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IMEX VARIABLE STEP-SIZE RUNGE-KUTTA METHODS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH NONSMOOTH INITIAL DATA*

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Abstract. We develop a class of implicit-explicit (IMEX) Runge-Kutta (RK) methods for solving parabolic integro-differential equations (PIDEs) with nonsmooth initial data, which describe several option pricing models in mathematical finance. Different from the usual IMEX RK methods, the proposed methods approximate the integral term explicitly by using an extrapolation operator based on the stage-values of RK methods, and we call them as IMEX stage-based interpolation RK (SBIRK) methods. It is shown that there exist arbitrarily high order IMEX SBIRK methods which are stable for abstract PIDEs under suitable time step restrictions. The consistency error and the global error bounds for this class of IMEX Runge-Kutta methods are derived for abstract PIDEs with nonsmooth initial data. The related higher time regularity analysis of the exact solution and stability estimates for IMEX SBIRK methods play key roles in deriving these error bounds. Numerical experiments for European options under jump-diffusion models and stochastic volatility model with jump verify and complement our theoretical results.

Keywords. Parabolic integro-differential equations; IMEX Runge-Kutta methods; stage-based interpolation Runge-Kutta methods; option pricing models; nonsmooth initial data; stability; error estimates.

AMS subject classifications. Primary 65M06; 65M12; 65L06; Secondary 91B24; 91G60; 65J10.

1. Introduction

In this paper, we study the high order time approximation of partial integro-differential equations (PIDEs)

$$\begin{cases} \partial_t u(t,x)+\mathcal{L}u(t,x)=f(t,x), \qquad t\in J:=(0,T], \quad x\in\Omega, \\ u(0,x)=u^0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

subject to proper boundary conditions, where $T>0$ is a fixed final time, $\Omega\subset\mathbb{R}^d$, $d\geq 1$, denotes an open regular domain with boundary $\partial\Omega$, $f:(0,T]\times\Omega\to H$ is a given function with H being a Hilbert space, u^0 is a given nonsmooth initial value, and $\mathcal{L}$ is a given operator with the general form

$$\mathcal{L}u(t,x):=\sum_{ij}\alpha_{ij}\partial_{ij}u(t,x)+\sum_i\beta_i\partial_i u(t,x)+\gamma u(t,x)+\int_\Omega u(t,x+y)g(y)dy, \tag{1.2}$$

which can be viewed as a generator of a d-dimensional Lévy process. The first term in (1.2) corresponds to the diffusion, the second to the drift, and the fourth to the jump part. Some special cases of (1.1) are the so-called jump-diffusion models and stochastic volatility model with jump for options pricing [6, 18, 31, 37, 44, 45]; see Section 7 for details.

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It is well known that implicit time discretization of PIDEs (1.1)-(1.2) always results in a system of algebraic equations with a full coefficient matrix, because of the nonlocal nature of the integral operator

$$\mathcal{J}u(t,x) := \int_{\Omega} u(t,x+y)g(y)dy. \tag{1.3}$$

The alternating direct implicit (ADI) method [3, 33], FFT [2, 22], iterative methods [2, 23, 54, 56] and other methods [17, 59] have been used to solve such systems with full coefficient matrices. In order to avoid having to solve the algebraic system with a full coefficient matrix, some researchers applied implicit-explicit (IMEX) scheme to solve the jump-diffusion option pricing model, a special case of PIDEs (1.1)-(1.2); see, for example, [9,12,16,19,24,34,34–36,38,47,49,57,58]. These IMEX schemes analyzed in the above literature are based on the uniform time grid. However, due to the nonsmoothness of the initial data u^0, the payoff function in option pricing models, singularities may arise at $t=0$. Considering this fact, Wang, Chen, and Fang employed the variable step-size IMEX BDF2 scheme to solve the PIDEs (1.1)-(1.2), proved its stability, and derived its error estimates based on the results on the time regularity of the solution [60]. Recently, the authors of [61] solved the jump-diffusion option pricing model by using the variable step-size IMEX mid-point scheme. The variable step-size IMEX Crank-Nicolson Adams Bashforth method was used to solve a stochastic volatility model with jump [43]. The a posteriori error estimates of the variable step-size IMEX BDF2 method for European option pricing under the jump-diffusion model were also derived in [62].

In this paper, we try to study high-order variable step-size IMEX Runge-Kutta (RK) methods for PIDEs (1.1)-(1.2) and derive the nonsmooth data error estimates. This will be done by analyzing higher time regularity of the exact solution and combining with the novel stability estimate techniques. Nonsmooth data error estimates have been widely explored by semigroup method for semidiscrete or fully discrete methods for parabolic problems; see, e.g., [8,20,41,42,46,65]. In these outstanding works for RK methods it is common to assume that the module of the stability function $R(z)$ of the RK method at infinity is strictly smaller than one. Then the Gauss RK methods are excluded by this assumption, since the stability function of the Gauss RK methods satisfies $R(\infty)=1$. In this paper, to obtain nonsmooth data error estimates for general RK methods which include the Gauss methods, the Radau IIA methods, and the Lobatto IIIC methods, we shall not use such an assumption and only require algebraic stability of the RK methods. We obtain higher time regularity results of the exact solution and the stability estimates for the numerical methods in a crucial way, and employ these to estimate the error by using energy technique.

It is important to realize that the IMEX RK methods have been applied to various problems (see, for example, [1,4,10,11,27,64]) because of their high order and effectiveness. Briani, Natalini, and Russo have also used them to solve jump-diffusion option pricing model on a uniform time grid and discussed the stability of these schemes under step-size restrictions by classical von Neumann analysis [13]. Different from the usual IMEX RK methods, the proposed methods in this paper approximate the integral term explicitly by using an extrapolation operator based on the stage-values of RK methods, and we call them as IMEX stage-based interpolation RK (SBIRK) methods, which have been introduced for solving nonlinear Allen-Cahn and Cahn-Hilliard equations [1] and nonlinear Volterra functional differential equations recently [63].

The paper is organized as follows. We start in Section 2 by introducing an abstract class of PIDEs and make some assumptions on the differential operators, the integral operator, and the initial data. The a priori bounds for the high order time derivations of the exact solutions to abstract PIDEs are also obtained in this section. In Section 3, we propose the variable step-size IMEX SBIRK methods for solving PIDEs and study the existence and uniqueness of the solution to the implicit algebraic equations. The stability of this class of methods is analyzed in Section 4. As a consequence of the stability analysis, arbitrarily high order stable IMEX SBIRK methods can be constructed by choosing a large number s of stages. Section 5 is devoted to the error estimates for this class of methods when they are applied to PIDEs with nonsmooth initial data. The fully discrete approximation based on finite difference method for space discretization will be discussed in Section 6. In Section 7, we give some numerical examples for jump-diffusion models and stochastic volatility model with jump to illustrate our theoretical results. In Section 8 we finally conclude with some remarks.

2. Abstract PIDEs

Let $\|\cdot\|$ denote the norm induced from the inner product $(\cdot,\cdot)$ on a Hilbert space H. Let the operator $\mathcal{L}$ be split into $\mathcal{L}=\mathcal{A}+\mathcal{B}+\mathcal{J}$ such that $\mathcal{J}$ has been defined in (1.3), $\mathcal{A}$: $D(\mathcal{A})\to H$ is a positive definite, self-adjoint, linear operator whose domain $D(\mathcal{A})$ is dense in H, and $\mathcal{B}: D(\mathcal{A})\to H$ is a linear operator. The operators $\mathcal{A}$ and $\mathcal{B}$ have different choices, for example, $\mathcal{A}$ can be $\sum_{ij}\alpha_{ij}\partial_{ij}u(t,x)$ or $\sum_{ij}\alpha_{ij}\partial_{ij}u(t,x)+\gamma u(t,x)$ in (1.2), and therefore $\mathcal{B}=\mathcal{L}-\mathcal{A}-\mathcal{J}$. Then the operator $\mathcal{A}$ is sectorial, and the spectral theory of the operator $\mathcal{A}$ allows us to define the powers $\mathcal{A}^{\theta/2}$ of $\mathcal{A}$ for $\theta\in\mathbb{R}$ (see [30,66]). For every $\theta>0$, $\mathcal{A}^{\theta/2}$ is an unbounded operator in H with a dense domain $\mathcal{D}(\mathcal{A}^{\theta/2})\subset H$. The space $\mathcal{D}(\mathcal{A}^{\theta/2})$, $0\le\theta\le 2$, is endowed with the norm $\|v\|_\theta:=\|\mathcal{A}^{\theta/2}v\|$. For $\theta=1$, we define $V:=\mathcal{D}(\mathcal{A}^{1/2})$ and the norm $\|\mathcal{A}^{1/2}v\|=\|v\|_1$, $v\in\mathcal{D}(\mathcal{A}^{1/2})$. Let V^* be the dual space of V, and denote by $\|\cdot\|_*$ the dual norm on V^*. We still denote by $(\cdot,\cdot)$ the duality pairing between V^* and V. As a consequence of $\mathcal{A}$ having the above properties, there exists a constant $C_P>0$ such that

$$\|v\|\le C_P\|v\|_1,\quad \forall v\in V,\qquad \|v\|_*\le C_P\|v\|,\quad \forall v\in H. \tag{2.1}$$

Now let us consider abstract PIDEs derived from (1.1)-(1.2)

$$u'(t)+\mathcal{A}u(t)+\mathcal{B}u(t)+\mathcal{J}(u(t))=F(t),\quad t\in(0,T], \tag{2.2}$$

$$u(0)=u^0, \tag{2.3}$$

where $F:(0,T]\to H$ and the initial data $u^0\in V$. For the linear operator $\mathcal{B}$, we assume that there exists a constant β such that

$$\|\mathcal{B}u\|\le\beta\|u\|_1,\qquad \forall u\in V. \tag{2.4}$$

As for the integral operator $\mathcal{J}$, we assume that it satisfies the condition

$$\|\mathcal{J}u\|\le C_J\|u\|,\quad \forall u\in H, \tag{2.5}$$

for constant C_J independent of t.

The operator $\mathcal{A}$ generates an analytic semi-group $E(t)=\exp(-t\mathcal{A})$ (see, e.g. [30,51]), and therefore, the solution $u(t)$ of the parabolic problem (2.2) becomes analytic with respect to t in the open interval $(0,T]$. It follows from the analyticity that $(D_t^l=\partial^l/\partial t^l, l=0,1,2,\ldots)$ [30]

$$\|D_t^lE(t)v\|=\|\mathcal{A}^lE(t)v\|\le C_lt^{-l}\|v\|,\quad 0<t\le T,\ v\in H. \tag{2.6}$$

We shall use the ensuing smoothing property

$$\|D_t^l E(t)v\|_s \le C_l t^{-l-(s-\theta)/2}\|v\|_\theta, \quad t>0,\ v\in\mathcal{D}(\mathcal{A}^{\theta/2}),\ 0\le\theta\le s\le 2,\ l=0,1. \tag{2.7}$$

Due to the non-smoothness of the initial data u^0, singularities may arise at $t=0$. In [60], on basis of the analyticity (2.6) and the smoothing property (2.7) of the analytic semigroup $E(t)$, the following lower time regularity has been provided.

THEOREM 2.1 (Lower time regularity [60]). *Let $\mathfrak{B}\subset V$ be a bounded set and $0<t^*\le T$ and $F\in C^2(0,T;H)$. If $u(t)\in\mathfrak{B}$ for $0\le t\le\min\{1,t^*\}$, then*

$$\begin{aligned}&\|u'(t)\|_\theta \le C(\mathfrak{B},t^*)t^{-1-(\theta-1)/2}, \quad t\in(0,t^*], \quad \theta\in[0,2),\\ &\|u^{(l)}(t)\|\le C(\mathfrak{B},t^*)t^{-l+1/2}, \quad t\in(0,t^*], \quad l=1,2,3.\end{aligned}$$

Since our IMEX RK methods can be arbitrarily high order, we need to provide higher time regularity results. To do this, we first show the following lemmas.

LEMMA 2.1. *Assume that the operators $\mathcal{B}$ and $\mathcal{J}$ satisfy (2.4) and (2.5). If*

$$\beta C_P + C_J C_P^2 < 1, \tag{2.8}$$

then the solution to (2.2)-(2.3) satisfies

$$\int_0^t \|u\|^2 ds \le \|u(0)\|_*^2 + C\int_0^t \|F\|^2 ds, \tag{2.9}$$

where C is a constant depending only on β, C_P and C_J.

