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# STABILITY FOR THE 2D MICROPOLAR EQUATIONS WITH PARTIAL DISSIPATION NEAR COUETTE FLOW*

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**Abstract.** In this paper, we will apply the Fourier multiplier method to explore the stability for the 2D micropolar equations with partial dissipation near Couette flow. The difficulty will be encountered due to the facts that one order derivative of the microtation appears on the right term of velocity equations and that the velocity equations only have vertical dissipation. To overcome the difficulty, we will make use of a Fourier multiplier to grasp the enhanced dissipation created by the special structure $y\partial_x - \nu\partial_y^2$ and obtain some new and higher-order estimates of the solution in an elegant way. Also, a time-dependent elliptic operator $\Lambda_t^b$ which commutes with linear part of the equations will be used to make our proof more clear.

**Keywords.** Two-dimensional micropolar equations; Stability; Couette flow.

**AMS subject classifications.** 35B65; 35Q35; 76D03.

## 1. Introduction

The micropolar equations in $\mathbb{R}^2$ are written as:

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P - (\nu+\kappa)\Delta U = 2\kappa\nabla^\perp W, \\ \mathrm{div}\, U = 0, \\ \partial_t W + U\cdot\nabla W - \gamma\Delta W + 4\kappa W = 2\kappa\nabla\times U, \end{cases} \tag{1.1}$$

where $(x,y)\in\mathbb{R}^2, t>0$. Here the unknown functions $U=(U^1(x,y,t),U^2(x,y,t))$, $P=P(x,y,t)$ and $W=W(x,y,t)$ represent the velocity, pressure and microtation, respectively. The parameters $\nu\geq 0$, $\kappa>0$ and $\gamma\geq 0$ are the Newtonian viscosity, the microrotation viscosity and the angular viscosity, respectively.

The micropolar equations were firstly studied in [14] by C.A. Eringen to model micropolar fluids. Micropolar fluids are fluids with microstructure which belong to a class of non-Newtonian fluids without symmetric stress tensor (called polar fluids). Furthermore, it describes phenomena such as fluids including particles suspended in a viscous medium. In fact, when $W=0$, the equations reduce to the classical incompressible Navier-Stokes equations. Due to the physical background and mathematical theoretical value, there has been much attention to well-posedness problem and large-time behavior issue (see [2, 3, 15] and [17]). Lukaszewicz in [17] explored the regularity result in three dimensions for both stationary and time-dependent cases. Global well-posedness and sharp algebraic decay estimates results were given in [11]. Furthermore, the decay estimates of linear micropolar fluids in three dimensions were investigated in [7]. The regularity criteria to the weak solutions in three dimensions can be found in [10]. Recently, micropolar equations with partial dissipation were studied in [12,13] and [23]. In addition, the regularity results in different domains applied to the micropolar equations were also solved (see [12, 13, 20] and [23]).

Recently, the stability of shear flows in the Navier-Stokes equations have been studied in a number of works (see [1–6], [18, 19, 21] and [22]). From the mathematical point

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of view, we need to choose an appropriate norm $\|\cdot\|_X$, and to determine the constant $\gamma$ (which can depend on space $X$) such that

$$\|f\|_X \lesssim \nu^{\gamma} \quad \Rightarrow \textit{stability},$$
$$\|f\|_X \gg \nu^{\gamma} \quad \Rightarrow \textit{possible instability},$$

where $f$ denotes the difference between the solution of the Navier-Stokes equations and a shear flow, and $\gamma$ is sometimes referred to as the transition threshold.

As mentioned in [8], the stability of Couette flow, which is a type of basic shear flow, can be solved due to the special structure $y\partial_x - \nu\partial_y^2$. This structure provides enhanced dissipation in comparison with the heat operator, which was first investigated by Hörmander in [16]. More precisely, consider the linear equations

$$\partial_t f + y\partial_x f = \nu\partial_y^2 f.$$

Taking the Fourier transform and changing the variables $\eta := \xi + kt$, one can get that

$$\widehat{f}(k,\xi,t) = \widehat{f}_0(k,\xi+kt)e^{-\nu\xi^2 t}e^{-\frac{1}{3}\nu k^2 t^3 - \nu k\xi t^2},$$

which implies that the dissipation time scale is $O(\nu^{-\frac{1}{3}})$. However, for the heat equations $\partial_t f = \nu\Delta f$, the dissipation time scale is $O(\nu^{-1})$. This reflects an enhanced dissipation due to the special structure $y\partial_x - \nu\partial_y^2$ in some sense. Deng-Wu-Zhang [9] constructed a Fourier multiplier to extract the enhanced dissipation to investigate the stability of Couette flow for the 2D Boussinesq equations. Moreover, a time-dependent elliptic operator $\Lambda_t^b = \left(1 - \partial_x^2 - (\partial_y + t\partial_x)^2\right)^{\frac{b}{2}}$ with $t \geq 0$ and $b > 0$ was used to obtain higher derivative estimates in a much more concise way.

Inspired by [9], we intend to explore the stability of the Couette flow for the 2D micropolar equations in this paper. The system we are concerned with reads as

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P - (\nu+\kappa)\Delta U = 2\kappa\nabla^{\perp}W, \\ \operatorname{div} U = 0, \\ \partial_t W + U\cdot\nabla W - \gamma\Delta W + 4\kappa W = 2\kappa\nabla\times U, \end{cases} \tag{1.2}$$

where $\nabla^{\perp} = (\partial_y, -\partial_x)$ and $(x,y) \in \mathbb{T}\times\mathbb{R}$ with $\mathbb{T} = [0,2\pi]$ being a periodic box, which means that the solution is $2\pi$-periodic along with the horizontal variable and defines on the whole line with respect to the vertical variable.

Denote the vorticity by $\overline{\Omega} = \nabla\times U$, then the system including vorticity equation, which corresponds to system $(1.2)_1$, is written as:

$$\begin{cases} \partial_t\overline{\Omega} + U\cdot\nabla\overline{\Omega} - (\nu+\kappa)\Delta\overline{\Omega} = -2\kappa\Delta W, \\ \operatorname{div} U = 0, \\ \partial_t W + U\cdot\nabla W - \gamma\Delta W + 4\kappa W = 2\kappa\overline{\Omega}. \end{cases} \tag{1.3}$$

It is clear that the Couette flow $\widetilde{u} = (y,0), \widetilde{w} = 0, \widetilde{p} = 0$ is a steady solution to (1.2), whose vorticity is $\widetilde{\Omega} = \nabla\times\widetilde{u} = -1$. Define $u = U - (y,0), \overline{W} = W, p = P$ and $\Omega = \overline{\Omega} + 1$, then system (1.3) turns into

$$\begin{cases} \partial_t\Omega + u\cdot\nabla\Omega + y\partial_x\Omega - (\nu+\kappa)\Delta\Omega = -2\kappa\Delta\overline{W}, \\ \operatorname{div} u = 0, \\ \partial_t\overline{W} + u\cdot\nabla\overline{W} + y\partial_x\overline{W} - \gamma\Delta\overline{W} + 4\kappa\overline{W} = 2\kappa\Omega - 2\kappa. \end{cases} \tag{1.4}$$

Letting $w=2\overline{W}+1$, we have

$$\begin{cases} \partial_t\Omega+u\cdot\nabla\Omega+y\partial_x\Omega-(\nu+\kappa)\Delta\Omega=-\kappa\Delta w, \\ \mathrm{div}\, u=0, \\ \partial_t w+u\cdot\nabla w+y\partial_x w-\gamma\Delta w+4\kappa w=4\kappa\Omega. \end{cases} \tag{1.5}$$

Particularly, we consider the following partial dissipation system:

$$\begin{cases} \partial_t\Omega+u\cdot\nabla\Omega+y\partial_x\Omega-(\nu+\kappa)\partial_y^2\Omega=-\kappa\Delta w, \\ \mathrm{div}\, u=0, \\ \partial_t w+u\cdot\nabla w+y\partial_x w-\gamma\Delta w+4\kappa w=4\kappa\Omega, \end{cases} \tag{1.6}$$

where the horizontal variable is periodic and vertical variable lies in the whole line, that is,

$$(x,y)\in\mathbb{T}\times\mathbb{R}. \tag{1.7}$$

The initial data is imposed as,

$$(\Omega,w)(x,t)|_{t=0}=(\Omega^0,w^0). \tag{1.8}$$

In what follows, we define

$$u_k(y):=\frac{1}{2\pi}\int_{\mathbb{T}}u(x,y)e^{-ixk}\mathrm{d}x, \quad k\in\mathbb{Z},$$

and

$$u_0=\frac{1}{2\pi}\int_{\mathbb{T}}u(x,y)\mathrm{d}x, \quad u_{\neq}=u-u_0.$$

It is noted that $u_0$ and $u_{\neq}$ stand for the projection of the function $u$ onto zero frequency and non-zero frequencies with respect to $x$, respectively. And it is easy to prove that $u_0$ and $u_{\neq}$ are orthogonal, that is

$$\|u\|_{L^2}=\|u_0\|_{L^2}+\|u_{\neq}\|_{L^2}.$$

In addition, the fractional derivative in the horizontal direction is defined as:

$$\widehat{|D_x|^\gamma f}(k,\xi)=|k|^\gamma\widehat{f}(k,\xi).$$

We first consider the partial dissipation system (1.6) with $\nu=\kappa=\gamma$. The main result is

