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GLOBAL SMOOTH SOLUTIONS TO THE TWO-DIMENSIONAL AXISYMMETRIC ZELDOVICH-VON NEUMANN-DÖRING COMBUSTION EQUATIONS WITH SWIRL*

HONGHUA CHEN[†], GENG LAI[‡], AND WANCHENG SHENG[§]

Abstract. This paper studies the two-dimensional (2D) Zeldovich-von Neumann-Döring (ZND) combustion equations with initial data, which are a combination of an axisymmetric flow in a ring and vacuum in the remaining domain. Existence of a global-in-time smooth solution to the initial value problem is obtained by the method of characteristic decomposition, provided that the initial data satisfy some sufficient conditions. The large-time behavior of the solution is also studied. As a result, at any time, the ring continues to expand until the gas burns out in infinite time for the system. The solution describes a phenomenon of the expansion of 2D reacting flows with swirl in vacuum or a phenomenon of "fire whirl".

Keywords. ZND combustion model; axial symmetric flow; characteristic decomposition; vacuum.

AMS subject classifications. 35L03; 35L65; 35L60; 35L67; 80A32.

1. Introduction

The Zeldovich-von Neumann-Döring (ZND) combustion model is an important and well-studied model to describe the propagation of combustion waves in one-step exothermic chemical reaction. This combustion model is formed by the Euler equations of gas dynamics and a combustion reaction equation for combustible gases. The two-dimensional (2D) inviscid ZND combustion equations can be written in the form

$$\begin{cases} \rho_t+(\rho u_1)_{x_1}+(\rho u_2)_{x_2}=0, \\ (\rho u_1)_t+(\rho u_1^2+p)_{x_1}+(\rho u_1u_2)_{x_2}=0, \\ (\rho u_2)_t+(\rho u_1u_2)_{x_1}+(\rho u_2^2+p)_{x_2}=0, \\ (\rho E)_t+(\rho u_1E+u_1p)_{x_1}+(\rho u_2E+u_2p)_{x_2}=0, \\ (\rho z)_t+(\rho u_1z)_{x_1}+(\rho u_2z)_{x_2}=-\phi\rho z, \end{cases} \tag{1.1}$$

where (u_1,u_2) is the velocity, ρ is the density, p is the pressure, $E=\frac{1}{2}(u_1^2+u_2^2)+\epsilon+zb_0$ is the total energy, ϵ is the internal energy, z is the fraction of unburnt gas in the mixture, b_0 is the binding energy per unit mass of unburnt gas, and ϕ is an ignition function. For convenience, we assume $b_0=1$. We assume that the ignition function ϕ has the Arrhenius kinetics mechanism

$$\phi=\begin{cases} ke^{-\frac{1}{T}}, & T>0, \\ 0, & T\le 0, \end{cases} \tag{1.2}$$

where T is the temperature and k is a positive constant. We refer the reader to [8,37,41] for more details about the inviscid ZND combustion model.

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[†]Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China; and School of Mathematics and Statistics Science, Ludong University, Yantai, 264025, P.R. China (huazhiye@shu.edu.cn).

[‡]Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China; Newtouch Center for Mathematics, Shanghai University, Shanghai, 200444, P.R. China (laigeng@shu.edu.cn).

[§]Corresponding author. Department of Mathematics, Shanghai University, Shanghai, 200444, P.R. China; Newtouch Center for Mathematics, Shanghai University, Shanghai, 200444, P.R. China (mathwcsheng@shu.edu.cn).

For polytropic gases,

$$\epsilon = \frac{e^s \rho^{\gamma-1}}{\gamma-1}, \tag{1.3}$$

where s is the specific entropy and γ is a positive constant. Then by the fundamental relation $\mathrm{d}\epsilon = T\mathrm{d}s - p\mathrm{d}\tau$ we have

$$p = e^s \rho^\gamma \quad \text{and} \quad T = \epsilon = \frac{c^2}{\gamma(\gamma-1)}, \tag{1.4}$$

where $c = \sqrt{\gamma e^s \rho^{\gamma-1}}$ represents the speed of sound. The ignition function ϕ can also be seen as a function of c, i.e., $\phi = \phi(c)$. Moreover, we have

$$\lim_{c\to 0} \frac{\phi}{c^n} = 0 \quad \text{and} \quad \lim_{c\to 0} \frac{\phi'(c)}{c^n} = 0 \tag{1.5}$$

for any fixed n.

There is a lot of literature on the ZND combustion model. The local and global existence of solutions to the 1D Cauchy problem for the ZND combustion model for initial data with small bounded variations were obtained in [11,43] and [3], respectively. Kuang and Zhao [18] obtained the global existence of weak solutions to a 1D piston problem for the ZND combustion model. Zumbrun [44, 45] studied the stability of detonation waves for the ZND combustion model. Costanzino et al. [7] obtained finite time existence of multi-D unsteady detonation waves for both ZND and CJ combustion equations. Chen et al. [2,4] constructed global entropy solutions to supersonic reacting flows past Lipschitz bending walls and supersonic reacting flows around sharp corners for 2D steady ZND combustion equations. There is also a lot of literature on the Cauchy problem for a scalar ZND combustion model proposed independently by Fickett [12] and Majda [36]; see, e.g., [1,23,29,30,42].

In this paper, we are concerned with axial symmetric flows to the inviscid ZND combustion Equations (1.1). That is, the flow has the property

$$\begin{gathered}\rho(x,\theta,t) = \rho(x,t), \quad s(x,\theta,t) = s(x,t), \quad z(x,\theta,t) = z(x,t), \\ \begin{pmatrix} u_1(x,\theta,t) \\ u_2(x,\theta,t) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}\end{gathered} \tag{1.6}$$

for all $t > 0$, $\theta \in [0, 2\pi)$, and $x > 0$, where (x,θ) are the polar coordinates of the (x_1,x_2)-plane. With this symmetry, system (1.1) can be reduced to

$$\begin{cases} \rho_t + (\rho u)_x + \dfrac{\rho u}{x} = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + \dfrac{\rho(u^2 - v^2)}{x} = 0, \\ (\rho v)_t + (\rho u v)_x + \dfrac{2\rho u v}{x} = 0, \\ (\rho E)_t + (\rho u E + u p)_x + \dfrac{\rho u E}{x} + \dfrac{u p}{x} = 0, \\ (\rho z)_t + (\rho u z)_x + \dfrac{\rho u z}{x} = -\phi \rho z. \end{cases} \tag{1.7}$$

Notice now that u and v in (1.7) represent the radial and pure rotational velocities in the flow, respectively.

We consider (1.7) with initial data

$$(u,v,c,s,z)(x,0)=\begin{cases}(u_0,v_0,c_0,s_0,z_0)(x), & a\le x\le b,\\ \text{vacuum}, & \text{otherwise},\end{cases} \tag{1.8}$$

where $0<a<b$, $(u_0,v_0,c_0,s_0)(x)\in C^1[a,b]$, $c_0(a)=c_0(b)=0$, $z_0(x)\equiv 1$, and

$$c_0(x)>0 \quad \text{and} \quad s_0(x)>0 \quad \text{for} \quad x\in[a,b].$$

The problem (1.7, 1.8) describes the expansion of a 2D axisymmetric reacting flow in vacuum or an interesting phenomena like "fire whirl". The aim of this paper is to find some sufficient conditions on the initial data to ensure that the problem (1.7, 1.8) admits a global-in-time smooth solution.

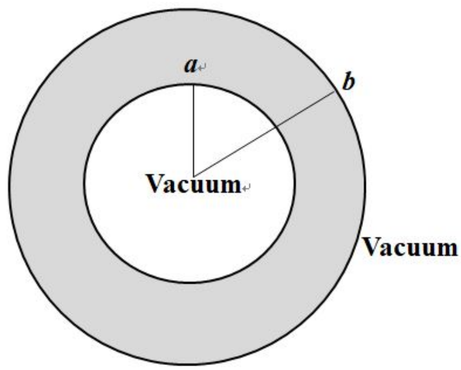

FIG. 1.1. *Initial data in the (x_1,x_2)-plane.*

We define the following constants

$$\begin{cases}\delta_1:=\sup\limits_{x\in(a,b)}\left\{\frac{{v_0}^2(x)}{c_0(x)}\right\}, \quad N:=\sup\limits_{x\in(a,b)}\left\{\frac{|{s_0}'(x)|}{c_0^{\kappa}(x)}\right\}, \quad s_*:=2\sup\limits_{x\in(a,b)} s_0(x),\\ c_M:=2\sup\limits_{x\in(a,b)} c_0(x), \quad u_a:=u_0(a), \quad u_b:=u_0(b), \quad \kappa:=\frac{2}{\gamma-1}.\end{cases} \tag{1.9}$$

The main result of the paper is stated as follows.

THEOREM 1.1. *Assume $5/3<\gamma<3$ and N is finite. Assume further that there exists a constant $\mathcal{C}\in(1,\kappa)$ such that the initial data (1.8) satisfy*

$$\kappa|{c_0}'(x)|+\frac{u_0(x)}{x} < {u_0}'(x) < -\kappa|{c_0}'(x)|+(2\mathcal{C}-1)\frac{u_0(x)}{x} \quad \textit{for } x\in[a,b]. \tag{1.10}$$

Then when δ_1 and c_M are sufficiently small the problem (1.7, 1.8) admits a classical solution in

$$\Omega=\{(x,t)\,|\,a+u_at<x<b+u_bt,\ 0<t<+\infty\}.$$

Moreover, the solution satisfies $\rho\in C(\overline{\Omega})$, $\rho>0$ in Ω, and $\rho=0$ on $C_v^a\cup C_v^b$, where $C_v^a:=\{(x,t)\,|\,x=a+u_at,t>0\}$ and $C_v^b:=\{(x,t)\,|\,x=a+u_bt,t>0\}$ are two vacuum boundaries.

