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ASYMPTOTIC STABILITY OF NONLINEAR WAVE FOR AN INFLOW PROBLEM TO THE COMPRESSIBLE NAVIER-STOKES-KORTEWEG SYSTEM*

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Abstract. In this paper, we are concerned with the inflow problem on the half line $(0,+\infty)$ for a one-dimensional compressible Navier-Stokes-Korteweg system, which is used to model compressible viscous fluids with internal capillarity, i.e., the liquid-vapor mixtures with phase interfaces. We first investigate that the asymptotic profile is a nonlinear wave: the superposition wave of a rarefaction wave and a boundary layer solution under the proper condition of the far fields and boundary values. The asymptotic stability on the nonlinear wave is shown under some conditions that the initial data are a small perturbation of the rarefaction wave and the strength of the stationary wave is small enough. The proofs are given by an elementary energy method.

Keywords. Compressible Navier-Stokes-Korteweg equation; Inflow problem; Rarefaction wave; Boundary layer solution; Asymptotic stability; Energy method.

AMS subject classifications. 76W05; 35B40.

1. Introduction

In this article, we are interested in the following inflow problem on the half space $\mathbb{R}^+$ for one dimensional compressible Navier-Stokes-Korteweg (denoted as NSK in the sequel) system, which reads in the Eulerian coordinates as:

$$\begin{cases} \rho_t+(\rho u)_{\tilde{x}}=0, \\ (\rho u)_t+(\rho u^2+p(\rho))_{\tilde{x}}=\mu u_{xx}+\kappa\rho\rho_{xxx}, \\ (\rho,u)(t=0,\tilde{x})=(\rho_0,u_0)(\tilde{x})\to(\rho_+,u_+), \quad \text{as } \tilde{x}\to+\infty, \\ (\rho,u)(t,\tilde{x}=0)=(\rho_-,u_-),\ \rho_{\tilde{x}}(t,\tilde{x}=0)=\rho_b. \end{cases} \tag{1.1}$$

Here, ρ,u are unknown functions in t and $\tilde{x}$, which stand for the density and the velocity, respectively. The time and space variables are $t>0$ and $\tilde{x}\in\mathbb{R}^+:=\{\tilde{x}\in\mathbb{R}:\tilde{x}>0\}$. The function $p(\rho)$ is the pressure defined by $p(\rho)=k\tilde{\rho}^\gamma$, where $k>0$ and $\gamma\geq 1$ are the gas constants. The positive constants μ,κ denote, respectively, the viscosity and the capillary coefficient, and κ is also called Weber number. ρ_+, ρ_-, ρ_b, u_+ and u_- are constants satisfying $\rho_\pm>0$ and $u_->0$. And $\rho_0(\tilde{x}),u_0(\tilde{x})$ are two given functions.

The model $(1.1)_{1,2}$ considered is supposed to govern the motion of compressible fluids such as liquid-vapor mixtures endowed with a variable internal capillarity, and is originated from the works by van der Waals [44] and Korteweg [25]. Its modern form is actually derived by using the second gradient theory (see for instance [11]). Recently, Heida and Màlek [17] also derived the compressible NSK system by the entropy production method which does not require to introduce any new or non-standard concepts such as multipolarity or interstitial working which are used in [11]. We point out that special cases of the model have also arisen in other contexts, e.g. in the water waves

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theory and more recently in quantum hydrodynamics. Finally, one can see easily that when $\kappa = 0$, the system $(1.1)_{1,2}$ is reduced to the compressible Navier-Stokes equation. The mathematical justification from the compressible NSK system to the compressible Navier-Stokes equation with well-prepared initial data was shown in [2]. When $\mu = \kappa = 0$, the system $(1.1)_{1,2}$ is reduced to the classical compressible Euler equation. Charve and Haspot [5] have shown the existence of global strong solution and vanishing capillarity-viscosity limit in one dimension for the compressible NSK system.

Recently, there have been a great number of mathematical studies about the compressible NSK system, possibly due to its many applications to compressible fluids endowed with a variable internal capillarity. About the existence and uniqueness of solutions to the isentropic compressible NSK system, we can refer to [1, 9, 10, 13–16, 21, 26] and some references therein. In what follows, let us focus on the large-time behavior of solutions to the isentropic compressible NSK system towards the nonlinear wave pattern, which is related to our interest. More precisely, Chen [3] and Li and Luo [32] discussed asymptotic stability of the rarefaction waves to Cauchy problem for the one-dimensional compressible NSK system, respectively. Chen, et al. [4] also showed asymptotic stability of the rarefaction waves for the one-dimensional compressible NSK system with large initial data. Li and Zhu [36] further showed asymptotic stability of the rarefaction wave with vacuum for the one-dimensional compressible NSK system. Chen, He and Zhao [7] studied nonlinear stability of traveling wave solutions to the Cauchy problem for the one-dimensional compressible NSK system. Li, Chen and Luo, and Li and Luo showed stability of the planar rarefaction wave to two- and three-dimensional compressible NSK system in [31, 33], respectively. The stability of stationary solutions of the multi-dimensional isentropic compressible NSK system was studied by Li [29], and Wang and Wang [46] in the case with a external force, respectively, under the assumption that the states at far fields $\pm\infty$ are equal. Moreover, we also mention that there are some studies about the large-time behavior and the optimal decay rates of the global classical solutions and of the global strong solutions for the isentropic compressible NSK system around the non-vacuum constant states, for example, see [43, 45, 46] and some references therein.

Next, for corresponding initial-boundary value problem of the isentropic compressible NSK system, there are some results about the large-time behavior of the solutions. Tsyganov [42] discussed the global existence and time-asymptotic behavior of weak solutions for an isothermal model with the viscosity coefficient $\mu(\rho) \equiv 1$, the capillarity coefficient $\kappa(\rho) = \rho^{-5}$ and large initial data on the interval $[0,1]$. The global existence and exponential decay of strong solutions with small initial data to the Korteweg system in a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ were also obtained by Kotschote in [27]. Chen, Li and Sheng [8] proved the nonlinear stability of viscous shock wave for an impermeable wall problem of the compressible NSK system with constant viscosity and capillarity coefficients and small initial data in the half space. Chen and Li [6] discussed the time-asymptotic behavior of strong solutions to the initial-boundary value problem of the compressible NSK system with density-dependent viscosity and capillarity on the half-line $\mathbb{R}^+$, and showed the strong solution converges to the rarefaction wave as $t \to \infty$ for the impermeable wall problem under large initial perturbation. Hong [18] and Li and Zhu [37] showed the existence and stability of stationary solution to an outflow problem of the compressible NSK system with constant viscosity and capillarity coefficient, respectively. Li, Tang and Yu [35] further obtained asymptotic stability of rarefaction wave for the out-flow problem to the one-dimensional compressible NSK system in the half space. However, to the best of our knowledge, there is little research about the stability of nonlinear wave patterns for the inflow problem on the compressible NSK system, which is the main interest in our paper. Hong [19] and Li and Chen [30] discussed the existence and stability of stationary solution to an inflow problem of the compressible NSK system in the half space, respectively. Moreover, Hong [19] also showed stability of viscous shock wave and the superposition of the stationary wave and the viscous shock wave in the inflow problem for isentropic NSK system as in [23]. Li, Qian and Yu [34] proved the asymptotic behavior toward rarefaction wave for an inflow problem of the compressible NSK equation in the half space. In this article, we are going to the asymptotic behavior toward the nonlinear wave: the superposition of the stationary wave and the rarefaction wave for an inflow problem of the compressible NSK system in the half space.

We now turn back to the inflow problem. First, consider the coordinate transformation

$$t=t,\ x=\int_{(0,0)}^{(\tilde{x},t)}\rho\,\mathrm{d}\tilde{x}-\rho u\,\mathrm{d}t,$$

to transform (1.1) to the problem in the Lagrangian coordinate as follows

$$\begin{cases} v_t-u_x=0,\ x>s_-t,\ t>0,\\ u_t+p(v)_x=\mu(\frac{u_x}{v})_x+\kappa(\frac{-v_{xx}}{v^5}+\frac{5v_x^2}{2v^6})_x,\ x>s_-t,\ t>0,\\ (v,u)|_{x=s_-t}=(v_-,u_-),\\ v_x|_{x=s_-t}=v_b,\\ (v,u)(t=0,x)=(v_0,u_0)(x)\to(v_+,u_+),\quad \text{as } x\to+\infty. \end{cases} \tag{1.2}$$

Here

$$v=\frac{1}{\rho},\quad s_-=-\frac{u_-}{v_-},\quad v_\pm=\frac{1}{\rho_\pm},\quad v_b=-\frac{1}{\rho_-^2}\rho_b.$$

Then we shall consider the inflow problem (1.2) from now on. First, the corresponding hyperbolic system without viscosity and capillarity is

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=0. \end{cases} \tag{1.3}$$

Its characteristic speeds are $\lambda_i=(-1)^i\sqrt{-p'(v)}(i=1,2)$, and the sound speed $c(v)$ is defined by

$$c(v)=v\sqrt{-p'(v)}=\sqrt{k\gamma}v^{-\frac{\gamma-1}{2}}.$$

By the relation of $|u|$ with $c(v)$, we can divide (v,u) into three regions:

$$\begin{aligned} \Omega_{\text{sub}}&=\{(u,v):|u|<c(v),v>0,u>0\},\\ \Gamma_{\text{trans}}&=\{(u,v):|u|=c(v),v>0,u>0\},\\ \Omega_{\text{super}}&=\{(u,v):|u|>c(v),v>0,u>0\}. \end{aligned}$$

We call them the subsonic, transonic, and supersonic regions, respectively. In the phase, the BL-solution line, the 2-shock curve and the $i(i=1,2)$-rarefaction wave curve through (v_-,u_-) are defined by

$$BL(v_-,u_-)=\{(v,u)\in\Omega_{\text{sub}}\cup\Gamma_{\text{trans}}:\frac{u}{v}=\frac{u_-}{v_-}=-s_-\}, \tag{1.4}$$

$$S_2(v_-,u_-)=\{(v,u)\in\mathbb{R}^+\times\mathbb{R}^+ : u=u_- - s(v-v_-), v_-<v\}, \tag{1.5}$$

$$\mathcal{R}_i(v_-,u_-)=\left\{(v,u)\in\omega \,\middle|\, u=u_- - \int_{v_-}^{v}\lambda_i(s)ds, u\geq u_-\right\} \tag{1.6}$$

with $s=\sqrt{\frac{p(v_-)-p(v)}{v-v_-}}$. Note that $BL(v_-,u_-)$ always intersects the transonic line Γ_{trans}, and let us denote the intersection point by (v_*,u_*).

In [19, 30], the authors showed stability of stationary solution, viscous shock wave and the superposition of the stationary wave and the viscous shock wave in the inflow problem for isentropic NSK system. Li, Qian and Yu [34] showed the asymptotic behavior toward rarefaction wave for an inflow problem of the compressible NSK equation when the boundary value and far field state satisfy $(v_-,u_-)\in\Omega_{\text{super}}$ and $(v_+,u_+)\in\mathcal{R}_2(v_-,u_-)$. Here, we are going to the asymptotic behavior toward the nonlinear wave: the superposition of the stationary wave and the rarefaction wave for an inflow problem of the compressible NSK system in the half space when the boundary value and far field state satisfy $(v_-,u_-)\in\Omega_{\text{sub}}\cup\Gamma_{\text{trans}}$ and $(v_+,u_+)\in BL\mathcal{R}_2(v_-,u_-)$. In this case, we can find $(\bar{v},\bar{u})\in BL(v_-,u_-)$ with $(\bar{v},\bar{u})\neq(v_*,u_*)$ or $(\bar{v},\bar{u})=(v_*,u_*)$, and $(v_+,u_+)\in\mathcal{R}_2(\bar{v},\bar{u})$ such that we can show that the solution (v,u) of (1.2) tends toward the combination of the stationary wave and the 2-rarefaction wave as in [22, 39]. More precisely, we firstly use the stationary solution $(V_0,U_0)(x-s_-t)$ which satisfies

$$\begin{cases} -s_-V_0'-U_0'=0, \\ -s_-U_0'+p(V_0)'=\mu\left(\frac{U_0'}{V_0}\right)'+\kappa\left(\frac{-V_0''}{V_0^5}+\frac{5(V_0')^2}{2V_0^6}\right)', \end{cases} \tag{1.7}$$

with the boundary data and the spatial asymptotic conditions

$$(V_0,U_0)(0)=(v_-,u_-),\ V_{0y}(0)=v_b, \tag{1.8}$$

$$(V_0,U_0)(+\infty)=(\bar{v},\bar{u}), \tag{1.9}$$

here $'=\frac{\mathrm{d}}{\mathrm{d}y}$ with $y=x-s_-t$. We call the solution (V_0,U_0) the boundary layer solution.