**Theorem 1.1** (The case $\nu=\kappa=\gamma$). *Given real numbers $\alpha\geq\frac{2}{3}$, $-\frac{2}{3}\leq\alpha-\beta\leq 0$ and $-\frac{1}{3}\leq\alpha-\delta\leq\frac{2}{3}$. Assume that $\|\Omega^0\|_{H^b}\leq\epsilon\nu^\alpha$ with $b>\frac{4}{3}$, $\|w^0\|_{H^b}\leq\epsilon\nu^\beta$ and $\||D_x|^{\frac{4}{3}}w^0\|_{H^b}\leq\epsilon\nu^\delta$ with $b>1$ for arbitrary small positive $\epsilon$. Then, there exists a global small solution $(\Omega,w)$ to system (1.6)-(1.8) with $\nu=\kappa=\gamma$, satisfying*

$$\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\alpha},$$

$$\begin{aligned}\|\Lambda_t^bw\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^bw\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2 \\ +\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^bw_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^bw\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\beta}\end{aligned}$$

*and*

$$\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^\infty L_x^2}+\nu\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}$$
$$+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^b w_{\neq}\|^2_{L_t^2L_x^2}+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}\le C\epsilon^2\nu^{2\delta}.$$

REMARK 1.1. In [9], the authors considered the Boussinesq system with vertical dissipations both on velocity and temperature. Here we consider the micropolar system with vertical dissipation on velocity but full dissipation on microtation. This is mainly due to the fact that on the right-hand side of the vorticity Equation $(1.6)_1$ there appears the term $-\nu\Delta w$, which is a "bad" term with higher derivatives in the sense of the energy estimate. While, to the Boussinesq system, on the right-hand side of the vorticity equation there appears the term $\partial_x\theta$, which is from buoyancy forcing and can be controlled by the enhanced dissipation.

Our second result is about the partial dissipation system (1.6) with $\nu=\kappa\neq\gamma$, which is

THEOREM 1.2 (The case $\nu=\kappa\neq\gamma$). *Given real numbers $\alpha$, $\beta$ and $\delta$. When $\alpha\ge\frac{2}{3}$ and $\nu^\alpha\le\gamma^{\frac{2}{3}}$, $\beta$ satisfies $\gamma^{\beta-\frac{1}{2}}\le\nu^{\alpha-\frac{1}{2}}$ and $\nu^{\alpha+\frac{5}{6}}\le\gamma^{\beta+\frac{1}{6}}$, $\delta$ satisfies $\gamma^{\delta-\frac{1}{6}}\le\nu^{\alpha-\frac{5}{6}}$ and $\nu^{\alpha+\frac{5}{6}}\le\gamma^{\delta+\frac{1}{2}}$. Assume that $\|\Omega^0\|_{H^b}\le\epsilon\nu^\alpha$ with $b>\frac{4}{3}$, $\|w^0\|_{H^b}\le\epsilon\gamma^\beta$ and $\||D_x|^{\frac{4}{3}}w^0\|_{H^b}\le\epsilon\gamma^\delta$ with $b>1$ for arbitrary small positive $\epsilon$. Then, there exists a global small solution $(\Omega,w)$ to system (1.6)-(1.8), satisfying*

$$\|\Lambda_t^b\Omega\|^2_{L_t^\infty L_x^2}+\nu\|\nabla_y\Lambda_t^b\Omega\|^2_{L_t^2L_x^2}+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|^2_{L_t^2L_x^2}+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|^2_{L_t^2L_x^2}\le C\epsilon^2\nu^{2\alpha},$$

$$\|\Lambda_t^b w\|^2_{L_t^\infty L_x^2}+\gamma\|\nabla\Lambda_t^b w\|^2_{L_t^2L_x^2}+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}$$
$$+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b w_{\neq}\|^2_{L_t^2L_x^2}+\nu\|\Lambda_t^b w\|^2_{L_t^2L_x^2}\le C\epsilon^2\gamma^{2\beta}$$

*and*

$$\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^\infty L_x^2}+\gamma\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}$$
$$+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^b w_{\neq}\|^2_{L_t^2L_x^2}+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}\le C\epsilon^2\gamma^{2\delta}.$$

More generally, if we denote $\widetilde{\nu}=\nu+\kappa$ and consider the partial dissipation system (1.6) with $\widetilde{\nu}\neq\gamma$, a similar result can be obtained. Our third result is stated as

THEOREM 1.3 (The case $\widetilde{\nu}=\nu+\kappa,\widetilde{\nu}\neq\gamma$). *Given real numbers $\alpha$, $\beta$ and $\delta$. When $\alpha\ge\frac{2}{3}$ and $\widetilde{\nu}^\alpha\le\gamma^{\frac{2}{3}}$, $\beta$ satisfies $\gamma^{\beta-\frac{1}{2}}\le\widetilde{\nu}^{\alpha-\frac{1}{2}}$ and $\widetilde{\nu}^{\alpha+\frac{5}{6}}\le\gamma^{\beta+\frac{1}{6}}$, $\delta$ satisfies $\gamma^{\delta-\frac{1}{6}}\le\widetilde{\nu}^{\alpha-\frac{5}{6}}$ and $\widetilde{\nu}^{\alpha+\frac{5}{6}}\le\gamma^{\delta+\frac{1}{2}}$. Assume that $\|\Omega^0\|_{H^b}\le\epsilon\widetilde{\nu}^\alpha$ with $b>\frac{4}{3}$, $\|w^0\|_{H^b}\le\epsilon\gamma^\beta$ and $\||D_x|^{\frac{4}{3}}w^0\|_{H^b}\le\epsilon\gamma^\delta$ with $b>1$ for arbitrary small positive $\epsilon$. Then, there exists a global small solution $(\Omega,w)$ to system (1.6)-(1.8), satisfying*

$$\|\Lambda_t^b\Omega\|^2_{L_t^\infty L_x^2}+\widetilde{\nu}\|\nabla_y\Lambda_t^b\Omega\|^2_{L_t^2L_x^2}+\frac{1}{4}\widetilde{\nu}^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|^2_{L_t^2L_x^2}+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|^2_{L_t^2L_x^2}\le C\epsilon^2\widetilde{\nu}^{2\alpha},$$

$$\|\Lambda_t^b w\|^2_{L_t^\infty L_x^2}+\gamma\|\nabla\Lambda_t^b w\|^2_{L_t^2L_x^2}+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|^2_{L_t^2L_x^2}$$
$$+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b w_{\neq}\|^2_{L_t^2L_x^2}+\kappa\|\Lambda_t^b w\|^2_{L_t^2L_x^2}\le C\epsilon^2\gamma^{2\beta}$$

*and*

$$\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\gamma\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L_t^2L_x^2}^2$$
$$+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\kappa\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L_t^2L_x^2}^2\leq C\epsilon^2\gamma^{2\delta}.$$

Lastly, we present a result on system (1.5) with $\nu=\kappa=\gamma$, which contains full dissipation. Our final result is stated as

THEOREM 1.4. *Given real numbers $\alpha\geq\frac{2}{3}$ and $-\frac{1}{3}\leq\alpha-\beta\leq 0$. Assume that $\|\Omega^0\|_{H^b}\leq\epsilon\nu^\alpha$, $\|w^0\|_{H^b}\leq\epsilon\nu^\beta$ with $b>1$ for arbitrary small positive $\epsilon$. Then, there exists a global small solution $(\Omega,w)$ to system (1.5), (1.7) and (1.8), satisfying*

$$\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\alpha},$$

$$\|\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|_{L_t^2L_x^2}^2$$
$$+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^b w\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\beta}.$$

REMARK 1.2. In Theorem 1.4, there is a horizontal dissipation in vorticity equation, thus we can directly estimate the term

$$\nu|\langle\Lambda_t^b\partial_{xx}^2 w,\mathcal{M}\Lambda_t^b\Omega\rangle|\leq\nu\|\partial_x\Lambda_t^b w\|_{L^2}\|\partial_x\Lambda_t^b\Omega\|_{L^2}.$$

Thanks to the dissipation terms "$\|\nabla\Lambda_t^b\Omega\|_{L^2}$" and "$\|\nabla\Lambda_t^b w\|_{L^2}$", Theorem 1.2 can be proved in a much more direct way.

REMARK 1.3. In Theorems 1.1-1.4, whether the numbers $\alpha,\beta$ and $\delta$ are transition thresholds is still an interesting question.

