_{k+1}), \quad \forall k \in \mathbb{N}, \tag{5.31}$$

$$\lim_{k\to\infty} F_i(v_k, p_k(\cdot)) = \lim_{k\to\infty} \left[\inf_{v \in \mathcal{B}_{i,p_k(\cdot)}} F_i(v, p_k) \right] = \inf_{v \in \mathcal{B}_{i,\widetilde{p}_i(\cdot)}} F_i(v, \widetilde{p}_i(\cdot)) = J_i(\widetilde{u}_i). \tag{5.32}$$

Proof. As immediately follows from Lemma 5.1, the sequence $\{(v_k, p_k)\}_{k\in\mathbb{N}}$ is compact with respect to the convergence (5.28)–(5.30). Let $(\widetilde{u}_i, \widetilde{p}_i)$ be its cluster point. In order to show that the function $\widetilde{u}_i$ is a weak solution to the problem (2.6)–(2.7), we assume the converse — namely, there is another function $z \in \mathcal{B}_{i,\widetilde{p}_i(\cdot)}$ such that

$$F_i(z, \widetilde{p}_i) = \inf_{v \in \mathcal{B}_{i,\widetilde{p}_i(\cdot)}} F_i(v, \widetilde{p}_i) < F_i(\widetilde{u}_i, \widetilde{p}_i) \equiv J_i(\widetilde{u}_i). \tag{5.33}$$

Using the procedure of the standard direct smoothing, we set

$$u_\varepsilon(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} K\left(\frac{x-s}{\varepsilon}\right) \widetilde{z}(s)\, ds,$$

where $\varepsilon > 0$ is a small parameter, K is a positive compactly supported smooth function with properties

$$K \in C_0^\infty(\mathbb{R}^2), \ \int_{\mathbb{R}^2} K(x)\, dx = 1, \ \text{and} \quad K(x) = K(-x),$$

and $\widetilde{z}$ is zero extension of z outside of Ω.

Since $z \in W^{1,\widetilde{p}(\cdot)}(\Omega)$ and $\widetilde{p}(x) \geq \alpha = 1 + \delta$ in Ω, it follows that $z \in W^{1,\alpha}(\Omega)$. Then

$$\begin{aligned} &u_\varepsilon \in C_0^\infty(\mathbb{R}^2) \text{ for each } \varepsilon > 0, \\ &u_\varepsilon \to z \ \text{ in } L^\alpha(\Omega), \quad \nabla u_\varepsilon \to \nabla z \ \text{ in } L^\alpha(\Omega;\mathbb{R}^2) \end{aligned} \tag{5.34}$$

by the classical properties of smoothing operators. From this, we deduce that

$$u_\varepsilon(x) \to z(x) \ \text{ a.e. in } \Omega. \tag{5.35}$$

Moreover, taking into account the estimates

$$\begin{aligned} u_\varepsilon(x) &= \int_{\mathbb{R}^2} K(y) \widetilde{z}(x - \varepsilon y)\, dy \leq \gamma_{1,i} \int_{\mathbb{R}^2} K(y)\, dy = \gamma_{1,i}, \\ u_\varepsilon(x) &\geq \int_{y \in \varepsilon^{-1}(x-\Omega)} K(y) \widetilde{z}(x - \varepsilon y)\, dy \geq \gamma_{0,i} \int_{y \in \varepsilon^{-1}(x-\Omega)} K(y)\, dy \geq \gamma_{0,i}, \end{aligned}$$

we see that each element u_ε is subjected to the pointwise constraints

$$\gamma_{0,i} \leq u_\varepsilon(x) \leq \gamma_{1,i} \ \text{ a.e. in } \Omega, \quad \forall \varepsilon > 0.$$

Since, for each $\varepsilon>0$, $u_\varepsilon\in W^{1,p_k(\cdot)}(\Omega)$ for all $k\in\mathbb{N}$, it follows that $u_\varepsilon\in\mathcal{B}_{i,p_k(\cdot)}$, i.e., each element of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is a feasible solution to all approximating problems

$$\inf_{v\in\mathcal{B}_{i,p_k(\cdot)}}\left\{\frac{1}{2\tau_k^u}\|v-v_k\|^2_{L^2(\Omega)}+F_i(v,p_k)\right\},\quad k\in\mathbb{N}. \tag{5.36}$$

Hence,

$$F_i(v_{k+1},p_k)+\frac{1}{2\tau_k^u}\|v_{k+1}-v_k\|^2_{L^2(\Omega)}\overset{\text{by (5.11)}}{\leq}F_i(u_\varepsilon,p_k)+\frac{1}{2\tau_k^u}\|u_\varepsilon-v_k\|^2_{L^2(\Omega)},$$
$$F_i(v_{k+1},p_{k+1})+\frac{1}{2\tau_k^q}\|p_{k+1}-p_k\|^2_{L^2(\Omega)}\overset{\text{by (5.15)}}{\leq}F_i(v_{k+1},p_k)$$

for all $\varepsilon>0$ and $k=0,1,\dots$. From this, we deduce that

$$F_i(v_{k+1},p_{k+1})\leq F_i(u_\varepsilon,p_k)+\frac{1}{2\tau_k^u}\|u_\varepsilon-v_k\|^2_{L^2(\Omega)},\quad\forall\varepsilon>0,\ \forall k=0,1,\dots \tag{5.37}$$

Further, we notice that

$$\liminf_{k\to\infty}F_i(v_k,p_k)\geq F_i(\widetilde{u}_i,\widetilde{p}_i) \tag{5.38}$$

by Proposition 3.1 and Fatou's lemma. Since

$$|\nabla u_\varepsilon(x)|^{p_k(x)}\to|\nabla u_\varepsilon(x)|^{\widetilde{p}_i(x)}\quad\text{uniformly in }\Omega\text{ as }\ k\to\infty,$$

it follows from the Lebesgue dominated convergence theorem that the objective functional $F_i(u_\varepsilon,\cdot)$ is continuous with respect to the norm convergence in $C(\overline{\Omega})$, i.e.