The main difficulty for the global existence is to establish a uniform a priori estimate for the C^1-norm of the solution. In this paper, we use the method of characteristic decomposition to establish the estimate. This method was first proposed by Li, Zhang, and Zheng [25] in investigating simple waves of the 2D compressible Euler equations. Recently, this method was extensively used to establish the global existence of smooth solutions for quasilinear hyperbolic systems with two variables, see, e.g., [5, 6, 15, 19, 20, 24–28]. Motivated by recent works of Lai et al. [21, 22], we derive a group of characteristic decompositions for (1.7); see (2.8), (2.13) and (2.14). These characteristic decompositions can be seen as a system of ODEs for the derivatives of the unknown functions. Using these decompositions we establish a uniform a priori C^1-norm estimate for the solution.

In the paper we also obtain that $|\nabla c|$ is bounded in the domain Ω; see (3.35) in Remark 3.1. This implies that $\tau p_x = 0$ on the vacuum boundaries, and hence there is no force to accelerate the vacuum boundaries. So the vacuum boundaries C_v^a and C_v^b are not physical vacuum boundaries introduced in [33, 34]. Recently, there is a lot of literature on compressible flows with a physical vacuum boundary. We refer the reader to [9, 10, 14, 16, 17, 27, 35, 39, 40] for this direction.

There are some other related works about global-in-time solutions to gas expansion in vacuum problem for the compressible Euler equations. Serre [38] established the global existence of classical solutions for the compressible Euler equations with compactly supported density in multi-dimensions, provided that the initial velocity is close to a linear field and the initial density is sufficiently small. Subsequently, Grassin [13] obtained the global existence of smooth solutions in multi-dimensions, provided the initial velocity forces particles to spread out and the initial density is sufficiently small in some norm. Li and Zheng [27] obtained the global existence of classical solutions to the expansion of a wedge of gas in vacuum.

The rest of the paper is organized as follows. In Section 2, we derive the characteristic decompositions for the system (1.7). The global existence of a classical solution to the problem (1.7, 1.8) is obtained in Section 3.

2. Characteristic equations for the axisymmetric ZND combustion equations

2.1. Characteristic equations. For smooth flow, system (1.7) can be changed into the form

$$\begin{cases} \rho_t + (\rho u)_x + \dfrac{\rho u}{x} = 0, \\ u_t + u u_x + \dfrac{p_x}{\rho} - \dfrac{v^2}{x} = 0, \\ v_t + u v_x + \dfrac{uv}{x} = 0, \\ s_t + u s_x = \dfrac{\phi z}{T}, \\ z_t + u z_x = -\phi z. \end{cases} \tag{2.1}$$

The eigenvalues of system (2.1) are

$$\lambda_+ = u + c, \quad \lambda_0 = u \text{ (triple)}, \quad \lambda_- = u - c.$$

The C_+, C_0, and C_- characteristic curves are defined as the integral curves of $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_+$, $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_0$, and $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_-$, respectively. The left eigenvectors corresponding to $\lambda_\pm$ are

$l_\pm = (c, \pm\rho, 0, \sqrt{\frac{e^s}{\gamma}}\rho^{\frac{\gamma+1}{2}}, 0)$. Multiplying (2.1) on the left by $l_\pm$ we get the characteristic equations

$$c\partial_\pm\rho \pm \rho\partial_\pm u \pm \rho^\gamma e^s s_x = -\frac{\rho c u}{x} \pm \frac{\rho v^2}{x}, \tag{2.2}$$

where

$$\partial_\pm = \partial_t + (u \pm c)\partial_x. \tag{2.3}$$

From $c^2 = \gamma e^s \rho^{\gamma-1}$ we have

$$\partial_\pm \rho = \frac{2c\partial_\pm c - \gamma\rho^{\gamma-1}e^s\partial_\pm s}{\gamma(\gamma-1)e^s\rho^{\gamma-2}}.$$

From the fourth equation of (2.1) we have

$$\partial_\pm s = \frac{\phi z}{T} \pm c s_x.$$

Hence, we get

$$c\partial_\pm\rho = \frac{2\rho\partial_\pm c}{\gamma-1} \mp \frac{\gamma\rho^\gamma e^s}{\gamma-1}s_x - \frac{c\rho\phi z}{T(\gamma-1)}.$$

Inserting this into (2.2), we get

$$\begin{cases} \partial_+ u = -\dfrac{2}{\gamma-1}\partial_+ c + \dfrac{c^2}{\gamma(\gamma-1)}s_x + \dfrac{c\phi z}{T(\gamma-1)} - \dfrac{uc}{x} + \dfrac{v^2}{x}, \\ \partial_- u = \dfrac{2}{\gamma-1}\partial_- c + \dfrac{c^2}{\gamma(\gamma-1)}s_x - \dfrac{c\phi z}{T(\gamma-1)} + \dfrac{uc}{x} + \dfrac{v^2}{x}. \end{cases} \tag{2.4}$$

From (2.3) we have

$$\partial_x = \frac{\partial_+ - \partial_-}{2c} \quad \text{and} \quad \partial_t = \frac{(u-c)\partial_+ - (u+c)\partial_-}{2c}. \tag{2.5}$$

2.2. Characteristic decompositions.

LEMMA 2.1. *For the system (1.7), we have the commutator relation*

$$\partial_+\partial_- - \partial_-\partial_+ = \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x} + \frac{\phi z}{T(\gamma-1)}\right)(\partial_+ - \partial_-), \tag{2.6}$$

where $\mu^2 = \frac{\gamma-1}{\gamma+1}$.

Proof. *Using (2.4) and* $\partial_x = \frac{\partial_+ - \partial_-}{2c}$, *we have*

$$\begin{aligned} \partial_+\partial_- - \partial_-\partial_+ &= (\partial_t + \lambda_+\partial_x)(\partial_t + \lambda_-\partial_x) - (\partial_t + \lambda_-\partial_x)(\partial_t + \lambda_+\partial_x) \\ &= (\partial_+\lambda_- - \partial_-\lambda_+)\partial_x = (\partial_+ u - \partial_+ c - \partial_- u - \partial_- c)\partial_x \\ &= \left(-\frac{1}{2c\mu^2}(\partial_+ c + \partial_- c) - \frac{u}{x} + \frac{\phi z}{T(\gamma-1)}\right)(\partial_+ - \partial_-). \end{aligned} \tag{2.7}$$

We then complete the proof of this lemma. ∎

LEMMA 2.2. *For smooth flow, we have*

$$\begin{cases} \partial_0\left(\frac{s_x}{c^\kappa}\right)=\frac{u}{x}\frac{s_x}{c^\kappa}-\frac{\gamma\phi z s_x}{c^{\frac{2\gamma}{\gamma-1}}}+\frac{2\phi' z c_x}{c^{\frac{\gamma+1}{\gamma-1}}}-\frac{2\gamma(\gamma-1)\phi z c_x}{c^{\frac{3\gamma-1}{\gamma-1}}}+\frac{\gamma(\gamma-1)\phi z_x}{c^{\frac{2\gamma}{\gamma-1}}}, \\ \partial_0\left(\frac{z_x}{c^\kappa}\right)=\frac{u}{x}\frac{z_x}{c^\kappa}-\frac{\gamma\phi z z_x}{c^{\frac{2\gamma}{\gamma-1}}}-\frac{2\phi' z c_x}{c^{\frac{3-\gamma}{\gamma-1}}\gamma(\gamma-1)}-\frac{\phi z_x}{c^\kappa}, \end{cases} \tag{2.8}$$

where $\partial_0=\partial_t+u\partial_x$ *and* $\phi'=\frac{\mathrm{d}\phi}{\mathrm{d}c}$.