Proof. Taking in (2.2) the inner product with $2\mathcal{A}^{-1}u$, using (2.1), (2.4) and (2.5), we have

$$\frac{d}{dt}(\mathcal{A}^{-1}u,u)+2\|u\|^2 \le \beta C_P\|u\|^2 + C_J C_P^2\|u\|^2 + C_P\|u\|\|F\|_*.$$

When the condition (2.8) holds, it follows after integration that

$$(\mathcal{A}^{-1}u(t),u(t)) + \int_0^t \|u\|^2 ds \le \|u(0)\|_*^2 + C\int_0^t \|F\|_*^2 ds.$$

Taking the positivity of the operator $\mathcal{A}^{-1}$ into account, we obtain the desired result (2.9). ∎

LEMMA 2.2. *Under the assumptions of Lemma 2.1, for any $\varepsilon>0$, we have*

$$\|u(t)\| \le \varepsilon \sup_{s\le t}(s\|F'(s)\|_*) + C_\varepsilon\left(t^{-1/2}\|u(0)\|_* + \sup_{s\le t}\|F(s)\|_*\right).$$

Proof. Taking in (2.2) the inner product with $2\mathcal{A}^{-1}u_t$ yields

$$\begin{aligned}2(\mathcal{A}^{-1}u_t,u_t)+\frac{d}{dt}\|u\|^2 &= -2(\mathcal{B}u,\mathcal{A}^{-1}u_t) - 2(\mathcal{J}u,\mathcal{A}^{-1}u_t) + 2(F,\mathcal{A}^{-1}u_t)\\ &\le 2\beta\|u\|\|u_t\|_* + 2C_J\|u\|_*\|u_t\|_* + 2\frac{d}{dt}(F,\mathcal{A}^{-1}u) - 2(F',\mathcal{A}^{-1}u),\end{aligned}$$

which implies

$$\frac{d}{dt}\|u\|^2 \le C_1\|u\|^2 + 2\frac{d}{dt}(F,\mathcal{A}^{-1}u) - 2(F',\mathcal{A}^{-1}u), \tag{2.10}$$

where $C_1:=\beta^2+C_J^2C_P^2$. Multiplying by t on both sides of (2.10), we obtain

$$\frac{d}{dt}\left(t\|u\|^2\right)\le C_1t\|u\|^2+2\frac{d}{dt}\left[t(F,\mathcal{A}^{-1}u)\right]-2t(F',\mathcal{A}^{-1}u)+\|u\|^2-2(F,\mathcal{A}^{-1}u). \quad (2.11)$$

Applying Gronwall lemma to (2.11) yields

$$t\|u(t)\|^2\le C\left(t\|F\|_*\|u\|_*+\int_0^t\left(s\|F'\|_*\|u\|_*+\|u\|^2+\|F\|_*\|u\|_*\right)ds\right).$$

By Lemma 2.1, we obtain, for any $\varepsilon>0$,

$$\|u(t)\|^2\le\frac{\varepsilon^2}{t}\int_0^t s^2\|F'\|_*^2ds+C_\varepsilon\left(\|F\|_*^2+\frac{1}{t}\|u(0)\|_*^2+\frac{1}{t}\int_0^t\|F\|_*^2ds\right)$$

and hence the desired result. ∎

Now we show the following theorem on higher time regularity of the solution to (2.2)-(2.3).

THEOREM 2.2 (Higher time regularity). *Let $\mathfrak{B}\subset V$ be a bounded set and $0<t^*\le T$ and $F\in C^2(0,T;H)$. If $u(t)\in\mathfrak{B}$ for $0\le t\le\min\{1,t^*\}$, then, under the condition (2.8),*

$$\|u^{(l)}(t)\|\le C(\mathfrak{B},t^*,l)t^{-l+1/2},\quad t\in(0,t^*],\quad l=1,2,\ldots. \quad (2.12)$$

Proof. We proceed by induction. We first note that (2.12) is satisfied for $l=1,2,3$ by Theorem 2.1. Let $k\ge3$ and assume that (2.12) holds for $l\le k-1$. We now show that (2.12) holds for $l=k$. To show that (2.12) holds for $l=k$, we set $v_k(t)=t^ku^{(k)}(t)$, $k=1,2,\ldots$, which satisfies

$$v_k'+\mathcal{A}v_k+\mathcal{B}v_k+\mathcal{J}(v_k)=kt^{k-1}u^{(k)}+t^kF^{(k)},\quad t\in(0,t^*];\qquad v_k(0)=0.$$

Noting $\mathcal{A}^{-1}u^{(k)}=\mathcal{A}^{-1}F^{(k-1)}-u^{(k-1)}-\mathcal{A}^{-1}\mathcal{B}u^{(k-1)}-\mathcal{J}(\mathcal{A}^{-1}u^{(k-1)})$, we use Lemma 2.2 to obtain, in view of $v_k(0)=0$,

$$\begin{aligned}\|v_k(t)\|\le&\varepsilon\sup_{s\le t}\Big(s\|s^kF^{(k+1)}+2ks^{k-1}F^{(k)}+k(k-1)s^{k-2}F^{(k-1)}\|_*\\&+sk\|(k-1)s^{k-2}u^{(k-1)}+s^{k-1}u^{(k)}\|\Big)\\&+C_\varepsilon\sup_{s\le t}\left(\|s^kF^{(k)}(s)+ks^{k-1}F^{(k-1)}(s)\|_*+ks^{k-1}\|u^{(k-1)}\|\right).\end{aligned}$$

Using our induction assumption, $\|u^{(k-1)}\|\le C(\mathfrak{B},t^*)t^{-k+3/2}$, choosing ε such that $\varepsilon k<1$, we obtain

$$\|v_k(t)\|\le C(\mathfrak{B},\bar\tau,l)t^{1/2}.$$

Then (2.12) holds for $l=k$ and the proof is completed. ∎

3. IMEX SBIRK methods for abstract PIDEs

In this section, we present the IMEX SBIRK methods for abstract PIDEs.

3.1. IMEX stage-based interpolation RK methods. For the given positive integers N, let the time interval $[0,T]$ be partitioned to $0=t_0<t_1<\ldots<t_{n-1}<t_n<t_{n+1}<\ldots<t_N=T$. Let $\tau_n=t_{n+1}-t_n, n=0,1,\ldots,N-1$, be the time step-sizes which in general will be variable. Let (A,b^T,c) denote a given RK method characterized by the $s\times s$ matrix $A=(a_{ij})$ and vectors $b=[b_1,\ldots,b_s]^T$, $c=[c_1,\ldots,c_s]^T$. In this paper we

always assume that $\sum_{j=1}^s b_j=1$ and $c_i=\sum_{j=1}^s a_{ij}\in[0,1]$. As one of the primary time discrete methods, the RK method (A,b^T,c) discretizes an initial value problem for ODEs

$$v'(t)=f(t,v(t)),\quad t\in(0,T],\qquad v(0)=v_0,$$

in the following way. For a given approximation v_n of the nodal value $v(t_n)$, one computes v_{n+1} by

$$\begin{aligned} V_i^{(n)}&=v_n+\tau_n\sum_{j=1}^s a_{ij}f(t_{n,j},V_j^{(n)}),\quad i=1,\dots,s,\\ v_{n+1}&=v_n+\tau_n\sum_{j=1}^s b_j f(t_{n,j},V_j^{(n)}),\end{aligned}$$

where $V_j^{(n)}$ are the internal stage value approximations of $v(t_{n,j})$ for $j=1,\dots,s$, with $t_{n,j}=t_n+c_j\tau_n$ being the internal RK nodes.

Now we construct IMEX SBIRK methods for solving the problem (2.2). To do this, let us consider Lagrange interpolation operator $\mathcal{I}^\tau$: for given internal stages $U_i^{(n-1)}$, $i=1,\dots,s$, the Lagrange interpolation polynomial $u_{n-1}^\tau(t):=\left(\mathcal{I}^\tau U^{(n-1)}\right)(t)$ of degree at most $s-1$ satisfies

$$u_{n-1}^\tau(t_{n-1,i})=U_i^{(n-1)},\quad i=1,\dots,s,$$

where $U^{(n-1)}=[U_1^{(n-1)},U_2^{(n-1)},\dots,U_s^{(n-1)}]^T$. We use the abbreviation $U_{n,j}^{n-1,\tau}:=u_{n-1}^\tau(t_{n,j})$, which approximates $u(t_{n,j})$ by the extrapolation method using the stage values $U_i^{(n-1)}$, $i=1,\dots,s$.

An s-stage RK method (A,b^T,c) for ODEs together with Lagrange interpolation operator $\mathcal{I}^\tau$ can now lead to an s-stage RK method $(A,b^T,c,\mathcal{I}^\tau)$ for solving problem (2.2) in PIDEs:

$$\begin{cases} U_i^{(n)}+\tau_n\sum_{j=1}^s a_{ij}[\mathcal{A}U_j^{(n)}+\mathcal{B}U_j^{(n)}+\mathcal{J}U_{n,j}^{n-1,\tau}]=u_n+\tau_n\sum_{j=1}^s a_{ij}F_j^n,\quad i=1,\dots,s,\\ u_{n+1}+\tau_n\sum_{j=1}^s b_j[\mathcal{A}U_j^{(n)}+\mathcal{B}U_j^{(n)}+\mathcal{J}U_{n,j}^{n-1,\tau}]=u_n+\tau_n\sum_{j=1}^s b_jF_j^n,\end{cases}$$

where $F_j^n:=F(t_{n,j})$. Using the Lagrange basis functions

$$l_j(t)=\prod_{m=1,m\neq j}^s\frac{(t-t_{n-1,m})}{(t_{n-1,j}-t_{n-1,m})},\quad t\in[t_n,t_{n+1}],$$

the s-stage RK method $(A,b^T,c,\mathcal{I}^\tau)$ can written as

$$\begin{cases} u_{n-1}^\tau(t)=\sum_{j=1}^s l_j(t)U_j^{(n-1)}, & t\in[t_n,t_{n+1}],\\ U^{(n)}+\tau_n(A\otimes I)[\mathcal{A}U^{(n)}+\mathcal{B}U^{(n)}+\mathcal{J}U^{n-1,\tau}]=(e\otimes I)u_n+\tau_n(A\otimes I)F^{(n)}, \\ u_{n+1}+\tau_n(b^T\otimes I)[\mathcal{A}U^{(n)}+\mathcal{B}U^{(n)}+\mathcal{J}U^{n-1,\tau}]=u_n+\tau_n(b^T\otimes I)F^{(n)},\end{cases}\tag{3.1}$$

where $e=[1,1,\dots,1]^T$, $U^{n-1,\tau}=[U_{n,1}^{n-1,\tau},U_{n,2}^{n-1,\tau},\dots,U_{n,s}^{n-1,\tau}]^T$, $F^{(n)}:=[F_1^n,F_2^n,\dots,F_s^n]^T$, $\otimes$ is the Kronecker product and I is the identity matrix.

It should be pointed out that to implement the IMEX SBIRK methods for PIDEs (2.2), we need the first step integral approximations u_1 and $U_i^{(0)}$, $i=1,\dots,s$, of (2.2), which will be obtained here by a low order method such as IMEX Euler method with small time stepsize (see, e.g., [60]) or a low order IMEX RK method with appropriate time stepsize (see, e.g., [13]).

For a vector $b=[b_1,b_2,\dots,b_s]^T$, the symbol $b>0$ (respectively, $b\geq 0$) means that each component $b_i>0$ (respectively, $b_i\geq 0$), $i=1,2,\dots,s$, and, for a matrix M, the symbol $M>0$ (respectively, $M\geq 0$) means that this matrix is positive definite (respectively, nonnegative definite).

For any positive definite diagonal matrix $D=\operatorname{diag}(d_1,d_2,\dots,d_s)>0$, for any vectors $U=[U_1,U_2,\dots,U_s]\in H^s$ and $W=[W_1,W_2,\dots,W_s]\in H^s$, we use the notations

$$\langle U,W\rangle_D=\sum_{j=1}^{s} d_j(U_j,W_j), \qquad \|U\|_D=\langle U,U\rangle_D^{1/2},$$

to represent an inner product and the corresponding norm, and $\|A\|_D$ for the corresponding matrix norm. We also use the notation $\|U\|_{1,D}$ to denote

$$\|U\|_{1,D}=\left(\sum_{j=1}^{s} d_j\|U_j\|_1^2\right)^{\frac{1}{2}}, \quad U\in V^s.$$

To analyze the stability and convergence of IMEX SBIRK methods for PIDEs (2.2), we need several definitions.

Definition 3.1 ([28]). *For the inner product $\langle U,V\rangle_D$ and matrix $D=\operatorname{diag}(d_1,d_2,\dots,d_s)>0$, we then denote by $\sigma_D(A^{-1})$ the largest number σ such that*

$$\langle U,A^{-1}U\rangle_D\geq\sigma\langle U,U\rangle_D, \qquad \text{for all } U\in H^s. \tag{3.2}$$

We also set

$$\sigma_0(A^{-1}):=\sup_{D>0}\sigma_D(A^{-1}).$$

Definition 3.2 ([14, 40]). *The method (A,b^T,c) is said to be algebraically stable if*

$$b\geq 0, \quad M=\operatorname{diag}(b)A+A^T\operatorname{diag}(b)-bb^T\geq 0.$$

3.2. Existence and uniqueness of IMEX SBIRK solutions. We first show that algebraic system (3.1) has at least one solution by following the approach of Hundsdorfer and Spijker [32] (see also [21]), which is based on the uniform monotonicity theorem. For this purpose, we first notice a fact that the interpolation operator $\mathcal{I}^\tau$ has the following property: For a positive definite diagonal matrix $D=\operatorname{diag}(d_1,d_2,\dots,d_s)>0$, there exists a positive constant $C_D^{\mathcal{I}}$ such that

$$\|\mathcal{I}^\tau U^{(n-1)}\|\leq C_D^{\mathcal{I}}\|U^{(n-1)}\|_D, \quad U^{(n-1)}\in H^s. \tag{3.3}$$

Obviously, the constant $C_D^{\mathcal{I}}$ depends on the matrix D and the Lagrange basis functions $l_j(t)$, and can be computed by

$$C_D^I=d_{\min}^{-1}\sup_{t\in[t_0,T]}\sum_{k=1}^{s}|l_k(t)|, \qquad d_{\min}=\min_{1\leq i\leq s} d_i.$$

THEOREM 3.1 (Existence). *If the RK matrix A is invertible and $\nu\tau_n<\sigma_0(A^{-1})$ for a positive diagonal matrix D and $\nu:=\frac{1}{2}(\beta^2-C_P^2)$, then the algebraic system (3.1) has a solution for any problem (2.2)-(2.3).*

Proof. The proof is identical with that of Theorem 5.3.12 in [21] and therefore is omitted. ∎

Before we give the uniqueness of the solution to Equations (3.1), we present the following general results.