$$\lim_{k\to\infty}F_i(u_\varepsilon,p_k)=F_i(u_\varepsilon,\widetilde{p}_i),\quad\forall\varepsilon>0. \tag{5.39}$$

As a result, passing to the limit in (5.37) and utilizing properties (5.38)–(5.39), we obtain

$$\lim_{k\to\infty}\frac{1}{2\tau_k^u}\|u_\varepsilon-v_k\|^2_{L^2(\Omega)}=\frac{1}{2\lim\limits_{k\to\infty}\tau_k^u}\|u_\varepsilon-\widetilde{u}_i\|^2_{L^2(\Omega)}=0.$$

Therefore,

$$\begin{aligned}F_i(\widetilde{u}_i,\widetilde{p}_i)\leq F_i(u_\varepsilon,\widetilde{p}_i)&=\int_\Omega|\nabla u_\varepsilon(x)|^{\widetilde{p}_i(x)}\,dx+\lambda\int_\Omega|\nabla u_\varepsilon(x)-\nabla\widetilde{S}_i(x)|\,dx\\&\quad+\mu\int_\Omega|T_Su_\varepsilon(x)-\widetilde{S}_i(x)|\,dx+\frac{1-\mu}{2}T_M\left(|(G_\sigma*u_\varepsilon)(\cdot)-\widetilde{M}_i(\cdot)|^2\right),\quad\forall\varepsilon>0.\end{aligned} \tag{5.40}$$

Taking into account the pointwise convergence (see (5.35) and property (5.34))

$$\begin{aligned}&|\nabla u_\varepsilon(x)|^{\widetilde{p}_i(x)}\to|\nabla z(x)|^{\widetilde{p}_i(x)},\quad\text{a.e. in }\ \Omega,\\&|T_Su_\varepsilon(x)-\widetilde{S}_i(x)|\to|T_Sz(x)-\widetilde{S}_i(x)|,\quad\text{a.e. in }\ \Omega,\\&\int_\Omega|\nabla u_\varepsilon(x)-\nabla\widetilde{S}_i(x)|\,dx\to\int_\Omega|\nabla z(x)-\nabla\widetilde{S}_i(x)|\,dx,\end{aligned}$$

$$(G_\sigma * u_\varepsilon)(x) - \widetilde{M}_i(x) \to (G_\sigma * z)(x) - \widetilde{M}_i(x), \quad \text{in } \Omega$$

as $\varepsilon \to 0$, and the fact that, for ε small enough,

$$|\nabla u_\varepsilon|^{\widetilde{p}_i(\cdot)} \le (1+|\nabla z|)^{\widetilde{p}_i(\cdot)} \in L^1(\Omega),$$
$$\left|T_S u_\varepsilon(\cdot) - \widetilde{S}_i(\cdot)\right| \le \left[\|T_S\|(1+|z(\cdot)|) + |\widetilde{S}_i(\cdot)|\right] \in L^1(\Omega),$$

we can pass to the limit in (5.40) as $\varepsilon \to 0$ by the Lebesgue dominated convergence theorem. This yields

$$F_i(\widetilde{u}_i, \widetilde{p}_i(\cdot)) \le \lim_{\varepsilon\to 0} F_i(u_\varepsilon, \widetilde{p}_i(\cdot)) = F_i(z, \widetilde{p}_i(\cdot)).$$

Combining this inequality with (5.40) and (5.33), we finally get

$$F_i(z,\widetilde{p}_i) = \inf_{v\in\mathcal{B}_{i,\widetilde{p}_i(\cdot)}} F_i(v,\widetilde{p}_i) < F_i(\widetilde{u}_i,\widetilde{p}_i) \le F_i(z,\widetilde{p}_i),$$

that leads us into conflict with the initial assumption. Thus,

$$J_i(\widetilde{u}_i) = F_i(\widetilde{u}_i,\widetilde{p}_i(\cdot)) = \inf_{v\in\mathcal{B}_{i,\widetilde{p}_i(\cdot)}} F_i(v,\widetilde{p}_i(\cdot)) \tag{5.41}$$

and, therefore, $\widetilde{u}_i$ is a weak solution to the original problem (2.6)–(2.7). As for the variational property (5.32) and property (5.31), they immediately follow from (5.41), (5.39), and Proposition 5.1. ∎

6. Optimality conditions

To characterize the solution $u^{0,p(\cdot)} \in \mathcal{B}_{i,p(\cdot)}$ of the approximating optimization problem $\left\langle \inf_{v\in\mathcal{B}_{i,p(\cdot)}} F_i(v,p(\cdot))\right\rangle$, we check whether the objective functional $\mathcal{J}_i(v,p)$

$$F_i(v,p) = \int_\Omega |\nabla v(x)|^{p(x)}\,dx + \lambda\int_\Omega |\nabla v(x) - \nabla\widetilde{S}_i(x)|\,dx$$
$$+\mu\int_\Omega |T_S v(x) - \widetilde{S}_i(x)|\,dx + \frac{1-\mu}{2}T_M\left(|(G_\sigma * v)(\cdot) - \widetilde{M}_i(\cdot)|^2\right) \tag{6.1}$$

is Gâteaux differentiable with respect to v. Namely, let us show that

$$\lim_{t\to 0}\frac{F_i(u^{0,p(\cdot)}+tv,p) - F_i(u^{0,p(\cdot)},p)}{t} = \int_\Omega p(x)\left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2}\nabla u^{0,p(\cdot)}(x), \nabla v(x)\right)dx$$
$$+\lambda\int_\Omega \frac{\left(\nabla u^{0,p(\cdot)}(x) - \nabla\widetilde{S}_i(x), \nabla v\right)}{|\nabla u^{0,p(\cdot)}(x) - \nabla\widetilde{S}_i(x)|}\,dx + \mu\int_\Omega \frac{T_S\left(u^{0,p(\cdot)}\right)}{|T_S\left(u^{0,p(\cdot)}\right) - \widetilde{S}_i|}T_S(v)\,dx$$
$$+(1-\mu)T_M\left(\left[\left(G_\sigma * u^{0,p(\cdot)}\right) - \widetilde{M}_i\right]G_\sigma * v\right), \quad \forall v\in W^{1,p(\cdot)}(\Omega). \tag{6.2}$$