We will mainly present details of proof of Theorem 1.1, and give a sketch of proof of Theorem 1.2. The proof of Theorem 1.3 is completely similar as that of Theorem 1.2 and we will omit it. Moreover, since the proof of Theorem 1.4 is direct (see Remark 1.2), we omit it as well. Now we explain the main ingredients of the proof of Theorem 1.1. First, since we only have vertical dissipation on velocity in (1.2) and hence also on the vorticity in (1.6), there will appear difficulties when we estimate higher derivatives of nonlinear terms such as $u\cdot\nabla w$ and $u\cdot\nabla\Omega$ in (1.6). To overcome these difficulties, we construct a Fourier multiplier denoted by $\mathcal{M}(k,\xi)$ which makes it available to obtain the horizontal $\frac{1}{3}$-order enhanced dissipation due to the special structure $y\partial_x-\nu\partial_y^2$ as in [9]. Moreover, to make full use of the horizontal $\frac{1}{3}$-order enhanced dissipation, we decompose our estimates into horizontal zeroth mode and non-zeroth modes, employ commutator estimates to shift derivatives and divide the frequency space into different subdomains to facilitate cancellations and derivative distribution. Second, the term $-\nu\Delta w$ is involved on the right-hand side of $(1.6)_1$. Therefore, when making $\|\Lambda_t^b\Omega\|_{L^2}$, we will encounter the following estimate

$$\nu|\langle\Lambda_t^b\partial_{xx}^2 w,\mathcal{M}\Lambda_t^b\Omega\rangle|\lesssim\nu\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2},$$

where $\mathcal{M}=\mathcal{M}(k,\xi)$ is a Fourier multiplier (See Section 2 for more details). To close the estimates, we need to make an estimate of $\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}$ in our approach, which together with the horizontal $\frac{1}{3}$-order enhanced dissipation deduces the desired estimates of $\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}$.

The paper is organized as follows. In Section 2, we introduce the Fourier multiplier which is mainly used to extract the horizontal $\frac{1}{3}$-order enhanced dissipation. In Section 3, we will present details of proof of Theorem 1.1, and give a sketch of proof of Theorem 1.2.

## 2. The Fourier multiplier $\mathcal{M}(k,\xi)$ and the elliptic operator $\Lambda_t^b$

As mentioned in introduction, Deng-Wu-Zhang [9] constructed a Fourier multiplier $\mathcal{M}(k,\xi)$ and a time-dependent elliptic operator $\Lambda_t^b$ to investigate the stability of Couette flow for the 2D Boussinesq equations. In this section, we will briefly introduce them. Before that, we present some notations.

Given $f,g$ two smooth functions, we define the $L^2-$ inner product as

$$\langle f,g\rangle_{L^2}=\int_{\mathbb{T}\times\mathbb{R}} f\overline{g}\mathrm{d}x\mathrm{d}y,$$

where $\overline{g}$ is the conjugate function of $g$. Then

$$\|f\|_{L^2}^2=\int_{\mathbb{T}\times\mathbb{R}}|f|^2\mathrm{d}x\mathrm{d}y=\sum_k\int_{\mathbb{R}}|f_k(y)|^2\mathrm{d}y=\sum_k\int_{\mathbb{R}}|\widehat{f}_k(\xi)|^2\mathrm{d}\xi,$$

where

$$f_k(y)=\frac{1}{2\pi}\int_{\mathbb{T}}f(x,y)e^{-ikx}\mathrm{d}x,\widehat{f}_k(\xi)=\int_{\mathbb{R}}f_k(y)e^{-i\xi y}\mathrm{d}y.$$

We also denote $\widehat{f}(k,\xi)=\widehat{f}_k(\xi)=\frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}}f(x,y)e^{-i(kx+\xi y)}\mathrm{d}x\mathrm{d}y$.

Now we introduce the Fourier multiplier briefly as follows. Choosing a real-valued, non-decreasing function $\phi\in C^\infty(\mathbb{R})$,

$$\phi(t)=\begin{cases}1, & t\in(-\infty,-2],\\ 0, & t\in[2,\infty),\end{cases}$$

and $\phi'=\frac{1}{4}$ on $[-1,1]$.

Define Fourier multiplier $\mathcal{M}(k,\xi)=\mathcal{M}_1(k,\xi)+\mathcal{M}_2(k,\xi)+1$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy:

$$\begin{aligned}&\mathcal{M}_1(k,\xi)=\phi(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}sgn(k)\xi),k\neq0,\\&\mathcal{M}_2(k,\xi)=\frac{1}{k^2}(arctan\frac{\xi}{k}+\frac{\pi}{2}),k\neq0,\\&\mathcal{M}_1(0,\xi)=\mathcal{M}_2(0,\xi)=0.\end{aligned}$$

It holds that $\mathcal{M}$ is self-adjoint and bounded with $1\leq\mathcal{M}\leq\pi+2$.

Noting that $y\partial_x$ is a non-self-adjoint operator and $\partial_{yy}$ is self-adjoint, one can obtain (see [9])

$$\begin{aligned}2Re\langle(y\partial_x-\nu\partial_{yy})\Omega,\mathcal{M}\Omega\rangle_{L^2}&=\langle([\mathcal{M},y\partial_x]+2\nu\mathcal{M}\xi^2)\Omega,\Omega\rangle_{L^2}\\&=\sum_k\int_{\mathbb{R}}(k\partial_\xi\mathcal{M}+2\nu\mathcal{M}\xi^2)|\widehat{\Omega}(k,\xi)|^2\mathrm{d}\xi.\end{aligned}$$

Using the expression of $\mathcal{M}$ defined above, we get

$$(k\partial_\xi\mathcal{M}+2\nu\mathcal{M}\xi^2)|\widehat{\Omega}(k,\xi)|^2\geq(\nu\xi^2+\frac{1}{4}\nu^{\frac{1}{3}}|k|^{\frac{2}{3}}+\frac{1}{k^2+\xi^2})|\widehat{\Omega}(k,\xi)|^2,$$

which yields that

$$\int_{\mathbb{R}}(k\partial_\xi\mathcal{M}+2\nu\mathcal{M}\xi^2)|\widehat{\Omega}(k,\xi)|^2\mathrm{d}\xi$$
$$\geq\nu\|\nabla_y\omega\|_{L^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Omega\|_{L^2}^2+\|(-\Delta)^{-\frac{1}{2}}\Omega_{\neq}\|_{L^2}^2. \tag{2.1}$$

Thus, the horizontal $\frac{1}{3}$−enhanced dissipation appears on the right-hand side of (2.1), which is $\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Omega\|_{L^2}^2$. Same structure can be found in the angular velocity equation.

Next, we introduce the time-dependent elliptic operator $\Lambda_t^b=(1-\partial_x^2-(\partial_y+t\partial_x)^2)^{\frac{b}{2}}$ for $t\geq 0$ and $b>0$, of which the symbol is $\Lambda_t^b(k,\xi)=(1+k^2+(\xi+tk)^2)^{\frac{b}{2}}$. The operator $\Lambda_t^b$ holds a few advantages and properties when obtaining the derivative estimates, which are collected as follows.

Lemma 2.1 ([9]). *For any two smooth functions $f$ and $g$, it holds that*

(1) *For any $b\in\mathbb{R}$, $\Lambda_t^b$ commutes with $\partial_t+y\partial_x$, in the following sense,*

$$\Lambda_t^b(\partial_t+y\partial_x)f=(\partial_t+y\partial_x)\Lambda_t^b f.$$

*Proof.* *We prove the equality in the form of Fourier transform*

$$\begin{aligned}&\mathcal{F}\Big((\partial_t+y\partial_x)(\Lambda_t^b f)\Big)\\=&\mathcal{F}\Big((\partial_t+y\partial_x)(\Lambda_t^b)f+\Lambda_t^b(\partial_t+y\partial_x)f\Big)\\=&\partial_t(\Lambda_t^b(k,\xi))\widehat{f}-k\partial_\xi(\Lambda_t^b(k,\xi))\widehat{f}+\mathcal{F}\Big(\Lambda_t^b(\partial_t+y\partial_x)f\Big)\\=&b\Lambda_t^{b-2}(k,\xi)(\xi+kt)\cdot k-k\cdot b\Lambda_t^{b-2}(k,\xi)(\xi+kt)+\mathcal{F}\Big(\Lambda_t^b(\partial_t+y\partial_x)f\Big)\\=&\mathcal{F}\Big(\Lambda_t^b(\partial_t+y\partial_x)f\Big).\end{aligned}$$

*This implies the ordinary equality we want.* □

(2) *For any $b>0$,*

$$\|\Lambda_t^b(fg)\|_{L^2}\leq\|f\|_{L^\infty}\|\Lambda_t^b g\|_{L^2}+\|g\|_{L^\infty}\|\Lambda_t^b f\|_{L^2}.$$

*Moreover, for $b>1$, we have*

$$\|f(t)\|_{L^\infty}\leq C\|\widehat{f(t)}\|_{L^1}\leq C\|\Lambda_t^b f(t)\|_{L^2},$$

*and consequently,*

$$\|\Lambda_t^b(fg)\|_{L^2}\leq C\|\Lambda_t^b f\|_{L^2}\|\Lambda_t^b g\|_{L^2}.$$

(3) *For any non-negative $s$ and $b>1$, there holds*

$$\||D_x|^s\Lambda_t^b(fg)\|_{L^2}\leq C(\||D_x|^s\Lambda_t^b f\|_{L^2}\|\Lambda_t^b g\|_{L^2}+\||D_x|^s\Lambda_t^b g\|_{L^2}\|\Lambda_t^b f\|_{L^2}).$$

Remark 2.1. According to (1) of Lemma 2.1, applying $\Lambda_t^b$ on both sides of the Equations (1.6) will not destroy the structure of the linear parts. Moreover, according to (2) and (3) of Lemma 2.1, $\Lambda_t^b$ shares similar properties as the standard fractional Laplacian operators.

## 3. Proof of main results

In this section, we will present details of proof of Theorem 1.1 and give a sketch of proof of Theorem 1.2. Since the proof of Theorem 1.3 is completely similar as that of Theorem 1.2 and the proof of Theorem 1.4 is much more direct (see Remark 1.2), we omit them here.