THEOREM 3.2. *Let u_{n-1}^τ, $U^{(n)}$ and u_{n+1} be given by (3.1) and consider perturbed values $\widehat{u}_{n-1}^\tau$, $\widehat{U}^{(n)}$ and $\widehat{u}^{n+1}$ satisfying*

$$\begin{cases} \widehat{u}_{n-1}^\tau(t)=\sum_{j=1}^{s} l_j(t)U_j^{(n-1)}+R_0(t), & t\in[t_n,t_{n+1}],\\ \widehat{U}^{(n)}+\tau_n(A\otimes I)[\mathcal{A}\widehat{U}^{(n)}+\mathcal{B}\widehat{U}^{(n)}+\mathcal{J}\widehat{U}^{n-1,\tau}]=(e\otimes I)u_n+\tau_n(A\otimes I)F^{(n)}+R_1, \\ \widehat{u}_{n+1}+\tau_n(b^T\otimes I)[\mathcal{A}\widehat{U}^{(n)}+\mathcal{B}\widehat{U}^{(n)}+\mathcal{J}\widehat{U}^{n-1,\tau}]=u_n+\tau_n(b^T\otimes I)F^{(n)}+R_2, \end{cases} \tag{3.4}$$

where $R_0\in C[t_n,t_{n+1}]$, $R_1=[R_{1,1},\dots,R_{1,s}]^T\in H^s$, $R_2\in H$ are any given perturbations, and $\widehat{U}^{n-1,\tau}=\widehat{u}_{n-1}^\tau(t_{n,j})$. If the RK matrix A is invertible, and $\nu\tau_n<\sigma_D(A^{-1})$ for a positive diagonal matrix D, then we have

$$||\widehat{U}^{(n)}-U^{(n)}||\le C_D(\tau_n)||R_1||_D+\sqrt{d}L_D(\tau_n)C_JC_D^{\mathcal{I}}\tau_n\max_{t\in[t_n,t_{n+1}]}||R_0(t)||, \tag{3.5}$$

$$\begin{aligned} ||\widehat{u}_{n+1}-u_{n+1}||\le& C_{BD}||A^{-1}||_D(1+C_D(\tau_n))||R_1||_D+||R_2||\\ &+C_{BD}\sqrt{d}C_D(\tau_n)C_JC_D^{\mathcal{I}}\tau_n\max_{t\in[t_n,t_{n+1}]}||R_0(t)||, \end{aligned} \tag{3.6}$$

where

$$d=\sum_{i=1}^{s} d_i,\qquad L_D(\tau_n):=\frac{1}{\sigma_D(A^{-1})-\nu\tau_n},$$
$$C_{BD}:=\max_{1\le i\le s}\sqrt{b_i/d_i},\quad C_D(\tau_n):=L_D(\tau_n)||A^{-1}||_D.$$

Proof. With the notation $\Delta U=\widehat{U}^{(n)}-U^{(n)}$, the difference of the second equations in both (3.1) and (3.4) can be written as

$$\Delta U+\tau_n(A\otimes I)\big[\mathcal{A}(\Delta U)+\mathcal{B}(\Delta U)+\mathcal{J}\widehat{U}^{n-1,\tau}-\mathcal{J}U^{n-1,\tau}\big]=R_1. \tag{3.7}$$

Multiplying both sides of the Equation (3.7) by $\Delta U^T(DA^{-1}\otimes I)$ yields

$$\begin{aligned} &\Delta U^T(DA^{-1}\otimes I)\Delta U+\tau_n\Delta U^T(D\otimes I)\big[\mathcal{A}(\Delta U)+\mathcal{B}(\Delta U)+\mathcal{J}\widehat{U}^{n-1,\tau}-\mathcal{J}U^{n-1,\tau}\big]\\ =&\Delta U^T(DA^{-1}\otimes I)R_1. \end{aligned} \tag{3.8}$$

Then it follows from (2.4) and (2.5) that

$$\begin{aligned} &|\Delta U^T(D\otimes I)\big[\mathcal{B}(\Delta U)+\mathcal{J}\widehat{U}^{n-1,\tau}-\mathcal{J}U^{n-1,\tau}\big]|\\ \le&\frac{\beta^2}{2}\|\Delta U\|_D^2+\frac{1}{2}\|\Delta U\|_{1,D}^2+||\Delta U||_D||\mathcal{J}\widehat{U}^{n-1,\tau}-\mathcal{J}U^{n-1,\tau}||_D \end{aligned}$$

$$\leq \frac{\beta^2}{2}\|\Delta U\|_D^2 + \frac{1}{2}\|\Delta U\|_{1,D}^2 + ||\Delta U||_D \sqrt{d} C_J C_D^{\mathcal{I}} \sup_{t\in[t_n,t_{n+1}]} ||R_0(t)||, \tag{3.9}$$

where we have used the inequality

$$\left(\sum_{j=1}^{s} d_j ||R_0(t_{n,j})||^2\right)^{\frac{1}{2}} \leq \sqrt{d} \sup_{t\in[t_n,t_{n+1}]} ||R_0(t)||.$$

In view of (2.1), we have

$$\Delta U^T (D\otimes I)\mathcal{A}(\Delta U) = \|\Delta U\|_{1,D}^2 \geq \frac{1}{2}\|\Delta U\|_{1,D}^2 + \frac{C_P^2}{2}\|\Delta U\|_D^2.$$

Then using (3.2) and combining (3.8) and (3.9) yield

$$\begin{aligned}\sigma_D(A^{-1})||\Delta U||_D^2 \leq & \nu\tau_n ||\Delta U||_D^2 + ||\Delta U||_D ||A^{-1}R_1||_D \\ & + \tau_n \sqrt{d} C_J C_D^{\mathcal{I}} ||\Delta U||_D \sup_{t\in[t_n,t_{n+1}]} ||R_0(t)||.\end{aligned}$$

Thus it follows that

$$\left(\sigma_D(A^{-1}) - \nu\tau_n\right)||\Delta U||_D \leq ||A^{-1}R_1||_D + \tau_n\sqrt{d} C_J C_D^{\mathcal{I}} \max_{t\in[t_n,t_{n+1}]} ||R_0(t)||,$$

which implies (3.5).

Similarly, it can be deduced from the difference of the third equations in both (3.1) and (3.4) that

$$\begin{aligned}\widehat{u}_{n+1} - u_{n+1} &= -\tau_n (b^T\otimes I)\left[\mathcal{A}(\Delta U) + \mathcal{B}(\Delta U) + \mathcal{J}\widehat{U}^{n-1,\tau} - \mathcal{J}U^{n-1,\tau}\right] + R_2 \\ &= -(b^T A^{-1}\otimes I)(\Delta U - R_1) + R_2 \\ &= -(b^T D^{-1} D A^{-1}\otimes I)(\Delta U - R_1) + R_2.\end{aligned}$$

Hence

$$||\widehat{u}_{n+1} - u_{n+1}|| \leq C_{BD}||A^{-1}||_D(||\Delta U||_D + ||R_1||_D) + ||R_2||. \tag{3.10}$$

Then substitute (3.5) into (3.10) to obtain (3.6). The proof is completed. □

As a direct consequence of Theorem 3.2, we have the following uniqueness result by putting $R_0 = 0$, $R_1 = 0$, and $R_2 = 0$.

THEOREM 3.3 (Uniqueness). *Suppose the problem (2.2)-(2.3) satisfies the conditions (2.4) and (2.5). If the RK matrix A is invertible and $\nu\tau_n < \sigma_0(A^{-1})$ for a positive diagonal matrix D, then the system (3.1) possesses at most one solution.*

Another important conclusion can be made from Theorem 3.2 that the SBIRK method $(A, b^T, c, \mathcal{I}^\tau)$ with invertible matrix A is BSI-stable and BS-stable (see, e.g., [21, 26, 28, 40, 63]) in the following sense.

DEFINITION 3.3. *The method $(A, b^T, c, \mathcal{I}^\tau)$ is said to be BSI-stable, if there exist coefficients $\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3$, which depend only on the method, such that*

$$||\widehat{U}^{(n)} - U^{(n)}|| \leq \widetilde{c}_1 ||R_1||_D + \widetilde{c}_2 \tau_n C_J \max_{t\in[t_n,t_{n+1}]} ||R_0(t)||, \quad \nu\tau_n < \widetilde{c}_3.$$

and is said to be BS-stable, if there exist coefficients $\widehat{c}_1,\widehat{c}_2,\widehat{c}_3$, which depend only on the method, such that

$$||\widehat{u}_{n+1}-u_{n+1}||\leq \widehat{c}_1(||R_1||_D+||R_2||)+\widehat{c}_2\tau_n C_J \max_{t\in[t_n,t_{n+1}]}||R_0(t)||, \quad \nu\tau_n<\widehat{c}_3. \tag{3.11}$$

As a consequence, we can choose $\bar{\tau}\in(0,\sigma_D(A^{-1})-\nu\tau_n)$ and set

$$\widetilde{c}_1=\frac{\sqrt{d_{\max}}}{\sqrt{d_{\min}}}\sup_{0\leq\tau\leq\bar{\tau}}C_D(\tau), \quad \widetilde{c}_2=\frac{\sqrt{d}}{\sqrt{d_{\min}}}\sup_{0\leq\tau\leq\bar{\tau}}L_D(\tau), \quad \widetilde{c}_3=\sigma_D(A^{-1}),$$

$$\widehat{c}_1=\max\{(1+\widetilde{c}_1)\sqrt{b_{\max}}\|A^{-1}\|_D,1\}, \quad \widehat{c}_2=C_{BD}\sqrt{d}\sup_{0\leq\tau\leq\bar{\tau}}C_D(\tau), \quad \widehat{c}_3=\widetilde{c}_3,$$

where

$$d_{\max}=\max_{1\leq i\leq s}d_i, \quad b_{\max}=\max_{1\leq i\leq s}b_i,$$

such that the method $(A,b^T,c,\mathcal{I}^\tau)$ is BSI-stable and BS-stable.

4. Stability of IMEX SBIRK time semidiscrete scheme

In this section, we use the algebraic stability of the ODEs RK methods to show the stability of IMEX SBIRK methods for PIDEs (2.2)-(2.3). We have the following stability result.

THEOREM 4.1 (Stability). *Let $\{u_n\}$ denote the approximation sequence which is produced by using algebraically stable method (3.1), with the first step approximations u_1 and $U_i^{(0)}$, $i=1,\ldots,s$, to solve problem (2.2)-(2.3). If the method (3.1) satisfies $\sigma_B(A^{-1})>0$, then under the condition, with a constant $\xi_1\in(0,\sigma_B(A^{-1}))$,*

$$\tau_{\max}(\nu+C_J C_B^{\mathcal{I}})\leq\xi_1, \quad \tau_{\max}=\max_{1\leq n\leq N-1}\tau_n, \tag{4.1}$$

we have the following inequality

$$\begin{aligned}||u_n||^2\leq&\varphi(t_n)\max\{\|u_0\|^2,\|u_1\|^2\}\\&+\varphi(t_n)(t_n-t_1)\left[\mu\|U^{(0)}\|_B^2+(1+\mu\bar{\gamma}^2)\max_{1\leq j\leq n}\|F^{(j)}\|^2\right], \quad \forall n\geq 1,\end{aligned} \tag{4.2}$$

where

$$\varphi(t_n)=\exp(\mu\bar{\gamma}^2(t_n-t_1)), \quad \mu=(2\nu+C_J C_B^{\mathcal{I}}+1)^+ + C_J C_B^{\mathcal{I}}, \quad \bar{\gamma}=\frac{||A^{-1}||_B}{\sigma_B(A^{-1})-\xi_1}.$$

Here we used the standard notation $z^+=z$ for $z\geq 0$ and $z^+=0$ for $z<0$.