To this end, we note that

$$\frac{|\nabla u^{0,p(\cdot)}(x) + t\nabla v(x)|^{p(x)} - |\nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t} \to \left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2}\nabla u^{0,p(\cdot)}(x), \nabla v(x)\right)$$

as $t\to 0$ almost everywhere in Ω. Indeed, by convexity,

$$|\xi|^p - |\eta|^p \le 2p\left(|\xi|p - 1 + |\eta|^{p-1}\right)|\xi-\eta|,$$

it follows that

$$\begin{aligned}&\Big|\frac{1}{p(x)t}\left(|\nabla u^{0,p(\cdot)}(x)+t\nabla v(x)|^{p(x)}-|\nabla u^{0,p(\cdot)}(x)|^{p(x)}\right)\Big|\\&\le 2\left(\|\nabla u^{0,p(\cdot)}(x)+t\nabla v(x)\|^{p(x)-1}+\|\nabla u^{0,p(\cdot)}(x)\|^{p(x)-1}\right)\|\nabla v(x)\|\\&\le \mathrm{const}\left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}+|\nabla v(x)|^{p(x)-1}\right)|\nabla v(x)|.\end{aligned}\tag{6.3}$$

Taking into account that

$$\begin{aligned}\int_\Omega |\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}|\nabla v(x)|\,dx &\le 2\||\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega)}\||\nabla v(x)|\|_{L^{p(\cdot)}(\Omega)}\\&\le 2\||\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega,\mathbb{R}^2)}\|\nabla v(x)\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^2)},\end{aligned}$$

and $\int_\Omega |\nabla v(x)|^{p(x)}\,dx \overset{by\ (5.21)}{\le} \|\nabla v\|^2_{L^{p(\cdot)}(\Omega,\mathbb{R}^2)}+1$, we see that the right-hand side of inequality (6.3) is an $L^1(\Omega)$-function. Therefore,

$$\begin{aligned}&\int_\Omega \frac{|\nabla u^{0,p(\cdot)}(x)+t\nabla v(x)|^{p(x)}-|\nabla u^{0,p(\cdot)}(x)|^{p(x)}}{t}\,dx\\&\to \int_\Omega p(x)\left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2}\nabla u^{0,p(\cdot)}(x),\nabla v(x)\right)dx \quad as \quad t\to 0\end{aligned}$$

by the Lebesgue dominated convergence theorem.

Utilizing similar arguments to the rest of the terms in (6.1), we deduce that the representation (6.2) for the Gâteaux differential of $F_i(\cdot,p(\cdot))$ at the point $u^{0,p(\cdot)}\in\mathcal{B}_{i,p(\cdot)}$ is valid.

Thus, in order to derive some optimality conditions for the minimizing element $v_{k+1}\in\mathcal{B}_{i,p_k(\cdot)}$ to the problem (5.36), we note that $\mathcal{B}_{i,p_k(\cdot)}$ is a nonempty convex subset of $W^{1,p_k(\cdot)}(\Omega)$ and the objective functional

$$\left\{\frac{1}{2\tau_k^u}\|\cdot - v_k\|^2_{L^2(\Omega)}+F_i(\cdot,p_k)\right\}:\mathcal{B}_{i,p_k(\cdot)}\to\mathbb{R}$$

is strictly convex. Hence, the well known classical result (see [30, Theorem 1.1.3]) and representation (6.2) lead us to the following conclusion.

THEOREM 6.1. *Let $p_k(\cdot)\in\mathfrak{S}$ be an exponent given by the iterative rule (5.12). Let $i\in\{1,\dots,m\}$ be the number of a fixed spectral channels. Then the unique minimizer $v_{k+1}\in\mathcal{B}_{i,p_k(\cdot)}$ to the approximating problem (5.36) is characterized by*

$$\begin{aligned}&\int_\Omega \left(p_k(x)|\nabla v_{k+1}(x)|^{p_k(x)-2}\nabla v_{k+1}(x),\nabla v(x)-\nabla v_{k+1}(x)\right)dx\\&\quad+\lambda\int_\Omega \frac{\left(\nabla v_{k+1}(x)-\nabla\widetilde{S}_i(x),\nabla v-\nabla v_{k+1}(x)\right)}{|\nabla v_{k+1}(x)-\nabla\widetilde{S}_i(x)|}\,dx\\&\quad+\mu\int_\Omega \frac{T_S(v_{k+1})}{|T_S(v_{k+1})-\widetilde{S}_i|}T_S(v-v_{k+1})\,dx\\&\quad+(1-\mu)T_M\left(\left[(G_\sigma * v_{k+1})-\widetilde{M}_i\right]G_\sigma*(v-v_{k+1})\right)\\&\quad+\frac{1}{\tau_k^u}(v_{k+1}-v_k,v-v_{k+1})_{L^2(\Omega)}\ge 0,\quad \forall v\in\mathcal{B}_{i,p_k(\cdot)}.\end{aligned}\tag{6.4}$$