Then we employ the following Euler equations

$$\begin{cases} v_t-u_x=0, \\ u_t+p(v)_x=0, \quad t>0, x\in\mathbb{R}, \end{cases} \tag{1.10}$$

with the initial data

$$(v,u)(t=0,x)=\begin{cases} (\bar{v},\bar{u}),\ x<0, \\ (v_+,u_+),\ x>0, \end{cases} \tag{1.11}$$

to construct a rarefaction wave $\left(v^R,u^R\right)\left(\frac{x}{t}\right)$. From gas dynamic theory in [40], we know that the problem (1.10)-(1.11) admits a weak entropy solution $\left(v_i^R,u_i^R\right)(t,x)\,(i=1,2)$ called the i-rarefaction wave if $(v_+,u_+)\in\mathcal{R}_i(\bar{v},\bar{u})$, where

$$\mathcal{R}_i(\bar{v},\bar{u})=\left\{(v,u)\in\omega \,\middle|\, u=\bar{u}-\int_{\bar{v}}^{v}\lambda_i(s)ds, u\geq\bar{u}\right\} \tag{1.12}$$

is the i-rarefaction wave curve, and $\left(v_i^R,u_i^R\right)(t,x)$ is expressed by:

$$\left(v_i^R,u_i^R\right)(t,x)=\begin{cases} (\bar{v},\bar{u}), & -\infty\leq\frac{x}{t}\leq\lambda_i(\bar{v}), \\ \left(\lambda_i^{-1}\left(\frac{x}{t}\right),\bar{u}-\int_{\bar{v}}^{\lambda_i^{-1}\left(\frac{x}{t}\right)}\lambda_i(s)\,ds\right), & \lambda_i(\bar{v})\leq\frac{x}{t}\leq\lambda_i(v_+), \\ (v_+,u_+), & \lambda_i(v_+)\leq\frac{x}{t}\leq+\infty. \end{cases}$$

In (1.12), ω is a suitable neighborhood of $(\bar v,\bar u)$ in $\mathbb{R}^2$. Here we assume $(v_+,u_+)\in\mathcal{R}_2(\bar v,\bar u)$, namely, $\left(v^R,u^R\right)\left(\frac{x}{t}\right)=\left(v_2^R,u_2^R\right)(t,x)$.

With a rarefaction wave $\left(v^R,u^R\right)\left(\frac{x}{t}\right)$ and a stationary wave $(V_0,U_0)(x-s_-t)$ in hand, we are able to define a nonlinear wave $(V,U)(t,x)$ as follows

$$(V,U)(t,x)=(V_0,U_0)(x)+(v^r,u^r)(t,x)-(\bar v,\bar u), \tag{1.13}$$

where $(v^r,u^r)(t,x)$ is a suitably smoothed function of $\left(v^R,u^R\right)\left(\frac{x}{t}\right)=\left(v_2^R,u_2^R\right)(t,x)$, and will be stated in Section 2. Moreover, we let $\delta=|v_+-\bar v|+|u_+-\bar u|$ and $\tilde\delta=|\bar v-v_-|+|v_b|$. Now the main results are stated as follows.

Theorem 1.1. *Let $(v_-,u_-)\in\Omega_{\text{sub}}\cup\Gamma_{\text{trans}}$ and $(v_+,u_+)\in BL\mathcal{R}_2(v_-,u_-)$. Assume that $v_0-v_0^r\in H_0^2(\mathbb{R}^+)$, $u_0-u_0^r\in H_0^1(\mathbb{R}^+)$. Then there exists $\varepsilon_0>0$ such that if $0<\varepsilon\le\varepsilon_0$ and $\tilde\delta+\|v_0-v_0^r\|_2+\|u_0-u_0^r\|_1\le\varepsilon_0$, then there exists a unique strong solution (v,u) of (1.2), which satisfies*

$$\begin{aligned}
&v-V\in C([0,\infty);H_0^2(\mathbb{R}^+)),\quad u-U\in C([0,\infty);H_0^1(\mathbb{R}^+)),\\
&(v-V)_x\in L^2([0,\infty);H^2(\mathbb{R}^+)),\quad (u-U)_x\in L^2([0,\infty);H^1(\mathbb{R}^+)).
\end{aligned}$$

Moreover, it holds that

$$\lim_{t\to+\infty}\sup_{x\ge s_-t}|(v,u)-(V,U)|=0. \tag{1.14}$$

Here ε, (v^r,u^r) and (v_0^r,u_0^r) are given by $(2.12)_2$, (2.14) and (2.15), respectively.

Remark 1.1. For the case $(v_-,u_-)\in\Omega_{\text{sub}}\cup\Gamma_{\text{trans}}$ and $(v_+,u_+)\in BL\mathcal{R}_1\mathcal{R}_2(v_-,u_-)$, we can find $(\bar v_1,\bar u_1)\in BL(v_-,u_-)$ $(\bar v_2,\bar u_2)\in\mathcal{R}_1(\bar v_1,\bar u_1)$ and $(v_+,u_+)\in\mathcal{R}_2(\bar v_2,\bar u_2)$ such that we can show that the solution (v,u) of (1.2) tends toward the combination of (V_0,U_0), (v_1^R,u_1^R) and (v_2^R,u_2^R) as in [22, 39, 41].

Remark 1.2. In this article we only consider the asymptotic behavior of the stationary wave for inflow problem to one-dimensional compressible NSK system with small initial perturbation, in fact, it is interesting and plausible that we can consider the corresponding results for large perturbation as in [12, 20] for the compressible Navier-Stokes equation. Moreover, here we only consider the inflow problem to one dimensional compressible NSK system in the half space. However we should mention that the corresponding initial boundary value problem such as the out-flow problem and the inflow problem for the multi-dimensional compressible NSK system is surely more difficult, thus more interesting. These are expected to be done in the forthcoming papers.

This article is a follow-up study of [19,30,34]. Now we give main ideas and arguments of the proof for Theorem 1.1. Applying L^2-energy method, some time-decay estimates in L^p-norm of the smoothed rarefaction wave and the spatial decay of the stationary wave as in [24, 39, 41], we prove the asymptotic stability of the nonlinear wave: the superposition of the stationary wave and the rarefaction wave in the case that the initial data are a small perturbation of the rarefaction wave and the strength of the stationary wave is small enough. The key ingredient in the proof of Theorem 1.1 is to deduce the a-priori estimates. Compared with [39] for the one-dimensional compressible Navier-Stokes system, the main difficulties are as follows. The first one is the occurrence of the third order dispersion term. The second is how to control the boundary terms in order to establish the dissipation of the density. To overcome the first difficulty, we need more regularities for the density and smooth rarefaction wave, which have been made in

[3,7,8,30,31,33,37]. We also note that the basic energy is obtained with the help of higher order estimates. For the second difficulty, we first have $\varphi(t,0)=\psi(t,0)=\varphi_y(t,0)=0$ from the boundary data $(3.10)_2$ and (iii) of Lemma 2.2. Next, similar as [41], we can establish the boundary dissipation of $\psi_y(t,0)$. Finally, we can obtain the boundary dissipation of $\varphi_{yy}(t,0)$ due to the Korteweg term, which is different from the out-flow problem in [35, 37]. With these boundary values and the boundary dissipations at hand, we can close the a-priori estimate.

The rest of the article is organized as follows. After stating some notations, in Section 2, we make some preliminaries. That is, we first recall the existence and properties of stationary solution. Next, we review a smooth approximation $(v^r(t,x),u^r(t,x))$ of the rarefaction wave $\left(v_2^R,u_2^R\right)(t,x)$ by (2.14), and list some basic properties of the smooth approximate rarefaction wave $(v^r(t,x),u^r(t,x))$. Finally, we are going to give the Poincaré-type inequality for later use. Then we reformulate the original problem in terms of the perturbation variables in Section 3. Section 4 is the key part of this article, in which we will establish the *a priori* estimates by the elaborate energy estimates. Finally, we complete the proof of Theorem 1.1 in Section 5.

Notations: Throughout this paper, two positive generic constants are denoted by C and c. For function space, $L^p(\mathbb{R}^+)(1\le p\le+\infty)$ is the usual Lebesgue space on $\Omega\subset\mathbb{R}=(-\infty,+\infty)$ with its norm

$$\|f\|_{L^p(\Omega)}=\left(\int_\Omega|f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}},1\le p<\infty,\ \|f\|_{L^\infty(\Omega)}=\sup_\Omega|f(x)|.$$

$H^l(\Omega)$ denotes the l-th order Sobolev Space with its norm

$$\|f\|_l=\left(\sum_{i=0}^l\|\partial_x^i f\|^2\right)^{\frac{1}{2}}\ \text{where}\ \|\cdot\|:=\|\cdot\|_{L^2(\Omega)}.$$

$H_0^l(\Omega)$ is a closure of $C_0^\infty(\Omega)$ with respect to $H^l(\Omega)$-norm, so that $f\in H_0^l(\Omega)$ satisfies $f(\partial\Omega)=0$. The domain Ω will be often abbreviated without any confusion. Finally, we denote by $C^0([0,T];H^k(\Omega))$ (resp. $L^2(0,T;H^k(\Omega))$) the space of continuous (resp. square integrable) functions on $[0,T]$ taking values in the space $H^k(\Omega)$.

2. Preliminaries

The aims of this section are to make some preliminaries. That is, we first recall the existence and properties of stationary solution. Next, we review a smooth approximation $(v^r(t,x),u^r(t,x))$ of the rarefaction wave $\left(v_2^R,u_2^R\right)(t,x)$ by (2.14), and list some basic properties of the smooth approximate rarefaction wave $(v^r(t,x),u^r(t,x))$. Finally, we give the Poincaré-type inequality. Firstly, we consider the stationary problem, and state the estimates for the solution to this problem, which have been derived in [19, 30] and those estimates will be used to deal with the stationary part $(V_0,U_0)(y=x-s_-t)$ in our time-asymptotic state (V,U). The stationary problem reads as

$$\begin{cases}-s_-V_0'-U_0'=0,\\ -s_-U_0'+p(V_)0)'=\mu\left(\frac{U_0'}{V_0}\right)'+\kappa\left(\frac{-V_0''}{V_0^5}+\frac{5(V_0')^2}{2V_0^6}\right)',\end{cases}\tag{2.1}$$

with the boundary data and the spatial asymptotic conditions

$$(V_0,U_0)(0)=(v_-,u_-),\ V_{0y}(0)=v_b,\quad (V_0,U_0)(+\infty)=(\bar v,\bar u).\tag{2.2}$$

Concerning the solution to (2.1)-(2.2), we have

LEMMA 2.1 (see [19,30]). *Assume that* $(u_-,v_-)\in\Omega_{\rm sub}$, $u_->0$, $\frac{\bar u}{\bar v}=\frac{u_-}{v_-}(=-s_-)$ *and that the boundary value* (v_-,v_b) *satisfies*

$$(v_-,v_b)\in\mathcal{M}^+:=\left\{(v_1,v_2)\in\mathbb{R}^2:|(v_1-\bar v,v_2)|<\varepsilon_0\right\} \tag{2.3}$$

for a certain positive constant ε_0.

(i) *If* $-s_-\bar v^{\frac{\gamma+1}{2}}>\sqrt{k\gamma}$, *then there is no solution to problem (2.1)-(2.2).*

(ii) *For* $-s_-\bar v^{\frac{\gamma+1}{2}}=\sqrt{k\gamma}$, *there exists a certain region* $\mathcal{M}^0\subset\mathcal{M}^+$ *such that if the boundary value* (v_-,v_b) *satisfies the condition*

$$(v_-,v_b)\in\mathcal{M}^0, \tag{2.4}$$

then there exists a unique smooth solution $(V_0,U_0)(\xi=x-s_-t)$ *to problem (2.1)-(2.2) which satisfies*

$$\left|\partial_y^k(V_0-\bar v,U_0-\bar u)\right|\le C\frac{\tilde\delta^{k+1}}{(1+\tilde\delta y)^{k+1}} \text{ for } k=0,1,2,\cdots, \tag{2.5}$$

and

$$(V_{0\xi},U_{0\xi})(\xi)=(a_1,a_2)z^2(y)+O(z^3(y)), \tag{2.6}$$

where $a_i>0(i=1,2)$ *are constants and* $z(\xi)$ *is a smooth function satisfying*

$$0<\frac{c\tilde\delta}{1+\tilde\delta y}\le z(y)\le\frac{C\tilde\delta}{1+\tilde\delta y},\ |\partial_y^k z(y)|\le\frac{C\tilde\delta^{k+1}}{(1+\tilde\delta y)^{k+1}},\ k=1,2,\cdots. \tag{2.7}$$

(iii) *For* $-s_-\bar v^{\frac{\gamma+1}{2}}<\sqrt{k\gamma}$, *there exists a certain curve* $\mathcal{M}^-\subset\mathcal{M}^+$ *such that if the boundary data* (v_-,v_b) *satisfies the condition*

$$(v_-,v_b)\in\mathcal{M}^-, \tag{2.8}$$

then there exists a unique smooth solution $(V_0,U_0)(\xi=x-s_-t)$ *to problem (2.1)-(2.2) such that*

$$\left|\partial_y^k(V_0-\bar v,U_0-\bar u)\right|\le C\tilde\delta e^{-cy} \text{ for } k=0,1,2,\cdots, \tag{2.9}$$

where C *and* c *are positive constants.*

Next, we use the same approach as in [38] to construct the smooth approximation of the rarefaction wave part $\left(v_2^R,u_2^R\right)\left(\frac{x}{t}\right)$ in our time-asymptotic state (V,U). Since the rarefaction wave $\left(v_2^R,u_2^R\right)\left(\frac{x}{t}\right)$ is not smooth, we need to construct a smooth approximation $(v^r,u^r)(t,x)$ of the rarefaction wave $\left(v_2^R,u_2^R\right)\left(\frac{x}{t}\right)$. As in [38], we start with the Riemann problem on $\mathbb{R}=(-\infty,+\infty)$ for the typical Burgers equation:

$$w_t+ww_x=0, \tag{2.10}$$

with initial data

$$w(0,x)=w_0^R(x)=\begin{cases} w_-,\ x<0\\ w_+,\ x>0,\end{cases} \tag{2.11}$$

where $w_- < w_+$. The weak solution of (2.10)-(2.11) is a rarefaction wave $w^R(\frac{x}{t})$ connecting w_- and w_+, namely,

$$w^R(\frac{x}{t}) = \begin{cases} w_-, \ x < w_- t, \\ \frac{x}{t}, \quad w_- t \le x \le w_+ t, \\ w_+, \ x > w_+ t. \end{cases}$$

From [40], it is well known that when $w_- = \lambda_2(\bar{v}) > 0$ and $w_+ = \lambda_2(v_+) > 0$, the centered rarefaction wave $\left(v_2^R, u_2^R\right)(\frac{x}{t})$ can be defined by

$$(v^R, u^R)(\frac{x}{t}) = \Big(\lambda_2^{-1}(w^R(\frac{x}{t})), \bar{u} - \int_{\bar{v}}^{\lambda_2^{-1}(w^R(\frac{x}{t}))} \lambda_2(s) ds\Big).$$

It is easy to check that $v_2^R(t,x)$ and $u_2^R(t,x)$ satisfy

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0 \end{cases}$$

with

$$(v,u)(0,x) = (v_0^R, u_0^R) = \begin{cases} (\bar{v}, \bar{u}), \quad x < 0, \\ (v_+, u_+), \ x > 0. \end{cases}$$

Now we approximate the rarefaction wave $w^R(\frac{x}{t})$ by the solution $w(t,x)$ of the following Cauchy problem:

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q \tilde{w} \int_0^{\varepsilon x} z^q e^{-z} \mathrm{d}z, & x \ge 0, \end{cases} \end{cases} \tag{2.12}$$

where $\tilde{w} = w_+ - w_-$, $C_q > 0$ is a constant satisfying $C_q \int_0^{+\infty} z^q e^{-z} dz = 1$ with $q \ge 12$ being a positive constant, and $\varepsilon \le 1$ is a positive constant to be determined later. Then the properties of $w(t,x)$ can be summarised in the following lemma.