Proof. For the simplicity, we write

$$Q_n=\tau_n\left[\mathcal{A}(U^{(n)})+\mathcal{B}(U^{(n)})+\mathcal{J}U^{n-1,\tau}\right].$$

It follows from (3.1) that

$$U^{(n)}+(A\otimes I)Q_n=(e\otimes I)u_n+(A\otimes I)F^{(n)}, \qquad u_{n+1}+(b^T\otimes I)Q_n=u_n+(b^T\otimes I)F^{(n)}.$$

Then we have

$$\begin{aligned}(u_{n+1},u_{n+1}) &= (u_n+(b^T\otimes I)[-Q_n+F^{(n)}],u_n+(b^T\otimes I)[-Q_n+F^{(n)}])\\ &= ||u_n||^2+2(U^{(n)},(B\otimes I)[-Q_n+F^{(n)}])\\ &\quad -([-Q_n+F^{(n)}],(M\otimes I)[-Q_n+F^{(n)}]).\end{aligned}$$

Using the algebraic stability of the method, conditions (2.4), (2.5) and (3.3), and Cauchy-Schwarz inequality, we further get

$$\begin{aligned}||u_{n+1}||^2 &\le ||u_n||^2+2\nu\tau_n||U^{(n)}||_B^2+2C_J\tau_n||U^{(n)}||_B||U^{n-1,\tau}||_B+2\tau_n||U^{(n)}||_B||F^{(n)}||_B\\ &\le ||u_n||^2+2\nu\tau_n||U^{(n)}||_B^2+C_JC_B^{\mathcal{I}}\tau_n||U^{(n)}||_B^2\\ &\quad +C_JC_B^{\mathcal{I}}\tau_n||U^{(n-1)}||_B^2+\tau_n||U^{(n)}||_B^2+\tau_n||F^{(n)}||_B^2,\end{aligned}$$

which implies

$$||u_{n+1}||^2\le||u_n||^2+\mu\tau_nX_n^2+\tau_n||F^{(n)}||_B^2, \tag{4.3}$$

where $X_n:=\max_{0\le m\le n}\|U^{(m)}\|_B$. Now let us consider the following two cases successively.

Case 1. $X_n=\|U^{(k)}\|_B$, $1\le k\le n$. By the same argument as in Theorem 3.2, we have

$$\begin{aligned}\left[\sigma_B(A^{-1})-\nu\tau_k\right]||U^{(k)}||_B &\le ||A^{-1}||_B\|u_k\|+||A^{-1}||_B\|F^{(k)}\|_B+\tau_nC_JC_B^{\mathcal{I}}||U^{(k-1)}||_B\\ &\le ||A^{-1}||_B\|u_k\|+||A^{-1}||_B\|F^{(k)}\|_B+\tau_nC_JC_B^{\mathcal{I}}||U^{(k)}||_B.\end{aligned}$$

Then under the condition (4.1) one gets

$$\begin{aligned}X_n=||U^{(k)}||_B &\le \frac{||A^{-1}||_B}{\sigma_B(A^{-1})-\tau_n(\nu+C_JC_B^{\mathcal{I}})}(||u_k||+\|F^{(k)}\|_B)\\ &\le \bar{\gamma}(||u_k||+\|F^{(k)}\|_B).\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) yields

$$||u_{n+1}||^2\le(1+\mu\bar{\gamma}^2\tau_n)Y_n^2+(\mu\bar{\gamma}^2+1)\tau_nZ_n^2,$$

where $Y_n:=\max_{0\le m\le n}\|u_m\|$ and $Z_n:=\max_{1\le m\le n}\|F^{(m)}\|_B$.

Case 2. $X_n=\|U^{(0)}\|_B$. In this case, from (4.3), we have

$$||u_{n+1}||^2\le||u_n||^2+\mu\tau_n\|U^{(0)}\|_B^2+\tau_n||F^{(n)}||_B^2.$$

In both cases, we have

$$||u_{n+1}||^2\le(1+\mu\bar{\gamma}^2\tau_n)Y_n^2+\mu\tau_n\|U^{(0)}\|_B^2+(1+\mu\bar{\gamma}^2)\tau_nZ_n^2.$$

By induction, we easily get

$$\begin{aligned}||u_n||^2\le Y_n^2 &\le \prod_{j=1}^{n-1}(1+\mu\bar{\gamma}^2\tau_j)Y_1^2+\sum_{j=1}^{n-1}\prod_{k=j}^{n-2}(1+\mu\bar{\gamma}^2\tau_{i+1})\mu\tau_j\|U^{(0)}\|_B^2\\ &\quad +\sum_{j=1}^{n-1}\prod_{k=j}^{n-2}(1+\mu\bar{\gamma}^2\tau_{i+1})(1+\mu\bar{\gamma}^2)\tau_jZ_j^2.\end{aligned}$$

Since $1+\mu\bar{\gamma}^2\tau_j\le\exp(\mu\bar{\gamma}^2\tau_j)$, we obtain (4.2) and thus complete the proof. ∎

Remark 4.1. It is worth noting that if $\nu+C_JC_B^{\mathcal{I}}\le0$, then the method is unconditionally stable, that is, for any $\tau_n>0$, the method is stable.

5. Nonsmooth data error estimates for IMEX SBIRK methods

In this section, we derive the error bounds for IMEX SBIRK methods for PIDEs (2.2)-(2.3) with nonsmooth initial data. For this purpose, besides the algebraic stability, we assume that the RK method (A,b^T,c) has order $p\geq s$ and stage order at least q, i.e.,

$$B(p): \qquad \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \qquad k=1,\dots,p,$$
$$C(q): \qquad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \qquad k=1,\dots,q, \quad i=1,\dots,s.$$

The order p and the stage order q of three popular families of algebraically stable RK methods, i.e., the Gauss methods, the Radau IIA methods and the Lobatto IIIC methods, are presented in Table 5.1. Note that arbitrarily high order IMEX SBIRK

	Gauss	Radau IIA	Lobatto IIIC
p	$2s$	$2s-1$	$2s-2$
q	s	s	$s-1$

Table 5.1. *The order p and the stage order q of three popular families of algebraically stable RK methods.*

methods can be constructed for these three families of methods. It is well known that the one-stage members of these families are the midpoint (or Crank-Nicolson) and backward Euler methods, respectively. The tableaus of the two- and three-stage members of the Gauss, Radau IIA and Lobatto IIIC methods are given in [28] and [40].

To bound the global error $e_{n+1}:=u(t_{n+1})-u_{n+1}$, we define $\widetilde{u}_{n+1}$ as follows

$$\begin{cases} \widetilde{u}_{n-1}^{\tau}(t)=\sum_{j=1}^{s} l_j(t)u(t_{n-1,j}), & t\in[t_n,t_{n+1}],\\ \widetilde{U}^{(n)}+\tau_n(A\otimes I)[\mathcal{A}\widetilde{U}^{(n)}+\mathcal{B}\widetilde{U}^{(n)}+\mathcal{J}\widetilde{U}^{n-1,\tau}]=(e\otimes I)u(t_n)+\tau_n(A\otimes I)F^{(n)}, \\ \widetilde{u}_{n+1}+\tau_n(b^T\otimes I)[\mathcal{A}\widetilde{U}^{(n)}+\mathcal{B}\widetilde{U}^{(n)}+\mathcal{J}\widetilde{U}^{n-1,\tau}]=u(t_n)+\tau_n(b^T\otimes I)F^{(n)}, \end{cases} \tag{5.1}$$

where $\widetilde{U}^{n-1,\tau}=[\widetilde{U}_{n,1}^{n-1,\tau},\dots,\widetilde{U}_{n,s}^{n-1,\tau}]^T$ with $\widetilde{U}_{n,j}^{n-1,\tau}=\widetilde{u}_{n-1}^{\tau}(t_{n,j})$, $j=1,\dots,s$. Then the global error e_{n+1} can be split into

$$e_{n+1}=u(t_{n+1})-\widetilde{u}_{n+1}+\widetilde{u}_{n+1}-u_{n+1}.$$

The term $u(t_{n+1})-\widetilde{u}_{n+1}$ in the splitting is related to the consistency of the method, while the term $\widetilde{u}_{n+1}-u_{n+1}$ can be bounded since the method is stable.

5.1. Consistency estimate. We first estimate the consistency error $u(t_{n+1})-\widetilde{u}_{n+1}$. To do this, we introduce the global interpolation residual $E_0^n\in C[t_n,t_{n+1}]$, the local stage residuals $E_1^n\in H^s$, and the local residual $E_2^n\in H$ defined as

$$\begin{cases} u(t) = \sum_{j=1}^{s} l_j(t)u(t_{n-1,j})+E_0^n(t), \qquad t\in[t_n,t_{n+1}],\\ \qquad U(t_n)+\tau_n(A\otimes I)[\mathcal{A}U(t_n)+\mathcal{B}U(t_n)+\mathcal{J}(\mathcal{I}^{\tau}U(t_{n-1}))]\\ \quad = (e\otimes I)u(t_n)+\tau_n(A\otimes I)F^{(n)}+E_1^n,\\ \qquad u(t_{n+1})+\tau_n(b^T\otimes I)[\mathcal{A}U(t_n)+\mathcal{B}U(t_n)+\mathcal{J}(\mathcal{I}^{\tau}U(t_{n-1}))]\\ \quad = u(t_n)+\tau_n(b^T\otimes I)F^{(n)}+E_2^n, \end{cases} \tag{5.2}$$

where $U(t_n):=[u(t_{n,1}),\ldots,u(t_{n,s})]^T$ and $E_1^n:=[E_{1,1}^n,\ldots,E_{1,s}^n]^T$.

Then we have the following estimates.

THEOREM 5.1 (Consistency error). *Suppose that an IMEX SBIRK method $(A,b^T,c,\mathcal{I}^\tau)$ satisfying the order condition $B(p)$ and the stage order condition $C(q)$ is applied to PIDEs (2.2)-(2.3) with $u^0\in\mathfrak{B}\subset V$. Then, under the condition (2.8), for $t_n\le\min\{1,t^*\}$ with $0<t^*\le T$, the following consistency estimate holds*

$$\tau_n\max_{t_n\le t\le t_{n+1}}\|E_0^n(t)\|+\|E_1^n\|+\|E_2^n\|\le d_0\tau_n^{q+1}t_n^{-q-1/2}, \tag{5.3}$$

with d_0 depending only on the method and $C(\mathfrak{B},t^*,q)$.

Proof. From the second equation in (5.2), we have

$$\begin{aligned}&u(t_{n,i})-u(t_n)+\tau_n\sum_{j=1}^s a_{ij}[\mathcal{L}u(t_{n,j})-F_j^n]\\ &=\tau_n\sum_{j=1}^s a_{ij}[\mathcal{J}u(t_{n,j})-\mathcal{J}(\mathcal{I}^\tau U(t_{n-1}))]+E_{1,i}^n.\end{aligned}\tag{5.4}$$

Let us denote by $\widetilde{E}_{1,i}^n$, $i=1,\ldots,s$, the quantity on the left-hand side of (5.4). By Taylor expansion about t_n, we obtain

$$\begin{aligned}\widetilde{E}_{1,i}^n=&\sum_{k=1}^s\frac{\tau_n^k}{(k-1)!}\left(\frac{c_i^k}{k}-\sum_{j=1}^s a_{ij}c_j^{k-1}\right)u^{(k)}(t_n)+\frac{1}{s!}\int_{t_n}^{t_{n,i}}(t_{n,i}-t)^s u^{(s+1)}(t)dt\\ &-\frac{\tau_n}{(s-1)!}\sum_{j=1}^s a_{ij}\int_{t_n}^{t_{n,j}}(t_{n,j}-t)^{s-1}u^{(s+1)}(t)dt,\quad i=1,\ldots,s.\end{aligned}$$

Using the stage order conditions $C(q)$, we find that leading terms of order up to q vanish, and $\widetilde{E}_{1,i}^n$ can be represented in the form

$$\widetilde{E}_{1,i}^n=\tau_n^s\int_{t_n}^{t_{n+1}}P_i\left(\frac{t-t_n}{\tau_n}\right)u^{(s+1)}(t)dt,$$

with the bounded Peano kernels

$$P_i(t):=\frac{1}{s!}((c_i-t)^+)^s-\frac{1}{(s-1)!}\sum_{j=1}^s a_{ij}((c_j-t)^+)^{s-1},\quad i=1,\ldots,s,\quad 0\le t\le 1.$$

In view of the time regularity (2.12) of the solution $u(t)$, $\widetilde{E}_{1,i}^n$ can be bounded by

$$\|\widetilde{E}_{1,i}\|\le C\tau_n^{q+1}t_n^{-q-1/2},\quad n\ge 1. \tag{5.5}$$

Similarly, taking the time regularity of the solution $u(t)$ into account, we see that the interpolation error $u(t_{n,j})-(\mathcal{I}^\tau U(t_{n-1}))(t_{n,j})$ due to s-point extrapolation is

$$\|u(t_{n,j})-(\mathcal{I}^\tau U(t_{n-1}))(t_{n,j})\|\le C\tau_n^q t_n^{-q+1/2},\quad j=1,\ldots,s, \tag{5.6}$$

which further implies that

$$\|\mathcal{J}u(t_{n,j})-\mathcal{J}(\mathcal{I}^\tau U(t_{n-1}))(t_{n,j})\|\le C\tau_n^q t_n^{-q+1/2},\quad j=1,\ldots,s. \tag{5.7}$$