Proof. By (2.4) we have

$$\begin{aligned}\partial_0\partial_x s&=(\partial_0\partial_x-\partial_x\partial_0)s+\partial_x\left(\frac{\phi z}{T}\right)=-\partial_x u\partial_x s+\partial_x\left(\frac{\phi z}{T}\right)\\ &=\frac{(\partial_+c+\partial_-c)}{(\gamma-1)c}s_x+\frac{u}{x}s_x-\frac{\phi z}{(\gamma-1)T}s_x+\partial_x\left(\frac{\phi z}{T}\right).\end{aligned} \tag{2.9}$$

Thus, by $\partial_0=\frac{\partial_++\partial_-}{2}$ *we have*

$$\begin{aligned}\partial_0\left(\frac{s_x}{c^\kappa}\right)&=\frac{\partial_0 s_x}{c^\kappa}-c^{-\frac{\gamma+1}{\gamma-1}}\frac{2s_x}{\gamma-1}\partial_0 c\\ &=c^{-\kappa}\left(\frac{\partial_+c+\partial_-c}{(\gamma-1)c}s_x+\frac{u}{x}s_x-\frac{\phi z}{(\gamma-1)T}s_x+\partial_x\left(\frac{\phi z}{T}\right)\right)-c^{-\frac{\gamma+1}{\gamma-1}}\frac{2s_x}{\gamma-1}\partial_0 c\\ &=c^{-\kappa}\frac{u}{x}s_x-c^{-\kappa}\frac{\phi z}{(\gamma-1)T}s_x+c^{-\kappa}\partial_x\left(\frac{\phi z}{T}\right)\\ &=\frac{u}{x}\frac{s_x}{c^\kappa}-\frac{\gamma\phi z s_x}{c^{\frac{2\gamma}{\gamma-1}}}+\frac{2\phi' z c_x}{c^{\frac{\gamma+1}{\gamma-1}}}-\frac{2\gamma(\gamma-1)\phi z c_x}{c^{\frac{3\gamma-1}{\gamma-1}}}+\frac{\gamma(\gamma-1)\phi z_x}{c^{\frac{2\gamma}{\gamma-1}}}.\end{aligned} \tag{2.10}$$

Similarly, we have

$$\begin{aligned}\partial_0\partial_x z&=(\partial_0\partial_x-\partial_x\partial_0)z-\partial_x(\phi z)=-\partial_x u\partial_x z-\partial_x(\phi z)\\ &=\frac{\partial_+c+\partial_-c}{(\gamma-1)c}z_x+\frac{u}{x}z_x-\frac{\gamma\phi z z_x}{c^2}-\phi z_x-\frac{2c\phi' z c_x}{\gamma(\gamma-1)}.\end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}\partial_0\left(\frac{z_x}{c^\kappa}\right)&=c^{-\kappa}\partial_0 z_x-c^{-\frac{\gamma+1}{\gamma-1}}\frac{(\partial_+c+\partial_-c)z_x}{\gamma-1}\\ &=c^{-\kappa}\left(\frac{\partial_+c+\partial_-c}{(\gamma-1)c}z_x+\frac{u}{x}z_x-\frac{\gamma\phi z z_x}{c^2}-\phi z_x-\frac{2c\phi' z c_x}{\gamma(\gamma-1)}\right)-c^{-\frac{\gamma+1}{\gamma-1}}\frac{(\partial_+c+\partial_-c)z_x}{\gamma-1}\\ &=\frac{u}{x}\frac{z_x}{c^\kappa}-\frac{\gamma\phi z z_x}{c^{\frac{2\gamma}{\gamma-1}}}-\frac{2\phi' z c_x}{c^{\frac{3-\gamma}{\gamma-1}}\gamma(\gamma-1)}-\frac{\phi z_x}{c^\kappa}.\end{aligned} \tag{2.12}$$

This completes the proof of the lemma. ∎

PROPOSITION 2.1. *We have the characteristic decompositions*

$$\begin{aligned}&c\partial_-\left(\partial_+c-\frac{(\gamma-1)v^2}{2x}-\frac{c^2}{2\gamma}s_x\right)\\ =&\frac{1}{2\mu^2}(\partial_+c+\partial_-c)\partial_+c+(\frac{3}{2}\partial_+c+\frac{1}{2}\partial_-c)\frac{uc}{x}+\frac{c^2}{2x}(\partial_+c-\partial_-c)+\frac{(\gamma-1)u^2c^2}{x^2}\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\gamma-1)c^2v^2}{2x^2}+\frac{3(\gamma-1)cuv^2}{2x^2}-\frac{c^4}{2\gamma x}s_x-\frac{c^3}{2\gamma}\frac{u}{x}s_x-\frac{\phi z}{2c}\Big((\gamma-1)^2\partial_+c\\
&+(\gamma+1)(\gamma-1)\partial_-c\Big)-\frac{c^2}{2(\gamma-1)}(\partial_+c+\partial_-c)s_x+\frac{\gamma(\gamma-1)\phi z}{4c}(\partial_+c+\partial_-c)\\
&-\frac{\gamma\phi z}{4c}\Big((\gamma+1)\partial_-c+(\gamma+5)\partial_+c\Big)+\frac{(\gamma+1)cz\phi'}{2\gamma}\partial_-c+\frac{(\gamma-1)cz\phi'}{2\gamma}\partial_+c-\frac{(\gamma^2-\gamma)\phi^2z}{2}\\
&+\frac{(1-\gamma)c\phi z_x}{2}+\frac{c\phi z}{2}s_x+\frac{\gamma^2(\gamma-1)\phi^2z^2}{2c^2}-\frac{\gamma(\gamma-1)u\phi z}{x}
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
&c\partial_+\Big(\partial_-c+\frac{(\gamma-1)v^2}{2x}+\frac{c^2}{2\gamma}s_x\Big)\\
=&\frac{1}{2\mu^2}(\partial_+c+\partial_-c)\partial_-c+\Big(\frac{3}{2}\partial_-c+\frac{1}{2}\partial_+c\Big)\frac{uc}{x}+\frac{c^2}{2x}(\partial_+c-\partial_-c)+\frac{(\gamma-1)u^2c^2}{x^2}\\
&-\frac{(\gamma-1)c^2v^2}{2x^2}-\frac{3(\gamma-1)cuv^2}{2x^2}-\frac{c^4}{2\gamma x}s_x+\frac{c^3}{2\gamma}\frac{u}{x}s_x-\frac{\phi z}{2c}\Big((\gamma-1)^2\partial_-c\\
&+(\gamma-1)(\gamma+1)\partial_+c\Big)+\frac{c^2}{2(\gamma-1)}(\partial_+c+\partial_-c)s_x+\frac{\gamma(\gamma-1)\phi z}{4c}(\partial_+c+\partial_-c)\\
&-\frac{\gamma\phi z}{4c}\Big((\gamma+1)\partial_+c+(\gamma+5)\partial_-c\Big)+\frac{(\gamma-1)cz\phi'}{2\gamma}\partial_-c+\frac{(\gamma+1)cz\phi'}{2\gamma}\partial_+c\\
&-\frac{(\gamma^2-\gamma)z\phi^2}{2}+\frac{(\gamma-1)c\phi}{2}z_x-\frac{c\phi z}{2}s_x+\frac{\gamma^2(\gamma-1)\phi^2z^2}{2c^2}-\frac{\gamma(\gamma-1)u\phi z}{x}.
\end{aligned}
\tag{2.14}
$$

Proof. Using the commutator relation (2.6) for the variable c, we have

$$
\partial_+\partial_-c-\partial_-\partial_+c=\Big(-\frac{1}{2c\mu^2}(\partial_+c+\partial_-c)-\frac{u}{x}+\frac{\phi z}{T(\gamma-1)}\Big)(\partial_+c-\partial_-c). \tag{2.15}
$$

Using the commutator relation (2.6) for the variable u, we have

$$
\partial_+\partial_-u-\partial_-\partial_+u=\Big(-\frac{1}{2c\mu^2}(\partial_+c+\partial_-c)-\frac{u}{x}+\frac{\phi z}{T(\gamma-1)}\Big)(\partial_+u-\partial_-u). \tag{2.16}
$$

Inserting (2.4) into (2.16), we get

$$
\begin{aligned}
&c\partial_+\partial_-c+c\partial_-\partial_+c\\
=&\frac{\gamma+1}{2(\gamma-1)}(\partial_+c+\partial_-c)^2+\frac{2cu}{x}(\partial_+c+\partial_-c)+\frac{c^2}{x}(\partial_+c-\partial_-c)-\frac{(\gamma+3)c\phi z}{2T(\gamma-1)}(\partial_+c+\partial_-c)\\
&+\frac{2(\gamma-1)c^2u^2}{x^2}-\frac{(\gamma-1)c^2v^2}{x^2}-\frac{(\gamma-1)c}{2}\partial_+\Big(\frac{c^2s_x}{\gamma(\gamma-1)}\Big)+\frac{(\gamma-1)c}{2}\partial_-\Big(\frac{c^2s_x}{\gamma(\gamma-1)}\Big)\\
&+\frac{(\gamma-1)c}{2}\Big(\partial_+\Big(\frac{c\phi z}{T(\gamma-1)}\Big)+\partial_-\Big(\frac{c\phi z}{T(\gamma-1)}\Big)\Big)-\frac{(\gamma-1)}{2}\partial_+\Big(\frac{v^2}{x}\Big)+\frac{(\gamma-1)}{2}\partial_-\Big(\frac{v^2}{x}\Big)\\
&-\frac{c^4}{\gamma x}s_x+\frac{(c\phi z)^2}{T^2(\gamma-1)}-\frac{2c^2\phi z}{T}\cdot\frac{u}{x}.
\end{aligned}
\tag{2.17}
$$

Combining (2.15) and (2.17), we get

$$
\begin{aligned}
&c\partial_-\left(\partial_+c-\frac{(\gamma-1)v^2}{2x}-\frac{c^2}{2\gamma}s_x\right)\\
=&\frac{\gamma+1}{2(\gamma-1)}(\partial_+c+\partial_-c)\partial_+c+(\frac{3}{2}\partial_+c+\frac{1}{2}\partial_-c)\frac{uc}{x}-\frac{c^4}{2\gamma x}s_x+\frac{c^2}{2x}(\partial_+c-\partial_-c)\\
&-\frac{c\phi z}{2T(\gamma-1)}(\partial_+c-\partial_-c)-\frac{c^2\phi z}{T}\cdot\frac{u}{x}-\frac{(\gamma+3)c\phi z}{4T(\gamma-1)}(\partial_+c+\partial_-c)+\frac{(\gamma-1)u^2c^2}{x^2}\\
&-\frac{(\gamma-1)c^2v^2}{2x^2}-\frac{c}{4}\left(\partial_+\left(\frac{c^2s_x}{\gamma}\right)+\partial_-\left(\frac{c^2s_x}{\gamma}\right)\right)-\frac{(\gamma-1)c}{4}\left(\partial_+\left(\frac{v^2}{x}\right)+\partial_-\left(\frac{v^2}{x}\right)\right)\\
&+\frac{(c\phi z)^2}{2T^2(\gamma-1)}+\frac{c}{4}\left(\partial_+\left(\frac{c\phi z}{T}\right)+\partial_-\left(\frac{c\phi z}{T}\right)\right). \qquad (2.18)
\end{aligned}
$$