7. Numerical scheme and settings

To illustrate the proposed algorithm for the simultaneous fusion and denoising of color images with different spatial resolution, we conduct the numerical simulations setting $T_S = Id$ for each spectral channel $i = R,G,B$ and extend the set of feasible solutions $\mathcal{B}_{i,p_k(\cdot)}$ to the form $\mathcal{B}_{i,p_k(\cdot)} = W^{1,p_k(\cdot)}(\Omega)$. In other words, we have dropped the two-side constraints $\gamma_{i,0} \le u(x) \le \gamma_{i,1}$ from the sets $\mathcal{B}_{i,p_k(\cdot)}$, and instead we control the fulfillment of this two-side constraints at each step of the numerical approximations. We also use the L^1-norm for the fidelity terms. As a result, it allows us to rewrite the variational problem (6.4) in the form of the following boundary value problem

$$\begin{aligned}
-\operatorname{div}\left(p_k|\nabla v_{k+1}|^{p_k(\cdot)-2}\nabla v_{k+1}\right) = \lambda \operatorname{div}\left(\frac{\nabla v_{k+1} - \nabla \widetilde{S}_i}{|\nabla v_{k+1} - \nabla \widetilde{S}_i|}\right) - \mu T_S^*\left(\frac{T_S(v_{k+1})}{|T_S(v_{k+1}) - \widetilde{S}_i|}\right)\\
-(1-\mu)\sum_{(x_i,y_j)\in S_L} \delta_{(x_i,y_j)}\left[(G_\sigma * v_{k+1}) - \widetilde{M}_i\right]\\
-\frac{1}{\tau_k^u}(v_{k+1} - v_k), \quad \text{in } \Omega,
\end{aligned} \tag{7.1}$$

$$\frac{\partial v_{k+1}}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{7.2}$$

with $p_k(\cdot)$ defined in (5.12), and $k = 1,2,\dots$.

Since, in practical implementations, it is reasonable to define the solution of the problem (7.1)–(7.2) using a “gradient descent” strategy, we can start with some initial image u_{k+1}^* and pass to the following initial-boundary value problem for a quasi-linear parabolic equation with Neumann boundary conditions

$$\begin{aligned}
\frac{\partial v_{k+1}}{\partial t} = \operatorname{div}\left(p_k|\nabla v_{k+1}|^{p_k(\cdot)-2}\nabla v_{k+1}\right) + \lambda \operatorname{div}\left(\frac{\nabla v_{k+1} - \nabla \widetilde{S}_i}{|\nabla v_{k+1} - \nabla \widetilde{S}_i|}\right)\\
-\mu T_S^*\left(\frac{T_S(v_{k+1})}{|T_S(v_{k+1}) - \widetilde{S}_i|}\right) - (1-\mu)\sum_{(x_i,y_j)\in S_L} \delta_{(x_i,y_j)}\left[(G_\sigma * v_{k+1}) - \widetilde{M}_i\right]\\
-\frac{1}{\tau_k^u}(v_{k+1} - v_k), \quad \text{in } (0,T)\times\Omega,
\end{aligned} \tag{7.3}$$

$$\frac{\partial v_{k+1}}{\partial n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{7.4}$$

$$v_{k+1}(0,\cdot) = u_{k+1}^*(\cdot), \quad k = 0,1,\dots, \quad v_0(0,\cdot) = \widetilde{S}_i(\cdot), \quad \text{in } \Omega. \tag{7.5}$$

There are numerous approaches to solving quasi-linear partial differential equations (see the references [3, 18] for various techniques). Since we are dealing with pixels in image processing, finite difference approaches and an explicit scheme of the forward Euler method are arguably the best options. Let Δt be a time step size. Then setting

$$t = n\Delta t,\ n = 0,1,2,\dots, \quad x = l\ (1 \le l \le N_x), \quad y = j\ (1 \le j \le N_y),$$

where (x,y) stands for image pixel and $N_x \times N_y$ is the original image size at the grid G_H, we define the following discrete notations

$$\Delta_\pm^x v_{lj}^n = \pm\left(v_{l\pm1,j}^n - v_{lj}^n\right), \quad \Delta_\pm^y v_{lj}^n = \pm\left(v_{l,j\pm1}^n - v_{lj}^n\right),$$
$$m(a,b) = \operatorname{minmod}(a,b) = \frac{\operatorname{sgn} a + \operatorname{sgn} b}{2}\min(|a|,|b|),$$

where $v_{l,j}^n$ denotes the approximation of $v_{k+1}(n\Delta t, l, j)$. Then the numerical approximation of the principal components of the boundary value problem (7.3)–(7.5) takes the form

$$\begin{aligned}
&\left(\frac{\partial v_{k+1}}{\partial t}\right)_{l,j}^n \approx \frac{v_{l,j}^{n+1}-v_{l,j}^n}{\Delta t},\\
&\left(\operatorname{div}\left(p_k|\nabla v_{k+1}|^{p_k(\cdot)-2}\nabla v_{k+1}\right)\right)_{l,j}^n \approx \Delta_-^x\left(P_{l,j}^n\right)+\Delta_-^y\left(Q_{l,j}^n\right),\\
&P_{l,j}^n = \frac{p_{l,j}^n}{\left(\sqrt{\varepsilon^2+\left(\Delta_+^x v_{l,j}^n\right)^2+\left(m\left(\Delta_+^y v_{l,j}^n, \Delta_-^y v_{l,j}^n\right)\right)^2}\right)^{2-p_{l,j}^n}}\Delta_+^x v_{l,j}^n,\\
&Q_{l,j}^n = \frac{p_{l,j}^n}{\left(\sqrt{\varepsilon^2+\left(\Delta_+^y v_{l,j}^n\right)^2+\left(m\left(\Delta_+^x v_{l,j}^n, \Delta_-^x v_{l,j}^n\right)\right)^2}\right)^{2-p_{l,j}^n}}\Delta_+^y v_{l,j}^n,\\
&\left(\operatorname{div}\frac{\nabla v_{k+1}-\nabla\widetilde{S}_i}{|\nabla v_{k+1}-\nabla\widetilde{S}_i|}\right)_{l,j}^n \approx \Delta_-^x\left(R_{l,j}^n\right)+\Delta_-^y\left(W_{l,j}^n\right),\\
&R_{l,j}^n = \frac{\Delta_+^x v_{l,j}^n-\Delta_+^x(\widetilde{S}_i)_{l,j}^n}{\sqrt{\varepsilon^2+\left(\Delta_+^x v_{l,j}^n-\Delta_+^x(\widetilde{S}_i)_{l,j}^n\right)^2+A_1^2}},\\
&W_{l,j}^n = \frac{\Delta_+^y v_{l,j}^n-\Delta_+^y(\widetilde{S}_i)_{l,j}^n}{\sqrt{\varepsilon^2+\left(\Delta_+^y v_{l,j}^n-\Delta_+^y(\widetilde{S}_i)_{l,j}^n\right)^2+B_1^2}},\\
&A_1 = m\left(\Delta_+^y v_{l,j}^n-\Delta_+^y(\widetilde{S}_i)_{l,j}^n, \Delta_-^y v_{l,j}^n-\Delta_-^y(\widetilde{S}_i)_{l,j}^n\right),\\
&B_1 = m\left(\Delta_+^x v_{l,j}^n-\Delta_+^x(\widetilde{S}_i)_{l,j}^n, \Delta_-^x v_{l,j}^n-\Delta_-^x(\widetilde{S}_i)_{l,j}^n\right),
\end{aligned}$$