LEMMA 2.2 (See [6,24,38]). *Let $0 < w_- < w_+$, then the Cauchy problem (2.12) admits a unique global smooth solution $w(t,x)$ satisfying:*

(i) $w_- \le w(t,x) \le w_+$, $w_x > 0$, $\quad x \ge 0, t \ge 0$.

(ii) *For any p with $1 \le p \le +\infty$, there exists a constant $C_{p,q} > 0$ such that for $t \ge 0$,*

$$\begin{aligned} \|w_x(t)\|_{L^p} &\le C_{p,q} \min\Big\{\tilde{w}\varepsilon^{1-\frac{1}{p}}, \ \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\Big\}, \\ \|w_{xx}(t)\|_{L^p} &\le C_{p,q} \min\Big\{\tilde{w}\varepsilon^{2-\frac{1}{p}}, \ \tilde{w}^{\frac{1}{q}} \varepsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\Big\}, \\ \|w_{xxx}(t)\|_{L^p} &\le C_{p,q} \min\Big\{\tilde{w}\varepsilon^{3-\frac{1}{p}}, \ \tilde{w}^{\frac{2}{q}} \varepsilon^{2-\frac{1}{p}+\frac{2}{q}} t^{-1+\frac{2}{q}}\Big\}, \\ \|w_{xxxx}(t)\|_{L^p} &\le C_{p,q} \min\Big\{\tilde{w}\varepsilon^{4-\frac{1}{p}}, \ \tilde{w}^{\frac{3}{q}} \varepsilon^{3-\frac{1}{p}+\frac{3}{q}} t^{-1+\frac{3}{q}}\Big\}. \end{aligned}$$

(iii) *When $x \le w_- t$, it holds that*

$$w(t,x) - w_- = w_x(t,x) = w_{xx}(t,x) = w_{xxx}(t,x) = 0.$$

(iv) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}} \big|w(t,x) - w^R(t,x)\big| = 0.$

Now, we should construct the smooth approximate rarefaction wave $(v^r,u^r)(t,x)$ of $(v_2^R,u_2^R)(t,x)$. As in [41], we define $(\tilde{v}^r,\tilde{u}^r)(t,x)$ as follows:

$$(\tilde{v}^r,\tilde{u}^r)(t,x)=\Big(\lambda_2^{-1}(w(t,x)),\bar{u}-\int_{\bar{v}}^{\lambda_2^{-1}(w(t,x))}\lambda_2(s)ds\Big), \tag{2.13}$$

here $w(x,t)$ is the solution of (2.12). Then we set

$$(v^r,u^r)(t,x)=(\tilde{v}^r,\tilde{u}^r)(t,x)\big|_{x\geq s_-t}, \tag{2.14}$$

which together with $(2.12)_2$ and (2.13) implies

$$v_0^r(x)=\lambda_2^{-1}(w_0),\ u_0^r(x)=\bar{u}-\int_{\bar{v}}^{\lambda_2^{-1}(w_0)}\lambda_2(s)ds, \tag{2.15}$$

here

$$w_0(x)=\begin{cases}\lambda_2(\bar{v}), & x<0,\\ \lambda_2(\bar{v})+(\lambda_2(v_+)-\lambda_2(\bar{v}))C_q\int_0^{\varepsilon x}z^q e^{-z}\mathrm{d}z, & x\geq 0.\end{cases}$$

It is easy to check from (2.13) and Lemma 2.1 that $(v^r,u^r)(t,x)$ has the following properties.

Lemma 2.3. *The smooth approximation $(v^r,u^r)(t,x)$ of (v_2^R,u_2^R) has the following properties:*

(i) $u_x^r\geq 0,\quad |u_x^r|\leq C\varepsilon,\quad \forall t\geq 0, x\geq s_-t.$

(ii) *For any p with $1\leq p\leq+\infty$, there exists a constant $C_{p,q}>0$ such that*

$$\begin{aligned}
&\|(v_x^r,u_x^r)(t)\|_{L^p}\leq C_{p,q}\min\Big\{\delta\varepsilon^{1-\frac1p},\ \delta^{\frac1p}(1+t)^{-1+\frac1p}\Big\},\\
&\|(v_{xx}^r,u_{xx}^r)(t)\|_{L^p}\leq C_{p,q}\min\Big\{\delta\varepsilon^{2-\frac1p},\ \delta^{\frac1q}\varepsilon^{1-\frac1p+\frac1q}(1+t)^{-1+\frac1q}\Big\},\\
&\|(v_{xxx}^r,u_{xxx}^r)(t)\|_{L^p}\leq C_{p,q}\min\Big\{\delta\varepsilon^{3-\frac1p},\ \delta^{\frac2q}\varepsilon^{2-\frac1p+\frac2q}(1+t)^{-1+\frac2q}\Big\},\\
&\|(v_{xxxx}^r,u_{xxxx}^r)(t)\|_{L^p}\leq C_{p,q}\min\Big\{\delta\varepsilon^{4-\frac1p},\ \delta^{\frac3q}\varepsilon^{3-\frac1p+\frac3q}(1+t)^{-1+\frac3q}\Big\}.
\end{aligned}$$

(iii) $(v_x^r,u_x^r)\big|_{x\leq s_-t}=(\bar{v},\bar{u}),\ \dfrac{\partial^j}{\partial x^j}(v_x^r,u_x^r)(t,x)\big|_{x\leq s_-t}=0,\quad j=1,2,3.$

(iv) $\lim\limits_{t\to+\infty}\sup\limits_{x\geq s_-t}\Big|(v^r,u^r)(t,x)-\big(v_2^R,u_2^R\big)\big(\dfrac{x}{t}\big)\Big|=0.$

Finally, we list some inequalities of the Poincaré type, which are proved in [28] and will be used in Section 4.

Lemma 2.4. *For any $f\in H_0^1(\mathbb{R}^+)$, and for any $\mathcal{A}$ in the set $\{V_0,U_0\}$, there hold*

(i) *If $-s_-\bar{v}^{\frac{\gamma+1}{2}}<\sqrt{k\gamma}$, i.e., $(\bar{v},\bar{u})\neq(v_*,u_*)$, then*

$$\int_0^\infty|\partial_y^k\mathcal{A}|^j|f|^2\leq C\tilde{\delta}\|f_y\|^2, \tag{2.16}$$

for $k,j=1,2,\cdots$.

(ii) *If $-s_-\bar{v}^{\frac{\gamma+1}{2}}=\sqrt{k\gamma}$, i.e., $(\bar{v},\bar{u})=(v_*,u_*)$, then*

$$\int_0^\infty|\partial_y^k\mathcal{A}|^j|f|^2\leq C\tilde{\delta}\|f_y\|^2, \tag{2.17}$$

for $k,j=1,2,\cdots$, except $k=j=1$.

3. Reformulation of the original problem

Since it is convenient to regard the solution $(v,u)(t,x)$ as the perturbation of the nonlinear wave $(V,U)(t,x)$, we are going to reformulate the original problem in terms of the perturbation variables in this section. To begin with, we recall the nonlinear wave (V,U), which is introduced in Section 1,

$$(V,U)(t,x)=(V_0,U_0)(x-s_-t)+(v^r,u^r)(t,x)-(\bar{v},\bar{u}), \tag{3.1}$$

here, $(V_0,U_0)(x-s_-t)$ is the stationary solution which satisfies for any $x>0$ that

$$\begin{cases} -s_-V_0'-U_0'=0,\\ -s_-U_0'+p(V_0)'=\mu\left(\frac{U_0'}{V_0}\right)'+\kappa\left(\frac{-V_0''}{V_0^5}+\frac{5(V_0')^2}{2V_0^6}\right)' \end{cases} \tag{3.2}$$

with the boundary data and the spatial asymptotic conditions

$$(V_0,U_0)(0)=(v_-,u_-),\ V_{0y}(0)=v_b,\ (V_0,U_0)(+\infty)=(\bar{v},\bar{u}). \tag{3.3}$$

And $(v^r,u^r)(t,x)$ is the smoothed rarefaction wave connecting $(\bar{v},\bar{u})$ and (ρ_+,u_+), and satisfying

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=0. \end{cases} \tag{3.4}$$

Further, we consider the coordinate transformation

$$t=t,\ y=x-s_-t, \tag{3.5}$$

by which, we rewrite the initial value problem (1.2) as follows

$$\begin{cases} v_t-s_-v_y-u_y=0,\\ u_t-s_-u_y+p(v)_y=\mu\left(\frac{u_y}{v}\right)_y+\kappa\left(\frac{-v_{yy}}{v^5}+\frac{5v_y^2}{2v^6}\right)_y,\\ (v,u)|_{t=0}=(v_0,u_0)(y)\to(v_+,u_+),\\ (v,u)|_{y=0}=(v_-,u_+),\quad v_y|_{y=0}=v_b. \end{cases} \tag{3.6}$$

Moreover, from (3.2) and (3.4), and using (3.5), we assert that (V,U) satisfies

$$\begin{cases} V_t-s_-V_y-U_y=0,\\ U_t-s_-U_y+p(V)_y=\mu(\frac{U_y}{V})_y+\kappa(-\frac{V_{yy}}{V^5}+\frac{5V_y^2}{2V^6})_y-f, \end{cases} \tag{3.7}$$

where f is defined by

$$\begin{aligned} f=&-(p'(V)-p'(V_0))V_{0y}+(p'(V)-p'(v^r))v_y^r-\mu(\frac{U_{0y}}{V_0})_y+\mu(\frac{U_y}{V})_y\\ &-\kappa\Big(-\frac{V_{0yy}}{V_0^5}+\frac{5V_{0y}^2}{2V_0^6}+\frac{V_{yy}}{V^5}-\frac{5V_y^2}{2V^6}\Big)_y. \end{aligned}$$

Now we define the new unknowns $(\varphi,\psi)(t,x)$ by

$$\varphi(t,x)=v(t,y)-V(t,y),\ \psi(t,x)=u(t,y)-U(t,y). \tag{3.8}$$

Then from system $(3.6)_{1,2}$ and system (3.7), it is easy to check that the perturbated variable $(\varphi,\psi)(t,x)$ satisfies the system in $\mathbb{R}^+\times\mathbb{R}^+$ below

$$\begin{cases} \varphi_t-s_-\varphi_y-\psi_y=0,\\ \varphi_t-s_-\psi_y+(p(V+\varphi)-p(V))y=\mu\left(\frac{U_y\psi_y}{V+\varphi}-\frac{U_y}{V}\right)_y+K_y+f, \end{cases} \tag{3.9}$$

and initial boundary values:

$$\begin{cases} (\varphi,\psi)|_{t=0}=(\varphi_0,\psi_0)(y)=(v_0-V_0,u_0-U_0), \\ (\varphi,\psi)|_{y=0}=(0,0), \quad \varphi_y|_{y=0}=0. \end{cases} \tag{3.10}$$

Here

$$K=\kappa\Big(\frac{-\varphi_{yy}-V_{yy}}{(V+\varphi)^5}+\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}+\frac{V_{yy}}{V^5}-\frac{5V_y^2}{2V^6}\Big).$$

Therefore, we are now in a position to restate our main results in terms of the perturbed variable $(\varphi,\psi)(t,y)$ as follows.

Theorem 3.1. *Suppose that all the assumptions of Theorem 1.1 are met. Then there exists a unique global solution $(\varphi,\psi)(t,\xi)$ to problem (3.9)-(3.10), satisfying*

$$\begin{aligned} &\varphi\in C([0,\infty);H_0^2(\mathbb{R}^+)),\psi\in C([0,\infty);H_0^1(\mathbb{R}^+)),\\ &\varphi_y\in L^2([0,\infty);H^2(\mathbb{R}^+)),\ \psi_y\in L^2([0,\infty);H^1(\mathbb{R}^+)), \end{aligned}$$

and

$$\lim_{t\to\infty}\sup_{y\in\mathbb{R}^+}|(\varphi,\psi)(t,y)|=0. \tag{3.11}$$

To prove this theorem, we shall employ the standard continuation argument based on a local existence theorem in the following lemma and on a priori estimates stated in the following proposition. First, the local existence of the solution (φ,ψ) to the initial-boundary value problem (3.9)-(3.10) is proved by the standard method, for example, the dual argument and iteration technique. We refer the details to [15, 16, 27, 42].