By a direct computation and recalling (2.9), we have

$$
\partial_+(\frac{v^2}{x})+\partial_-(\frac{v^2}{x})=\frac{2v}{x}(\partial_+v+\partial_-v)-\frac{2uv^2}{x^2}=-\frac{6uv^2}{x^2} \qquad (2.19)
$$

and

$$
\begin{aligned}
&\partial_+\left(\frac{c^2s_x}{\gamma}\right)+\partial_-\left(\frac{c^2s_x}{\gamma}\right)=\frac{2c}{\gamma}(\partial_+c+\partial_-c)s_x+\frac{2c^2}{\gamma}\partial_0s_x\\
=&\frac{2c}{\gamma}(\partial_+c+\partial_-c)s_x+\frac{2c^2}{\gamma}\left(\frac{(\partial_+c+\partial_-c)s_x}{c(\gamma-1)}+\frac{u}{x}s_x-\frac{\phi z}{(\gamma-1)T}s_x+\partial_x\left(\frac{\phi z}{T}\right)\right)\\
=&\frac{2c}{\gamma-1}(\partial_+c+\partial_-c)s_x-2\phi zs_x+\frac{2c^2us_x}{\gamma x}+2(\gamma-1)\phi z_x\\
&+\left(\frac{2\phi' z}{\gamma}-\frac{2(\gamma-1)\phi z}{c^2}\right)(\partial_+c-\partial_-c). \qquad (2.20)
\end{aligned}
$$

A direct computation yields

$$
\begin{aligned}
&\partial_+\left(\frac{c\phi z}{T}\right)+\partial_-\left(\frac{c\phi z}{T}\right)\\
=&\frac{\gamma(\gamma-1)\phi z}{c^2}(\partial_+c+\partial_-c)+c\left(\partial_+\left(\frac{\phi z}{T}\right)+\partial_-\left(\frac{\phi z}{T}\right)\right)\\
=&\frac{\gamma(\gamma-1)\phi z}{c^2}(\partial_+c+\partial_-c)+c\left(\frac{2\phi\partial_0 z}{T}+\left(\frac{2z\phi' c}{T\gamma(\gamma-1)}-\frac{2\gamma(\gamma-1)\phi z}{c^3}\right)(\partial_+c+\partial_-c)\right)\\
=&-\frac{2\gamma(\gamma-1)\phi^2 z}{c}+\left(2z\phi'-\frac{\gamma(\gamma-1)\phi z}{c^2}\right)(\partial_+c+\partial_-c). \qquad (2.21)
\end{aligned}
$$

Inserting (2.19), (2.20) and (2.21) into (2.18) we get the Equation (2.13). The Equation (2.14) can be proved similarly. This completes the proof of the proposition. □

From (2.13) and (2.14) we have

$$
\begin{aligned}
&\partial_-\left(\frac{\partial_+c}{c}-\frac{(\gamma-1)v^2}{2cx}-\frac{c}{2\gamma}s_x\right)\\
=&\frac{1}{2\mu^2}\left(\frac{\partial_+c}{c}+\frac{\partial_-c}{c}\right)\frac{\partial_+c}{c}+\left(\frac{3}{2}\frac{\partial_+c}{c}+\frac{1}{2}\frac{\partial_-c}{c}\right)\frac{u}{x}+\frac{c}{2x}\left(\frac{\partial_+c}{c}-\frac{\partial_-c}{c}\right)+\frac{(\gamma-1)u^2}{x^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(\gamma-1)v^2}{2x^2}+\frac{3(\gamma-1)uv^2}{2cx^2}+\frac{(\gamma-1)v^2}{2cx}\frac{\partial_- c}{c}-\frac{\partial_+ c\partial_- c}{c^2}-\frac{c^2}{2\gamma x}s_x-\frac{c}{2\gamma}\frac{u}{x}s_x+\frac{cs_x}{2\gamma}\frac{\partial_- c}{c}\\
&-\frac{(\gamma-1)\phi}{2c}z_x-\frac{c}{2(\gamma-1)}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)s_x+\frac{\phi z}{2c}s_x+\frac{\gamma^2\phi^2z^2(\gamma-1)}{2c^4}-\frac{\gamma(\gamma-1)\phi z}{c^2}\cdot\frac{u}{x}\\
&-\frac{\gamma\phi z}{4c^2}\left((\gamma+1)\frac{\partial_- c}{c}+(\gamma+5)\frac{\partial_+ c}{c}\right)+\frac{(\gamma+1)\phi' z}{2\gamma}\frac{\partial_- c}{c}+\frac{(\gamma-1)\phi' z}{2\gamma}\frac{\partial_+ c}{c}-\frac{\gamma(\gamma-1)\phi^2 z}{2c^2}\\
&-\frac{\phi z}{2c^2}\left((\gamma^2-1)\frac{\partial_- c}{c}+(\gamma-1)^2\frac{\partial_+ c}{c}\right)+\frac{\gamma(\gamma-1)\phi z}{4c^2}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)
\end{aligned}
\tag{2.22}
$$

and

$$
\begin{aligned}
&\partial_+\Big(\frac{\partial_- c}{c}+\frac{(\gamma-1)v^2}{2cx}+\frac{c}{2\gamma}s_x\Big)\\
=&\frac{1}{2\mu^2}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)\frac{\partial_- c}{c}+\Big(\frac{3}{2}\frac{\partial_- c}{c}+\frac{1}{2}\frac{\partial_+ c}{c}\Big)\frac{u}{x}+\frac{c}{2x}\Big(\frac{\partial_+ c}{c}-\frac{\partial_- c}{c}\Big)+\frac{(\gamma-1)u^2}{x^2}\\
&-\frac{(\gamma-1)v^2}{2x^2}-\frac{3(\gamma-1)uv^2}{2cx^2}-\frac{(\gamma-1)v^2}{2cx}\frac{\partial_+ c}{c}-\frac{\partial_+ c\partial_- c}{c^2}-\frac{c^2}{2\gamma x}s_x+\frac{c}{2\gamma}\frac{u}{x}s_x-\frac{cs_x}{2\gamma}\frac{\partial_+ c}{c}\\
&+\frac{(\gamma-1)\phi}{2c}z_x+\frac{c}{2(\gamma-1)}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big)s_x-\frac{\phi z}{2c}s_x+\frac{\gamma^2\phi^2z^2(\gamma-1)}{2c^4}-\frac{\gamma(\gamma-1)\phi z}{c^2}\cdot\frac{u}{x}\\
&-\frac{\gamma\phi z}{4c^2}\left((\gamma+1)\frac{\partial_+ c}{c}+(\gamma+5)\frac{\partial_- c}{c}\right)+\frac{(\gamma+1)\phi' z}{2\gamma}\frac{\partial_+ c}{c}+\frac{(\gamma-1)\phi' z}{2\gamma}\frac{\partial_- c}{c}-\frac{\gamma(\gamma-1)\phi^2 z}{2c^2}\\
&-\frac{\phi z}{2c^2}\left((\gamma^2-1)\frac{\partial_+ c}{c}+(\gamma-1)^2\frac{\partial_- c}{c}\right)+\frac{\gamma(\gamma-1)\phi z}{4c^2}\Big(\frac{\partial_+ c}{c}+\frac{\partial_- c}{c}\Big).
\end{aligned}
\tag{2.23}
$$

We define

$$
\begin{cases}
R_+=\dfrac{\partial_+ c}{c}-\dfrac{(\gamma-1)v^2}{2cx}-\dfrac{cs_x}{2\gamma}+\dfrac{\kappa(\gamma-1)u}{x},\\
R_-=\dfrac{\partial_- c}{c}+\dfrac{(\gamma-1)v^2}{2cx}+\dfrac{cs_x}{2\gamma}+\dfrac{\kappa(\gamma-1)u}{x}.
\end{cases}
\tag{2.24}
$$