where

$$\begin{aligned}
&p_{l,j}^n = 1+\frac{a^2}{a^2+[(|(\nabla G_\sigma * v_k)(x)|)^2]_{l,j}^n},\\
&(|(\nabla G_\sigma * v_k)(x)|)_{l,j}^n = \sum_{k_1=-5}^{5}\sum_{k_2=-5}^{5} G_\sigma(k_1,k_2)v_{l-k_1,j-k_2}^n.
\end{aligned}$$

As a result, utilizing the formulas given above and associating each step $k=1,2,\dots$ of the iterative procedure (5.10)–(5.12) with the corresponding time step $n\Delta t$ in the numerical approximation of the parabolic problem (7.3)–(7.5), we arrive at the following numerical scheme associated with the initial boundary problem (7.3)–(7.5):

$$\begin{aligned}
v_{l,j}^{n+1} &= v_{l,j}^n+\Delta_-^x\left[P_{l,j}^n\right]\Delta t+\Delta_-^y\left[Q_{l,j}^n\right]\Delta t+\lambda\Delta_-^x\left[R_{l,j}^n\right]\Delta t+\lambda\Delta_-^y\left[W_{l,j}^n\right]\Delta t\\
&\quad+\mu\frac{v_{l,j}^n}{\sqrt{\varepsilon^2+\left(v_{l,j}^n\right)^2+\left((\widetilde{S})_{l,j}^n\right)^2}}+(1-\mu)\left\{\begin{matrix}\left[v_{l,j}^n-\widetilde{M}_{l,j}\right] & \text{if } (l,j)\in S_L,\\ 0 & \text{otherwise}\end{matrix}\right\} \qquad (7.6)\\
&\forall l=1,\dots,N_x, \quad \forall j=1,\dots,N_y, \quad \forall n=0,1,\dots
\end{aligned}$$

with the initial conditions

$$v_{l,j}^0 = \left(\widetilde{S}_i\right)_{l,j}, \quad \forall l = 1,\dots,N_x, \quad \forall j = 1,\dots,N_y \tag{7.7}$$

and boundary conditions

$$v_{0,j}^n = v_{1,j}^n, \quad v_{N_x,j}^n = v_{N_x-1,j}^n, \quad v_{l,0}^n = v_{l,1}^n, \quad v_{l,N_y}^n = v_{l,N_y-1}^n, \quad \forall l = 1,\dots,N_x, \quad \forall j = 1,\dots,N_y. \tag{7.8}$$

To conclude this section, we note that the step size Δt should be small enough to guarantee the stability of the numerical scheme (7.6)–(7.8). As for the stopping condition

$$v_{l,j}^{n+1} \approx v_{l,j}^n \quad \text{for all } l \text{ and } j$$

it can be formalized as follows

$$\max_{1\le l\le N_x} \max_{1\le j\le N_y} \left| v_{l,j}^{n+1} - v_{l,j}^n \right| \le \varepsilon.$$

8. Numerical results

For numerical simulations in this section, we set: $\sigma = 0.5$, $\varepsilon = 0.001$, $\tau_k^u = 100 * k$, $\lambda = 0.01$, $\mu = 0.1$. As for the noise estimator $a > 0$, we use the choice of Black et al. [4] , i.e.

$$a = \frac{1.4826}{\sqrt{2}} MAD(\nabla \widetilde{S}_i),$$

where MAD denotes the median absolute deviation of the corresponding spectral channel $S_i : G_H \to \mathbb{R}$ of original image $S : G_H \to \mathbb{R}^m$ that can be computed as

$$MAD(\nabla \widetilde{S}_i) = \text{median}\left[\left| \nabla \widetilde{S}_i - \text{median}\left(\left| \nabla \widetilde{S}_i \right| \right) \right| \right]$$

and $\text{median}\left(\left| \nabla \widetilde{S}_i \right| \right)$ represents the median over the band $S_i : G_H \to \mathbb{R}$ to the gradient amplitude. To guarantee the stability of the proposed algorithm, we make use of the following condition

$$2\left[\frac{1}{\varepsilon} + \lambda\right] \Delta t < 1,$$

where ε comes from the approximation formulae for $P_{l,j}^n$ and $Q_{l,j}^n$, and we set $\varepsilon = 0.001$.

To illustrate the proposed approach we have used three images $S^I : G_H^I \to \mathbb{R}^3$ (Dog), $S^{II} : G_H^{II} \to \mathbb{R}^1$ (Barbara), and $S^{III} : G_H^{III} \to \mathbb{R}^3$ (Christmas Tree) with the resolutions in pixels $G_H^I = 342 \times 458$, $G_H^{II} = 512 \times 512$, and $G_H^{III} = 1200 \times 800$, respectively. Each of these images has been previously corrupted by the additive zero-mean Gaussian white noise with variance 0.01 (see Figures 8.1–8.3).

As for the images of the same scenes with low resolution and with some extra objects, we have considered two collections. The first one is defined on the grids $G_L^I = 114 \times 152$, $G_L^{II} = 170 \times 170$, and $G_L^{III} = 400 \times 266$, respectively, and the second one has the resolution $G_L^I = 68 \times 91$, $G_L^{II} = 102 \times 102$, and $G_L^{III} = 240 \times 160$, respectively (see Figures 8.4–8.5).