Lemma 3.1 (Local existence). *Assume that the conditions in Theorem 1.1 hold. Then there exists a positive constant T_0 such that the initial-boundary value problem (3.9)-(3.10) has a unique solution $(\varphi,\psi)(t,y)$ that has the following properties:*

$$\begin{aligned} &\varphi\in C([0,T_0];H_0^2(\mathbb{R}^+)),\ \psi\in C([0,T_0];H_0^1(\mathbb{R}^+)),\\ &\varphi_y\in L^2([0,T_0];H^2(\mathbb{R}^+)),\ \psi_y\in L^2([0,T_0];H^1(\mathbb{R}^+)),\\ &\inf_{t,y\in\mathbb{R}^+} v(t,y)>0. \end{aligned}$$

Next, we should prove the following *a priori* estimates in Sobolev spaces, which are stated in Proposition 3.1.

Proposition 3.1. *Let (φ,ψ) be a solution to the initial-boundary value problem (3.9)-(3.10) in a time interval $[0,T]$, which has same regularities as in Theorem 3.1. Then there exist constants $\chi>0$ and $C>0$ such that if*

$$N(T):=\sup_{t\in[0,T]}\{\|\varphi(t)\|_2+\|\psi(t)\|_1\}\le\chi, \tag{3.12}$$

and $\tilde\delta+\varepsilon+\chi\ll1$, then the following estimate holds for any $t\in[0,T]$

$$\begin{aligned} &\|\varphi(t)\|_2^2+\|\psi(t)\|_1^2+\int_0^t\big(\psi_y(\tau,0)^2+\varphi_{yy}(\tau,0)^2+\|\varphi_y(\tau)\|_2^2+\|\psi_y(\tau)\|_1^2\big)\,d\tau\\ &\le C(\|\varphi_0\|_2^2+\|\psi_0\|_1^2+\tilde\delta+\varepsilon^{\frac{1}{10}}). \end{aligned} \tag{3.13}$$

4. A priori estimates

This section is devoted to the derivation of *a priori* estimates for the unknown function $(\varphi,\psi)(t,y)$ and their derivatives, we then show that Proposition 3.1 is valid. Since the decay rates of the non-degenerate stationary solution and the degenerate stationary solution in the nonlinear wave are different, we will derive *a priori* estimates in Subsections 4.1 and 4.2, separately. Moreover, in establishing *a priori* estimates, we shall employ a mollifier with respect to time variable t to resolve an insufficiency of regularity of the solution obtained in Proposition 3.1. As this argument is standard, we omit the details and proceed with a derivation of those estimates formally. To derive these *a priori* estimates, we assume that there exists a solution $(\varphi,\psi)(t,y)$ to problem (3.9)-(3.10), such that

$$(\varphi,\psi)(t,y)\in C([0,T];H_0^2(\mathbb{R}^+))\times C([0,T];H_0^1(\mathbb{R}^+)),$$
$$\inf_{(t,y)\in[0,T]\times\mathbb{R}^+}(\varphi+V)(t,y)>0$$

for any $T>0$. From (3.12), one can see easily that there exist two positive constants c and C such that

$$0<c\le v\le C \text{ for } t\in[0,T], \tag{4.1}$$

since $V\ge c>0$ for a positive constant c.

4.1. Estimates for the case $(\bar v,\bar u)\neq(v_*,u_*)$.

In this subsection, we shall obtain the uniform *a priori* estimates of the perturbation from the nonlinear wave with the non-degenerate stationary solution. Namely, we will show (3.13) holds in the case that $(\bar v,\bar u)\neq(v_*,u_*)$. First, we are going to establish the first energy estimate for the solution $(\varphi,\psi)(t,x)$ to problem (3.6)-(3.10). To this end, we introduce

$$\Phi(V,\varphi)=p(V)\varphi-\int_V^{V+\varphi}p(\eta)d\eta.$$

Combining this with (4.1) yields

$$c\varphi^2\le\Phi(V,\varphi)\le C\varphi^2. \tag{4.2}$$

Now let us derive the basic energy estimate. First, utilizing $(3.9)_1$, we see that

$$\begin{aligned}
K_y\psi=&-\kappa\Big\{\Big[\frac{\varphi_{yy}+V_{yy}}{(V+\varphi)^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}-\frac{V_{yy}}{V^5}+\frac{5V_y^2}{2V^6}\Big]\psi\Big\}_y+\kappa\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\psi_y\\
&+\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_y-\frac{5\kappa}{2(V+\varphi)^6}(2V_y\varphi_y+\varphi_y^2)\psi_y-\frac{5\kappa}{2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\psi_y\\
=&-\kappa\Big\{\Big[\frac{\varphi_{yy}+V_{yy}}{(V+\varphi)^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}-\frac{V_{yy}}{V^5}+\frac{5V_y^2}{2V^6}\Big]\psi\Big\}_y+\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}(\varphi_t-s_-\varphi_y)\\
&+\kappa\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\psi_y-\frac{5\kappa}{2(V+\varphi)^6}(2V_y\varphi_y+\varphi_y^2)\psi_y-\frac{5\kappa}{2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\psi_y\\
=&-\kappa\Big\{\Big[\frac{\varphi_{yy}+V_{yy}}{(V+\varphi)^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}-\frac{V_{yy}}{V^5}+\frac{5V_y^2}{2V^6}\Big]\psi\Big\}_y+\Big(\frac{\kappa}{(V+\varphi)^5}\varphi_y\varphi_t\Big)_y\\
&-\Big(\frac{5\kappa s_-}{2(V+\varphi)^5}\varphi_y^2\Big)_y-\Big(\frac{\kappa}{2(V+\varphi)^5}\varphi_y^2\Big)_t-\frac{5\kappa}{2(V+\varphi)^6}U_y\varphi_y^2+\kappa\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\psi_y\\
&-\frac{5\kappa}{2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\psi_y.
\end{aligned}$$

Further, from (3.9) and using above equality, a straightforward but tedious computation gives

$$\Big[\frac{1}{2}\psi^2+\Phi(V,\varphi)+\frac{\kappa}{2(V+\varphi)^5}\varphi_y^2\Big]_t+R_{1y}+R_2=R_3+R_4+R_5. \tag{4.3}$$

Here

$$\begin{aligned} R_1=&-s_-\Big[\frac{1}{2}\psi^2+\Phi(V,\varphi)\Big]+[p(V+\varphi)-p(V)]\psi-\mu\Big(\frac{U_y+\psi_y}{V+\varphi}-\frac{U_y}{V}\Big)\psi\\ &+\kappa\Big[\frac{\varphi_{yy}+V_{yy}}{(V+\varphi)^5}-\frac{V_{yy}}{V^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}+\frac{5V_y^2}{2V^6}\Big]\psi-\frac{\kappa}{(V+\varphi)^5}\varphi_y\varphi_t+\frac{5\kappa s_-}{2(V+\varphi)^5}\varphi_y^2,\\ R_2=&\mu\frac{\psi_y^2}{V+\varphi}-\frac{\mu u_y^r\varphi\psi_y}{V(V+\varphi)}+[p(V+\varphi)-p(V)-p'(V)\varphi]u_y^r,\ R_3=f\psi,\\ R_4=&-\frac{\mu}{V(V+\varphi)}U_{0y}\varphi\psi_y+[p(V+\varphi)-p(V)-p'(V)\varphi]U_{0y}, \end{aligned}$$

and

$$R_5=-\frac{5\kappa}{2(V+\varphi)^6}U_y\varphi_y^2+\kappa\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\psi_y-\frac{5\kappa}{2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\psi_y.$$

LEMMA 4.1. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.9)-(3.10), satisfying the conditions in Proposition 3.1, then the following estimate holds*

$$\begin{aligned} &\|\varphi(t)\|_1^2+\|\psi(t)\|^2+\int_0^t\Big(\|(u_y^r)^{\frac{1}{2}}\varphi\|^2+\|\psi_y\|^2\Big)\,\mathrm{d}\tau\\ &\le C(\|\varphi_0\|_1+\|\varphi_0\|^2+\tilde{\delta}+\varepsilon^{\frac{1}{10}})+C(\delta+\varepsilon)\int_0^t\|\varphi_y\|^2\,\mathrm{d}\tau \end{aligned} \tag{4.4}$$

for all $t\in[0,T]$.

Proof. Integrating (4.3) with respect to y over $(0,\infty)$ yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\Big[\frac{1}{2}\psi^2+\Phi(v^r,\varphi)+\frac{\kappa}{2(v^r+\varphi)^2}\varphi_y^2\Big]\mathrm{d}y-R_1\Big|_{y=0}+\int_0^\infty R_2\mathrm{d}y\\ &=\int_0^\infty R_3dy+\int_0^\infty R_4dy+\int_0^\infty R_5dy. \end{aligned} \tag{4.5}$$

First, noting (4.1) and using (4.2), we obtain easily

$$\int_0^\infty\Big[\frac{1}{2}\psi^2+\Phi(V,\varphi)+\frac{\kappa}{2(V+\varphi)^2}\varphi_y^2\Big]\mathrm{d}y\ge c(\|\varphi\|^2+\|\psi\|^2+\|\varphi_y\|^2). \tag{4.6}$$

Due to $\varphi(t,0)=\psi(t,0)=\varphi_y(t,0)=0$, it is easy to see

$$R_1|_{y=0}=0. \tag{4.7}$$

Moreover, in [41], the authors have showed

$$R_2\ge c(\|\psi_y\|^2+\|(u_y^r)^{\frac{1}{2}}\varphi\|^2). \tag{4.8}$$

Since

$$
\begin{aligned}
f \sim O[&(v^r-\bar{v})V_{0y}+(V_0-\bar{v})v_y^r+u_{yy}^r+(v^r-\bar{v})U_{0yy}+v_y^rU_{0y}+v_y^ru_y^r+(v^r-\bar{v})V_{0y}U_{0y}\\
&+v_{yyy}^r+(v^r-\bar{v})V_{0y}V_{0yy}+v_y^rV_{0yy}+v_{yy}^rV_{0y}+v_y^rv_{yy}^r+(v_y^r)^3+(v_y^r)^2V_{0y}\\
&+v_y^rV_{0y}^2+(v^r-\bar{v})V_{0y}^3],
\end{aligned}
\tag{4.9}
$$

we have

$$
\begin{aligned}
\int_0^\infty R_3dy \le & C\Big[|\int_0^\infty u_{yy}^r\psi dy|+|\int_0^\infty v_y^ru_y^r\psi dy|+|\int_0^\infty v_{yyy}^r\psi dy|+|\int_0^\infty v_y^rv_{yy}^r\psi dy|\\
&+|\int_0^\infty (v_y^r)^3\psi dy|\Big]+C\Big[|\int_0^\infty (v^r-\bar{v})V_{0y}\psi dy|+|\int_0^\infty (V_0-\bar{v})v_y^r\psi dy|\\
&+|\int_0^\infty (v^r-\bar{v})U_{0yy}\psi dy|+|\int_0^\infty (v^r-\bar{v})V_{0y}U_{0y}\psi dy|+|\int_0^\infty (v^r-\bar{v})V_{0y}^3\psi dy|\\
&+|\int_0^\infty (v^r-\bar{v})V_{0y}V_{0yy}\psi dy|\Big]+C\Big[|\int_0^\infty v_y^rU_{0y}\psi dy|+|\int_0^\infty v_y^rV_{0yy}\psi dy|\\
&+|\int_0^\infty v_{yy}^rV_{0y}\psi dy|+|\int_0^\infty (v_y^r)^2V_{0y}U_{0y}\psi dy|+|\int_0^\infty v_y^rV_{0y}^2V_{0yy}\psi dy|\Big]\\
=:& R_{31}+R_{32}+R_{33}.
\end{aligned}
$$

Now, let us estimate the terms R_{31}, R_{32} and R_{33} one by one. First, from Lemma 2.3, we have

$$
\|u_{yy}^r\|_{L^1}^{\frac{4}{3}} \le C\|u_{yy}^r\|_{L^1}^{\frac{1}{6}}\|u_{yy}^r\|_{L^1}^{\frac{7}{6}} \le C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{77}{72}},
\tag{4.10}
$$

and

$$
\|u_y^r\|^{\frac{4}{3}} \le C\|u_y^r\|^{\frac{1}{6}}\|u_y^r\|^{\frac{7}{6}} \le C\varepsilon^{\frac{1}{12}}(1+t)^{-\frac{7}{12}}.
\tag{4.11}
$$

Then it follows from the Hölder inequality, the Sobolev inequality, and the Young inequality and using (4.10)-(4.11) that

$$
\begin{aligned}
&|\int_0^\infty u_{yy}^r\psi \mathrm{d}y|+|\int_0^\infty v_y^ru_y^r\psi \mathrm{d}y|\\
\le & C\|\psi\|_{L^\infty}\left(\|u_{yy}^r\|_{L^1}+\|v_y^r\|\|u_y^r\|\right)\\
\le & C\|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}\left(\|u_{yy}^r\|_{L^1}+\|v_y^r\|\|u_y^r\|\right)\\
\le & \frac{c}{16}\|\psi_y\|^2+C\|\psi\|^{\frac{2}{3}}(\|u_{yy}^r\|_{L^1}^{\frac{4}{3}}+\|v_y^r\|^{\frac{4}{3}}\|u_y^r\|^{\frac{4}{3}})\\
\le & \frac{c}{16}\|\psi_y\|^2+C\varepsilon^{\frac{1}{6}}\|\psi\|^{\frac{2}{3}}\left[(1+t)^{-\frac{77}{72}}+(1+t)^{-\frac{7}{6}}\right]\\
\le & \frac{c}{16}\|\psi_y\|^2+C\varepsilon^{\frac{1}{6}}\left[(1+t)^{-\frac{9}{8}}+(1+t)^{-\frac{5}{4}}\right]\|\psi\|^2+C\varepsilon^{\frac{1}{6}}\left[(1+t)^{-\frac{25}{24}}+(1+t)^{-\frac{9}{8}}\right].
\end{aligned}
$$