Combining (5.4) with (5.5) and (5.7) yields the desired final estimates for $E_{1,i}^n$,

$$\|E_{1,i}^n\| \le C\tau_n^{q+1} t_n^{-q-1/2}, \quad j=1,\dots,s. \tag{5.8}$$

Now we turn to estimate $\|E_2^n\|$. From the third equation in (5.2), we obtain

$$u(t_{n+1})-u(t_n)+\tau_n\sum_{j=1}^{s} b_j[\mathcal{L}u(t_{n,j})-F_j^n] = \tau_n\sum_{j=1}^{s} b_j[\mathcal{J}u(t_{n,j})-\mathcal{J}(\mathcal{I}^\tau U(t_{n-1}))]+E_2^n. \tag{5.9}$$

Denoting the quantity on the left-hand side of (5.9) by $\widetilde{E}_2^n$, employing Taylor expansion about t_n, we get

$$\widetilde{E}_2^n = \sum_{k=1}^{p}\frac{\tau_n^k}{(k-1)!}\left(\frac{1}{k}-\sum_{j=1}^{s} b_j c_j^{k-1}\right)u^{(k)}(t_n)+\frac{1}{p!}\int_{t_n}^{t_{n+1}}(t_{n+1}-t)^p u^{(p+1)}(t)dt$$
$$-\frac{\tau_n}{(p-1)!}\sum_{j=1}^{s} b_j\int_{t_n}^{t_{n,j}}(t_{n,j}-t)^{p-1}u^{(p+1)}(t)dt.$$

Because of the order conditions $B(p)$, leading terms of order up to p vanish, and $\widetilde{E}_2^n$ can be represented in the form

$$\widetilde{E}_2^n = \tau_n^p\int_{t_n}^{t_{n+1}} P\left(\frac{t-t_n}{\tau_n}\right)u^{(p+1)}(t)dt,$$

with the bounded Peano kernels

$$P(t) := \frac{1}{p!}(1-t)^p - \frac{1}{(p-1)!}\sum_{j=1}^{s} b_j((c_j-t)^+)^{p-1}, \quad 0\le t\le 1.$$

Using the result on the time regularity of solution $u(t)$, we obtain the estimates for $\widetilde{E}_2^n$,

$$\|\widetilde{E}_2^n\| \le C\tau_n^{p+1}t_n^{-p-1/2}, \quad n\ge 1. \tag{5.10}$$

Then the desired final estimate for E_2^n can be obtained by combining (5.9) with (5.7) and (5.10)

$$\|E_2^n\| \le C\tau_n^{q+1}t_n^{-q-1/2}, \quad n\ge 1. \tag{5.11}$$

Now the estimate (5.3) is a direct result of (5.6), (5.8) and (5.11). The proof is complete. ∎

It is useful to note the case when $q=s-1$, i.e., the method is a Lobatto IIIC method, we have from (5.3)

$$\tau_n \max_{t_n\le t\le t_{n+1}}\|E_0^n(t)\|+\|E_1^n\|+\|E_2^n\| \le d_0\tau_n^s t_n^{-s+1/2};$$

and the case when $q=s$, i.e., the method is a Gauss method or a Radau IIA method, we get similarly

$$\tau_n \max_{t_n\le t\le t_{n+1}}\|E_0^n(t)\|+\|E_1^n\|+\|E_2^n\| \le d_0\tau_n^{s+1}t_n^{-s-1/2}.$$

It should be also pointed out that when $t_n\ge 1$, from (5.3) we obtain the estimate

$$\tau_n \max_{t_n\le t\le t_{n+1}}\|E_0^n(t)\|+\|E_1^n\|+\|E_2^n\| \le d_0\tau_n^{q+1}.$$

Now we are ready to give the estimates of $||u(t_{n+1})-\widetilde{u}_{n+1}||$.

THEOREM 5.2. *Suppose that a BS-stable IMEX SBIRK method* $(A,b^T,c,\mathcal{I}^\tau)$ *satisfying the order condition* $B(p)$ *and the stage order condition* $C(q)$ *is applied to PIDEs* (2.2)-(2.3) *with* $u^0\in\mathfrak{B}\subset V$. *Then, under the condition* (2.8), *for* $t_n\le\min\{1,t^*\}$ *with* $0<t^*\le T$, *the following consistency estimate holds*

$$||u(t_{n+1})-\widetilde{u}_{n+1}||\le\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2},\quad \tau_n\le\tau_{\max}, \tag{5.12}$$

where the constant $\widetilde{d}$, $\tau_{\max}$ *depends only on the method and* $C(\mathfrak{B},t^*,q)$.

Proof. Because the method $(A,b^T,c,\mathcal{I}^\tau)$ is BS-stable, there exist constants $\widehat{c}_1,\widehat{c}_2,\widehat{c}_3>0$, which depend only on the method, such that all the requirements of Definition 3.3 are satisfied. Then it follows from (5.2), (3.11), and (5.3) that

$$\begin{aligned}||u(t_{n+1})-\widetilde{u}_{n+1}||&\le\widehat{c}_1(||E_1^n||_D+||E_2^n||)+\widehat{c}_2C_J\tau_n\max_{t_n\le t\le t_{n+1}}||E_0^n(t)||\\&\le\widehat{c}_1\left(d_0\tau_n^{q+1}t_n^{-q-1/2}\right)+\widehat{c}_2C_J\tau_nd_0\tau_n^qt_n^{-q+1/2}\\&\le\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2},\end{aligned}$$

where

$$\widetilde{d}=\widehat{c}_1d_0+\widehat{c}_2C_Jd_0.$$

This completes the proof of the theorem. □

5.2. Global error estimate. The following theorem provides an estimate of $||\widetilde{u}_{n+1}-u_{n+1}||$, whose proof is different from the usual one, and plays a key role in the nonsmooth data error estimates.

THEOREM 5.3. *Suppose that an IMEX SBIRK method* $(A,b^T,c,\mathcal{I}^\tau)$, *which is algebraically stable and satisfies* $\sigma_B(A^{-1})>0$, *is applied to PIDEs* (2.2)-(2.3). *Then for any constant* $\xi_2\in(0,\sigma_B(A^{-1}))$ *such that*

$$\nu\tau_{\max}\le\xi_2,\qquad 0<\tau_{\max}<1, \tag{5.13}$$

we have

$$||\widetilde{u}_{n+1}-u_{n+1}||\le(1+\mu_1\tau_n)^{\frac{1}{2}}||u(t_n)-u_n||+\mu_2\tau_n||U(t_{n-1})-U^{(n-1)}||_B,\quad\forall n\ge1, \tag{5.14}$$

where

$$\begin{aligned}&\mu_1:=4\nu^+\gamma_1^2,\quad \mu_2:=\max\left\{C_JC_B^{\mathcal{I}}\gamma_1,\left(2C_JC_B^{\mathcal{I}}\gamma_2+4\nu^+\gamma_2^2\right)^{\frac{1}{2}}\right\},\\&\gamma_1:=\frac{||A^{-1}||_B}{\sigma_B(A^{-1})-\xi_2},\quad \gamma_2:=\frac{C_JC_B^{\mathcal{I}}}{\sigma_B(A^{-1})-\xi_2}.\end{aligned}$$

Proof. Proceeding as in the proof of Theorem 4.1, we first have

$$\begin{aligned}||\widetilde{u}_{n+1}-u_{n+1}||^2\le&||u(t_n)-u_n||^2+2\nu\tau_n||\widetilde{U}^{(n)}-U^{(n)}||_B^2\\&+2C_JC_B^{\mathcal{I}}\tau_n||\widetilde{U}^{(n)}-U^{(n)}||_B||U(t_{n-1})-U^{(n-1)}||_B^2.\end{aligned}\tag{5.15}$$

By the same argument as in Theorem 3.2, we have

$$[\sigma_B(A^{-1})-\nu\tau_n]||\widetilde{U}^{(n)}-U^{(n)}||_B$$

$$\leq ||A^{-1}||_B \|u(t_n)-u_n\| + \tau_n C_J C_B^{\mathcal{I}} ||U(t_{n-1})-U^{(n-1)}||_B.$$

Then under the condition (5.13) one gets

$$\|\widetilde{U}^{(n)}-U^{(n)}\|_B \leq \frac{||A^{-1}||_B}{\sigma_B(A^{-1})-\xi_2}||u(t_n)-u_n|| + \frac{\tau_n C_J C_B^{\mathcal{I}}}{\sigma_B(A^{-1})-\xi_2}||U(t_{n-1})-U^{(n-1)}||_B. \tag{5.16}$$

Substitute (5.16) into (5.15) to obtain

$$\begin{aligned}||\widetilde{u}_{n+1}-u_{n+1}||^2 \leq &(1+4\nu^+\gamma_1^2\tau_n)||u(t_n)-u_n||^2 + 4\nu^+\gamma_2^2\tau_n^3||U(t_{n-1})-U^{(n-1)}||_B^2\\ &+2C_J C_B^{\mathcal{I}}\gamma_1\tau_n||u(t_n)-u_n||||U(t_{n-1})-U^{(n-1)}||_B\\ &+2C_J C_B^{\mathcal{I}}\gamma_2\tau_n^2||U(t_{n-1})-U^{(n-1)}||_B^2.\end{aligned} \tag{5.17}$$

Using the definition of μ_2 and the fact that the right-hand side of (5.17) is smaller than the square of the right-hand side of (5.14), we get the desired estimate (5.14) and complete the proof. ∎

Combining Theorems 5.2 and 5.3, we get following error estimate.

THEOREM 5.4 (Global error estimate). *Suppose that an IMEX SBIRK method $(A,b^T,c,\mathcal{I}^\tau)$ satisfying the order condition $B(p)$ and the stage order condition $C(q)$ is applied to PIDEs (2.2)-(2.3) with $u^0\in\mathfrak{B}\subset V$. If the method (A,b^T,c) is algebraically stable and satisfies $\sigma_B(A^{-1})>0$, and $t_n\leq\min\{1,t^*\}$ with $0<t^*\leq T$, then under the conditions (2.8), (4.1) and (5.13), we have the following estimate*

$$||u(t_n)-u_n|| \leq \mathcal{E}_n, \tag{5.18}$$

where

$$\begin{aligned}\mathcal{E}_n =: &\widetilde{\varphi}(t_n-t_1)\|u(t_1)-u_1\| + \mu_2\widehat{\varphi}(t_n-t_1)\|U(t_0)-U^{(0)}\|_B\\ &+\sum_{j=1}^{n-1}\widetilde{\varphi}(t_n-t_{j+1})\left(d_2H_j\tau_j + \widetilde{d}\tau_j^{q+1}t_j^{-q-\frac{1}{2}}\right),\end{aligned}$$

and

$$\begin{aligned}&\widetilde{\varphi}(z):=\exp(d_1z/2),\quad d_1=\mu_1+2(1+\mu_1)^{\frac{1}{2}}\mu_2\bar{\gamma}+\mu_2^2\bar{\gamma}^2,\quad d_2=\mu_2(\gamma_3+\bar{\gamma})d_0,\\ &\widehat{\varphi}(s)=\frac{2}{d_1}[\widetilde{\varphi}(s)-1],\quad H_n=\max_{1\leq i\leq n-1}\tau_i^{q+1}t_i^{-q-1/2},\quad \gamma_3:=\frac{C_J C_B^{\mathcal{I}}}{\sigma_B(A^{-1})-\xi_1}.\end{aligned}$$

Proof. Combining (5.12) and (5.14) leads to

$$\begin{aligned}||u(t_{n+1})-u_{n+1}|| \leq &\|u(t_{n+1})-\widetilde{u}_{n+1}\| + \|\widetilde{u}_{n+1}-u_{n+1}\|\\ &\leq (1+\mu_1\tau_n)^{\frac{1}{2}}||u(t_n)-u_n|| + \mu_2\tau_n||U(t_{n-1})-U^{(n-1)}||_B\\ &+\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2}.\end{aligned} \tag{5.19}$$

Now let us define $\widetilde{X}_{n-1}:=\max_{0\leq m\leq n-1}||U(t_m)-U^{(m)}||_B$ and consider the following two cases successively.

Case 1. $\widetilde{X}_{n-1}=\|U(t_i)-U^{(i)}\|_B$, $1\leq i\leq n-1$. Then using the same argument as in Theorem 3.2, one gets

$$\left(\sigma_B(A^{-1})-\nu\tau_i\right)||U^{(i)}-U(t_i)||_B \leq ||A^{-1}||_B\left(||u(t_i)-u_i||+||E_1^i||_B\right)$$

$$+\tau_i C_J C_B^{\mathcal{I}}\big(||U^{(i-1)}-U(t_{i-1})||_B+\max_{t\in[t_i,t_{i+1}]}||E_0^i(t)||\big).$$

In view of the condition (4.1), it follows from Theorem 5.1 that

$$\begin{aligned}||U^{(i)}-U(t_i)||_B &\le \bar{\gamma}\big(||E_1||+||u(t_i)-u_i||\big)+\tau_i\gamma_3\max_{t\in[t_i,t_{i+1}]}||E_0(t)||\\ &\le \bar{\gamma}||u(t_i)-u_i||+\big(\gamma_3+\bar{\gamma}\big)d_0H_n.\end{aligned} \tag{5.20}$$

Substituting (5.20) into (5.19) yields

$$||u(t_{n+1})-u_{n+1}||\le(1+d_1\tau_n)^{\frac{1}{2}}\widetilde{Y}_n+d_2H_n\tau_n+\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2},$$

where $\widetilde{Y}_n:=\max_{1\le m\le n}\|u(t_m)-u_m\|$.