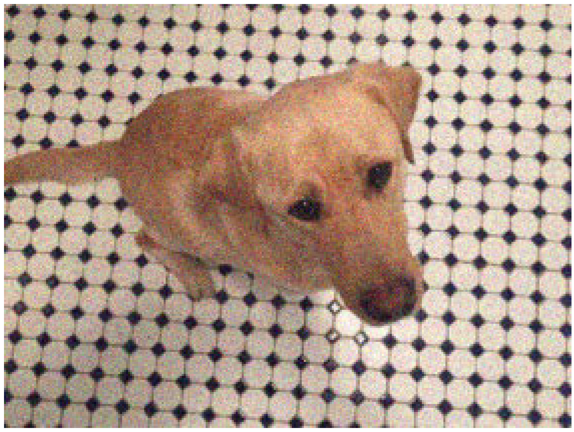

FIG. 8.1. *Noisy image* $S^I : G_H^I \to \mathbb{R}^3$ *(Dog) defined on the grid* $G_H^I = 342 \times 458$.

FIG. 8.2. *Noisy image* $S^{II} : G_H^{II} \to \mathbb{R}^1$ *(Barbara) defined on the grid* $G_H^{II} = 512 \times 512$.

Then following the proximal alternating minimization algorithm described in Section 5, we realize the fusion procedure of given images with a simultaneous denoising procedure. Obtained results are depicted in Figures 8.6–8.8.

It is worth emphasizing that the proposed algorithm is rather sensitive to the choice of parameter μ (see Figure 8.9 for illustration). As for the running time of processing, it takes for the Matlab realization about 30, 95, and 280 sec for the images depicted in Figures 8.6–8.8, respectively.

FIG. 8.3. *Noisy image* $S^{III}: G_H^{III} \to \mathbb{R}^3$ *(Christmas Tree) defined on the grid* $G_H^{III} = 1200 \times 800$.

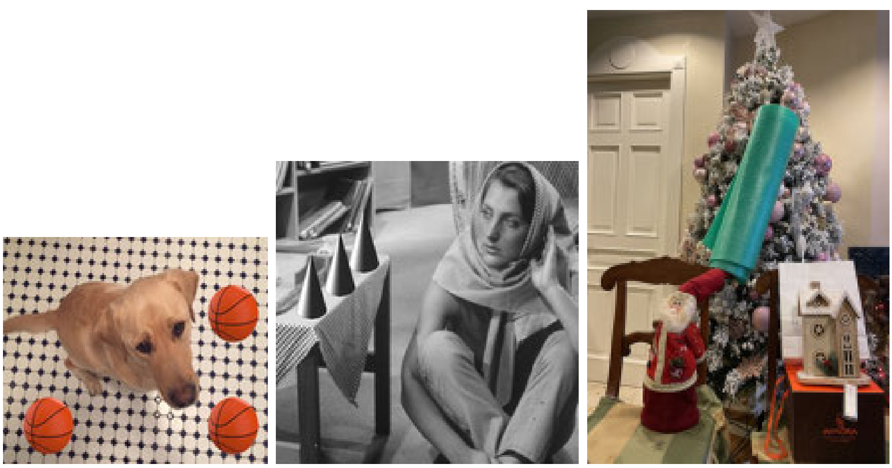

FIG. 8.4. *Images with extra objects and which are defined on the grids with low resolution* ($G_L^I = 114 \times 152$, $G_L^{II} = 170 \times 170$, *and* $G_L^{III} = 400 \times 266$)*, respectively.*

FIG. 8.5. *Images with extra objects and which are defined on the grids with low resolution* ($G_L^I = 68\times 91$, $G_L^{II} = 102\times 102$, *and* $G_L^{III} = 240\times 160$), *respectively.*

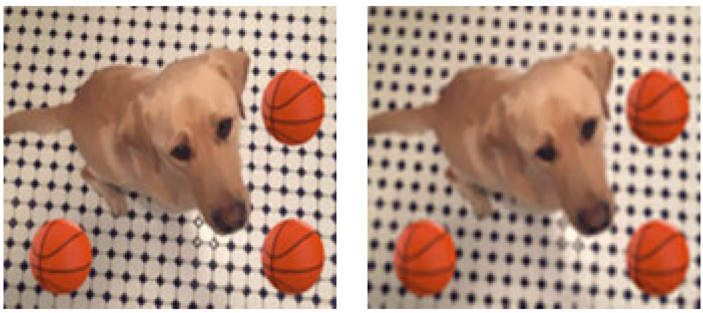

FIG. 8.6. *Result of Simultaneous Fusion and Denoising of* $S^I : G_H^I \to \mathbb{R}^3$ *with* $M^I : (114\times 152) \to \mathbb{R}^3$ *(left) and* $S^I : G_H^I \to \mathbb{R}^3$ *with* $M^I : (68\times 91) \to \mathbb{R}^3$ *(right).*

FIG. 8.7. *Result of Simultaneous Fusion and Denoising of* $S^{II} : G_H^{II} \to \mathbb{R}^3$ *with* $M^{II} : (170\times 170) \to \mathbb{R}^3$ *(up) and* $S^{II} : G_H^{II} \to \mathbb{R}^3$ *with* $M^{II} : (102\times 102) \to \mathbb{R}^3$ *(bottom).*

FIG. 8.8. *Result of Simultaneous Fusion and Denoising of* $S^{III}:G_H^{III}\to\mathbb{R}^3$ *with* $M^{III}:(400\times 266)\to\mathbb{R}^3$ *(left) and* $S^{III}:G_H^{III}\to\mathbb{R}^3$ *with* $M^{III}:(240\times 160)\to\mathbb{R}^3$ *(right).*

The next portion of numerical simulations shows that the proposed technique can be successfully applied to the well-known spatial increasing resolution problem of MODIS-like multi-spectral satellite images via their fusion with the Lansat-like imagery at higher resolution. As input data, we have used a MODIS (the Moderate Resolution Imaging Spectroradiometer) image of some regions with a resolution $350m/pixel$ (see Figure 8.10). This region represents a typical agricultural area with medium-sized fields of various shapes.