Similarly, we can show

$$
\begin{aligned}
&|\int_0^\infty v_{yyy}^r\psi dy|+|\int_0^\infty v_y^rv_{yy}^r\psi dy|+|\int_0^\infty (v_y^r)^3\psi dy|\\
\le & C\|\psi\|_{L^\infty}\left(\|v_{yyy}^r\|_{L^1}+\|v_y^r\|_{L^\infty}\|v_{yy}^r\|_{L^1}+\|v_y^r\|_{L^3}^3\right)\\
\le & C\|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}\left(\|v_{yyy}^r\|_{L^1}+\|v_y^r\|_{L^\infty}\|v_{yy}^r\|_{L^1}+\|v_y^r\|_{L^3}^3\right)
\end{aligned}
$$

$$\le \frac{c}{16}\|\psi_y\|^2 + C\|\psi\|^{\frac{2}{3}}\left(\|v^r_{yyy}\|^{\frac{1}{12}}_{L^1}\|v^r_{yyy}\|^{\frac{5}{4}}_{L^1} + \|v^r_y\|^{\frac{4}{3}}_{L^\infty}\|v^r_{yy}\|^{\frac{4}{3}}_{L^1} + \|v^r_y\|^{\frac{1}{4}}_{L^3}\|v^r_y\|^{\frac{15}{4}}_{L^3}\right)$$
$$\le \frac{c}{16}\|\psi_y\|^2 + C\|\psi\|^{\frac{2}{3}}\left[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{25}{24}} + \varepsilon^{\frac{1}{9}}(1+t)^{-\frac{23}{9}} + \varepsilon^{\frac{1}{6}}(1+t)^{-\frac{5}{2}}\right]$$
$$\le \frac{c}{16}\|\psi_y\|^2 + C\left[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{17}{16}} + \varepsilon^{\frac{1}{6}}(1+t)^{-2} + \varepsilon^{\frac{1}{9}}(1+t)^{-2}\right]\|\psi\|^2$$
$$+ C\left[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{33}{32}} + \varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{4}} + \varepsilon^{\frac{1}{9}}(1+t)^{-\frac{17}{6}}\right].$$

Then we have

$$R_{31} \le \frac{c}{8}\|\psi_y\|^2 + C\{\varepsilon^{\frac{1}{6}}[(1+t)^{-\frac{17}{16}} + (1+t)^{-2} + (1+t)^{-\frac{9}{8}} + (1+t)^{-\frac{5}{4}}] + \varepsilon^{\frac{1}{9}}(1+t)^{-2}\}\|\psi\|^2$$
$$+ C\left\{\left[\varepsilon^{\frac{1}{6}}[(1+t)^{-\frac{33}{32}} + (1+t)^{-\frac{11}{4}} + (1+t)^{-\frac{25}{24}} + (1+t)^{-\frac{9}{8}}\right] + \varepsilon^{\frac{1}{9}}(1+t)^{-\frac{17}{6}}\right\}. \quad (4.12)$$

We now turn to estimate R_{32}. First, applying the estimates in Lemmas 2.1 and 2.3, we arrive at

$$\int_0^\infty (v^r - \bar{v})V_{0y}dy = \int_0^t (v^r - \bar{v})V_{0y}dy + \int_t^\infty (v^r - \bar{v})V_{0y}dy$$
$$= (v^r - \bar{v})(V_0 - \bar{v})\Big|_0^t - \int_0^t v^r_y(V_0 - \bar{v})dy + \int_t^\infty (v^r - \bar{v})V_{0y}dy$$
$$\le C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{10}}\ln(1+\tilde{\delta}t) + C\tilde{\delta}(1+\tilde{\delta}t)^{-1}$$
$$\le C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{4}{5}} + C\tilde{\delta}(1+t)^{-1},$$

and using again the Sobolev inequality, we then obtain

$$\left|\int_0^\infty (v^r - \bar{v})V_{0y}\psi dy\right|$$
$$\le C\|\psi(t)\|_{L^\infty}\int_0^\infty (v^r - \bar{v})V_{0y}dy$$
$$\le C\|\psi(t)\|^{\frac{1}{2}}\|\psi_y(t)\|^{\frac{1}{2}}\int_0^\infty (v^r - \bar{v})V_{0y}dy$$
$$\le C\varepsilon^{\frac{1}{10}}\left(\|\psi_y(t)\|^2 + (1+t)^{-\frac{21}{20}} + (1+t)^{-\frac{11}{10}}\|\psi(t)\|^2\right)$$
$$+ C\tilde{\delta}\left(\|\psi_y(t)\|^2 + (1+t)^{-\frac{17}{12}} + \tilde{\delta}(1+t)^{-\frac{7}{6}}\|\psi(t)\|^2\right).$$

In a similar manner, one can estimate the remaining terms in R_{32} and conclude that

$$R_{32} \le C(\tilde{\delta} + \varepsilon^{\frac{1}{10}})\|\psi_y(t)\|^2 + C\left(\tilde{\delta}(1+t)^{-\frac{17}{12}} + \varepsilon^{\frac{1}{10}}(1+t)^{-\frac{21}{20}}\right)$$
$$+ C\left(\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{11}{10}} + \tilde{\delta}(1+t)^{-\frac{7}{6}}\right)\|\psi(t)\|^2. \quad (4.13)$$

Similarly, we have

$$R_{33} \le C(\|v^r_y\|_{L^\infty}\|U_{0y}\|_{L^1} + \|v^r_y\|_{L^\infty}\|V_{0yy}\|_{L^1} + \|v^r_{yy}\|_{L^\infty}\|V_{0y}\|_{L^1}$$
$$+ \|v^r_y\|^2_{L^\infty}\|V_{0y}\|_{L^\infty}\|U_{0y}\|_{L^1} + \|v^r_y\|_{L^\infty}\|V_{0y}\|^2_{L^\infty}\|V_{0yy}\|_{L^1})\|\psi\|_{L^\infty}$$
$$\le C(\|v^r_y\|_{L^\infty}\|U_{0y}\|_{L^1} + \|v^r_y\|_{L^\infty}\|V_{0yy}\|_{L^1} + \|v^r_{yy}\|_{L^\infty}\|V_{0y}\|_{L^1}$$
$$+ \|v^r_y\|^2_{L^\infty}\|V_{0y}\|_{L^\infty}\|U_{0y}\|_{L^1} + \|v^r_y\|_{L^\infty}\|V_{0y}\|^2_{L^\infty}\|V_{0yy}\|_{L^1})\|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq C\tilde{\delta}\|\psi_y\|^2+C\tilde{\delta}\Big[(1+t)^{-\frac{4}{3}}+C\tilde{\delta}(1+t)^{-\frac{8}{3}}+C\tilde{\delta}(1+t)^{-\frac{11}{9}}\Big]\|\psi_y\|^{\frac{2}{3}}\\
&\leq C\tilde{\delta}\|\psi_y\|^2+C\tilde{\delta}\Big[(1+t)^{-\frac{4}{3}}+(1+t)^{-\frac{8}{3}}\Big]\|\psi(t)\|^2\\
&\quad +C\tilde{\delta}\Big[(1+t)^{-\frac{4}{3}}+(1+t)^{-\frac{8}{3}}+(1+t)^{-\frac{7}{6}}\Big]. \qquad (4.14)
\end{aligned}
$$

From the Young inequality and Lemma 2.4, it follows that

$$
\int_0^\infty R_4 \mathrm{d}y \leq \frac{c}{8}\|\psi_y\|^2+C\int_0^\infty (U_{0y}^2\varphi^2+U_{0y}\varphi^2)\mathrm{d}y \leq \frac{c}{8}\|\psi_y\|^2+C\tilde{\delta}\|\varphi_y\|^2. \qquad (4.15)
$$

Finally, let us deal with the term $\int_0^\infty R_5 \mathrm{d}y$. First, it is easy to find

$$
\begin{aligned}
\int_0^\infty R_5 \mathrm{d}y \leq & C\Big(|\int_0^\infty u_y^r\varphi_y^2\mathrm{d}y|+|\int_0^\infty U_{0y}\varphi_y^2\mathrm{d}y|\Big)+C\Big(|\int_0^\infty v_{yy}^r\varphi\psi_y\mathrm{d}y|+|\int_0^\infty (v_y^r)^2\varphi\psi_y\mathrm{d}y|\Big)\\
&+C\Big(|\int_0^\infty V_{0yy}\varphi\psi_y\mathrm{d}y|+|\int_0^\infty V_{0y}^2\varphi\psi_y\mathrm{d}y|\Big)+C|\int_0^\infty v_y^r V_{0y}\varphi\psi_y\mathrm{d}y|\\
=:& R_{51}+R_{52}+R_{53}+R_{54}.
\end{aligned}
$$

From Lemmas 2.1 and 2.3, it is easy to obtain

$$
R_{51}\leq C(\tilde{\delta}+\varepsilon)\|\varphi_y\|^2. \qquad (4.16)
$$

Next, utilizing the Hölder inequality, the Young inequality, and Lemma 2.3, we have

$$
\begin{aligned}
R_{52}&\leq C\|v_{yy}^r\|_{L^\infty}\|\varphi\|\|\psi_y\|+C\|v_y^r\|_{L^\infty}^2\|\varphi\|\|\psi_y\|\\
&\leq \frac{c}{8}\|\psi_y\|^2+C\Big[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}+\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}}\Big]\|\varphi\|^2. \qquad (4.17)
\end{aligned}
$$

Similar to (4.15), we have

$$
R_{53}\leq \frac{c}{8}\|\psi_y\|^2+C\tilde{\delta}\|\varphi_y\|^2. \qquad (4.18)
$$

Finally, similar to (4.14), one gets

$$
\begin{aligned}
R_{54}&\leq C\|v_y^r\|_{L^\infty}\|V_{0y}\|_{L^\infty}\|\varphi\|\|\psi_y\|\leq C\|v_y^r\|_{L^\infty}^2\|\varphi\|^2+C\|V_{0y}\|_{L^\infty}^2\|\psi_y\|^2\\
&\leq C\tilde{\delta}\|\psi_y\|^2+C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}\|\varphi\|^2. \qquad (4.19)
\end{aligned}
$$

Therefore, combining (4.5), (4.6)-(4.8), (4.12)-(4.18) and (4.19), and integrating the resultant inequality with respect to t, then implies (4.4) provided that $C\tilde{\delta}, C\varepsilon^{\frac{1}{10}}<\frac{1}{4}$. This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.9)-(3.10), satisfying the conditions in Proposition 3.1, then the following estimate holds*

$$
\|\varphi_y\|^2+\int_0^t (\varphi_y^2+\varphi_{yy}^2)\,\mathrm{d}\tau \leq C(\|\varphi_0\|_1^2+\|\psi_0\|_1^2+\tilde{\delta}+\varepsilon^{\frac{1}{10}}) \qquad (4.20)
$$

for all $t\in[0,T]$.

Proof. Rewriting Equation $(3.9)_2$ as

$$
\Big(\mu\frac{\varphi_y}{V+\varphi}-\psi\Big)_t-s_-\Big(\mu\frac{\varphi_y}{V+\varphi}-\psi\Big)_y-p'(V+\varphi)\varphi_y
$$

$$
\begin{aligned}
&=\frac{\mu}{(V+\varphi)^2}V_y\psi_y+[p'(V+\varphi)-p'(V)]V_y+\mu\Big[\frac{1}{(V+\varphi)^2}-\frac{1}{V^2}\Big]V_yU_y\\
&\quad-\mu\Big(\frac{1}{V+\varphi}-\frac{1}{V}\Big)U_{yy}-K_y-f.
\end{aligned}\tag{4.21}
$$

Multiplying (4.21) by $\frac{\varphi_y}{v^r+\varphi}$, we obtain

$$
\begin{aligned}
&\Big(\frac{\mu\varphi_y^2}{2(V+\varphi)^2}-\frac{\psi\varphi_y}{V+\varphi}\Big)_t+\Big\{\frac{1}{V+\varphi}\psi\psi_y-\frac{\mu s_-}{2(V+\varphi)^2}\varphi_y^2+\frac{s_-}{V+\varphi}\psi\varphi_y-\kappa\Big[\frac{V_{yy}+\varphi_{yy}}{(V+\varphi)^5}\\
&\quad-\frac{V_{yy}}{V^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}+\frac{5V_y^2}{2V^6}\Big]\frac{\varphi_y}{V+\varphi}\Big\}_y-\frac{p'(V+\varphi)}{V+\varphi}\varphi_y^2+\frac{\kappa}{(V+\varphi)^6}\varphi_{yy}^2\\
&=\frac{1}{V+\varphi}\psi_y^2-\frac{1}{(V+\varphi)^2}V_y\psi\psi_y+\frac{1}{(V+\varphi)^2}U_y\psi\varphi_y+\frac{\mu}{(V+\varphi)^3}V_y\varphi_y\psi_y+\frac{1}{V+\varphi}[p'(V+\varphi)\\
&\quad-p'(V)]V_y\varphi_y+\frac{\mu}{V+\varphi}\Big[\frac{1}{(V+\varphi)^2}-\frac{1}{V^2}\Big]V_yU_y\varphi_y-\frac{\mu}{V+\varphi}\Big(\frac{1}{V+\varphi}-\frac{1}{V}\Big)U_{yy}\varphi_y\\
&\quad+\frac{\kappa}{(V+\varphi)^7}(V_y+\varphi_y)\varphi_y\varphi_{yy}-\frac{\kappa}{V+\varphi}\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\varphi_{yy}\\
&\quad+\frac{\kappa}{(V+\varphi)^2}\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}(V_y+\varphi_y)\varphi_y+\frac{5\kappa}{2(V+\varphi)^7}(\varphi_y^2+2V_y\varphi_y)\varphi_{yy}\\
&\quad-\frac{5\kappa}{2(V+\varphi)^8}(\varphi_y^2+2V_y\varphi_y)(V_y+\varphi_y)\varphi_y+\frac{5\kappa}{2(V+\varphi)}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\varphi_{yy}\\
&\quad+\frac{5\kappa}{2(V+\varphi)^2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2(V_y+\varphi_y)\varphi_y-f\frac{\varphi_y}{V+\varphi}.
\end{aligned}\tag{4.22}
$$