Case 2. $\widetilde{X}_{n-1}=\|U(t_0)-U^{(0)}\|_B$. In this case, from (5.19), we have

$$||u(t_{n+1})-u_{n+1}||\le(1+\mu_1\tau_n)^{\frac{1}{2}}||u(t_n)-u_n||+\mu_2\tau_n||U(t_0)-U^{(0)}||_B+\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2}.$$

In both cases, we have

$$\begin{aligned}||u(t_{n+1})-u_{n+1}||\le&(1+d_1\tau_n)^{\frac{1}{2}}\widetilde{Y}_n+\mu_2\tau_n||U(t_0)-U^{(0)}||_B\\ &+d_2H_n\tau_n+\widetilde{d}\tau_n^{q+1}t_n^{-q-1/2}.\end{aligned}$$

By induction, we obtain

$$\begin{aligned}||u(t_n)-u_n||\le&\prod_{j=1}^{n-1}(1+d_1\tau_j)^{\frac{1}{2}}\|u(t_1)-u_1\|+\sum_{j=1}^{n-1}\prod_{m=j+1}^{n-1}(1+d_1\tau_m)^{\frac{1}{2}}\left(d_2H_j\tau_j+\widetilde{d}\tau_j^{q+1}t_j^{-q-\frac{1}{2}}\right)\\ &+\sum_{j=1}^{n-1}\prod_{m=j+1}^{n-1}(1+d_1\tau_m)^{\frac{1}{2}}\mu_2\tau_j\|U(t_0)-U^{(0)}\|_B\\ \le&\widetilde{\varphi}(t_n-t_1)\|u(t_1)-u_1\|+\sum_{j=1}^{n-1}\widetilde{\varphi}(t_n-t_{j+1})\left(d_2H_j\tau_j+\widetilde{d}\tau_j^{q+1}t_j^{-q-\frac{1}{2}}\right)\\ &+\sum_{j=1}^{n-1}\widetilde{\varphi}(t_n-t_{j+1})\mu_2\tau_j\|U(t_0)-U^{(0)}\|_B.\end{aligned}$$

Employing

$$\sum_{j=1}^{n-1}\widetilde{\varphi}(t_n-t_{j+1})\tau_j\le\sum_{j=1}^{n-1}\int_{t_j}^{t_{j+1}}\widetilde{\varphi}(t_n-s)ds\le\frac{2}{d_1}\left[\widetilde{\varphi}(t_n-t_1)-1\right], \tag{5.21}$$

we obtain (5.18) and complete the proof. □

Theorem 5.4 reveals that the global error estimator $\mathcal{E}_n$ depends on the first step error $\mathcal{E}_n^1=\widetilde{\varphi}(t_n-t_1)\|u(t_1)-u_1\|+\mu_2\widehat{\varphi}(t_n-t_1)\|U(t_0)-U^{(0)}\|_B$ and thereafter the SBIRK approximation error $\mathcal{E}_n^2=\sum_{j=1}^{n-1}\widetilde{\varphi}(t_n-t_{j+1})\left(d_2H_j\tau_j+\widetilde{d}\tau_j^{q+1}t_j^{-q-\frac{1}{2}}\right)$. As a consequence, to obtain a optimal global error, we should balance the two errors, $\mathcal{E}_n^1$ and $\mathcal{E}_n^2$. Then if the first step integral approximations u_1 and $U_i^{(0)}$, $i=1,\ldots,s$, of (2.2), are obtained by a low order method such as IMEX Euler method (see, e.g., [60]) and IMEX RK method (see, e.g., [13]), it is beneficial to take smaller time steps near $t=0$ due to the nonsmoothness of the initial data u^0.

Several possible choices for time grids were thus provided in the literature (see, e.g., [7, 56, 60]). Since the graded mesh $t_n = T(n/N)^{\varpi}$ with $\varpi \geq 1$ has been widely used for nonsmooth initial data (see, e.g., [7, 55, 60]), we will discuss the error bounds of IMEX SBIRK methods for PIDEs (2.2)-(2.3) on a graded mesh in next subsection.

5.3. Error estimates on a graded mesh. In this subsection we consider a special time grid on which the graded time steps are defined by $t_n = T(n/N)^{\varpi}$ with $\varpi \geq 1$, and specialize the error estimates obtained in Theorem 5.4. Note that when $\varpi = 1$, it is a uniform time grid. We first consider the term $\mathcal{E}_n^{2,2} =: \sum_{j=1}^{n-1} \widetilde{\varphi}(t_n - t_{j+1}) \widetilde{d} \tau_j^{q+1} t_j^{-q-\frac{1}{2}}$ in (5.18). On the one hand, because of $\tau_n = t_{n+1} - t_n \leq T\varpi N^{-1}(n/N)^{\varpi-1}$ in this case, it can be simplified to

$$\begin{aligned}
\mathcal{E}_n^{2,2} \leq & \widetilde{\varphi}(t_n - t_2)\widetilde{d} \sum_{j=1}^{n-1} \left[T\varpi N^{-1}(j/N)^{\varpi-1}\right]^{q+1} \left[T(j/N)^{\varpi}\right]^{-q-\frac{1}{2}} \\
\leq & \widetilde{\varphi}(t_n - t_2)\widetilde{d} T^{1/2} \varpi^{q+1} N^{-q} \sum_{j=1}^{n-1} N^{-1}(j/N)^{\frac{1}{2}\varpi-(q+1)} \\
\leq & \begin{cases} CN^{-\varpi/2}, & \varpi/2 < q, \\ CN^{-q}\log N, & \varpi/2 = q, \\ CN^{-q}, & \varpi/2 > q. \end{cases}
\end{aligned} \tag{5.22}$$

On the other hand, observing that $\widetilde{\varphi}(t_n - t_{j+1}) \geq 1$, we find easily that

$$\begin{aligned}
\mathcal{E}_n^{2,2} \geq & \widetilde{d} T^{1/2} N^{-\frac{1}{2}\varpi} \sum_{j=1}^{n-1} \left[(1+j)^{\varpi} - j^{\varpi}\right]^{q+1} j^{-\frac{1}{2}\varpi - q\varpi)} \\
\geq & \widetilde{d} T^{1/2} N^{-\frac{1}{2}\varpi}.
\end{aligned} \tag{5.23}$$

Now we turn to estimate $\mathcal{E}_n^{2,1} =: \sum_{j=1}^{n-1} \widetilde{\varphi}(t_n - t_{j+1}) d_2 H_j \tau_j$. After performing a bit of algebra, we can show that $\tau_j^{q+1} t_j^{-q-\frac{1}{2}}$ is decreasing on a graded mesh and thus get $H_j = \tau_1^{q+1} t_1^{-q-\frac{1}{2}} = TN^{-\frac{1}{2}\varpi}$. In view of (5.21), we have the estimate

$$\mathcal{E}_n^{2,1} \leq d_2 \widehat{\varphi}(t_n - t_1) H_n \leq d_2 \widehat{\varphi}(t_n - t_1) TN^{-\frac{1}{2}\varpi}. \tag{5.24}$$

For the first step error $\mathcal{E}_n^1$ on a graded mesh, if the first step integral approximations u_1 and $U_i^{(0)}$, $i = 1, \ldots, s$, of (2.2) are obtained by a numerical method of order q_1, then as shown in [60] for IMEX Euler method, we obtain

$$\mathcal{E}_n^1 \leq C(\widetilde{\varphi}(t_n - t_1) + \mu_2 \widehat{\varphi}(t_n - t_1)) N^{-\frac{1}{2}q_1\varpi}. \tag{5.25}$$

These results can be summed up as Corollary 5.1.

Corollary 5.1 (Error estimate on a graded mesh). *Under the conditions of Theorem 5.4, if the time steps are defined by $t_n = T(n/N)^{\varpi}$ with $\varpi \geq 1$ and the first step integral* *approximations u_1 and $U_i^{(0)}$, $i=1,\ldots,s$, of (2.2) are obtained by a numerical method with convergence order $q_1 \geq 1$, then we have the following estimate*

$$\widetilde{d}T^{1/2}N^{-\frac{1}{2}\varpi} \leq ||u(t_n)-u_n|| \leq \begin{cases} CN^{-\varpi/2}, & \varpi/2<q, \\ CN^{-q}\log N, & \varpi/2=q, \\ CN^{-q}, & \varpi/2>q. \end{cases} \tag{5.26}$$

We end this section with some remarks.

Firstly, we note that in Theorem 5.4 and Corollary 5.1, we assume that $t_n<1$. This is because singularities may arise at $t=0$, and the time regularity results in Theorems 2.1 and 2.2 are valid for $t<1$. As for the numerical solution on $t_n \geq 1$, it can be shown from Theorem 5.4 that the variable step-size IMEX SBIRK methods have qth-order convergence, since the solution to PIDEs (2.2)-(2.3) generally has higher regularity for $t_n>1$.

The second remark is about the step-sizes τ_n. We note that if $\nu+C_JC_B^{\mathcal{I}} \leq 0$, then algebraically stable IMEX SBIRK methods are unconditionally convergent, that is, for any $\tau_n>0$, they are convergent.

The third remark is about the graded mesh. Corollary 5.1 reveals that when $t_n = T(n/N)^{\varpi}$, the convergence rate of the IMEX SBIRK method behaves explicitly as a function of the number of time steps N in terms of ϖ and q, and the size of ϖ acts as a limiter on the convergence rate. Corollary 5.1 also suggests that for a graded mesh with $t_1=TN^{-\varpi}$, already a single time step is sufficient and the error achieved by taking the first step using this low order scheme such as IMEX Euler method is $O(N^{-\frac{1}{2}\varpi})$. This means that for a sufficiently large value of ϖ the convergence order in N of the overall scheme is not affected by taking this low-order method in the first step.

6. Fully discrete approximation

The analysis above can be carried over to the fully discretized case with either finite difference or finite element or spectral methods. Here we consider finite difference method since it is one of the most commonly used methods in computational finance; See, e.g., [5, 12, 17, 24, 25, 29, 39, 48, 52, 60, 61]. Without loss of generality, we take the two-dimensional case as an example.

We first describe the discretization of the spatial derivatives terms, that is

$$\begin{aligned}\mathcal{A}u(t,x,y)+\mathcal{B}u(t,x,y) =&\alpha_{11}\partial_{xx}u(t,x,y)+\alpha_{12}\partial_{xy}u(t,x,y)+\alpha_{22}\partial_{yy}u(t,x,y)\\ &+\beta_1\partial_x u(t,x,y)+\beta_2\partial_y u(t,x,y)+\gamma u(t,x,y),\\ &x\in(X_l,X_r), \quad y\in(0,Y_r).\end{aligned} \tag{6.1}$$

To do this, we use the spatial mesh $X_l=x_0<\ldots<x_{m-1}<x_m<x_{m+1}<\ldots<x_{M_x}=X_r$ and $0=y_0<\ldots<y_{l-1}<y_l<y_{l+1}<\ldots<y_{M_y}=Y_r$. Let $h_m=x_m-x_{m-1}$ and $k_l=y_l-y_{l-1}$. Then the derivatives in (6.1) are approximated by central difference quotients

$$\begin{aligned}&\frac{\partial u}{\partial x}(t,x_m,y_l) \approx \delta_x u_{m,l}(t) := \frac{u_{m+1,l}(t)-u_{m-1,l}(t)}{h_m+h_{m+1}},\\ &\frac{\partial^2 u}{\partial x^2}(t,x_m,y_l) \approx \delta_{xx} u_{m,l}(t) := \frac{2[h_m u_{m+1,l}(t)-(h_m+h_{m+1})u_{m,l}(t)+h_{m+1}u_{m-1,l}(t)]}{h_m h_{m+1}(h_m+h_{m+1})},\\ &\frac{\partial^2 u}{\partial x\partial y}(t,x_m,y_l) \approx \delta_{xy} u_{m,l}(t) := \frac{u_{m+1,l+1}(t)-u_{m+1,l-1}(t)-u_{m-1,l+1}(t)+u_{m-1,l-1}(t)}{(h_m+h_{m+1})(k_l+k_{l+1})},\end{aligned}$$

where $u_{m,l}(t)$ is the approximation of $u(t,x_m,y_l)$. The approximations to $\frac{\partial}{\partial y}u(t,x,y)$ and $\frac{\partial^2}{\partial y^2}u(t,x,y)$ could be defined in the same fashion. As a consequence, the operators

$\mathcal{A}$ and $\mathcal{B}$ in (6.1) at (t,x_m,y_l) can be approximated by

$$\mathcal{A}_h u_{m,l}(t)=\alpha_{11}\delta_{xx}u_{m,l}(t)+\alpha_{12}\delta_{xy}u_{m,l}(t)+\alpha_{22}\delta_{yy}u_{m,l}(t)+\gamma u_{m,l}(t);$$

and

$$\mathcal{B}_h u_{m,l}(t)=\beta_1\delta_x u_{m,l}(t)+\beta_2\delta_y u_{m,l}(t).$$