We also have the image of the same territory with resolution $25m/pixel$ that was delivered from Landsat satellite at a higher resolution. Figure 8.11 shows the RGB spectral channels of this image.

Figure 8.12 displays the result of image fusion corresponding to the data given by Figures 8.10 and 8.11.

To validate the obtained result for satellite images, we have provided the following calculations.

- Closednees of the means $\rho_2=|\mathrm{Mean}\,I-\mathrm{Mean}\,L|=0$;
- Closedness of the variances $\rho_3=100\frac{|\mathrm{Var}\,I-\mathrm{Var}\,L|}{\mathrm{Var}\,L}\approx 6\%$;
- ERGAS metric

$$ERGAS=100\frac{h}{l}\sqrt{\frac{1}{3}\sum_{k=1}^{3}\left(\frac{\mathrm{RMSE(k)}}{\mu_0(k)}\right)^2}=2.24,$$

FIG. 8.9. *Data Fusion of* $S^{III}: G_H^{III} \to \mathbb{R}^3$ *with* $M^{III}: (400 \times 266) \to \mathbb{R}^3$ *with a semi-transparency effect* ($\mu = 0.8$ *left) and* $\mu = 0.4$ *(right).*

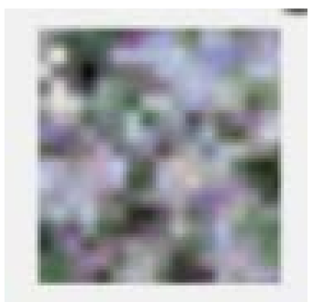

FIG. 8.10. *The MODIS image with resolution* $350m/pixel$.

where h/l is the ratio between the size of the high spatial resolution image pixel and the size of the pixel in the MODIS-like image.

It is worth noticing that in view of the suggestions of Prof. L. Wald if the ERGAS value is less than 3, the spectral quality of an image is satisfactory.

FIG. 8.11. *The Landsat image with resolution* 25*m/pixel.*

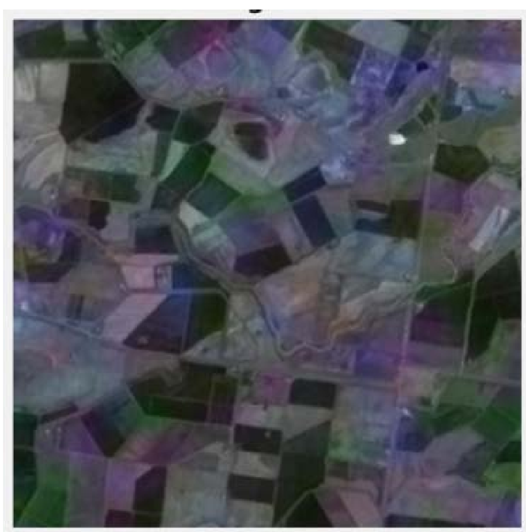

FIG. 8.12. *The retrieved image at high resolution* 25*m/pixel as a result of simultaneous fusion and denoising of the MODIS and Landsat images.*

Declarations.

- Funding: No funds, grants, or other support was received.
- Conflict of interest/Competing interests: The authors have no competing interests to declare that are relevant to the content of this article.
- Availability of data and materials: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
- Authors' contributions: The authors declare that all of them have contributed to the realization of the results.

REFERENCES

[1] H. Alaa, N.E. Alaa, A. Bouchriti, and A. Charkaoui, *An improved nonlinear anisotropic PDE with p(x)-growth conditions applied to image restoration and enhancement*, Authorea, 07, 2022. 2

[2] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, *Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the Kurdyka-Lojasiewicz inequality*, Math. Oper. Res., 35(2):438-457, 2010. 5, 5

[3] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, New York: Springer, 147, 2006. 7

[4] M.J. Black, G. Sapiro, D.H. Marimont, and D. Heeger, *Robust anisotropic diffusion*, IEEE Trans. Image Process., 7(3):421–432, 1998. 8

[5] P. Blomgren, T.F. Chan, P. Mulet, and C. Wong, *Total variation image restoration: Numerical methods and extensions*, Proceedings of the 1997 IEEE International Conference on Image Processing, III, 384–387, 1997. 1, 3

[6] R.S. Blum, Z. Xue, and Z. Zhang, *An overview of image fusion*, in R.S. Blum (ed.), Multi-Sensor Image Fusion and its Applications, CRC Press, Boca Raton, 2006. 1

[7] L. Bungert, D.A. Coomes, M.J. Ehrhardt, J. Rasch, R. Reisenhofer, and C.D. Schönlieb, *Blind image fusion for hyperspectral imaging with the directional total variation*, Inverse Probl., 34(4):044003, 2018. 2, 3

[8] N. Burgos, M.J. Cardoso, M. Modat, S. Pedemonte, J. Dickson, A. Barnes, J.S. Duncan, D. Atkinson, S.R. Arridge, B.F. Hutton, and S. Ourselin, *Attenuation correction synthesis for hybrid PET-MR scanners*, IEEE Trans. Med. Imaging, 33(12):2332–2341, 2014. 1

[9] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., 66(4):1383–1406, 2006. 1, 2, 3

[10] Y. Chen, S. Levine, and J. Stanich, *Image restoration via nonstandard diffusion*, Technical Report, Duquesne University, 13, 2004. 2

[11] D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, New York: Birkhäuser, 2013. 4.1, 5

[12] C. D'Apice, U. De Maio, and P.I. Kogut, *Thermistor problem: Multi-dimensional modeling, optimization, and approximation*, Proceedings of the 32nd European conference on modeling and simulation, May 22nd - May 25th Wilhelmshaven Germany, 348–356, 2018. 2, 3