In what follows, we denote $D=(D_x,D_y)=\frac{1}{i}(\partial_x,\partial_y)$.

### 3.1. Proof of Theorem 1.1.

*Proof.* Since the operator $\Lambda_t^b$ is commutable with $\partial_t+y\partial_x$, we apply $\Lambda_t^b$ on both sides of (1.6) to get that

$$\begin{cases} \partial_t\Lambda_t^b\Omega+\Lambda_t^b(u\cdot\nabla\Omega)+y\partial_x\Lambda_t^b\Omega-2\nu\partial_y^2\Lambda_t^b\Omega=-\nu\Lambda_t^b\Delta w,\\ \partial_t\Lambda_t^b w+\Lambda_t^b(u\cdot\nabla w)+y\partial_x\Lambda_t^b w-\nu\Delta\Lambda_t^b w+4\nu\Lambda_t^b w=4\nu\Lambda_t^b\Omega. \end{cases} \tag{3.1}$$

Taking $L^2$ inner product of $(3.1)_1$ with $\mathcal{M}\Lambda_t^b\Omega$ and applying the property of multiplier, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t^b\Omega\|_{L^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}^2+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}^2\\ =&-2\nu Re\langle\Lambda_t^b\Delta w,\mathcal{M}\Lambda_t^b\Omega\rangle+2Re\langle\Lambda_t^b(u\cdot\nabla\Omega),\mathcal{M}\Lambda_t^b\Omega\rangle. \end{aligned} \tag{3.2}$$

Define

$$\nu\langle\Lambda_t^b\Delta w,\mathcal{M}\Lambda_t^b\Omega\rangle+\langle\Lambda_t^b(u\cdot\nabla\Omega),\mathcal{M}\Lambda_t^b\Omega\rangle:=I_1+I_2.$$

Similarly, taking $L^2$ inner product of $(3.1)_2$ with $\mathcal{M}\Lambda_t^b w$ and applying the property of multiplier, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_t^b w\|_{L^2}^2+\nu\|\nabla\Lambda_t^b w\|_{L^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|_{L^2}^2\\ &\quad+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b w_{\neq}\|_{L^2}^2+\nu\|\Lambda_t^b w\|_{L^2}^2\\ =&8\nu Re\langle\Lambda_t^b\Omega,\mathcal{M}\Lambda_t^b w\rangle+2Re\langle\Lambda_t^b(u\cdot\nabla w),\mathcal{M}\Lambda_t^b w\rangle. \end{aligned} \tag{3.3}$$

Define

$$4\nu\langle\Lambda_t^b\Omega,\mathcal{M}\Lambda_t^b w\rangle+\langle\Lambda_t^b(u\cdot\nabla w),\mathcal{M}\Lambda_t^b w\rangle:=I_3+I_4.$$

Moreover, taking $L^2$ inner product of $(3.1)_2$ with $\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^b w$ and applying the property of multiplier, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}^2+\nu\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}^2\\ &\quad+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^b w_{\neq}\|_{L^2}^2+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}^2\\ =&8\nu Re\langle\Lambda_t^b\Omega,\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^b w\rangle+2Re\langle\Lambda_t^b(u\cdot\nabla w),\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^b w\rangle. \end{aligned} \tag{3.4}$$

Similarly, define

$$4\nu\langle\Lambda_t^b\Omega,\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^b w\rangle+\langle\Lambda_t^b(u\cdot\nabla w),\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^b w\rangle:=I_5+I_6.$$

In what follows, define $\mathcal{M}_t^b=\sqrt{\mathcal{M}}\Lambda_t^b$.

We deal with terms $I_1,I_3$ and $I_5$ firstly. Divide the term $I_1$ into two parts, that is

$$\begin{aligned} I_1=&\nu\langle\mathcal{M}_t^b\partial_x^2 w,\mathcal{M}_t^b\Omega\rangle+\nu\langle\mathcal{M}_t^b\partial_y^2 w,\mathcal{M}_t^b\Omega\rangle\\ :=&I_{11}+I_{12}. \end{aligned}$$

Then, we get

$$|I_{11}|\leq \nu \||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}$$

and

$$|I_{12}|\leq \nu\|\nabla \Lambda_t^b w\|_{L^2}\|\nabla_y \Lambda_t^b \Omega\|_{L^2}.$$

Combining the estimates on $I_{11}$ and $I_{12}$, we obtain

$$|I_1|\leq \nu\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}+\nu\|\nabla\Lambda_t^b w\|_{L^2}\|\nabla_y\Lambda_t^b\Omega\|_{L^2}.$$

By Hölder inequality, we get estimates of $I_3$ and $I_5$ directly,

$$|I_3|\leq 4\nu\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|_{L^2},$$

$$|I_5|\leq 4\nu\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}.$$

In the following, we focus on the terms $I_2$, $I_4$ and $I_6$. The term $I_4$ can be written as

$$\begin{aligned}I_4=&\langle \mathcal{M}_t^b(u^1\partial_x w),\mathcal{M}_t^b w\rangle+\langle \mathcal{M}_t^b(u^2\partial_y w),\mathcal{M}_t^b w\rangle\\ =&\langle \mathcal{M}_t^b(u_0^1\partial_x w),\mathcal{M}_t^b w\rangle+\langle \mathcal{M}_t^b(u_{\neq}^1\partial_x w),\mathcal{M}_t^b w\rangle+\langle \mathcal{M}_t^b(u^2\partial_y w),\mathcal{M}_t^b w\rangle\\ :=&I_{41}+I_{42}+I_{43}.\end{aligned}$$

By Biot-Savart law, we have

$$u=(u^1,u^2)^t=\nabla^{\perp}(-\Delta)^{-1}\Omega=(\partial_y(-\Delta)^{-1}\Omega,-\partial_x(-\Delta)^{-1}\Omega)^t. \tag{3.5}$$

Due to Plancherel's theorem, we have

$$\||D|^{\alpha}\Lambda_t^b u_{\neq}^i\|_{L^2}\leq \|\Lambda_t^b\Omega_{\neq}\|_{L^2}\leq\|\Lambda_t^b\Omega\|_{L^2}\qquad (0\leq\alpha\leq 1, i=1,2).$$

Note that

$$\langle \mathcal{M}_t^b(u_0^1\partial_x w_{\neq}),\mathcal{M}_t^b w_0\rangle=0$$

and

$$\langle u_0^1\partial_x\mathcal{M}_t^b w_{\neq},\mathcal{M}_t^b w_{\neq}\rangle=0.$$

Then, we write $I_{41}$ as

$$\begin{aligned}I_{41}=&\langle \mathcal{M}_t^b(u_0^1\partial_x w_{\neq})-u_0^1\partial_x\mathcal{M}_t^b w_{\neq},\mathcal{M}_t^b w_{\neq}\rangle\\ =&\sum_k\int_{R^2}\mathcal{M}_t^b(k,\xi)\widehat{u_0^1}(0,\eta)ik\widehat{w_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{w_{\neq}}}(k,\xi)\\ &\qquad-\widehat{u_0^1}(0,\eta)\mathcal{M}_t^b(k,\xi-\eta)ik\widehat{w_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\ =&-\sum_k\int_{R^2}\Big(\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k,\xi-\eta)\Big)k\eta^{-1}\widehat{\Omega_0}(0,\eta)\\ &\qquad\widehat{w_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{3.6}$$

By the mean value formula,

$$|\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k,\xi-\eta)|\le\left|\int_0^1\partial_\xi\mathcal{M}_t^b(k,\xi-s\eta)\eta\mathrm{d}s\right|.$$

Moreover, it holds that

$$|\partial_\xi\mathcal{M}_t^b(k,\xi)|\lesssim(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\Lambda_t^b(k,\xi). \tag{3.7}$$

Then, by Young's inequality, for $b>1$, estimate $I_{41}$ as

$$\begin{aligned}|I_{41}|\lesssim&\Big|\sum_k(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\int_{R^2}|k|\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big)\widehat{\Omega_0}(0,\eta)\\&\qquad\qquad\widehat{w_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\Big|\\\lesssim&\Big|\sum_k(\nu^{\frac{1}{3}}|k|^{\frac{2}{3}}+1)\int_{R^2}\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big)\widehat{\Omega_0}(0,\eta)\\&\qquad\qquad\widehat{w_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\Big|\\\lesssim&\nu^{\frac{1}{3}}\|\Lambda_t^b\Omega_0\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L^2}^2+\|\Lambda_t^b\Omega_0\|_{L^2}\|\Lambda_t^bw_{\neq}\|_{L^2}^2\\\lesssim&\|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L^2}^2.\end{aligned}$$

For the term $I_{42}$, by Lemma 2.1, for $b>1$, we have

$$\begin{aligned}|I_{42}|\lesssim&\|\Lambda_t^bu_{\neq}^1\|_{L^2}\|\Lambda_t^b\partial_xw\|_{L^2}\|\Lambda_t^bw\|_{L^2}\\\lesssim&\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}\|\nabla\Lambda_t^bw\|_{L^2}\|\Lambda_t^bw\|_{L^2}.\end{aligned}$$

For the term $I_{43}$, we observe that

$$u^2=-\partial_x(-\Delta)^{-1}\Omega=-\partial_x(-\partial_{yy}^2)^{-1}\Omega_{\neq}=u_{\neq}^2.$$