Here we used

$$
\begin{aligned}
K_y\frac{\varphi_y}{V+\varphi}&=-\kappa\Big\{\Big[\frac{V_{yy}+\varphi_{yy}}{(V+\varphi)^5}-\frac{V_{yy}}{V^5}-\frac{5(V_y+\varphi_y)^2}{2(V+\varphi)^6}+\frac{5V_y^2}{2V^6}\Big]\frac{\varphi_y}{V+\varphi}\Big\}_y\\
&\quad+\frac{\kappa}{(V+\varphi)^6}\varphi_{yy}^2-\frac{\kappa}{(V+\varphi)^7}(V_y+\varphi_y)\varphi_y\varphi_{yy}+\frac{\kappa}{V+\varphi}\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}\varphi_{yy}\\
&\quad-\frac{\kappa}{(V+\varphi)^2}\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yy}(V_y+\varphi_y)\varphi_y-\frac{5\kappa}{2(V+\varphi)^7}(\varphi_y^2+2V_y\varphi_y)\varphi_{yy}\\
&\quad+\frac{5\kappa}{2(V+\varphi)^8}(\varphi_y^2+2V_y\varphi_y)(V_y+\varphi_y)\varphi_y-\frac{5\kappa}{2(V+\varphi)}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2\varphi_{yy}\\
&\quad+\frac{5\kappa}{2(V+\varphi)^2}\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_y^2(V_y+\varphi_y)\varphi_y.
\end{aligned}
$$

Integrating (4.22) with respect to y over $\mathbb{R}^+$ and taking into account the boundary condition $(3.10)_2$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\Big(\frac{\mu\varphi_y^2}{2(V+\varphi)^2}-\frac{\psi\varphi_y}{V+\varphi}\Big)\mathrm{d}y-\int_0^\infty\frac{p'(V+\varphi)}{V+\varphi}\varphi_y^2\,\mathrm{d}y+\int_0^\infty\frac{\kappa}{(V+\varphi)^6}\varphi_{yy}^2\,\mathrm{d}y\\
&=\int_0^\infty\frac{1}{V+\varphi}\psi_y^2\,\mathrm{d}y+\int_0^\infty\frac{1}{(v^r+\varphi)^2}v_y^r\psi\psi_y\,\mathrm{d}y-\int_0^\infty\frac{1}{(v^r+\varphi)^2}u_y^r\psi\varphi_y\,\mathrm{d}y\\
&\quad+\int_0^\infty\frac{\mu}{(v^r+\varphi)^3}v_y^r\varphi_y\psi_y\,\mathrm{d}y+\int_0^\infty\frac{1}{v^r+\varphi}[p'(v^r+\varphi)-p'(v^r)]v_y^r\varphi_y\,\mathrm{d}y\\
&\quad+\int_0^\infty\Big[\frac{\mu}{(v^r+\varphi)^3}v_y^ru_y^r\varphi_y-\frac{\mu}{(v^r+\varphi)^2}u_{yy}^r\varphi_y-\frac{\kappa}{(v^r+\varphi)^6}v_{yy}^r\varphi_{yy}\Big]\mathrm{d}y
\end{aligned}
$$

$$+\int_0^\infty [\frac{5\kappa}{2(v^r+\varphi)^7}(v_y^r+\varphi_y)^2\varphi_{yy}+\frac{\kappa}{(v^r+\varphi)^7}(v_y^r+\varphi_y)\varphi_y\varphi_{yy}$$
$$+\frac{\kappa}{(v^r+\varphi)^7}(v_y^r+\varphi_y)v_{yy}^r\varphi_y-\frac{5\kappa}{2(v^r+\varphi)^8}(v_y^r+\varphi_y)^3\varphi_y]\,\mathrm{d}y, \tag{4.23}$$

which together with (4.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty(\varphi_y^2-\psi\varphi_y)\,\mathrm{d}y+\int_0^\infty(\varphi_y^2+\varphi_{yy}^2)\,\mathrm{d}y\le C\int_0^\infty\psi_y^2\,\mathrm{d}y+C\sum_{i=1}^7 H_i, \tag{4.24}$$

where

$$H_1=|\int_0^\infty v_y^r\psi\psi_y\,\mathrm{d}y|+|\int_0^\infty u_y^r\psi\varphi_y\,\mathrm{d}y|+|\int_0^\infty v_y^r\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty u_{yy}^r\varphi\varphi_y\,\mathrm{d}y|$$
$$+|\int_0^\infty v_{yy}^r\varphi\varphi_{yy}\,\mathrm{d}y|+|\int_0^\infty v_y^ru_y^r\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty v_y^rv_{yy}^r\varphi\varphi_y\,\mathrm{d}y|$$
$$+|\int_0^\infty (v_y^r)^3\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty (v_y^r)^2\varphi\varphi_{yy}\,\mathrm{d}y|,$$
$$H_2=|\int_0^\infty V_{0y}\psi\psi_y\,\mathrm{d}y|+|\int_0^\infty U_{0y}\psi\varphi_y\,\mathrm{d}y|+|\int_0^\infty V_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty U_{0yy}\varphi\varphi_y\,\mathrm{d}y|$$
$$+|\int_0^\infty V_{0yy}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty V_{0y}U_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty V_{0y}V_{0yy}\varphi\varphi_y\,\mathrm{d}y|$$
$$+|\int_0^\infty V_{0y}^2\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty V_{0y}^3\varphi\varphi_y\,\mathrm{d}y|,$$
$$H_3=|\int_0^\infty v_y^rU_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty u_y^rV_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty v_y^rV_{0yy}\varphi\varphi_y\,\mathrm{d}y|$$
$$+|\int_0^\infty v_{yy}^rV_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty v_y^rV_{0y}\varphi\varphi_{yy}\,\mathrm{d}y|$$
$$+|\int_0^\infty (v_y^r)^2V_{0y}\varphi\varphi_y\,\mathrm{d}y|+|\int_0^\infty v_y^rV_{0y}^2\varphi\varphi_y\,\mathrm{d}y|,$$
$$H_4=|\int_0^\infty v_y^r\varphi_y\psi_y\,\mathrm{d}y|+|\int_0^\infty v_y^r\varphi_y\varphi_{yy}\,\mathrm{d}y|+|\int_0^\infty v_{yy}^r\varphi_y^2\,\mathrm{d}y|+|\int_0^\infty v_y^r\varphi_y\varphi_{yy}\,\mathrm{d}y|$$
$$+|\int_0^\infty (v_y^r)^2\varphi_y^2\,\mathrm{d}y|+|\int_0^\infty V_{0y}\varphi_y\psi_y\,\mathrm{d}y|+|\int_0^\infty V_{0y}\varphi_y\varphi_{yy}\,\mathrm{d}y|+|\int_0^\infty V_{0yy}\varphi_y^2\,\mathrm{d}y|$$
$$+|\int_0^\infty V_{0y}\varphi_y\varphi_{yy}+|\int_0^\infty V_{0y}^2\varphi_y^2\,\mathrm{d}y|,$$

and

$$H_5=|\int_0^\infty v_y^r\varphi_y^3\,\mathrm{d}y|+|\int_0^\infty V_{0y}\varphi_y^3\,\mathrm{d}y|,\quad H_6=|\int_0^\infty \varphi_y^2\varphi_{yy}\,\mathrm{d}y|\quad H_7=|\int_0^\infty f\varphi_y\,\mathrm{d}y|.$$

Now let us estimate the terms on the right-hand side of (4.24). First, using Lemma 2.2, the Hölder inequality and the Young inequality, we have

$$H_1\le C[\|v_y^r\|_{L^\infty}\|\psi\|\|\psi_y\|+\|u_y^r\|_{L^\infty}\|\psi\|\|\varphi_y\|+(\|v_y^r\|_{L^\infty}+\|v_y^r\|_{L^\infty}\|u_y^r\|_{L^\infty}+\|u_{yy}^r\|_{L^\infty}$$
$$+\|v_y^r\|_{L^\infty}^3+\|v_y^r\|_{L^\infty}\|v_{yy}^r\|_{L^\infty})\|\varphi\|\|\varphi_y+(\|v_{yy}^r\|_{L^\infty}+\|v_y^r\|_{L^\infty}^2)\|\varphi\|\|\varphi_{yy}\|\|]$$
$$\le C\|\psi_y\|^2+\frac{1}{8}(\|\varphi_y\|^2+\|\varphi_{yy}\|^2)+C(\|v_y^r\|_{L^\infty}^2+\|u_y^r\|_{L^\infty}^2)\|\psi\|^2+C(\|v_y^r\|_{L^\infty}^2+\|v_y^r\|_{L^\infty}^4$$

$$
\begin{aligned}
&+\|v^r_y\|^2_{L^\infty}\|u^r_y\|^2_{L^\infty}+\|u^r_{yy}\|^2_{L^\infty}+\|v^r_y\|^6_{L^\infty}+\|v^r_y\|^2_{L^\infty}\|v^r_{yy}\|^2_{L^\infty}+\|v^r_{yy}\|^2_{L^\infty})\|\varphi\|^2\\
\le& C\|\psi_y\|^2+\frac{1}{8}(\|\varphi_y\|^2+\|\varphi_{yy}\|^2)+C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}\|\psi\|^2\\
&+C[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}+\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{11}{3}}+\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{11}{2}}]\|\varphi\|^2. \qquad (4.25)
\end{aligned}
$$

Next, utilizing the Young inequality and Lemma 2.4, one gets

$$
\begin{aligned}
H_2\le& C\|\psi_y\|^2+\frac{1}{8}(\|\varphi_y\|^2+\|\varphi_{yy}\|^2)+C\Big(\int_0^\infty V_{0y}^2\psi^2\,\mathrm{d}y+\int_0^\infty U_{0y}^2\psi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^2\varphi^2\,\mathrm{d}y\\
&+\int_0^\infty U_{0yy}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0yy}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^2U_{0y}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^2V_{0yy}^2\varphi^2\,\mathrm{d}y\\
&+\int_0^\infty V_{0y}^4\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^6\varphi^2\,\mathrm{d}y\Big)\\
\le& C\|\psi_y\|^2+\frac{1}{8}(\|\varphi_y\|^2+\|\varphi_{yy}\|^2)+C\tilde{\delta}(\|\psi_y\|^2+\|\varphi_y\|^2). \qquad (4.26)
\end{aligned}
$$

Using the Young and the Hölder inequality, and Lemmsa 2.1 and 2.4, we have

$$
\begin{aligned}
H_3\le& C(\int_0^\infty (v^r_y)^2\varphi_y^2\,\mathrm{d}y+\int_0^\infty (u^r_y)^2\varphi_y^2\,\mathrm{d}y+\int_0^\infty (v^r_{yy})^2\varphi_y^2\,\mathrm{d}y+\int_0^\infty (v^r_{yy})^2\varphi_{yy}^2\,\mathrm{d}y\\
&+\int_0^\infty (v^r_y)^4\varphi_y^2\,\mathrm{d}y)+C(\int_0^\infty U_{0y}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0yy}^2\varphi^2\,\mathrm{d}y+\int_0^\infty V_{0y}^4\varphi^2\,\mathrm{d}y)\\
\le& C(\tilde{\delta}+\varepsilon)(\|\varphi_y\|^2+\|\varphi_{yy}\|^2). \qquad (4.27)
\end{aligned}
$$

Similarly, we can obtain

$$
H_4\le C(\tilde{\delta}+\varepsilon)(\|\varphi_y\|^2+\|\psi_y\|^2+\|\varphi_{yy}\|^2). \qquad (4.28)
$$

Moreover, using the Cauchy inequality, the Sobolev inequality and the assumption (3.12), we get

$$
H_5\le C\|v^r_y\|_{L^\infty}\|\varphi_y\|_{L^\infty}\|\varphi_y\|^2+C\|V_{0y}\|_{L^\infty}\|\varphi_y\|_{L^\infty}\|\varphi_y\|^2\le C(\tilde{\delta}\chi+\varepsilon\chi)\|\varphi_y\|^2, \qquad (4.29)
$$

and

$$
H_6\le C\|\varphi_y\|_{L^\infty}(\|\varphi_y\|^2+\|\varphi_{yy}\|^2)\le C\chi(\|\varphi_y\|^2+\|\varphi_{yy}\|^2). \qquad (4.30)
$$