[13] C. D'Apice, U. De Maio, and P.I. Kogut, *An indirect approach to the existence of quasi-optimal controls in coefficients for multi-dimensional thermistor problem*, in V.A. Sadovnichiy and M.Z. Zgurovsky (eds.), Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics, Springer, 489–522, 2020. 2, 3

[14] C. D'Apice, P.I. Kogut, R. Manzo, and M. Uvarov, *Variational model with nonstandard growth conditions for the restoration of satellite optical images using synthetic aperture radar*, Eur. J. Appl. Math., 34:77–105, 2023. 3

[15] C. D'Apice, P.I. Kogut, O. Kupenko, and R. Manzo, *On variational problem with nonstandard growth functional and its applications to image processing*, J. Math. Imaging Vis., 65:472–491, 2023. 3, 3, 3

[16] L. Diening, P. Harjulehto, M. Hästö, and P. Růžičk, *Lebesgue and Sobolev Spaces with Variable Exponents*, New York: Springer, 2011. 5

[17] C. Huang, M.K. Ng, T. Wu, and T. Zeng, *Quaternion-based dictionary learning and saturation-value total variation regularization for color image restoration*, IEEE Trans. Multimedia, 24:3769–3781, 2021. 1

[18] F. Karami, D. Meskine, and K. Sadik, *A new nonlocal model for the restoration of textured images*, J. Appl. Anal. Comput., 9(6):2070–2095, 2019. 7

[19] S. Goyal, V. Singh, A. Rani, and N. Yadav, *Multimodal image fusion and denoising in NSCT domain using CNN and FOTGV*, Biomed. Signal Process. Control, 71:103214, 2022. 1

[20] P.I. Kogut, *On optimal and quasi-optimal controls in coefficients for multi-dimensional thermistor problem with mixed Dirichlet-Neumann boundary conditions*, Control Cybern., 48(1):31–68, 2019. 2, 3

[21] P.I. Kogut and O.P. Kupenko, *Approximation Methods in Optimization of Nonlinear Systems*, Walter de Gruyter GmbH, Berlin, Boston, 2019. 4

[22] P.I. Kogut, O.P. Kupenko, and R. Manzo, *On regularization of an optimal control problem for ill-posed nonlinear elliptic equations*, Abstr. Appl. Anal., 2020:7418707, 2020. 4

[23] P.I. Kogut and R. Manzo, *On vector-valued approximation of state constrained optimal control problems for nonlinear hyperbolic conservation laws*, J. Dyn. Control Syst., 19(3):381–404, 2013. 4

[24] P. Kogut, Ya. Kohut, and R. Manzo, *Fictitious controls and approximation of an optimal control problem for Perona-Malik equation*, J. Optim. Differ. Equ. Appl., 30(1):42–70, 2022. 4

[25] H. Kohr, *Total variation regularization with variable Lebesgue prior*, arXiv preprint, arXiv:1702.08807, 2017. 1

[26] M. Kumar and S. Dass, *A total variation-based algorithm for pixel-level image fusion*, IEEE Trans. Image Process., 18(9):2137–2143, 2009. 1

[27] O.P. Kupenko and R. Manzo, *Approximation of an optimal control problem in the coefficient for variational inequality with anisotropic p-Laplacian*, Nonlinear Differ. Equ. Appl., 23:35, 2016. 4

[28] F. Li, Zh. Li, and L. Pi, *Variable exponent functionals in image restoration*, Appl. Math. Comput., 216:870–882, 2010. 2

[29] F. Li and T. Zeng, *Variational image fusion with first and second-order gradient information*, J. Comput. Math., 34(2):200–222, 2016. 1

[30] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Berlin: Springer, 1971. 6

[31] Q. Ma, Y. Wang, and T. Zeng, *Retinex-based variational framework for low-light image enhancement and denoising*, IEEE Trans. Multimedia, 25:5580–5588, 2022. 1

[32] J.-J. Mei, Y. Dong, and T.-Z. Huang, *Simultaneous image fusion and denoising by using fractional-order gradient information*, J. Comput. Appl. Math., 351:212–227, 2001. 1

[33] V. Naidu and J. Raol, *Pixel-level image fusion using wavelets and principal component analysis*, Def. Sci. J., 58(3):338–348, 2008. 1

[34] G. Piella, *Image fusion for enhanced visualization: A variational approach*, J. Comput. Vis., 83(1):1–11, 2009. 1

[35] W. Ring, *Structural properties of solutions to total variation regularization problems*, ESAIM: Math. Model. Numer. Anal., 34(4):799–810, 2000. 1

[36] L.I. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica, 60(D):259–268, 1992. 2

[37] P. Shah, S.N. Merchant, and U.B. Desai, *An efficient spatial domain fusion scheme for multifocus images using statistical properties of neighborhood*, IEEE Int. Conf. Multimedia Expo., 18(9):1–6, 2011. 1

[38] F. Šroubek and J. Flusser, *Fusion of blurred images*, in R.S. Blum (ed.), Multi-Sensor Image Fusion and Its Applications, CRC Press, Boca Raton, 2006. 1

[39] W.W. Wang, P.L. Shui, and X.C. Feng, *Variational models for fusion and denoising of multifocus images*, IEEE Signal Process. Lett., 15:65–68, 2008. 1

[40] T. Wunderli, *On time flows of minimizers of general convex functionals of linear growth with variable exponent in BV space and stability of pseudosolutions*, J. Math. Anal. Appl., 364(2):591–598, 2010. 1

[41] A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Series: Springer Optimization and Its Applications, New York: Springer, 44, 2010. 4

[42] V.V. Zhikov, *Solvability of the three-dimensional thermistor problem*, Proc. Steklov Inst. Math., 281:98–111, 2008. 5

[43] V.V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci., 173:463–570, 2011. 4.1

[44] Z. Zhou, S. Li, and B. Wang, *Multi-scale weighted gradient-based fusion for multi-focus images*, Inf. Fusion, 20:60–72, 2014. 1