Using Lemma 2.1, for $b>1$, we get

$$\begin{aligned}|I_{43}|\lesssim&\|\Lambda_t^b(u^2\partial_yw)\|_{L^2}\|\Lambda_t^bw\|_{L^2}\\\lesssim&\|\Lambda_t^bu_{\neq}^2\|_{L^2}\|\nabla_y\Lambda_t^bw\|_{L^2}\|\Lambda_t^bw\|_{L^2}\\\lesssim&\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}\|\nabla\Lambda_t^bw\|_{L^2}\|\Lambda_t^bw\|_{L^2}.\end{aligned}$$

Combining estimates of $I_{41}$, $I_{42}$ and $I_{43}$ above, $I_4$ is estimated as follows:

$$\begin{aligned}|I_4|\lesssim&\|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L^2}^2+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L^2}\|\Lambda_t^bw\|_{L^2}\\&+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}\|\nabla\Lambda_t^bw\|_{L^2}\|\Lambda_t^bw\|_{L^2}.\end{aligned}$$

For the term $I_6$, integrating by parts and decomposing $u^1=u_0^1+u_{\neq}^1$ yield that

$$\begin{aligned}I_6=&\langle\Lambda_t^bu^1\partial_xw,\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^bw\rangle+\langle\Lambda_t^bu^2\partial_yw,\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^bw\rangle\\=&\langle\Lambda_t^b(u_0^1\partial_xw),\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^bw\rangle+\langle\Lambda_t^b(u_{\neq}^1\partial_xw),\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^bw\rangle\\&+\langle\Lambda_t^b(u^2\partial_yw),\mathcal{M}|D_x|^{\frac{8}{3}}\Lambda_t^bw\rangle\end{aligned}$$

$$
\begin{aligned}
=&\langle |D_x|^{\frac{4}{3}}\Lambda_t^b(u_0^1\partial_x w),\mathcal{M}|D_x|^{\frac{4}{3}}\Lambda_t^b w\rangle+\langle |D_x|^{\frac{1}{3}}\Lambda_t^b(u_{\neq}^1\partial_x w),\mathcal{M}|D_x|^{\frac{7}{3}}\Lambda_t^b w\rangle\\
&+\langle |D_x|^{1}\Lambda_t^b(u^2\partial_y w),\mathcal{M}|D_x|^{\frac{5}{3}}\Lambda_t^b w\rangle\\
:=&I_{61}+I_{62}+I_{63}.
\end{aligned}
$$

For the term $I_{61}$, note that

$$
\langle |D_x|^{\frac{4}{3}}\mathcal{M}_t^b(u_0^1\partial_x w_{\neq}),|D_x|^{\frac{4}{3}}\mathcal{M}_t^b w_0\rangle=0
$$

and

$$
\langle u_0^1\partial_x|D_x|^{\frac{4}{3}}\mathcal{M}_t^b w_{\neq},|D_x|^{\frac{4}{3}}\mathcal{M}_t^b w_{\neq}\rangle=0.
$$

Denote $\lambda_t^b(k,\xi)=|D_x|^{\frac{4}{3}}\mathcal{M}_t^b(k,\xi)$. Then we have

$$
\begin{aligned}
I_{61}=&\langle \lambda_t^b(u_0^1\partial_x w_{\neq})-u_0^1\partial_x\lambda_t^b w_{\neq},\lambda_t^b w_{\neq}\rangle\\
=&\sum_k\int_{R^2}\lambda_t^b(k,\xi)\widehat{u_0^1}(0,\eta)ik\widehat{w_{\neq}}(k,\xi-\eta)\overline{\lambda_t^b\widehat{w_{\neq}}}(k,\xi)\\
&\qquad-\widehat{u_0^1}(0,\eta)\lambda_t^b(k,\xi-\eta)ik\widehat{w_{\neq}}(k,\xi-\eta)\overline{\lambda_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\
=&-\sum_k\int_{R^2}\Big(\lambda_t^b(k,\xi)-\lambda_t^b(k,\xi-\eta)\Big)k\eta^{-1}\widehat{\Omega_0}(0,\eta)\\
&\qquad\widehat{w_{\neq}}(k,\xi-\eta)\overline{\lambda_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}\tag{3.8}
$$

By the mean value formula,

$$
|\lambda_t^b(k,\xi)-\lambda_t^b(k,\xi-\eta)|\le\left|\int_0^1\partial_\xi\lambda_t^b(k,\xi-s\eta)\eta\mathrm{d}s\right|.
$$

It holds that

$$
\partial_\xi\lambda_t^b(k,\xi)\lesssim(\nu^{\frac{1}{3}}|k|^1+|k|^{\frac{1}{3}})\Lambda_t^b(k,\xi).
$$

Then we can estimate the term $I_{61}$ as

$$
\begin{aligned}
|I_{61}|\lesssim&\sum_k(\nu^{\frac{1}{3}}|k|^2+|k|^{\frac{4}{3}})\int_{R^2}\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big)\widehat{\Omega_0}(0,\eta)\\
&\qquad\widehat{w_{\neq}}(k,\xi-\eta)\overline{\lambda_t^b\widehat{w_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\
\lesssim&\|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}^2.
\end{aligned}
$$

Using same argument to term $I_{62}$, by Lemma 2.1, for $b>1$, we get

$$
\begin{aligned}
|I_{62}|\lesssim&\Big(\||D_x|^{\frac{1}{3}}\Lambda_t^b u_{\neq}^1\|_{L^2}\|\Lambda_t^b\partial_x w\|_{L^2}+\|\Lambda_t^b u_{\neq}^1\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\partial_x w\|_{L^2}\Big)\|\Lambda_t^b w\|_{L^2}\\
\lesssim&\|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|_{L^2}\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|_{L^2}.
\end{aligned}
$$

At last, due to

$$
u^2=-\partial_x(-\Delta)^{-1}\Omega=-\partial_x(-\Delta)^{-1}\Omega_{\neq}=u_{\neq}^2.
$$

By Lemma 2.1, for $b>1$, we obtain

$$\begin{aligned}|I_{63}| \lesssim& \| |D_x| \Lambda_t^b (u^2 \partial_y w) \|_{L^2} \| |D_x|^{\frac{5}{3}} \Lambda_t^b w \|_{L^2} \\ \lesssim& \Big( \| |D_x| \Lambda_t^b u^2_{\neq} \|_{L^2} \| \nabla_y \Lambda_t^b w \|_{L^2} \| \Lambda_t^b w \|_{L^2} \\ &+ \| \Lambda_t^b u^2_{\neq} \|_{L^2} \| \nabla_y |D_x| \Lambda_t^b w \|_{L^2} \| \Lambda_t^b w \|_{L^2} \Big) \| |D_x|^{\frac{5}{3}} \Lambda_t^b w \|_{L^2} \\ \lesssim& \| \Lambda_t^b \Omega \|_{L^2} \| \nabla |D_x|^{\frac{4}{3}} \Lambda_t^b w \|_{L^2} \| |D_x|^{\frac{5}{3}} \Lambda_t^b w \|_{L^2}.\end{aligned}$$

Combining estimates on $I_{61}$, $I_{62}$ and $I_{63}$ above, we obtain

$$|I_6| \lesssim \| \Lambda_t^b \Omega \|_{L^2} \| |D_x|^{\frac{5}{3}} \Lambda_t^b w \|_{L^2}^2 + \| \Lambda_t^b \Omega \|_{L^2} \| \nabla |D_x|^{\frac{4}{3}} \Lambda_t^b w \|_{L^2} \| |D_x|^{\frac{5}{3}} \Lambda_t^b w \|_{L^2}.$$

Due to the fact that there is only vertical dissipation and horizontal $\frac{1}{3}$-order enhanced dissipation in the vorticity equation, the nonlinear term $I_2$ seems to be the most difficult one to be dealt with.

By the decomposition of function, we have

$$u = u_0 + u_{\neq} = \nabla^{\perp} (-\Delta)^{-1} (\Omega_0 + \Omega_{\neq}). \tag{3.9}$$

Combining (3.5) with (3.9), we write $I_2$ as

$$\begin{aligned}I_2 =& \langle \mathcal{M}_t^b ((u_0 + u_{\neq}) \cdot \nabla \Omega), \mathcal{M}_t^b \Omega \rangle \\ =& \langle \mathcal{M}_t^b (u_0^1 \partial_x \Omega), \mathcal{M}_t^b \Omega \rangle + \langle \mathcal{M}_t^b (u_{\neq}^1 \partial_x \Omega), \mathcal{M}_t^b \Omega \rangle \\ &+ \langle \mathcal{M}_t^b (u_{\neq}^2 \partial_y \Omega), \mathcal{M}_t^b \Omega \rangle \\ :=& I_{21} + I_{22} + I_{23}.\end{aligned}$$

By (3.9) and Lemma 2.1, for $b>1$, we bound the term $I_{23}$ directly as follows

$$\begin{aligned}|I_{23}| =& |\langle \Lambda_t^b (\partial_x (\Delta)^{-1} \Omega_{\neq} \partial_y \Omega), \mathcal{M} \Lambda_t^b \Omega \rangle| \\ \lesssim& \| \Lambda_t^b (\partial_x (-\Delta)^{-1} \Omega_{\neq} \partial_y \Omega) \|_{L^2} \| \Lambda_t^b \Omega \|_{L^2} \\ \lesssim& \| (-\Delta)^{-\frac{1}{2}} \Lambda_t^b \Omega_{\neq} \|_{L^2} \| \nabla_y \Lambda_t^b \Omega \|_{L^2} \| \Lambda_t^b \Omega \|_{L^2}.\end{aligned}$$