Then by (2.22) and (2.23) we have

$$
\begin{cases}
\partial_+ R_-=a_{11}R_-^2+a_{12}R_+R_-+a_{13}R_-+a_{14}R_++a_{15},\\
\partial_- R_+=a_{21}R_+^2+a_{22}R_+R_-+a_{23}R_++a_{24}R_-+a_{25},
\end{cases}
\tag{2.25}
$$

where

$$
a_{14}=\left(\frac{\gamma-3}{2}\kappa+\frac{1}{2}-\left(2\kappa-\frac{1}{2}\right)\frac{c}{u}-\frac{(\gamma+1)v^2}{4cu}+\frac{xcs_x}{4\gamma u}-\frac{\gamma^2+\gamma-1}{2u}\frac{x\phi z}{c^2}+\frac{(\gamma+1)x\phi' z}{2\gamma u}\right)\frac{u}{x},
\tag{2.26}
$$

$$
a_{24}=\left(\frac{\gamma-3}{2}\kappa+\frac{1}{2}+\left(2\kappa-\frac{1}{2}\right)\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}-\frac{xcs_x}{4\gamma u}-\frac{\gamma^2+\gamma-1}{2u}\frac{x\phi z}{c^2}+\frac{(\gamma+1)x\phi' z}{2\gamma u}\right)\frac{u}{x},
\tag{2.27}
$$

$$
\begin{aligned}
a_{15}=&\Bigg(2\kappa^2-3\kappa+1-2\kappa(1-\kappa)\frac{c}{u}+(\gamma\kappa-2)\frac{v^2}{cu}+\frac{x^2\phi z_x}{2cu^2}+\frac{x\phi' z}{2\gamma u^2}\frac{v^2}{c}-\frac{\kappa x\phi' z}{u}\\
&+\tfrac{x\phi z}{c^2}\left(\tfrac{\gamma^2}{2u^2}\tfrac{x\phi z}{c^2}-\tfrac{\gamma x\phi}{2u^2}+\tfrac{1}{2u^2}\tfrac{v^2}{c}+\tfrac{\gamma(\gamma+1)\kappa}{u}-\tfrac{xcs_x}{2\gamma u^2}-\tfrac{\gamma}{u}+\tfrac{\gamma\kappa}{u}\tfrac{c}{u}\right)+\tfrac{x\phi' z\cdot xcs_x}{2\gamma^2(\gamma-1)u^2}\Bigg)\tfrac{(\gamma-1)u^2}{x^2},
\end{aligned}
\tag{2.28}
$$

and

$$a_{25}=\Big(2\kappa^2-3\kappa+1+2\kappa(1-\kappa)\frac{c}{u}-(\gamma\kappa-2)\frac{v^2}{cu}-\frac{x^2\phi z_x}{2cu^2}-\frac{x\phi' z}{2\gamma u^2}\frac{v^2}{c}-\frac{\kappa x\phi' z}{u}$$
$$+\frac{x\phi z}{c^2}\Big(\frac{\gamma(\gamma+1)\kappa}{u}-\frac{1}{2u^2}\frac{v^2}{c}-\frac{\gamma x\phi}{2u^2}+\frac{\gamma^2}{2u^2}\frac{x\phi z}{c^2}+\frac{xcs_x}{2\gamma u^2}-\frac{\gamma}{u}-\frac{\gamma\kappa}{u}\frac{c}{u}\Big)$$
$$-\frac{x\phi' z\cdot xcs_x}{2\gamma^2(\gamma-1)u^2}\Big)\frac{(\gamma-1)u^2}{x^2}. \tag{2.29}$$

We define

$$\begin{cases}\widehat{R}_+=\dfrac{\partial_+ c}{c}-\dfrac{(\gamma-1)v^2}{2cx}-\dfrac{c}{2\gamma}s_x+\dfrac{2(\gamma-1)u}{(\gamma+1)x},\\ \widehat{R}_-=\dfrac{\partial_- c}{c}+\dfrac{(\gamma-1)v^2}{2cx}+\dfrac{c}{2\gamma}s_x+\dfrac{2(\gamma-1)u}{(\gamma+1)x}.\end{cases} \tag{2.30}$$

Then by (2.22) and (2.23), we have

$$\begin{cases}\partial_+\widehat{R}_-=\hat{a}_{11}\widehat{R}_-^2+\hat{a}_{12}\widehat{R}_+\widehat{R}_-+\hat{a}_{13}\widehat{R}_-+\hat{a}_{14}\widehat{R}_++\hat{a}_{15},\\ \partial_-\widehat{R}_+=\hat{a}_{21}\widehat{R}_+^2+\hat{a}_{22}\widehat{R}_+\widehat{R}_-+\hat{a}_{23}\widehat{R}_++\hat{a}_{24}\widehat{R}_-+\hat{a}_{25},\end{cases} \tag{2.31}$$

where

$$\hat{a}_{14}=\Big(\frac{3\gamma-5}{2(\gamma+1)}-\frac{7-\gamma}{2(\gamma+1)}\frac{c}{u}-\frac{(\gamma+1)v^2}{4cu}+\frac{xcs_x}{4\gamma u}-\frac{\gamma^2+\gamma-1}{2u}\frac{x\phi z}{c^2}+\frac{(\gamma+1)x\phi' z}{2\gamma u}\Big)\frac{u}{x}, \tag{2.32}$$

$$\hat{a}_{24}=\Big(\frac{3\gamma-5}{2(\gamma+1)}+\frac{7-\gamma}{2(\gamma+1)}\frac{c}{u}+\frac{(\gamma+1)v^2}{4cu}-\frac{xcs_x}{4\gamma u}-\frac{\gamma^2+\gamma-1}{2u}\frac{x\phi z}{c^2}+\frac{(\gamma+1)x\phi' z}{2\gamma u}\Big)\frac{u}{x}, \tag{2.33}$$

$$\hat{a}_{15}=\Big(\frac{(\gamma-3)(\gamma-1)}{(\gamma+1)^2}-\frac{4(\gamma-1)}{(\gamma+1)^2}\frac{c}{u}-\frac{2}{\gamma+1}\frac{v^2}{cu}+\frac{x\phi' z\cdot xcs_x}{2\gamma^2(\gamma-1)u^2}+\frac{x^2\phi z_x}{2cu^2}+\frac{x\phi' z}{2\gamma u^2}\frac{v^2}{c}$$
$$+\frac{x\phi z}{c^2}\Big(\frac{\gamma^2}{2u^2}\frac{x\phi z}{c^2}-\frac{\gamma x\phi}{2u^2}+\frac{1}{2u^2}\frac{v^2}{c}+\frac{2\gamma}{u}-\frac{xcs_x}{2\gamma u^2}-\frac{\gamma}{u}+\frac{2\gamma}{(\gamma+1)u}\frac{c}{u}\Big)-\frac{2x\phi' z}{(\gamma+1)u}\Big)\frac{(\gamma-1)u^2}{x^2}, \tag{2.34}$$

and

$$\hat{a}_{25}=\Big(\frac{(\gamma-3)(\gamma-1)}{(\gamma+1)^2}+\frac{4(\gamma-1)}{(\gamma+1)^2}\frac{c}{u}+\frac{2}{\gamma+1}\frac{v^2}{cu}-\frac{x\phi' z\cdot xcs_x}{2\gamma^2(\gamma-1)u^2}-\frac{x^2\phi z_x}{2cu^2}-\frac{x\phi' z}{2\gamma u^2}\frac{v^2}{c}$$
$$+\frac{x\phi z}{c^2}\Big(\frac{\gamma^2}{2u^2}\frac{x\phi z}{c^2}-\frac{\gamma x\phi}{2u^2}-\frac{1}{2u^2}\frac{v^2}{c}+\frac{2\gamma}{u}+\frac{xcs_x}{2\gamma u^2}-\frac{\gamma}{u}-\frac{2\gamma}{(\gamma+1)u}\frac{c}{u}\Big)$$
$$-\frac{2x\phi' z}{(\gamma+1)u}\Big)\frac{(\gamma-1)u^2}{x^2}. \tag{2.35}$$

LEMMA 2.3. *Assume $\frac{5}{3}<\gamma<3$. Then there exist small positive constants $\delta_1,\cdots,\delta_6$ and $c_{_M}$ such that the following inequalities hold*

(P1) $$\frac{3\gamma-5}{\gamma+1}-\frac{7-\gamma}{\gamma+1}\frac{c_M}{u_a}-\frac{(\gamma+1)\delta_1}{2u_a}-\frac{\delta_2}{2\gamma u_a}-\frac{(\gamma^2+\gamma-1)\delta_3}{u_a}-\frac{(\gamma+1)\delta_4}{\gamma u_a}>0,$$

(P2) $$\frac{(\gamma-3)(\gamma-1)}{(\gamma+1)^2}+\frac{\delta_6}{2u_a^2}+\frac{\delta_1\cdot\delta_4}{2\gamma u_a^2}+\frac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2}+\delta_3\Big(\frac{\gamma^2\delta_3}{2u_a^2}+\frac{\delta_2}{2\gamma u_a^2}+\frac{\delta_1}{2u_a^2}$$
$$+\frac{2\gamma c_{_M}}{(\gamma+1)u_a^2}+\frac{\gamma}{u_a}\Big)<0,$$

(P3) $\dfrac{3\gamma-5}{2(\gamma+1)}-\dfrac{\delta_2}{4\gamma u_a}-\dfrac{(\gamma^2+\gamma-1)\delta_3}{2u_a}>0,$

(P4) $\dfrac{(\gamma-3)(\gamma-1)}{(\gamma+1)^2}+\dfrac{4(\gamma-1)}{(\gamma+1)^2}\dfrac{c_M}{u_a}+\dfrac{2}{\gamma+1}\dfrac{\delta_1}{u_a}+\dfrac{\delta_6}{2u_a^2}+\dfrac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2}+\delta_3\Big(\dfrac{\gamma^2\delta_3}{2u_a^2}$ $+\dfrac{\gamma}{u_a}+\dfrac{\delta_2}{2\gamma u_a^2}\Big)<0,$