As for the integral operator $\mathcal{J}$,

$$\mathcal{J}u_{m,l}(t)=\int_0^{Y_r}\int_{X_l}^{X_r} u(t,x_m+z_1,y_l+z_2)g(z_1,z_2)dz_1dz_2,$$

we use compound trapezoidal rule to approximate it

$$\mathcal{J}_h u_{m,l}(t)=\sum_{i=0}^{M_x}{}''\sum_{j=0}^{M_y}{}'' g_{i,j}u_{m,l},$$

where

$$\sum_{j=0}^{M_y}{}'' g_{i,j}u_{m,l}:=\frac{1}{2}\Bigg(g_{i,0}u_{m+i,l}(t)k_1+2\sum_{j=1}^{M_y-1}g_{i,j}u_{m+i,l+j}(t)(k_j+k_{j+1}) \\ +g_{i,M_y}u_{m+i,l+M_y}(t)k_{M_y}\Bigg).$$

Then applying the IMEX SBIRK methods to the spatial semi-discrete systems

$$u'_{m,l}(t)+\mathcal{A}_h u_{m,l}(t)+\mathcal{B}_h u_{m,l}(t)+\mathcal{J}_h(u_{m,l}(t))=F_{m,l}(t),\quad t\in(0,T], \\ m=1,\ldots,M_x-1,\quad l=1,\ldots,M_y-1,$$

where $F_{m,l}(t)=f(t,x_m,y_l)+S_h(t,x_m,y_l)$ with $S_h(t,x_m,y_l)=O(h_x^2+k_y^2)$, $h_x=\max_m h_m$, $k_y=\max_l k_l$, being space discrete error, will lead to fully discrete schemes for (1.1). Letting $H=\mathbb{R}^{M_x\times M_y}$, then the corresponding results presented in Sections 3, 4 and 5, are also available for these fully discrete schemes.

7. Numerical experiments.

We now illustrate the theoretical results of the previous sections by using several numerical examples.

7.1. Jump-diffusion option pricing model.

We first price European option under Merton's and Kou's jump-diffusion models. Assuming that the underlying asset price S satisfies a stochastic process, the price $W(\tau,S)$ of European options, depending on time τ and underlying asset price S, satisfies a final value problem defined by the following PIDE (see, e.g., [18,31,45]):

$$\frac{\partial W}{\partial \tau}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 W}{\partial S^2}+(r-\lambda\kappa)S\frac{\partial W}{\partial S}-(r+\lambda)W+\lambda J(W(\tau,S))=0, \tag{7.1}$$

where λ is the Poisson arrival intensity, $\kappa=\mathbb{E}(\eta-1)$ denotes the average relative jump size, and $J(W(\tau,S))$ denotes the integral

$$J(W(\tau,S))=\int_0^\infty W(\tau,S\eta)\hat{g}(\eta)d\eta.$$

Here $g(\eta)$ is the probability density function of the jump amplitude η, satisfies $\hat{g}(\eta)\geq 0$ for all η, and $\int_0^\infty \hat{g}(\eta)d\eta = 1$.

The value W at expiry date is given by

$$W(T,S)=\phi(S), \quad S\in[0,\infty),$$

where $\phi(S)$ is the payoff function for the option contract. In the case of European option, the payoff function is

$$W(T,S)=\phi(S)=\begin{cases}(S-K)^+, & \text{in the case of call option,}\\ (K-S)^+, & \text{in the case of put option.}\end{cases} \tag{7.2}$$

By introducing new variables $x=\ln(S/K)$, $t=T-\tau$, $y=\ln\eta$ $(0<\eta<\infty)$, $W(T-\tau,S)=u(t,x)$, evaluation of the option values requires solving the PIDE

$$\frac{\partial u}{\partial t}-\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}-\left(r-\frac{1}{2}\sigma^2-\lambda\kappa\right)\frac{\partial u}{\partial x}+(r+\lambda)u-\lambda\int_{-\infty}^{+\infty}u(t,x+y)g(y)dy=0, \tag{7.3}$$

subject to the corresponding initial and boundary value conditions (see, for example, [60] for details). It can be seen from (7.2) that the corresponding initial function is weakly discontinuous at $x=0$ and belongs to the space $V:=\mathcal{D}(\mathcal{A}^{1/2})$ with H being the usual L^2 space and $\mathcal{A}=-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}+(r+\lambda)I$, where I is the identity operator. To construct a numerical scheme for approximation of the PIDE (7.3), we need to truncate the infinite domain $\mathbb{R}$ for x to be $\Omega:=(X_l,X_r)$ with a sufficiently small X_l and a sufficiently large X_r. Then on the truncated domain Ω, we solve the PIDE (1.1) with

$$f(t,x)=\lambda R(t,x)=\lambda\int_{\mathbb{R}\setminus\Omega}u(t,x+y)g(y)dy.$$

Because of the asymptotic behaviour of the option, for Merton's model and Kou's model, the remainder $R(t,x)$ can be computed directly (See, for example, [60]). It is also easy to verify that the conditions (2.4) and (2.5) are satisfied for this model (see, for example, [15, 18, 36, 60]).

In the following numerical examples, we consider uniform time-space grid and variable time-space grid [60, 61]. The variable time grid is accomplished by choosing the graded time steps $t_n=T(n/N)^{\varpi}$ with $\varpi\geq 1$.

For the space grid, we choose, for eliminating the singularity at $x=0$,

$$x(\zeta=0)=X_l, \quad x(\zeta=1)=X_r, \quad x(\zeta)=\widetilde{x}+\delta\sinh(a_2\zeta+a_1(1-\zeta)), \tag{7.4}$$

where δ is a prescribed uniformity parameter, $x_m:=x(\zeta_m)$, $\zeta_m=\frac{m-1}{M}$, $m=1,2,\cdot,M+1$, $a_1=\sinh^{-1}(\frac{X_l-\widetilde{x}}{\delta})$, $a_2=\sinh^{-1}(\frac{X_r-\widetilde{x}}{\delta})$, and $\widetilde{x}$ corresponds to the singular point $x=0$ here.

For the first step approximations u_1 and $U_i^{(0)}$, $i=1,\ldots,s$, of (2.2), we use IMEX Euler method and 2-stage second order IMEX RK method [13]

$$\begin{cases}U_i^{(0)}+\tau_1\sum_{j=1}^{i}a_{ij}[\mathcal{A}U_j^{(0)}+\mathcal{B}U_j^{(0)}]+\tau_1\sum_{j=1}^{i-1}\tilde{a}_{ij}\mathcal{J}U_j^{(0)}=u_0+\tau_1\sum_{j=1}^{i}a_{ij}F_j^0, \quad i=1,2,\\ u_1+\tau_1\sum_{j=1}^{2}b_j[\mathcal{A}U_j^{(0)}+\mathcal{B}U_j^{(0)}]+\tau_1\sum_{j=1}^{2}\tilde{b}_j\mathcal{J}U_j^{(0)}]=u_0+\tau_1\sum_{j=1}^{2}b_jF_j^0,\end{cases} \tag{7.5}$$

where

$$A := (a_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \tilde{A} := (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \quad (b_1, b_2) = (0,1), \quad (\tilde{b}_1, \tilde{b}_2) = (0,1).$$

Firstly, we price European options under Merton's jump-diffusion model. Let the parameters in the Merton's model be

$$\begin{array}{lllll} \sigma = 0.15, & r = 0.05, & \mu_{Me} = -0.9, & \sigma_{Me} = 0.45, & \\ \lambda = 0.1, & T = 0.25, & K = 100, & X_l = -1.5, & X_r = 1.5. \end{array}$$

It is well-known that when there are no jumps, the option value V_{BS} of Black-Scholes model can be computed by the formula:

$$V_{BS}(S,t,K,\zeta_n,\sigma_n) = \begin{cases} S\mathcal{N}(\varpi_1) - Ke^{-\zeta_n t}\mathcal{N}(\varpi_2), & \text{in the case of a call option,} \\ Ke^{-\zeta_n t}\mathcal{N}(-\varpi_2) - S\mathcal{N}(-\varpi_1), & \text{in the case of a put option,} \end{cases}$$

where $t \in [0,T]$, $S \in [S_{\min}, S_{\max}]$, $\sigma_n^2 = \sigma^2 + \frac{n\sigma_{\mathrm{Me}}^2}{t}$, $\zeta_n = r - \lambda\kappa + \frac{n}{t}(\mu_{\mathrm{Me}} + \frac{1}{2}\sigma_{\mathrm{Me}}^2)$, and

$$\varpi_1 = \frac{\ln(\frac{S}{K}) + (\zeta_n + \frac{\sigma_n^2}{2})t}{\sigma_n\sqrt{t}}, \qquad \varpi_2 = \frac{\ln(\frac{S}{K}) + (\zeta_n - \frac{\sigma_n^2}{2})t}{\sigma_n\sqrt{t}} = \varpi_1 - \sigma_n\sqrt{t}.$$

Based on this Black-Scholes formula, for Merton's model, the price of a European option can be expressed as an infinite sum [45]:

$$W(\tau,S) = \sum_{n=0}^{\infty} \frac{(\lambda' t)^n}{n!} e^{-\lambda' t} V_{BS}(S,t,K,\zeta_n,\sigma_n), \tag{7.6}$$

where $t = T - \tau$, $t \in [0,T]$, and $\lambda' = \lambda(1+\kappa)$. Then the reference solution can be calculated by the formula (7.6) with the first six terms in the sum from which we can obtain six digits of accuracy in the option price.

With the reference value given by the series solution (7.6), we present the pricing errors and the convergence orders of the IMEX SBIRK methods derived from 2-stage Radau IIA and Gauss methods with constant step-size (for short CS-Radau IIA and CS-Gauss, respectively) and variable step-size (for short VS-Radau IIA and VS-Gauss, respectively) in Tables 7.1 and 7.2, respectively. Here the space grid is given by (7.4) with $\delta = 0.5$ and the variable time steps are given by $t_n = T(n/N)^{\varpi}$ with $\varpi = 6$. The convergence order is calculated by

$$\text{Order} = \log_2(E_i^{N,M} / E_i^{2N,\sqrt{8}M}),$$

where $E_i^{N,M}$ denotes the error computed at the maturity date T and $x = x_i$ with N time sub-intervals and M spatial sub-intervals. The numerical results in Tables 7.1 and 7.2 indicate that the variable step-size 2-stage IMEX Radau IIA and Gauss methods with both first step integrator, IMEX Euler method and 2-stage second order IMEX RK method (7.5), have 3rd-order convergence at these points and there is little, if any, difference between the numerical results obtained by variable step-size methods with two different first step integrators. The constant step-size 2-stage IMEX Radau IIA and Gauss methods with IMEX Euler method for the first step integral only have 1rd-order convergence at these points.

First step	M	N	S=90 Error	Order	S=100 Error	Order	S=110 Error	Order
IMEX Euler	48	4	6.6415E-02		4.4765E-01		5.4593E-01	
	136	8	3.4523E-02	0.9439	1.8175E-01	1.3004	2.6783E-01	1.0274
	384	16	1.7433E-02	0.9857	8.5540E-02	1.0873	1.3339E-01	1.0057
	1086	32	8.7904E-03	0.9879	4.1976E-02	1.0270	6.6695E-02	1.0000
IMEX Euler	48	4	7.9028E-03		7.2744E-02		2.3247E-02	
	136	8	3.2880E-03	1.2652	8.7191E-03	3.0606	3.8594E-03	2.5906
	384	16	5.1668E-04	2.6699	1.0250E-03	3.0886	5.1768E-04	2.8982
	1086	32	7.0513E-05	2.8733	1.2252E-04	3.0645	6.8340E-05	2.9213
IMEX RK	48	4	4.2611E-03		9.7781E-02		1.9631E-02	
	136	8	9.6251E-04	2.1463	1.1614E-02	3.0738	2.6733E-03	2.8765
	384	16	1.2979E-04	2.8906	1.4527E-03	2.9990	3.4500E-04	2.9540
	1086	32	1.6957E-05	2.9363	1.8244E-04	2.9932	4.4715E-05	2.9478
IMEX RK	48	4	7.8336E-03		7.2420E-02		2.2724E-02	
	136	8	3.2869E-03	1.2529	8.7140E-03	3.0550	3.8512E-03	2.5609
	384	16	5.1666E-04	2.6694	1.0249E-03	3.0878	5.1756E-04	2.8955
	1086	32	7.0513E-05	2.8733	1.2252E-04	3.0644	6.8338E-05	2.9209

TABLE 7.1. *The pricing error of European call option under Merton model obtained by IMEX SBIRK Radau IIA method with nonuniform space grid ($\delta = 0.5$). Upper: uniform time grid; Second: graded time mesh with $\varpi = 6$; Third: uniform time grid; Bottom: graded time mesh with $\varpi = 6$.*