Then the term $I_{21}$ can be written as

$$I_{21} = \langle \mathcal{M}_t^b (u_0^1 \partial_x \Omega_{\neq}), \mathcal{M}_t^b \Omega \rangle.$$

Due to the equality

$$\langle \mathcal{M}_t^b (u_0^1 \partial_x \Omega_{\neq}), \mathcal{M}_t^b \Omega_0 \rangle = 0$$

and

$$\langle u_0^1 \partial_x \mathcal{M}_t^b \Omega_{\neq}, \mathcal{M}_t^b \Omega_{\neq} \rangle = 0.$$

$I_{21}$ can be rewritten as

$$I_{21} = \langle \mathcal{M}_t^b (u_0^1 \partial_x \Omega_{\neq}) - u_0^1 \partial_x \mathcal{M}_t^b \Omega_{\neq}, \mathcal{M}_t^b \Omega_{\neq} \rangle.$$

By Plancherel's theorem,

$$
\begin{aligned}
I_{21}=&\sum_{k\neq 0}\int_{R^2}\mathcal{M}_t^b(k,\xi)\widehat{u_0^1}(0,\eta)ik\widehat{\Omega_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\\
&\qquad -\widehat{u_0^1}(0,\eta)\mathcal{M}_t^b(k,\xi-\eta)ik\widehat{\Omega_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\
=&-\sum_{k\neq 0}\int_{R^2}(\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k,\xi-\eta))k\eta^{-1}\widehat{\Omega_0}(0,\eta)\\
&\qquad \widehat{\Omega_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}\tag{3.10}
$$

Then, by the mean value formula, it holds that

$$
|\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k,\xi-\eta)|\leq\left|\int_0^1\partial_\xi\mathcal{M}_t^b(k,\xi-s\eta)\eta\mathrm{d}s\right|.
$$

Using (3.7), we have

$$
\begin{aligned}
|\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k,\xi-\eta)|\lesssim&|\eta|(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\Big(\Lambda_t^b(k,\xi)+\Lambda_t^b(k,\xi-\eta)\Big)\\
\lesssim&|\eta|(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big).
\end{aligned}
$$

Then, by Young's inequality, for $b>1$, we have

$$
\begin{aligned}
|I_{21}|\lesssim&\sum_k(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\int_{R^2}\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big)\widehat{\Omega_0}(0,\eta)\\
&\qquad\widehat{\Omega_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\
\lesssim&\sum_k(\nu^{\frac{1}{3}}|k|^{\frac{2}{3}}+1)\int_{R^2}\Big(\Lambda_t^b(0,\eta)+\Lambda_t^b(k,\xi-\eta)\Big)\widehat{\Omega_0}(0,\eta)\\
&\qquad\widehat{\Omega_{\neq}}(k,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta\\
\lesssim&\nu^{\frac{1}{3}}\|\Lambda_t^b\Omega_0\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}^2+\|\Lambda_t^b\Omega_0\|_{L^2}\|\Lambda_t^b\Omega_{\neq}\|_{L^2}^2\\
\lesssim&\|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}^2.
\end{aligned}
$$

Now we deal with the term $I_{22}$. Notice that

$$
\mathrm{div}u_{\neq}=0,
$$

and

$$
\langle u_{\neq}\cdot\nabla\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega\rangle=0.
$$

We rewrite $I_{22}$ as

$$
\begin{aligned}
I_{22}=&\langle\mathcal{M}_t^b(u_{\neq}^1\partial_x\Omega)-u_{\neq}\cdot\nabla\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega\rangle\\
=&\langle\mathcal{M}_t^b(u_{\neq}^1\partial_x\Omega)-u_{\neq}^1\partial_x\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega\rangle-\langle u_{\neq}^2\partial_y\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega\rangle\\
=&\langle\mathcal{M}_t^b(u_{\neq}^1\partial_x\Omega)-u_{\neq}^1\partial_x\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega_{\neq}\rangle+\langle\mathcal{M}_t^b(u_{\neq}^1\partial_x\Omega)-u_{\neq}^1\partial_x\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega_0\rangle\\
&-\langle u_{\neq}^2\partial_y\mathcal{M}_t^b\Omega,\mathcal{M}_t^b\Omega\rangle\\
:=&K_1+K_2+K_3.
\end{aligned}
$$

For the term $K_2$, by Plancherel's theorem and Young's inequality, we have

$$\begin{aligned}|K_2| &= \Big|\sum_{l\neq 0}\int_{R^2} \mathcal{M}_t^b(0,\xi)\widehat{u^1_{\neq}}(l,\eta)i(-l)\widehat{\Omega_{\neq}}(-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_0}}(0,\xi) \\ &\qquad - \widehat{u^1_{\neq}}(l,\eta)\mathcal{M}_t^b(-l,\xi-\eta)i(-l)\widehat{\Omega_{\neq}}(-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_0}}(0,\xi)\mathrm{d}\xi\mathrm{d}\eta\Big| \\ &= \Big|\sum_{l\neq 0}\int_{R^2}\Big(\mathcal{M}_t^b(0,\xi)-\mathcal{M}_t^b(-l,\xi-\eta)\Big)i(-l)\widehat{u^1_{\neq}}(l,\eta) \\ &\qquad \widehat{\Omega_{\neq}}(-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_0}}(0,\xi)\mathrm{d}\xi\mathrm{d}\eta\Big| \\ &\lesssim \sum_{l\neq 0}\int_{R^2}\Big(\mathcal{M}_t^b(l,\eta)+\mathcal{M}_t^b(-l,\xi-\eta)\Big)i\frac{|l||\eta|}{l^2+\eta^2}\widehat{\Omega^1_{\neq}}(l,\eta) \\ &\qquad \widehat{\Omega_{\neq}}(-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_0}}(0,\xi)\mathrm{d}\xi\mathrm{d}\eta \\ &\lesssim \|\Lambda_t^b\Omega\|_{L^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L^2}^2.\end{aligned}$$

For the term $K_3$, we have

$$\begin{aligned}|K_3| &\lesssim \|u^2_{\neq}\|_{L^\infty}\|\nabla_y\mathcal{M}_t^b\Omega\|_{L^2}\|\mathcal{M}_t^b\Omega\|_{L^2} \\ &\lesssim \|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L^2}\|\nabla_y\Lambda_t^b\Omega\|_{L^2}\|\Lambda_t^b\Omega\|_{L^2}.\end{aligned}$$

For the term $K_1$, we apply same method as $I_{21}$, which is

$$\begin{aligned}K_1 &= \sum_{k,l}\int_{R^2}\mathcal{M}_t^b(k,\xi)\widehat{u^1_{\neq}}(l,\eta)i(k-l)\widehat{\Omega_{\neq}}(k-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi) \\ &\qquad - \widehat{u^1_{\neq}}(l,\eta)\mathcal{M}_t^b(k-l,\xi-\eta)i(k-l)\widehat{\Omega_{\neq}}(k-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta \\ &= -\sum_{k,l}\int_{R^2}\Big(\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k-l,\xi-\eta)\Big)\frac{(k-l)\eta}{l^2+\eta^2}\widehat{\Omega_{\neq}}(l,\eta) \\ &\qquad \widehat{\Omega_{\neq}}(k-l,\xi-\eta)\overline{\mathcal{M}_t^b\widehat{\Omega_{\neq}}}(k,\xi)\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{3.11}$$

Computing of operator $\mathcal{M}_t^b$ yields that

$$\begin{aligned}&\partial_k\mathcal{M}_t^b(k,\xi)\lesssim(\frac{1}{k}+\frac{|\xi|}{k^2})\Lambda_t^b(k,\xi), \quad k>0, \\ &\partial_\xi\mathcal{M}_t^b(k,\xi)\lesssim(\nu^{\frac{1}{3}}|k|^{-\frac{1}{3}}+\frac{1}{|k|})\Lambda_t^b(k,\xi).\end{aligned}$$

According to the estimate of $\partial_k\mathcal{M}_t^b$, we divide the different range of $k$ and $(k-l)$ as follows. Define

$$\begin{aligned}&D_1=\{k>0,k-l>0\}, \qquad D_2=\{k<0,k-l<0\}, \\ &D_3=\{k>0,k-l<0\}, \qquad D_4=\{k<0,k-l>0\}.\end{aligned}$$