(P5) $\dfrac{\gamma+1}{2}\kappa^2-\dfrac{5}{2}\kappa+1-\dfrac{3\kappa c_M}{2u_a}-\dfrac{2\delta_1}{u_a}-\dfrac{\kappa\delta_2}{4\gamma u_a}-\dfrac{\gamma\delta_3}{u_a}-\dfrac{\kappa(\gamma-1)\delta_4}{2\gamma u_a}-\dfrac{\gamma\delta_5}{2u_a^2}-\dfrac{\delta_6}{2u_a^2}$ $-\dfrac{\delta_1\cdot\delta_3}{2u_a^2}-\dfrac{\delta_1\cdot\delta_4}{2\gamma u_a^2}-\dfrac{\delta_2\cdot\delta_3}{2\gamma u_a^2}-\dfrac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2}-\dfrac{\kappa\gamma\delta_3 c_M}{u_a^2}>0,$

(P6) $2\kappa^2-3\kappa+1-\dfrac{(2\kappa^2+2\kappa)c_M}{u_a}-\dfrac{(\kappa\gamma-2)\delta_1}{u_a}-\dfrac{\gamma\delta_3}{u_a}-\dfrac{\kappa\delta_4}{u_a}-\dfrac{\gamma\delta_5}{2u_a^2}-\dfrac{\delta_6}{2u_a^2}-\dfrac{\delta_1\cdot\delta_3}{2u_a^2}$ $-\dfrac{\delta_1\cdot\delta_4}{2\gamma u_a^2}-\dfrac{\delta_2\cdot\delta_3}{2\gamma u_a^2}-\dfrac{\kappa\gamma\delta_3 c_M}{u_a^2}-\dfrac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2}>0.$

In view of $\frac{5}{3}<\gamma<3$ and $(P1)-(P4)$, we have

LEMMA 2.4. *If* $0<c<c_M$, $u>u_a$, $\frac{v^2}{c}<\delta_1$, $|xcs_x|<\delta_2$, $\frac{x\phi}{c^2}<\delta_3$, $x\phi'<\delta_4$, *and* $\frac{x^2\phi}{c}<\delta_6$, *then* $\hat{a}_{14}>0$, $\hat{a}_{24}>0$, $\hat{a}_{15}<0$, $\hat{a}_{25}<0$.

3. Global smooth solution to the initial value problem

3.1. A priori C^1 estimate. We now consider (1.7) with initial data

$$(u,v,c,s,z)(x,0)=(u_0,v_0,c_0,s_0,z_0)(x),\quad a+\delta\le x\le b-\delta, \tag{3.1}$$

where $\delta>0$ may be arbitrarily small.

The existence and uniqueness of a local C^1 solution to the problem (1.7) with (3.1) are known by the method of characteristics (cf. [32]). That is to say there exists a small $\mathcal{T}>0$ such that the problem admits a classical solution (u,v,c,s,z) in a domain $\Omega(\delta,\mathcal{T})$ closed by $\{t=0\}$, $\{t=\mathcal{T}\}$, a C_+ characteristic curve issuing from $(a+\delta,0)$ (see C_+^δ in Figure 3.1), and a C_- characteristic curve issuing from $(b-\delta,0)$ (see C_-^δ in Figure 3.1). The characteristic curves $C_\pm^\delta$ can be represented by $x=x_\pm^\delta(t)$ which satisfy

$$\begin{cases}\dfrac{\mathrm{d}x_\pm^\delta(t)}{\mathrm{d}t}=(u\pm c)(x_\pm^\delta(t),t), & t>0,\\ x_+^\delta(0)=a+\delta,\quad x_-^\delta(0)=b-\delta.\end{cases} \tag{3.2}$$

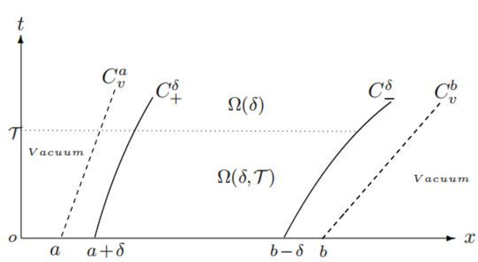

FIG. 3.1. *Characteristic curves*

In order to extend the local solution to a whole domain of determinacy, one needs to establish an a priori C^1 estimate for the solution.

Lemma 3.1. *Assume that the Cauchy problem (1.7), (3.1) admits a classical solution in $\Omega(\delta,\mathcal{T})$ for some $\mathcal{T}>0$. Then the solution satisfies*

$$-\frac{(\gamma-1)\delta_1}{2a}-\frac{\delta_2}{2\gamma a}-\frac{2u_b}{a}<\frac{\partial_\pm c}{c}<0,\quad \left|\frac{s_x}{c^\kappa}\right|<\frac{(N+1)x}{a},\quad \left|\frac{z_x}{c^\kappa}\right|<\frac{x}{a},$$
$$\widehat{R}_\pm<0,\quad R_\pm>0,\quad 0<s<s_*,\quad 0<c<c_{\scriptscriptstyle M}\quad \text{and}\quad u_a<u<u_b. \tag{3.3}$$

Proof. We shall prove this lemma by the method of continuity. The proof proceeds in two steps.

Step 1. We first prove that the inequalities in (3.3) hold on $\{(x,t)\,|\,t=0,a<x<b\}$. From $c^2=\gamma e^s\rho^{\gamma-1}$ we have

$$\rho_t=\frac{2cc_t-\gamma e^s\rho^{\gamma-1}s_t}{\gamma(\gamma-1)e^s\rho^{\gamma-2}},\quad \rho_x=\frac{2cc_x-\gamma e^s\rho^{\gamma-1}s_x}{\gamma(\gamma-1)e^s\rho^{\gamma-2}}.$$

Inserting this into the first equation of (1.7) and using the fourth equation of (1.7), we get

$$c_t=-uc_x-\frac{(\gamma-1)cu_x}{2}-\frac{(\gamma-1)cu}{2x}+\frac{\gamma(\gamma-1)\phi z}{2c}. \tag{3.4}$$

Hence, we have

$$\partial_\pm c=c_t+(u\pm c)c_x=\pm cc_x-\frac{(\gamma-1)cu_x}{2}-\frac{(\gamma-1)cu}{2x}+\frac{\gamma(\gamma-1)\phi z}{2c}. \tag{3.5}$$

Consequently, by Assumption (1.10) and (1.5) we have that when $c_{\scriptscriptstyle M}$ is sufficiently small,

$$-\frac{2u_b}{a}<\left(\frac{\partial_\pm c}{c}\right)(x,0)=\pm c_0{}'(x)-\frac{(\gamma-1)u_0{}'(x)}{2}-\frac{(\gamma-1)u_0(x)}{2x}$$
$$+\frac{\gamma(\gamma-1)\phi(c_0(x))}{2c_0{}^2(x)}<0\qquad \text{for } a<x<b. \tag{3.6}$$

By (3.5) we get

$$\widehat{R}_+(x,0)=c_0{}'(x)-\frac{(\gamma-1)u_0{}'(x)}{2}+\left(\frac{2}{\gamma+1}-\frac{1}{2}\right)\frac{(\gamma-1)u_0(x)}{x}$$
$$-\frac{(\gamma-1)v_0{}^2(x)}{2c_0(x)x}-\frac{c_0(x)s_0{}'(x)}{2\gamma}+\frac{\gamma(\gamma-1)\phi(c_0(x))}{2c_0{}^2(x)} \tag{3.7}$$

and

$$\widehat{R}_-(x,0)=-c_0{}'(x)-\frac{(\gamma-1)u_0{}'(x)}{2}+\frac{(\gamma-1)v_0{}^2(x)}{2c_0(x)x}+\frac{c_0(x)s_0{}'(x)}{2\gamma}$$
$$+\frac{\gamma(\gamma-1)\phi(c_0(x))}{2c_0{}^2(x)}. \tag{3.8}$$

Then, by Assumption (1.10) and (1.5) we know that when δ_1 and $c_{\scriptscriptstyle M}$ are sufficiently small,

$$\widehat{R}_\pm(x,0)<0\quad \text{for}\quad a<x<b. \tag{3.9}$$

Similarly, by (3.5) we have

$$R_{\pm}(x,0)=\pm c_0{}'(x)-\frac{(\gamma-1)u_0{}'(x)}{2}+\left(\kappa-\frac{1}{2}\right)\frac{(\gamma-1)u_0(x)}{x}\mp\frac{(\gamma-1)v_0{}^2(x)}{2c_0(x)x}$$
$$\mp\frac{c_0(x)s_0{}'(x)}{2\gamma}+\frac{\gamma(\gamma-1)\phi(c_0(x))}{2c_0{}^2(x)}. \tag{3.10}$$

Thus, by Assumption (1.10) and (1.5) we know that when δ_1 and c_M are sufficiently small,

$$R_{\pm}(x,0)>0 \quad \text{for} \quad a<x<b. \tag{3.11}$$

The other inequalities in (3.3) are obviously true on $\{(x,t)\,|\,t=0,a<x<b\}$.
Step 2. Let $P:=(x_p,t_p)$ be an arbitrary point in the domain. The backward C_+ and C_- characteristic curves issuing from P intersect the x-axis at some points P_+ and P_-, respectively. The backward stream line C_0 issuing from P intersects the x-axis at some point P_0. Let Ω_P be a closed triangle domain closed by $\widehat{P_+P}$, $\widehat{P_-P}$, and $\overline{P_+P_-}$ (see Figure 3.2). We are going to prove that if the inequalities in (3.3) hold for all points in $\Omega_P\setminus\{P\}$, then they also hold at P.