Finally, recalling (4.9), we have

$$
\begin{aligned}
H_7\le& C\Big[|\int_0^\infty u^r_{yy}\varphi_y\mathrm{d}y|+|\int_0^\infty v^r_yu^r_y\varphi_y\mathrm{d}y|+|\int_0^\infty v^r_{yyy}\varphi_y\mathrm{d}y|+|\int_0^\infty v^r_yv^r_{yy}\varphi_y\mathrm{d}y|\\
&+|\int_0^\infty (v^r_y)^3\varphi_y\mathrm{d}y|\Big]+C\Big[|\int_0^\infty (V_0-\bar{v})v^r_y\varphi_y\mathrm{d}y|+|\int_0^\infty v^r_yU_{0y}\varphi_y\mathrm{d}y|\\
&+|\int_0^\infty v^r_yV_{0yy}\varphi_y\mathrm{d}y|+|\int_0^\infty v^r_{yy}V_{0y}\varphi_y\mathrm{d}y|+|\int_0^\infty (v^r_y)^2V_{0y}U_{0y}\varphi_y\mathrm{d}y|\\
&+|\int_0^\infty v^r_yV_{0y}^2V_{0yy}\varphi_y\mathrm{d}y|\Big]+C\Big[|\int_0^\infty (v^r-\bar{v})V_{0y}\varphi_y\mathrm{d}y|+|\int_0^\infty (v^r-\bar{v})U_{0yy}\varphi_y\mathrm{d}y|\\
&+|\int_0^\infty (v^r-\bar{v})V_{0y}U_{0y}\varphi_y\mathrm{d}y|+|\int_0^\infty (v^r-\bar{v})V_{0y}^3\varphi_y\mathrm{d}y|\\
&+|\int_0^\infty (v^r-\bar{v})V_{0y}V_{0yy}\varphi_y\mathrm{d}y|\Big]. \qquad (4.31)
\end{aligned}
$$

Utilizing the Cauchy inequality, the Young inequality and Lemma 2.3, we have

$$|\int_0^\infty u^r_{yy}\varphi_y \mathrm{d}y| \le \frac{1}{40}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}.$$

Moreover, using the Cauchy inequality, the Young inequality, Lemma 2.1 and Lemma 2.3, one gets

$$\begin{aligned}|\int_0^\infty (V_0-\bar v)v^r_y\varphi_y \mathrm{d}y| &\le C\|v^r_y\|_{L^\infty}\|V_0-\bar v\|\|\varphi_y\| \le C\|v^r_y\|^2_{L^\infty} + C\|V_0-\bar v\|^2\|\varphi_y\|^2\\ &\le C\tilde\delta\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}}.\end{aligned}$$

Next, from the mean-value theorem, and using the Cauchy inequality, the Young inequality, Lemma 2.1 and Lemma 2.3, we can obtain

$$\begin{aligned}|\int_0^\infty (v^r-\bar v)V_{0y}\varphi_y \mathrm{d}y| &\le C\int_0^\infty |v^r_y yV_{0y}\varphi_y|\mathrm{d}y \le C\|v^r_y\|^2_{L^\infty} + C\|yV_{0y}\|^2\|\varphi_y\|^2\\ &\le C\tilde\delta\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}}.\end{aligned}$$

In the same way, we can deal with the remaining terms on the right-hand side of (4.31). Then we have

$$\begin{aligned}H_7 \le &\frac{1}{8}\|\varphi_y\|^2 + C\tilde\delta\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}} + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\\ &+C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{5}{2}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{23}{6}}.\end{aligned} \tag{4.32}$$

Hence, combining (4.24), (4.25)-(4.27) and (4.28), then integrating the resultant inequality with respect to t and using (4.4), we can obtain (4.20) provided that $C\varepsilon, \tilde\delta$ and χ are small enough. This completes the proof of Lemma 4.2. □

With Lemmas 4.1 and 4.2 in hand, we can show the following fundamental energy estimate.

Corollary 4.1. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.9)- (3.10), satisfying the conditions in Proposition 3.1, then it holds that*

$$\|\varphi(t)\|_1^2 + \|\psi(t)\|^2 + \int_0^t \Big[\|\varphi_y(\tau)\|_1^2 + \|\psi_y(\tau)\|^2\Big]\mathrm{d}\tau \le C\Big(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + \tilde\delta + \varepsilon^{\frac{1}{10}}\Big) \tag{4.33}$$

for any $t \in [0,T]$.

Lemma 4.3. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.9)-(3.10), satisfying the conditions in Proposition 3.1, then the following estimate holds*

$$\begin{aligned}&\|\psi_y(t)\|^2 + \|\varphi_{yy}(t)\|^2 + \int_0^t \left(\|\psi_{yy}\|^2 + \psi_y(\tau,0)^2 + \varphi_{yy}(\tau,0)^2\right)\mathrm{d}\tau\\ &\le C\Big(\|\psi_0\|_1^2 + \|\varphi_0\|_2^2 + \tilde\delta + \varepsilon^{\frac{1}{10}}\Big)\end{aligned} \tag{4.34}$$

for all $t \in [0,T]$.

Proof. Multiplying $(3.9)_2$ by $-\psi_{yy}$, one has

$$\Big(\frac{1}{2}\psi_y^2 + \frac{\kappa}{2(V+\varphi)^5}\varphi_{yy}^2\Big)_t - \Big(\psi_t\psi_y - \frac{s_-}{2}\psi_y^2 + \frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_{yy} + \frac{\kappa s_-}{2(V+\varphi)^5}\varphi_{yy}^2\Big)_y$$

$$
\begin{aligned}
&+\frac{\mu}{V+\varphi}\psi_{yy}^2=-\mu\Big(\frac{1}{V+\varphi}-\frac{1}{V}\Big)U_{yy}\psi_{yy}+\frac{\mu}{(V+\varphi)^2}(V_y\psi_y+U_y\varphi_y+\varphi_y\psi_y)\psi_{yy}\\
&+\mu\Big[\frac{1}{(V+\varphi)^2}-\frac{1}{V^2}\Big]V_yU_y\psi_{yy}+p'(V+\varphi)\varphi_y\psi_{yy}+[p'(V+\varphi)-p'(V)]V_y\psi_{yy}\\
&-\frac{5\kappa}{(V+\varphi)^6}(U_y+\psi_y)\varphi_{yy}^2+\kappa\Big[\frac{1}{(V+\varphi)^5}-\frac{1}{V^5}\Big]V_{yyy}\psi_{yy}-\frac{5\kappa}{(V+\varphi)^6}V_{yy}\varphi_y\psi_{yy}\\
&-10\kappa\Big[\frac{1}{(V+\varphi)^6}-\frac{1}{V^6}\Big]V_yV_{yy}\psi_{yy}-\frac{5\kappa}{(V+\varphi)^6}(V_y\varphi_{yy}+V_{yy}\varphi_y+\varphi_y\varphi_{yy})\psi_{yy}\\
&+\frac{15\kappa}{2(V+\varphi)^7}(3V_y\varphi_y^2+3V_y^2\varphi_y+\varphi_y^3)\psi_{yy}+15\kappa\Big[\frac{1}{(V+\varphi)^7}-\frac{1}{V^7}\Big]V_y^3\psi_{yy}. \qquad (4.35)
\end{aligned}
$$

Here we used

$$-\psi_t\psi_{yy}=-(\psi_t\psi_y)_y+\Big(\frac{1}{2}\psi_y^2\Big)_t,$$

and

$$
\begin{aligned}
\Big(\frac{\kappa\varphi_{yy}}{(V+\varphi)^5}\Big)_y\psi_{yy}&=\Big(\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_{yy}\Big)_y-\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_{yyy}\\
&=\Big(\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_{yy}\Big)_y-\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}(\varphi_{tyy}-s_-\varphi_{yyy})\\
&=\Big(\frac{\kappa}{(V+\varphi)^5}\varphi_{yy}\psi_{yy}\Big)_y-\Big(\frac{\kappa}{2(V+\varphi)^5}\varphi_{yy}^2\Big)_t\\
&\quad+\Big(\frac{\kappa s_-}{2(V+\varphi)^5}\varphi_{yy}^2\Big)_y-\frac{5\kappa}{2(V+\varphi)^6}(U_y+\psi_y)\varphi_{yy}^2
\end{aligned}
$$

with the help of $\varphi_{tyy}-s_-\varphi_{yyy}-\psi_{yyy}=0$. Moreover, from $\varphi_{ty}-s_-\varphi_{yy}-\psi_{yy}=0$ and $\varphi_{ty}(0)=0$, it is easy to see

$$\psi_{yy}(0)=-s_-\varphi_{yy}(0). \qquad (4.36)$$

Then integrating the equality (4.35) with respect to y over $\mathbb{R}^+$ and taking into account the boundary condition $(3.10)_2$ and (4.36), and (4.1), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty(\psi_y^2+\varphi_{yy}^2)\,\mathrm{d}y-\frac{s_-}{2}\psi_y(t,0)^2-\frac{\kappa s_-}{2v_-^5}\varphi_{yy}(t,0)^2+\int_0^\infty\psi_{yy}^2\,\mathrm{d}y\\
&\le C(I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8). \qquad (4.37)
\end{aligned}
$$

Here

$$
\begin{aligned}
I_1&=\Big|\int_0^\infty\varphi_y\psi_{yy}\,\mathrm{d}y\Big|,\\
I_2&=\Big|\int_0^\infty\varphi_y\psi_y\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty\psi_y\varphi_{yy}^2\,\mathrm{d}y\Big|+\Big|\int_0^\infty\varphi_y^3\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty\varphi_y\varphi_{yy}\psi_{yy}\,\mathrm{d}y\Big|,\\
I_3&=\Big|\int_0^\infty v_y^r\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty u_{yy}^r\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty v_{yyy}^r\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty v_y^ru_y^r\varphi\psi_{yy}\,\mathrm{d}y\Big|\\
&\quad+\Big|\int_0^\infty v_y^rv_{yy}^r\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty(v_y^r)^3\varphi\psi_{yy}\,\mathrm{d}y\Big|,\\
I_4&=\Big|\int_0^\infty V_{0y}\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty U_{0yy}\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty V_{0yyy}\varphi\psi_{yy}\,\mathrm{d}y\Big|\\
&\quad+\Big|\int_0^\infty V_{0y}U_{0y}\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty V_{0y}V_{0yy}\varphi\psi_{yy}\,\mathrm{d}y\Big|+\Big|\int_0^\infty V_{0y}^3\varphi\psi_{yy}\,\mathrm{d}y\Big|,
\end{aligned}
$$

$$
\begin{aligned}
I_5 =& |\int_0^\infty v_y^r U_{0y}\varphi\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty u_y^r V_{0y}\varphi\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_y^r V_{0yy}\varphi\psi_{yy}\,\mathrm{d}y| \\
&+ |\int_0^\infty v_{yy}^r V_{0y}\varphi\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^2 V_{0y}\varphi\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_y^r V_{0y}^2\varphi\psi_{yy}\,\mathrm{d}y|, \\
I_6 =& |\int_0^\infty v_y^r \varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty u_y^r \varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_y^r \varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty v_y^r \varphi_{yy}\psi_{yy}\,\mathrm{d}y| \\
&+ |\int_0^\infty v_{yy}^r \varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^2 \varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty U_{0y}\varphi_y\psi_{yy}\,\mathrm{d}y| \\
&+ |\int_0^\infty V_{0y}\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_{yy}\psi_{yy} + |\int_0^\infty V_{0yy}\varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}^2\varphi_y\psi_{yy}\,\mathrm{d}y| \\
&+ |\int_0^\infty v_y^r V_{0y}\varphi_y\psi_{yy}\,\mathrm{d}y|,
\end{aligned}
$$

and

$$
I_7 = |\int_0^\infty v_y^r \varphi_y^2\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_y^2\psi_{yy}\,\mathrm{d}y|, \quad I_8 = |\int_0^\infty f\psi_{yy}\,\mathrm{d}y|.
$$

Now let us estimate the terms on the right-hand side of (4.24). First, from the Cauchy inequality and the Young inequality, it is easy to obtain

$$
I_1 \le \frac{1}{8}\|\psi_{yy}\|^2 + C\|\varphi_y\|^2. \tag{4.38}
$$

Next, from the Holder inequality, the Sobolev inequality and Young inequality, we have

$$
\begin{aligned}
I_2 \le& C(\|\varphi_y\|_{L^\infty}\|\psi_y\|\|\psi_{yy}\| + \|\psi_y\|_{L^\infty}\|\varphi_{yy}\|^2 + \|\varphi_y\|_{L^\infty}^2\|\varphi_y\|\|\psi_{yy}\| + \|\varphi_y\|_{L^\infty}\|\varphi_{yy}\|\|\psi_{yy}\|) \\
\le& C\chi(\|\varphi_y\|^2 + \|\psi_y\|^2 + \|\varphi_{yy}\|^2 + \|\psi_{yy}\|^2).
\end{aligned} \tag{4.39}
$$

Moreover, on one hand, similar to (4.25) and (4.26), we have

$$
I_3 \le \tfrac{1}{8}\|\psi_{yy}\|^2 + C[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}} + \varepsilon^{\frac{1}{3}}(1+t)^{-\frac{11}{3}} + \varepsilon^{\frac{1}{2}}(1+t)^{-\frac{11}{2}} + \varepsilon^{\frac{1}{2}}(1+t)^{-\frac{5}{3}}\|\varphi\|^2, \tag{4.40}
$$

and

$$
I_4 \le \frac{1}{8}\|\psi_{yy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2. \tag{4.41}
$$

On the other hand, similar to (4.27) and (4.28), we obtain

$$
I_5 \le C\varepsilon\|\psi_{yy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2, \tag{4.42}
$$

and

$$
I_6 \le C(\tilde{\delta}+\varepsilon)\left(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2 + \|\psi_{yy}\|^2\right). \tag{4.43}
$$

Finally, similar to (4.29) and (4.32), we have

$$
I_7 \le C(\tilde{\delta}\chi + \varepsilon\chi)\left(\|\varphi_y\|^2 + \|\psi_{yy}\|^2\right) \tag{4.44}
$$

and

$$
\begin{aligned}
I_8 \le& \frac{1}{8}\|\psi_{yy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}} + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} \\
&+ C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{5}{2}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{23}{6}}.
\end{aligned} \tag{4.45}
$$

Hence, inserting (4.38)-(4.42) into (4.37), then integrating the resultant inequality with respect to t and using (4.33), we can obtain (4.34). This completes the proof of Lemma 4.3. ∎

Finally, we are going to establish the dissipation for φ_{yyy}.