In region $D_1$, we apply the mean value formula in dimension two to obtain

$$|\mathcal{M}_t^b(k,\xi)-\mathcal{M}_t^b(k-l,\xi-\eta)| \le \Big|\int_0^1\partial_k\mathcal{M}_t^b(k-sl,\xi-s\eta)l\mathrm{d}s\Big| + \Big|\int_0^1\partial_\xi\mathcal{M}_t^b(k-sl,\xi-s\eta)\eta\mathrm{d}s\Big|$$

$$\lesssim \int_0^1 \left( \frac{\nu^{\frac{1}{3}}|\eta|}{(k-sl)^{\frac{1}{3}}} + \frac{|\eta|+|l|}{k-sl} + \frac{|\xi-s\eta||l|}{(k-sl)^2} \right) \Lambda_t^b(k-sl,\xi-s\eta)\mathrm{d}s$$

$$\lesssim \left( \frac{\nu^{\frac{1}{3}}|\eta|}{\min(k-l,k)^{\frac{1}{3}}} + \frac{|\eta|+|l|}{\min(k-l,k)} + \frac{(|\xi|+|\xi-\eta|)|l|}{k(k-l)} \right) \times \left( \Lambda_t^b(k-l,\xi-\eta) + \Lambda_t^b(l,\eta) \right),$$

where $s\in(0,1)$, divide $D_1$ into $D_{11}$ and $D_{12}$ to compare $k, k-l$ as

$$D_{11} = \{k>0, l>0\}, \qquad D_{12} = \{k>0, l<0\}.$$

Then, it follows from (3.11),

$$\begin{aligned}
K_1 = & \sum_{(k,l)\in D_1} \int_{R^2} \left( \mathcal{M}_t^b(k,\xi) - \mathcal{M}_t^b(k-l,\xi-\eta) \right) \frac{(k-l)\eta}{l^2+\eta^2} \widehat{\Omega_{\neq}}(l,\eta) \\
& \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \\
& + \sum_{(k,l)\in D_2} \cdots + \sum_{(k,l)\in D_3} \cdots + \sum_{(k,l)\in D_4} \cdots \\
= & \sum_{(k,l)\in D_{11}} \int_{R^2} \left( \mathcal{M}_t^b(k,\xi) - \mathcal{M}_t^b(k-l,\xi-\eta) \right) \frac{(k-l)\eta}{l^2+\eta^2} \widehat{\Omega_{\neq}}(l,\eta) \\
& \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \\
& + \sum_{(k,l)\in D_{12}} \cdots + \sum_{(k,l)\in D_2} \cdots + \sum_{(k,l)\in D_3} + \sum_{(k,l)\in D_4} \cdots \\
:= & J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{3.12}$$

Putting the estimates of $\mathcal{M}_t^b$ into the term $J_1$, we get

$$\begin{aligned}
|J_1| \lesssim & \Bigg| \sum_{(k,l)\in D_{11}} \int_{R^2} \left( \frac{\nu^{\frac{1}{3}}|\eta|}{(k-l)^{\frac{1}{3}}} + \frac{|\eta|+|l|}{k-l} + \frac{(|\xi|+|\xi-\eta|)|l|}{k(k-l)} \right) \Big( \Lambda_t^b(k-l,\xi-\eta) \\
& + \Lambda_t^b(l,\eta) \Big) \times \frac{|\eta|(k-l)}{l^2+\eta^2} \widehat{\Omega_{\neq}}(l,\eta) \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg| \\
\lesssim & \Bigg| \sum_{(k,l)\in D_{11}} \int_{R^2} \left( \nu^{\frac{1}{3}}(k-l)^{\frac{2}{3}} + 1 + \frac{|\xi|+|\xi-\eta|}{(l^2+\eta^2)^{\frac{1}{2}}} \right) \Big( \Lambda_t^b(k-l,\xi-\eta) \\
& + \Lambda_t^b(l,\eta) \Big) \widehat{\Omega_{\neq}}(l,\eta) \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg|.
\end{aligned}$$

By Young's inequality and Lemma 2.1, for $b>1$, we get

$$|J_1| \lesssim \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2 + \|\Lambda_t^b \Omega\|_{L^2} \|(-\Delta)^{-\frac{1}{2}} \Lambda_t^b \Omega_{\neq}\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2}.$$

For the term $J_2$, it holds that

$$\begin{aligned}
|J_2| \lesssim & \Bigg| \sum_{(k,l)\in D_{12}} \int_{R^2} \left( \frac{\nu^{\frac{1}{3}}|\eta|}{k^{\frac{1}{3}}} + \frac{|\eta|+|l|}{k} + \frac{(|\xi|+|\xi-\eta|)|l|}{k(k-l)} \right) \Big( \Lambda_t^b(k-l,\xi-\eta) \\
& + \Lambda_t^b(l,\eta) \Big) \times \frac{|\eta|(k-l)}{l^2+\eta^2} \widehat{\Omega_{\neq}}(l,\eta) \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg|
\end{aligned}$$

$$\lesssim \Bigg| \sum_{(k,l)\in D_{12}} \int_{R^2} \left( \frac{\nu^{\frac{1}{3}}(k-l)}{k^{\frac{1}{3}}} + \frac{k-l}{k} + |\xi| + |\xi-\eta| \right) \Bigg( \Lambda_t^b(k-l,\xi-\eta) + \Lambda_t^b(l,\eta) \Bigg) \widehat{\Omega_{\neq}}(l,\eta) \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg|.$$

When $k \geq 1$ and $l < 0$, exists

$$\frac{k-l}{k^{\frac{1}{3}}} \leq \min\{(k-l)^{\frac{1}{3}} k^{\frac{1}{3}} + (k-l)^{\frac{1}{3}} |l|^{\frac{2}{3}}, 2(k-l)^{\frac{2}{3}} |l|^{\frac{1}{3}}\},$$
$$\frac{k-l}{k} \leq 2\min\{(k-l)^{\frac{1}{3}} |l|^{\frac{2}{3}}, (k-l)^{\frac{2}{3}} |l|^{\frac{1}{3}}\}.$$

Combining inequalities above with Young's inequality, for $b > \frac{4}{3}$, we get

$$\begin{aligned} |J_2| \lesssim & \Bigg| \sum_{(k,l)\in D_{12}} \int_{R^2} \left( \nu^{\frac{1}{3}} (k-l)^{\frac{1}{3}} k^{\frac{1}{3}} + |\xi| + |\xi-\eta| \right) \left( \Lambda_t^b(k-l,\xi-\eta) + \Lambda_t^b(l,\eta) \right) \\ & + \left( (k-l)^{\frac{1}{3}} |l|^{\frac{2}{3}} \Lambda_t^b(k-l,\xi-\eta) + (k-l)^{\frac{2}{3}} |l|^{\frac{1}{3}} \Lambda_t^b(l,\eta) \right) \widehat{\Omega_{\neq}}(l,\eta) \\ & \times \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg| \\ \lesssim & \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2 + \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2}. \end{aligned}$$

The term $J_3$ has same estimates with the terms $J_1$ and $J_2$.

Next, we observe that it always holds $|k-l| < |l|$ in the domains $D_3$ and $D_4$. Therefore, $J_4$ and $J_5$ also have same estimates.

$$\begin{aligned} |J_4| \lesssim & \Bigg| \sum_{(k,l)\in D_3} \int_{R^2} \left( \Lambda_t^b(k-l,\xi-\eta) + \Lambda_t^b(l,\eta) \right) \frac{|\eta|(k-l)}{l^2+\eta^2} \widehat{\Omega_{\neq}}(l,\eta) \\ & \widehat{\Omega_{\neq}}(k-l,\xi-\eta) \overline{\mathcal{M}_t^b \widehat{\Omega_{\neq}}}(k,\xi) \mathrm{d}\xi \mathrm{d}\eta \Bigg| \\ \lesssim & \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2. \end{aligned}$$

Combining all estimates of terms $J_1 - J_5$, we get

$$\begin{aligned} |K_1| \lesssim & \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2 + \|\Lambda_t^b \Omega\|_{L^2} \|(-\Delta)^{-\frac{1}{2}} \Lambda_t^b \Omega_{\neq}\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2} \\ & + \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2}. \end{aligned}$$

Collecting estimates of $K_1, K_2, K_3$, for $b > 1$, we obtain

$$\begin{aligned} |I_{22}| \lesssim & \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2 + \|\Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2} \|(-\Delta)^{-\frac{1}{2}} \Lambda_t^b \Omega_{\neq}\|_{L^2} \\ & + \|\Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}. \end{aligned}$$

Combining the estimates of terms $I_{21}, I_{22}$ and $I_{23}$, we get

$$\begin{aligned} |I_2| \lesssim & \|\Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}^2 + \|\Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2} \|(-\Delta)^{-\frac{1}{2}} \Lambda_t^b \Omega_{\neq}\|_{L^2} \\ & + \|\Lambda_t^b \Omega\|_{L^2} \|\nabla_y \Lambda_t^b \Omega\|_{L^2} \||D_x|^{\frac{1}{3}} \Lambda_t^b \Omega\|_{L^2}. \end{aligned}$$

Putting all estimates of terms $I_1-I_6$ into (3.2), (3.3) and (3.4), respectively, then integrating with respect to time, we get

$$\begin{aligned}&\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\\&\leq 2\|\Lambda_0^b\Omega^0\|_{L_x^2}^2+C_1\nu\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}\\&\quad+C_1\nu\|\nabla\Lambda_t^bw\|_{L_t^2L_x^2}\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2\\&\quad+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}\\&\quad+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2},\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\|\Lambda_t^bw\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^bw\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^bw_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\leq 2\|\Lambda_0^bw^0\|_{L_x^2}^2+C_1\nu\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}\\&\quad+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\quad+C_2\|\Lambda_t^bw\|_{L_t^\infty L_x^2}\|\nabla\Lambda_t^bw\|_{L_t^2L_x^2}\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}&\||D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^\infty L_x^2}^2+\nu\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^bw_{\neq}\|_{L_t^2L_x^2}^2+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\leq 2\||D_x|^{\frac{3}{4}}\Lambda_0^bw^0\|_{L_x^2}^2+C_1\nu\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}\\&\quad+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}\\&\quad+C_2\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2.\end{aligned}\tag{3.15}$$