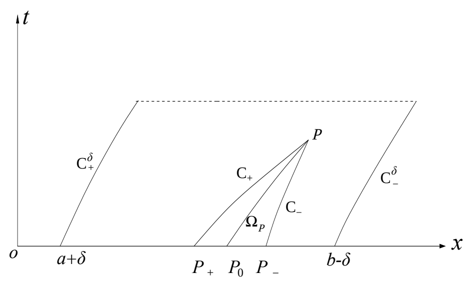

FIG. 3.2. *Domain* Ω_p

Since $\widehat{R}_{\pm}<0$ in $\Omega_P\setminus\{P\}$, we have

$$\frac{\partial_0 c}{c}<-\frac{2(\gamma-1)}{\gamma+1}\frac{u}{x}=-\frac{2(\gamma-1)}{\gamma+1}\partial_0\ln x \quad \text{in} \quad \Omega_P\setminus\{P\}.$$

Integrating it along $\widehat{P_0P}$ from P_0 to P, we get

$$c(P)<c(P_0)\left(\frac{x_P}{x_{P_0}}\right)^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{3.12}$$

In addition, in view of $u>0$ in $\Omega_P\setminus\{P\}$, one has $x_P>x_{P_0}$ and hence $c(P)<c(P_0)<c_M$. Actually, we have

$$c<c_M\left(\frac{x}{a}\right)^{-\frac{2(\gamma-1)}{\gamma+1}} \quad \text{in} \quad \Omega_P. \tag{3.13}$$

From $R_{\pm}>0$ in $\Omega_P\setminus\{P\}$, we have

$$\frac{\partial_0 c}{c}>-\frac{\kappa(\gamma-1)u}{x} \quad \text{in} \quad \Omega_P\setminus\{P\}.$$

Integrating it along $\widehat{P_0P}$ from P_0 to P, we have

$$c(P)>c(P_0)\left(\frac{x_P}{x_{P_0}}\right)^{-\kappa(\gamma-1)}>0. \tag{3.14}$$

From the third equation of (2.1) we have

$$\partial_0 \ln|v| = -\partial_0 \ln x. \tag{3.15}$$

Integrating it along $\widehat{P_0P}$ from P_0 to P, we get

$$|v(P)| = \frac{x_{P_0}}{x_P}|v(P_0)|. \tag{3.16}$$

Combining (3.14) and (3.16) and $x_P > x_{P_0}$, $\kappa = \frac{2}{\gamma-1}$, we get

$$\left(\frac{v^2}{c}\right)(P) < \left(\frac{v^2}{c}\right)(P_0)\left(\frac{x_{P_0}}{x_P}\right)^{\kappa(\gamma-1)-2} \le \delta_1. \tag{3.17}$$

In view of (2.4), we have

$$\partial_- u = \frac{2}{\gamma-1}c\widehat{R}_- - \frac{\gamma\phi z}{c} < 0, \quad \partial_+ u = -\frac{2}{\gamma-1}c\widehat{R}_+ + (\frac{2}{\gamma+1} - \frac{1}{2})\frac{2uc}{x} + \frac{\gamma\phi z}{c} > 0 \tag{3.18}$$

in $\Omega_P \setminus \{P\}$. Thus, by integration we have

$$u_a < u(P_+) < u(P) < u(P_-) < u_b.$$

In view of (1.5), (3.13), (2.5), and the assumption that the inequalities in (3.3) hold for all points in $\Omega_P \setminus \{P\}$, we know that there exist sufficiently small constants ε, ε_1, ε_2 depending only on c. When c_M is sufficiently small, such that

$$\left|\frac{\phi}{c^2}\right| < \varepsilon, \quad \left|\frac{\phi}{c^6}\right| < \varepsilon_1, \quad \left|\frac{\phi'}{c^4}\right| < \varepsilon_2.$$

Then, we get

$$\begin{aligned}
&\left|-\frac{\gamma\phi z}{c^2}\frac{s_x}{c^{\frac{2}{\gamma-1}}} + \frac{2\phi' z}{c}\frac{c_x}{c^{\frac{2}{\gamma-1}}} - \frac{2\gamma(\gamma-1)\phi z}{c^3}\frac{c_x}{c^{\frac{2}{\gamma-1}}} + \frac{\gamma(\gamma-1)\phi}{c^2}\frac{z_x}{c^{\frac{2}{\gamma-1}}}\right| \\
&< \left|\frac{\gamma(\gamma-1)\phi}{c^2}\frac{z_x}{c^{\frac{2}{\gamma-1}}}\right| + \left|\frac{\gamma\phi z}{c^2}\frac{s_x}{c^{\frac{2}{\gamma-1}}}\right| + \left|\frac{2\phi' z}{c}\frac{c_x}{c^{\frac{2}{\gamma-1}}}\right| + \left|\frac{2\gamma(\gamma-1)\phi z}{c^3}\frac{c_x}{c^{\frac{2}{\gamma-1}}}\right| \\
&< \gamma(\gamma-1)\varepsilon\frac{x}{a} + \gamma\varepsilon(N+1)\frac{x}{a} + 2\varepsilon_2 c_M^{\frac{3\gamma-5}{\gamma-1}}\frac{2u}{x} + 2\gamma(\gamma-1)\varepsilon_1 c_M^{\frac{3\gamma-5}{\gamma-1}}\frac{2u}{x} \\
&< \gamma(\gamma-1)\varepsilon\frac{b}{a} + \gamma\varepsilon(N+1)\frac{b}{a} + 4\varepsilon_2 c_M^{\frac{3\gamma-5}{\gamma-1}}\frac{u}{x} + 4\gamma(\gamma-1)\varepsilon_1 c_M^{\frac{3\gamma-5}{\gamma-1}}\frac{u}{x} < \frac{u}{x}.
\end{aligned} \tag{3.19}$$

Similarly, we have

$$\left|-\frac{\gamma\phi z z_x}{c^{\frac{2\gamma}{\gamma-1}}} - \frac{2\phi' z c_x}{c^{\frac{3-\gamma}{\gamma-1}}\gamma(\gamma-1)} - \frac{\phi z_x}{c^{\frac{2}{\gamma-1}}}\right| < \frac{u}{x}$$

in $\Omega_P \setminus \{P\}$. Thus, integrating (2.8) from P_0 to P one gets

$$\left(\frac{|s_x|}{c^{\frac{2}{\gamma-1}}}\right)(P) \le \frac{x_p}{x_{p_0}}\left(\int_{x_{P_0}}^{x_P}\frac{x_{P_0}}{x^2}\mathrm{d}x + \left(\frac{|s_x|}{c^{\frac{2}{\gamma-1}}}\right)(P_0)\right) < \frac{(N+1)x_P}{a} \tag{3.20}$$

and

$$\left(\frac{|z_x|}{c^{\frac{2}{\gamma-1}}}\right)(P) \le \frac{x_p}{x_{p_0}} \int_{x_{P_0}}^{x_P} \frac{x_{P_0}}{x^2}\mathrm{d}x < \frac{x_P}{a}. \tag{3.21}$$

Using (3.20) and $x_p > x_{p_0}$ we know that when c_M is small,

$$|xcs_x| = \left|xc^{\frac{\gamma+1}{\gamma-1}} \frac{s_x}{c^{\frac{2}{\gamma-1}}}\right| < \left(|xcs_x|(p_0) + bc_M^{\frac{\gamma+1}{\gamma-1}}\right)\left(\frac{x_p}{x_{p_0}}\right)^{2-2} < \delta_2 \quad \text{at} \quad P. \tag{3.22}$$

From (1.5) and (3.12) we know that when c_M is small, there hold

$$\frac{x\phi}{c^2} < \delta_3, \quad x\phi' < \delta_4, \quad \frac{x^2\phi^2}{c^2} < \delta_5 \quad \text{and} \quad \frac{x^2\phi}{c} < \delta_6 \quad \text{at} \quad P. \tag{3.23}$$

In what follows, we are going to prove $\widehat{R}_\pm < 0$ and $R_\pm > 0$ at P.

Suppose $\widehat{R}_- = 0$ and $\widehat{R}_+ \le 0$ at P. Then by the assumption that the inequalities in (3.3) hold in $\Omega_P \setminus \{P\}$, we have $\partial_+\widehat{R}_- \ge 0$ at P. While, by the first equation of (2.31), (P1) and (P2) we have

$$\partial_+\widehat{R}_- = \underbrace{\hat{a}_{14}}_{>0}\underbrace{\widehat{R}_+}_{\le 0} + \underbrace{\hat{a}_{15}}_{<0} < 0 \quad \text{at} \quad P. \tag{3.24}$$

This leads to a contradiction. We then get $\widehat{R}_-(P) < 0$. Similarly, we have $\widehat{R}_+(P) < 0$.