LEMMA 4.4. *Let $(\varphi,\psi)(t,y)$ be a solution to (3.9)-(3.10), satisfying the conditions in Proposition 3.1, then it holds that*

$$\int_0^t \|\varphi_{yyy}(\tau)\|^2 \mathrm{d}\tau \le C(\|\varphi_0\|_2^2 + \|\psi_0\|_1^2 + \tilde{\delta} + \varepsilon^{\frac{1}{10}}) \tag{4.46}$$

for an arbitrary $t\in[0,T]$.

Proof. We first recall that

$$\varphi_{ty} - s_-\varphi_{yy} - \psi_{yy} = 0, \tag{4.47}$$

and

$$\varphi_{tyy} - s_-\varphi_{yyy} - \psi_{yyy} = 0, \tag{4.48}$$

further, we have

$$(\psi_t - s_-\psi_y)\varphi_{yyy} = (\psi_t\varphi_{yy})_y - (\psi_y\varphi_{yy})_t + (\psi_y\psi_{yy})_y - \psi_{yy}^2, \tag{4.49}$$

and

$$\begin{aligned}
\frac{\mu}{V+\varphi}\psi_{yy}\varphi_{yyy} &= \frac{\mu}{V+\varphi}(\varphi_{ty} - s_-\varphi_{yy})\varphi_{yyy}\\
&= \Big(\frac{\mu}{V+\varphi}\varphi_{ty}\varphi_{yy} - \frac{\mu s_-}{2(V+\varphi)}\varphi_{yy}^2\Big)_y - \Big(\frac{\mu}{2(V+\varphi)}\varphi_{yy}^2\Big)_t\\
&\quad - \frac{\mu}{2(V+\varphi)^2}(U_y+\psi_y)\varphi_{yy}^2 + \frac{\mu}{(V+\varphi)^2}(V_y+\varphi_y)\varphi_{yy}\psi_{yy}.
\end{aligned} \tag{4.50}$$

Then multiply $(3.9)_2$ by φ_{yyy} and use (4.49)-(4.50) to obtain

$$\begin{aligned}
&\Big[\frac{\mu}{2(V+\varphi)}\varphi_{yy}^2 - \psi_y\varphi_{yy}\Big]_t + \Big[p'(V+\varphi)\varphi_y\varphi_{yy} + \frac{\mu s_-}{2(V+\varphi)}\varphi_{yy}^2 - \frac{\mu}{V+\varphi}\varphi_{ty}\varphi_{yy} + \psi_y\psi_{yy}\Big]_y\\
&-p'(V+\varphi)\varphi_{yy}^2 + \frac{\kappa}{(V+\varphi)^5}\varphi_{yyy}^2 = \psi_{yy}^2 - [p'(V+\varphi) - p'(V)]V_y\varphi_{yyy} + p''(V+\varphi)(V_y\\
&+\varphi_y)\varphi_y\varphi_{yy} + \mu\Big(\frac{1}{V+\varphi} - \frac{1}{V}\Big)U_{yy}\varphi_{yyy} - \frac{\mu}{(V+\varphi)^2}U_y\varphi_y\varphi_{yyy} - \mu\Big[\frac{1}{(V+\varphi)^2} - \frac{1}{V^2}\Big]V_yU_y\varphi_{yyy}\\
&+f\varphi_{yyy} - \kappa\Big[\frac{1}{(V+\varphi)^5} - \frac{1}{V^5}\Big]V_{yyy}\varphi_{yyy} + 10\kappa\Big[\frac{1}{(V+\varphi)^6} - \frac{1}{V^6}\Big]V_yV_{yy}\varphi_{yyy}\\
&+\frac{10\kappa}{(V+\varphi)^6}(V_y\varphi_{yy} + V_{yy}\varphi_y + \varphi_y\varphi_{yy})\varphi_{yyy} - \frac{15\kappa}{(V+\varphi)^7}(3V_y\varphi_y^2 + 3V_y^2\varphi_y + \varphi_y^3)\varphi_{yyy}\\
&-15\kappa\Big[\frac{1}{(V+\varphi)^7} - \frac{1}{V^7}\Big]V_y^3\varphi_{yyy}.
\end{aligned}$$

Integrating the above equality with respect to y over $\mathbb{R}^+$ and taking into account the boundary condition $(3.10)_2$ and (4.36), and (4.1), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\varphi_{yy}^2 - \psi_y\varphi_{yy})\mathrm{d}y - \frac{\mu s_-}{4v_-}\varphi_{yy}(t,0)^2 + \int_0^\infty \varphi_{yy}^2\mathrm{d}y + \int_0^\infty \varphi_{yyy}^2\mathrm{d}y\\
&\le C\Big(\int_0^\infty \psi_{yy}^2\mathrm{d}y + \psi_y(t,0)^2 + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7\Big).
\end{aligned} \tag{4.51}$$

Here

$$
\begin{aligned}
J_1 =& |\int_0^\infty \varphi_y\varphi_{yy}\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty \varphi_y^3\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty \varphi_y^2\varphi_{yyy}\,\mathrm{d}y|,\\
J_2 =& |\int_0^\infty v_y^r\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty u_{yy}^r\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_{yyy}^r\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r u_y^r\varphi\varphi_{yyy}\,\mathrm{d}y|\\
&+ |\int_0^\infty v_y^r v_{yy}^r\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^3\varphi\varphi_{yyy}\,\mathrm{d}y|,\\
J_3 =& |\int_0^\infty V_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty U_{0yy}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0yyy}\varphi\varphi_{yyy}\,\mathrm{d}y|\\
&+ |\int_0^\infty V_{0y}U_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}V_{0yy}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}^3\varphi\varphi_{yyy}\,\mathrm{d}y|,\\
J_4 =& |\int_0^\infty v_y^r U_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty u_y^r V_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r V_{0yy}\varphi\varphi_{yyy}\,\mathrm{d}y|\\
&+ |\int_0^\infty v_{yy}^r V_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^2 V_{0y}\varphi\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r V_{0y}^2\varphi\varphi_{yyy}\,\mathrm{d}y|,\\
J_5 =& |\int_0^\infty v_y^r\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty u_y^r\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi_{yy}\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_{yy}^r\varphi_y\varphi_{yyy}\,\mathrm{d}y|\\
&+ |\int_0^\infty (v_y^r)^2\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty U_{0y}\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_{yy}\varphi_{yyy}\,\mathrm{d}y|\\
&+ |\int_0^\infty V_{0yy}\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}^2\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r V_{0y}\varphi_y\varphi_{yyy}\,\mathrm{d}y|,
\end{aligned}
$$

and

$$
J_6 = |\int_0^\infty v_y^r\varphi_y^2\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty V_{0y}\varphi_y^2\varphi_{yyy}\,\mathrm{d}y|, \quad J_7 = |\int_0^\infty f\varphi_{yyy}\,\mathrm{d}y|.
$$

Now let us deal with the terms on the right-hand side of (4.24). First, similar to (4.38), we have

$$
J_1 \le C\chi(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2 + \|\varphi_{yyy}\|^2. \tag{4.52}
$$

Next, similar to (4.25)-(4.28), one gets

$$
\begin{aligned}
&J_2 \le \tfrac{1}{8}\|\varphi_{yyy}\|^2 + C[\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}} + \varepsilon^{\frac{1}{3}}(1+t)^{-\frac{11}{3}} + \varepsilon^{\frac{1}{2}}(1+t)^{-\frac{11}{2}} + \varepsilon^{\frac{1}{2}}(1+t)^{-\frac{5}{3}}\|\varphi\|^2, && (4.53)\\
&J_3 \le \frac{1}{8}\|\varphi_{yyy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2, && (4.54)\\
&J_4 \le C\varepsilon\|\varphi_{yyy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2, && (4.55)
\end{aligned}
$$

and

$$
J_5 \le C(\tilde{\delta} + \varepsilon)\left(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2 + \|\varphi_{yyy}\|^2\right). \tag{4.56}
$$

Finally, similar to (4.29) and (4.32), we have

$$
J_6 \le C(\tilde{\delta}\chi + \varepsilon\chi)\left(\|\varphi_y\|^2 + \|\varphi_{yyy}\|^2\right) \tag{4.57}
$$

and

$$
\begin{aligned}
J_7 \le& \frac{1}{8}\|\varphi_{yyy}\|^2 + C\tilde{\delta}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{3}}(1+t)^{-\frac{5}{3}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}} + C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\\
&+ C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{5}{2}} + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{23}{6}}.
\end{aligned} \tag{4.58}
$$

Therefore, inserting (4.52)-(4.55) into (4.51) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty(\varphi_{yy}^2-\psi_y\varphi_{yy})dy+\varphi_{yy}(t,0)^2+\|\varphi_{yy}(t)\|^2+\|\varphi_{yyy}(t)\|^2\\ \le& C\|\psi_{yy}(t)\|^2+C\psi_y(t,0)^2+C(\chi+\varepsilon)\Big(\|\varphi_y\|^2+\|\psi_y\|^2\Big),\end{aligned}$$

further, integrating the above inequality with respect to t, and using (4.33) and (4.34), we obtain (4.46). This completes the proof. □

Proof. **(Proof of Proposition 3.1 for** $(\bar v,\bar u)\neq(v_*,u_*)$**.)** Summing up the estimates (4.33), (4.34) and (4.46), we immediately have (3.13). □

4.2. Estimates for the case $(\bar v,\bar u)=(v_*,u_*)$**.** In this subsection, we will obtain the uniform *a priori* estimates for the perturbation from the nonlinear wave with the degenerate stationary solution. Namely, we show (3.13) for the case $(\bar v,\bar u)=(v_*,u_*)$. In the following, we only need to show (4.4) in this case, the estimates (4.20), (4.33), (4.34) and (4.46) can be obtained same as those in the Subsection 4.1.

Indeed, we rewrite R_4 as

$$R_4=[p(V+\varphi)-p(V)-p'(V)\varphi]U_{0y}-\frac{\mu}{V(V+\varphi)}U_{0y}\varphi\psi_y=:R_{41}+R_{42}. \tag{4.59}$$

Since

$$p(V+\varphi)-p(V)-p'(V)\varphi=\frac{p''(V)}{2}\varphi^2+O(\varphi^3),$$

from Lemma 2.1, we have

$$\begin{aligned}\int_0^\infty R_{41}\mathrm{d}y=&\int_0^\infty a_2z^2(y)\frac{p''(V)}{2}\varphi^2\mathrm{d}y+\int_0^\infty O(\varphi^3)U_{0y}\mathrm{d}y+\int_0^\infty O(z^3(y))\frac{p''(V)}{2}\varphi^2\mathrm{d}y\\ \le& C(\tilde\delta+\chi)\|\varphi_y\|^2.\end{aligned}\tag{4.60}$$

Similar to (4.15), one gets

$$\int_0^\infty R_{42}\mathrm{d}y\le\frac{c}{8}\|\psi_y\|^2+C\tilde\delta\|\varphi_y\|^2. \tag{4.61}$$

Therefore, putting (4.6)-(4.8), (4.12)-(4.14), (4.60), (4.61), (4.16)-(4.18) and (4.19) into (4.5), and integrating the resultant inequality with respect to t, then also implies (4.4) provided that $\tilde\delta,\varepsilon$ and χ are small enough.

Proof. **(Proof of Proposition 3.1 for** $(\bar v,\bar u)=(v_*,u_*)$**.)** Summing up the estimates (4.33), (4.34) and (4.46), we immediately have (3.13). □

5. The proof of Theorem 1.1

This section is concerned with the proof of our main theorem. To prove Theorem 1.1, we employ the standard continuation argument based on a local existence theorem and the *a priori* estimates. Therefore, to complete the proof of Theorem 1.1, we need only to investigate the large-time behavior of the solution $(v,u)(t,x)$ to the initial boundary value problem (1.2) as time tends to infinity.

Proof. **(The completion of the proof of Theorem 1.1.)** Based upon the energy estimates derived in the previous sections, we will complete the proof of Theorem 1.1. To this end, we first prove that

$$\sup_{x\ge s_-t}|(v-V,u-U)(t,x)|\to 0, \tag{5.1}$$

namely,

$$\sup_{y\in\mathbb{R}^+}|(\varphi,\psi)(t,y)|\to 0, \tag{5.2}$$

as $t\to\infty$.

This is obvious supposing that we have proved the following assertion

$$\lim_{t\to+\infty}\|(\varphi_y,\psi_y)(t)\|=0. \tag{5.3}$$

As a matter of fact, if it is true, we infer from the Sobolev inequality that

$$\|(\varphi,\psi)\|_{L^\infty}\to 0, \text{ as } t\to+\infty. \tag{5.4}$$

Hence, it remains to show (5.3). To this end, from the relations (4.23) and (4.37), and Corollary 4.1, Lemmas 4.3 and 4.4, one can show that

$$\int_0^\infty \left(\|\varphi_y\|^2+\|\psi_y\|^2\right)\mathrm{d}t<+\infty, \tag{5.5}$$

and that

$$\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_y\|^2\right|\mathrm{d}t<+\infty, \quad \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_y\|^2\right|\mathrm{d}t<+\infty. \tag{5.6}$$

Then (5.3) follows from inequalities (5.5)-(5.6). Consequently, from (5.1), we prove (1.14) and complete the proof of Theorem 1.1. □

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