It follows from the standard bootstrap procedure to get a global small solution. More precisely, assume that $\|\Omega^0\|_{H^b}\leq\epsilon\nu^\alpha$ with $b>\frac{4}{3}$, $\|w^0\|_{H^b}\leq\epsilon\nu^\beta$ and $\||D_x|^{\frac{4}{3}}w^0\|_{H^b}\leq\epsilon\nu^\delta$ with $b>1$. The solution $(\omega,w)$ of system (1.6)-(1.8) satisfies that

$$\begin{aligned}&\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\alpha},\end{aligned}\tag{3.16}$$

$$\begin{aligned}&\|\Lambda_t^bw\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^bw\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^bw_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^bw\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\beta}\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}&\||D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^\infty L_x^2}^2+\nu\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\\&\quad+\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^bw_{\neq}\|_{L_t^2L_x^2}^2+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^bw\|_{L_t^2L_x^2}^2\leq C\epsilon^2\nu^{2\delta}.\end{aligned}\tag{3.18}$$

Based on the assumptions above, we get the estimates from inequality (3.13), (3.14) and (3.15) that

$$\begin{aligned}&\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac{1}{4}\nu^{\frac13}\||D_x|^{\frac13}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\|(-\Delta)^{-\frac12}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\\ \le& C_1\epsilon^2(\nu^{2\alpha}+\nu\nu^{\alpha-\frac16}\nu^{\delta-\frac16}+\nu\nu^{\alpha-\frac12}\nu^{\beta-\frac12})\\ &+C_2\epsilon^3(\nu^\alpha\nu^{2(\alpha-\frac16)}+\nu^\alpha\nu^{\alpha-\frac12}\nu^{\alpha-\frac16}+\nu^\alpha\nu^{\alpha-\frac12}\nu^\alpha),\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\|\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\nu\|\nabla\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac14\nu^{\frac13}\||D_x|^{\frac13}\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ &\quad+\|(-\Delta)^{-\frac12}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ \le& 2C_1\epsilon^2(\nu^{2\beta}+\nu\nu^{\alpha-\frac16}\nu^{\beta-\frac16})+C_2\epsilon^3(\nu^\alpha\nu^{2(\beta-\frac16)}+\nu^\beta\nu^{\beta-\frac12}\nu^\alpha)\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\||D_x|^{\frac43}\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\nu\|\nabla|D_x|^{\frac43}\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac14\nu^{\frac13}\||D_x|^{\frac53}\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ &\quad+\|(-\Delta)^{-\frac12}|D_x|^{\frac43}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\nu\||D_x|^{\frac43}\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ \le& 2C_1\epsilon^2(\nu^{2\delta}+\nu\nu^{\alpha-\frac16}\nu^{\delta-\frac12})+C_2\epsilon^3(\nu^\alpha\nu^{\delta-\frac12}\nu^{\delta-\frac16}+\nu^\alpha\nu^{2(\delta-\frac16)}).\end{aligned}\tag{3.21}$$

Thus, to use the standard bootstrap method, we choose

$$16C_1\le C,\quad \epsilon\le\frac{C^2}{64C_2},$$

and we have that, when $\alpha\ge\frac23$, $-\frac23\le\alpha-\beta\le0$, and $-\frac13\le\alpha-\delta\le\frac23$, the constant $C$ in (3.16), (3.17) and (3.18) can be replaced by $\frac{C}{2}$, which implies that there exists the global small solution. The proof of Theorem 1.1 is complete. ∎

### 3.2. Proof of Theorem 1.2.

*Proof.* Due to similar dissipation term, we have similar estimates with respect to terms $I_1-I_6$ in the proof of Theorem 1.1. Assume that the solution $(\Omega,w)$ satisfies that

$$\begin{aligned}&\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac14\nu^{\frac13}\||D_x|^{\frac13}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2\\ &\quad+\|(-\Delta)^{-\frac12}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2\le C\epsilon^2\nu^{2\alpha},\end{aligned}\tag{3.22}$$

$$\begin{aligned}&\|\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\gamma\|\nabla\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac14\gamma^{\frac13}\||D_x|^{\frac13}\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ &\quad+\|(-\Delta)^{-\frac12}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\nu\|\Lambda_t^b w\|_{L_t^2L_x^2}^2\le C\epsilon^2\gamma^{2\beta}\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}&\||D_x|^{\frac43}\Lambda_t^b w\|_{L_t^\infty L_x^2}^2+\gamma\|\nabla|D_x|^{\frac43}\Lambda_t^b w\|_{L_t^2L_x^2}^2+\frac14\gamma^{\frac13}\||D_x|^{\frac53}\Lambda_t^b w\|_{L_t^2L_x^2}^2\\ &\quad+\|(-\Delta)^{-\frac12}|D_x|^{\frac43}\Lambda_t^b w_{\neq}\|_{L_t^2L_x^2}^2+\nu\||D_x|^{\frac43}\Lambda_t^b w\|_{L_t^2L_x^2}^2\le C\epsilon^2\gamma^{2\delta}.\end{aligned}\tag{3.24}$$

Then we can obtain that

$$\|\Lambda_t^b\Omega\|_{L_t^\infty L_x^2}^2+\nu\|\nabla_y\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\frac14\nu^{\frac13}\||D_x|^{\frac13}\Lambda_t^b\Omega\|_{L_t^2L_x^2}^2+\|(-\Delta)^{-\frac12}\Lambda_t^b\Omega_{\neq}\|_{L_t^2L_x^2}^2$$

$$\leq C_1\epsilon^2(\nu^{2\alpha}+\nu\nu^{\alpha-\frac{1}{6}}\gamma^{\delta-\frac{1}{6}}+\nu\nu^{\alpha-\frac{1}{2}}\gamma^{\beta-\frac{1}{2}}) \\ +C_2\epsilon^3(\nu^{\alpha}\nu^{2(\alpha-\frac{1}{6})}+\nu^{\alpha}\nu^{\alpha-\frac{1}{2}}\nu^{\alpha-\frac{1}{6}}+\nu^{\alpha}\nu^{\alpha-\frac{1}{2}}\nu^{\alpha}), \tag{3.25}$$

$$\|\Lambda_t^b w\|^2_{L^\infty_t L^2_x}+\gamma\|\nabla\Lambda_t^b w\|^2_{L^2_t L^2_x}+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{1}{3}}\Lambda_t^b w\|^2_{L^2_t L^2_x} \\ +\|(-\Delta)^{-\frac{1}{2}}\Lambda_t^b w_{\neq}\|^2_{L^2_t L^2_x}+\nu\|\Lambda_t^b w\|^2_{L^2_t L^2_x} \\ \leq 2C_1\epsilon^2(\gamma^{2\beta}+\nu\nu^{\alpha-\frac{1}{6}}\gamma^{\beta-\frac{1}{6}})+C_2\epsilon^3(\nu^{\alpha}\gamma^{2(\beta-\frac{1}{6})}+\gamma^{\beta}\gamma^{\beta-\frac{1}{2}}\nu^{\alpha}) \tag{3.26}$$

and

$$\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L^\infty_t L^2_x}+\gamma\|\nabla|D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L^2_t L^2_x}+\frac{1}{4}\gamma^{\frac{1}{3}}\||D_x|^{\frac{5}{3}}\Lambda_t^b w\|^2_{L^2_t L^2_x} \\ +\|(-\Delta)^{-\frac{1}{2}}|D_x|^{\frac{4}{3}}\Lambda_t^b w_{\neq}\|^2_{L^2_t L^2_x}+\nu\||D_x|^{\frac{4}{3}}\Lambda_t^b w\|^2_{L^2_t L^2_x} \\ \leq 2C_1\epsilon^2(\gamma^{2\delta}+\nu\nu^{\alpha-\frac{1}{6}}\gamma^{\delta-\frac{1}{2}})+C_2\epsilon^3(\nu^{\alpha}\gamma^{\delta-\frac{1}{2}}\gamma^{\delta-\frac{1}{6}}+\nu^{\alpha}\gamma^{2(\delta-\frac{1}{6})}). \tag{3.27}$$

To use the bootstrap method, we choose $\alpha$, $\beta$ and $\gamma$ such that

$$\alpha\geq\frac{2}{3}, \qquad \nu^{\alpha}\leq\gamma^{\frac{2}{3}}, \\ \gamma^{\beta-\frac{1}{2}}\leq\nu^{\alpha-\frac{1}{2}}, \qquad \nu^{\alpha+\frac{5}{6}}\leq\gamma^{\beta+\frac{1}{6}}$$

and

$$\gamma^{\delta-\frac{1}{6}}\leq\nu^{\alpha-\frac{5}{6}}, \qquad \nu^{\alpha+\frac{5}{6}}\leq\gamma^{\delta+\frac{1}{2}},$$

and take

$$16C_1\leq C, \quad \epsilon\leq\frac{C^2}{64C_2},$$

Then the constant $C$ in (3.22), (3.23) and (3.24) can be replaced by $\frac{C}{2}$, which implies that there exists the global small solution. The proof of Theorem 1.2 is finished. ∎

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