Suppose $R_- = 0$ and $R_+ \ge 0$ at P. Then by the assumption we have $\partial_+R_- \le 0$ at P. While, by the first equation of (2.25), (P5) and (P6) we have that at the point P,

$$\begin{cases} \partial_+R_- > \dfrac{a_{14}\kappa(\gamma-1)u}{x} + a_{15} > \Big\{\dfrac{\gamma+1}{2}\kappa^2 - \dfrac{5}{2}\kappa + 1 \\ \qquad - \dfrac{3\kappa c_M}{2u_a} - \dfrac{2\delta_1}{u_a} - \dfrac{\kappa\delta_2}{4\gamma u_a} - \dfrac{\gamma\delta_3}{u_a} - \dfrac{\kappa(\gamma-1)\delta_4}{2\gamma u_a} - \dfrac{\gamma\delta_5}{2u_a^2} - \dfrac{\delta_6}{2u_a^2} - \dfrac{\delta_1\cdot\delta_3}{2u_a^2} - \dfrac{\delta_1\cdot\delta_4}{2\gamma u_a^2} \\ \qquad - \dfrac{\delta_2\cdot\delta_3}{2\gamma u_a^2} - \dfrac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2} - \dfrac{\kappa\gamma\delta_3 c_M}{u_a^2}\Big\}\dfrac{(\gamma-1)u^2}{x^2} > 0 & \text{if } a_{14} < 0, \\ \partial_+R_- > a_{15} > \big\{2\kappa^2 - 3\kappa + 1 \\ \qquad - \dfrac{(2\kappa^2+2\kappa)c_M}{u_a} - \dfrac{(\kappa\gamma-2)\delta_1}{u_a} - \dfrac{\gamma\delta_3}{u_a} - \dfrac{\kappa\delta_4}{u_a} - \dfrac{\gamma\delta_5}{2u_a^2} - \dfrac{\delta_6}{2u_a^2} - \dfrac{\delta_1\cdot\delta_3}{2u_a^2} - \dfrac{\delta_1\cdot\delta_4}{2\gamma u_a^2} \\ \qquad - \dfrac{\delta_2\cdot\delta_3}{2\gamma u_a^2} - \dfrac{\delta_2\cdot\delta_4}{2\gamma^2(\gamma-1)u_a^2} - \dfrac{\kappa\gamma\delta_3 c_M}{u_a^2}\Big\}\dfrac{(\gamma-1)u^2}{x^2} > 0 & \text{if } a_{14} \ge 0. \end{cases}$$

This leads to a contradiction. We then get $R_-(P) > 0$. Similarly, we have $R_+(P) > 0$.

Using $\widehat{R}_\pm(P) < 0$, $R_\pm(P) > 0$, (3.17), and (3.22) we have $-\frac{(\gamma-1)\delta_1}{2a} - \frac{\delta_2}{2\gamma a} - \frac{2u_b}{a} < \frac{\partial_\pm c}{c} < 0$ at P. We then prove that if the inequalities in (3.3) hold for all points in $\Omega_P \setminus \{P\}$, then they also hold at P. Therefore, by an argument of continuity we know that the classical solution of the Cauchy problem satisfies (3.3).

This completes the proof of the lemma. ∎

As in (3.14), we know that the solution to the problem (1.7) with (3.1) satisfies

$$c > c_\delta b^2 x^{-2}, \tag{3.25}$$

where $c_\delta := \min_{x\in[a+\delta,b-\delta]} c_0(x)$. Moreover, according to (3.2) we know that $x < b-\delta+u_b t$. Combining this with (3.25) we know that the solution satisfies

$$c > c_\delta b^2(b-\delta+u_b t)^{-2}. \tag{3.26}$$

Combining (2.5), (3.3), (3.17), and (3.26) we obtain an a priori C^0 estimate for (u,v,c) and gradient estimates for c, z and s. The gradient estimate for u can be obtained by (2.4). In order to estimate $|\nabla v|$, we use the commutator relation

$$\partial_0\partial_x v - \partial_x\partial_0 v = -\partial_x u \partial_x v.$$

Inserting the third equation of (2.1) into this, we have

$$\partial_0(\partial_x v) + \left(\frac{u}{x} + \partial_x u\right)\partial_x v = \frac{uv}{x^2} - \frac{v\partial_x u}{x}. \tag{3.27}$$

The gradient estimate for v can be obtained by (3.27) and the third equation of (2.1). We then establish an a priori C^1 estimate for the solution. Thus, the existence of global classical solution can be obtained by the classical extension method (cf. Li [31]). We then get the following global existence.

LEMMA 3.2. *The initial value problem (1.7) with (3.1) admits a global classical solution in a domain* $\Omega(\delta)$ *bounded by* $\{(x,t)\,|\,a+\delta \le x \le b-\delta, t=0\}$, C_+^δ *and* C_-^δ *(see Figure 1.1). Moreover, the solution satisfies (3.3) and (3.26).*

LEMMA 3.3. *By Lemmas 3.1 and 3.2, we obtain that the solution satisfies* $\rho \in C(\overline{\Omega})$.

Proof. In what follows, we are going to prove that for any fixed $\mathcal{T} > 0$,

$$\|x_+^\delta(t) - (a+u_a t)\|_{0;[0,\mathcal{T}]} \to 0 \quad \text{and} \quad \|x_-^\delta(t) - (b+u_b t)\|_{0;[0,\mathcal{T}]} \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.28}$$

Firstly, by $\partial_- c < 0$ and $\partial_- u < 0$ we have

$$c < c_0(b-\delta) \quad \text{and} \quad u < u_0(b-\delta) \quad \text{on} \quad C_-^\delta. \tag{3.29}$$

By a direct computation, we have

$$c^2 s_x = s\partial_- c - \partial_-(cs) + \frac{c\phi z}{T}.$$

Hence, we get

$$\partial_- u = \frac{2}{\gamma-1}\partial_- c - \frac{\partial_-(cs)}{\gamma(\gamma-1)} + \frac{s\partial_- c}{\gamma(\gamma-1)} - (\gamma-1)\frac{\phi z}{c} + \frac{uc}{x} + \frac{v^2}{x}. \tag{3.30}$$

As in (3.12) we know that the solution satisfies

$$c < c_{\scriptscriptstyle M}\left(\frac{x}{a}\right)^{-\frac{2(\gamma-1)}{\gamma+1}}. \tag{3.31}$$

Combining this with (1.5) we know that when $\delta > 0$ is small,

$$\frac{\phi}{c} < \frac{(u-c)c}{x^2} \quad \text{on} \quad C_-^\delta. \tag{3.32}$$

Thus, by integrating (3.30) along C_-^δ we get

$$u > u_0(b-\delta) - \frac{2}{\gamma-1}\Big(1+s_*+\frac{1}{b-\delta}\Big)c_0(b-\delta) \quad \text{on} \quad C_-^\delta. \tag{3.33}$$

Combining (3.29) and (3.33), one gets

$$\|(u-c)-u_b\|_{0;C_-^\delta} \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.34}$$

Thus, for any fixed $\mathcal{T}>0$, we obtain $\|x_-^\delta(t)-(b+u_b t)\|_{0;[0,\mathcal{T}]} \to 0$ as $\delta \to 0$. Similarly, we have $\|x_+^\delta(t)-(a+u_a t)\|_{0;[0,\mathcal{T}]} \to 0$ as $\delta \to 0$. ◻

Consequently, we obtain a solution in Ω. The property that $\rho>0$ in Ω and $\rho=0$ on $C_v^a \cup C_v^b$ is obvious. This completes the proof of Theorem 1.1.

COROLLARY 3.1. *$\|z(\cdot,t)\|_{0,\mathcal{R}} \to 0$, as $t \to +\infty$.*

Proof. From the last equation of (2.1), we get

$$z = \exp\Big(-\int_0^t \phi(c)\mathrm{d}t\Big) = \exp\Big(-k\int_0^t e^{-\frac{\gamma(\gamma-1)}{c^2}}\mathrm{d}t\Big).$$

Like the proof of the Theorem 1.1, we have

$$c(P_0)\Big(\frac{a}{b}\Big)^2 < c(P_0)\Big(\frac{x_P}{x_{P_0}}\Big)^{-2} < c.$$

A direct computation yields

$$\exp\Big(-\frac{\gamma(\gamma-1)}{c^2}\Big) > \exp\Big(-\frac{\gamma(\gamma-1)}{c^2(P_0)}\Big)\Big(\frac{a}{b}\Big)^{-4}.$$

Then, we have

$$\int_0^t \phi(c)\ \mathrm{d}t > \int_0^t k\exp\Big(-\frac{\gamma(\gamma-1)}{c^2(P_0)}\Big)\Big(\frac{a}{b}\Big)^{-4}\mathrm{d}t.$$

Thus, we get $\int_0^t \phi(c)\ \mathrm{d}t \to +\infty$ for $t \to +\infty$, i.e., $\exp\Big(-\int_0^t \phi(c)\mathrm{d}t\Big) \to 0$, as $t \to +\infty$. ◻

REMARK 3.1. From Lemma 3.1, the solution of the initial value problem (1.7) with (1.8) satisfies

$$|\nabla c| < \frac{(\gamma-1)\delta_1}{2a} + \frac{\delta_2}{2\gamma a} + \frac{2u_b}{a} \quad \text{in} \quad \Omega. \tag{3.35}$$

4. Summary

With axial symmetry, we have obtained that the problem (1.7) with (1.8) admits a classical solution in the domain Ω. We propose to construct a global-in-time smooth reacting flow solution expanding in vacuum. The solution describes a phenomenon of the expansion of 2D reacting flows with swirl in vacuum or a phenomenon of "fire whirl". As a result, we get for any t, that the ring continues to expand until the gas burns out in infinite time for the system (1.1).

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