First step	M	N	S=90 Error	Order	S=100 Error	Order	S=110 Error	Order
IMEX Euler	48	4	6.6258E-02		4.4850E-01		5.4577E-01	
	136	8	3.4472E-02	0.9427	1.8182E-01	1.3026	2.6780E-01	1.0271
	384	16	1.7426E-02	0.9842	8.5552E-02	1.0877	1.3339E-01	1.0055
	1086	32	8.7894E-03	0.9874	4.1983E-02	1.0270	6.6695E-02	1.0000
IMEX Euler	48	4	6.4224E-03		1.2339E-01		2.0594E-02	
	136	8	6.3129E-04	3.3467	1.3756E-02	3.1652	2.2949E-03	3.1657
	384	16	7.5197E-05	3.0696	1.5521E-03	3.1478	3.0739E-04	2.9003
	1086	32	1.1197E-05	2.7476	1.9002E-04	3.0299	4.1303E-05	2.8958
IMEX RK	48	4	3.9889E-03		9.8279E-02		1.9468E-02	
	136	8	9.0863E-04	2.1342	1.1676E-02	3.0734	2.6484E-03	2.8779
	384	16	1.2240E-04	2.8920	1.3834E-03	3.0773	3.4172E-04	2.9542
	1086	32	1.6009E-05	2.9347	4.9632E-04	1.4788	4.4317E-05	2.9469
IMEX RK	48	4	6.3509E-03		1.2307E-01		2.0073E-02	
	136	8	6.3020E-04	3.3331	1.3750E-02	3.1619	2.2868E-03	3.1338
	384	16	7.5180E-05	3.0674	1.5520E-03	3.1473	3.0727E-04	2.8957
	1086	32	1.1196E-05	2.7473	1.9002E-04	3.0299	4.1301E-05	2.8952

TABLE 7.2. *The pricing error of European call option under Merton model obtained by IMEX SBIRK Gauss method with nonuniform space grid ($\delta = 0.5$). Upper: uniform time grid; Second: graded time mesh with $\varpi = 6$; Third: uniform time grid; Bottom: graded time mesh with $\varpi = 6$.*

Fist step	M	N	uniform time grid RMSE	Order	graded time mesh ($\varpi = 6$) RMSE	Order
IMEX Euler	48	4	4.0940E-01		4.4327E-02	
	136	8	1.8794E-01	1.1233	5.8232E-03	2.9283
	384	16	9.2041E-02	1.0299	7.2699E-04	3.0018
	1086	32	4.5780E-02	1.0075	9.0652E-05	3.0035
IMEX RK	48	4	5.7633E-02		4.4054E-02	
	136	8	6.9028E-03	3.0616	5.8186E-03	2.9205
	384	16	8.6527E-04	2.9960	7.2692E-04	3.0008
	1086	32	1.0889E-04	2.9903	9.0651E-05	3.0034

TABLE 7.3. *The* RMSE *of European call option under Merton model obtained by IMEX SBIRK Radau IIA method with nonuniform space grid ($\delta = 0.5$).*

			uniform time grid		graded time mesh ($\varpi = 6$)	
First step	M	N	RMSE	Order	RMSE	Order
IMEX Euler	48	4	3.6975E-01		4.1370E-02	
	136	8	1.8332E-01	1.0122	3.8512E-03	3.4252
	384	16	9.1477E-02	1.0029	3.1428E-04	3.6152
	1086	32	4.5712E-02	1.0008	2.7466E-05	3.5164
IMEX RK	48	4	5.9591E-02		4.0604E-02	
	136	8	3.0638E-02	0.9598	3.8482E-03	3.3994
	384	16	1.5487E-02	0.9843	3.1425E-04	3.6142
	1086	32	7.7727E-03	0.9946	2.7465E-05	3.5162

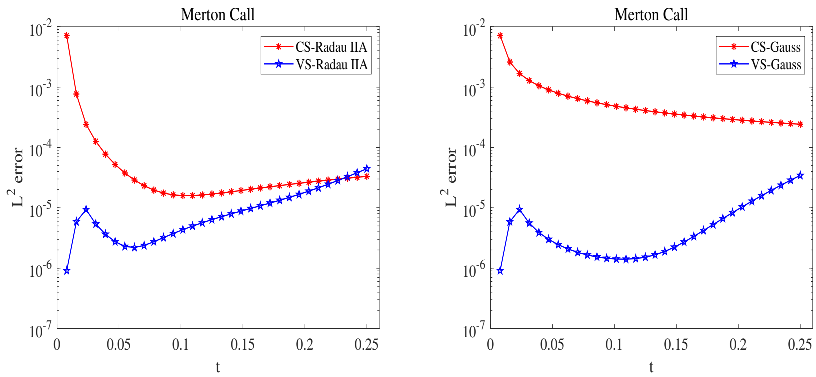
TABLE 7.4. *The* RMSE *of European call option under Merton model obtained by IMEX SBIRK Gauss method with nonuniform space grid ($\delta = 0.01$).*

To further demonstrate the impact of the first step integral on computational accuracy, we compute the root mean square error (RMSE) of the three points $S = \{90, 100, 110\}$, which is calculated by the formula

$$\text{RMSE} = \sqrt{\frac{1}{3}\sum_{i=1}^{3}\left(E_i^{N,M}\right)^2}.$$

Since the space singularity leads to the larger errors near the execution price $S = K$, the RMSE can be used to measure the numerical accuracy of the numerical methods. The RMSEs of the 2-stage Radau IIA and Gauss methods are presented in Tables 7.3 and 7.4, respectively. From Tables 7.3 and 7.4, we observe that for both constant step-size IMEX RK methods, the numerical results obtained by them with the first step 2-stage second order IMEX RK integral (7.5) have much higher accuracy than those obtained by them with the first step IMEX Euler integral. However, the distinction between the numerical results obtained by variable step-size methods with two different first step integrators is not sharp. These further confirm our theoretical result that for a sufficiently large value of ϖ the numerical results of the overall scheme is not affected by taking a low-order method in the first step. We also see that the errors of the variable step-size IMEX SBIRK methods are smaller than those of the constant step-size methods. To clearly illustrate that the proposed variable step-size IMEX SBIRK methods are more accurate, we also present the time evolution of the discrete L^2 errors for these methods in Figure 7.1. Here and after, taking into account the previous theoretical analysis and numerical observation, we only present the numerical results obtained by IMEX numerical methods with the 2-stage IMEX RK method (7.5) being the first step integrator.

It is interesting to compare the numerical results obtained by the IMEX SBIRK methods proposed here and the IMEX BDF2 method discussed in [60]. The errors at the singularity point $S = K$ for the Merton call option generated by the three IMEX methods under a non-uniform space-time grid ($\varpi = 3$ and $\delta = 0.2$) are shown in Figure 7.2. The numerical data show that the variable-step IMEX SBIRK methods have smaller errors at $S = K$ than the variable-step IMEX BDF2 method. For the sake of confirming this more clearly, we also present the RMSE and the time evolution of the discrete L^2 errors of Merton's call option produced by the three IMEX methods with graded time mesh ($\varpi = 6$) and nonuniform space grid ($\delta = 0.5$) in Figure 7.3. It can be inferred from the above comparison that the two variable step-size IMEX SBIRK methods have much higher accuracy than the variable step-size IMEX BDF2 method discussed in [60] when they are applied to Merton's option pricing model with nonsmooth payoff function.



(a) 2-stage IMEX Radau IIA method. (b) 2-stage IMEX Gauss method.

FIG. 7.1. *The times evolution of the discrete L^2 errors of Merton's call option produced by IMEX SBIRK methods with graded time mesh ($\varpi = 6$) and nonuniform space grid ($\delta = 0.01$), where $M = 1086$ and $N = 32$. Left: IMEX SBIRK Radau IIA method; Right: IMEX SBIRK Gauss method.*

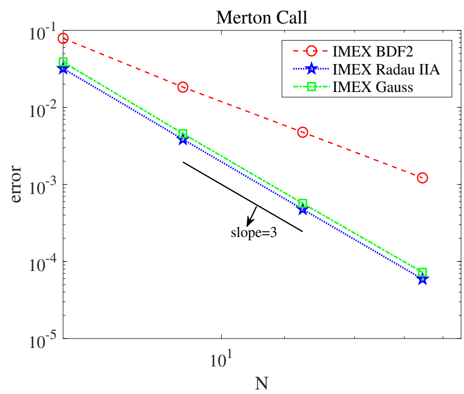
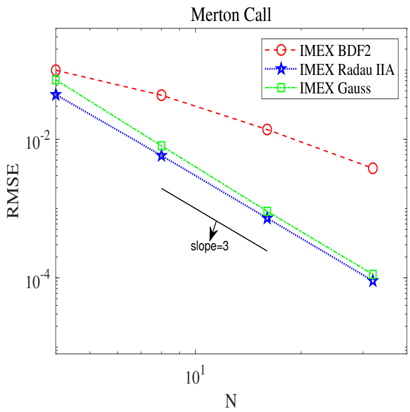


FIG. 7.2. *Errors at $S = K$ of Merton's call option produced by three variable time step-size IMEX methods ($\varpi = 3$) with nonuniform space grid ($\delta = 0.2$), where $N = \{4, 8, 16, 32\}$.*

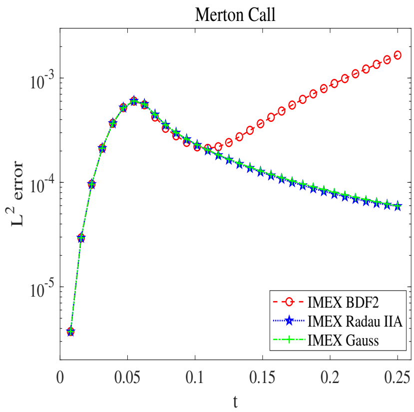
7.2. Stochastic volatility model with jump. We consider Bates model under European put options in which the asset prices S and its variance w satisfy stochastic differential equations [6, 31, 33, 43, 50, 53],

$$dS = \nu S d\tau + \sqrt{w} S dW_S + S dJ_S,$$
$$dw = \kappa(\theta - w)d\tau + \sigma\sqrt{w} dW_w,$$

where $0 \leq \tau \leq T$, $S(0), w(0) > 0$, $\nu = r_I - \lambda\xi_B$ is the drift rate, $\xi_B = e^{\mu_{\mathrm{Me}} + \sigma_{\mathrm{Me}}^2/2} - 1$, r_I is the risk-free interest rate, v represents the volatility of asset prices, σ is the volatility of w, θ represents the average level of w, and κ represents the mean reversion rate of w, the Wiener processes W_S and W_w have the correlation ρ, and J_S represents the compound Poisson process with the jumping strength λ satisfied by the asset price.



(a) The RMSE of numerical methods.



(b) The time evolution of the L^2 errors.

FIG. 7.3. *The RMSE and the time evolution of the discrete L^2 errors of Merton's call option produced by three IMEX methods with graded time mesh ($\varpi = 6$) and nonuniform space grid ($\delta = 0.5$). Left: The RMSE with $N = \{4,8,16,32\}$; Right: The time evolution of the discrete L^2 errors with $M = 1086$ and $N = 32$.*

The price function $W(\tau,S,w)$ of European option is given by PIDE

$$\frac{\partial W}{\partial \tau} + \frac{1}{2} w S^2 \frac{\partial^2 W}{\partial S^2} + \rho\sigma w S \frac{\partial^2 W}{\partial S \partial w} + \frac{1}{2}\sigma^2 w \frac{\partial^2 W}{\partial w^2} + (r_I - \lambda \xi_B) S \frac{\partial W}{\partial S} + \kappa(\theta - w)\frac{\partial W}{\partial w}$$
$$- (r_I + \lambda) W + \lambda \int_0^\infty W(\tau, S\eta, w) p(\eta) d\eta = 0,$$

subject to the boundary conditions

$$W(\tau,0,w) \to K e^{-r_I(T-\tau)}, \qquad \text{and} \quad W(\tau,S,w) \to 0, \ \textit{as} \ S \to +\infty,$$
$$\frac{\partial W(\tau,S,w)}{\partial w} \to 0, \ \textit{as} \ w \to 0, \quad \text{and} \quad \frac{\partial W(\tau,S,w)}{\partial w} \to 0, \ \textit{as} \ w \to +\infty,$$

where $p(\eta) = \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{Me}}\eta} e^{-[\ln \eta - \mu_{\mathrm{Me}}]^2 / 2\sigma_{\mathrm{Me}}^2}$. The payoff function at expiry date is given by

$$W(T,S,w) = (K-S)^+.$$

By introducing new variables $x = \ln(S/K)$, $y = w/\sigma$, $t = T - \tau$, $u(t,x,y) = e^{(r_I+\lambda)t} W(\tau,S,w)/K$, $z = \ln(\eta)$, we obtain

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma y \frac{\partial^2 u}{\partial x^2} - \rho\sigma y \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\sigma y \frac{\partial^2 u}{\partial y^2} - \left(r_I - \lambda\xi_B - \frac{1}{2}\sigma y \right) \frac{\partial u}{\partial x}$$
$$- \frac{\kappa(\theta - \sigma y)}{\sigma} \frac{\partial u}{\partial y} - \lambda \int_{-\infty}^{+\infty} u(t, x+z, y) g(z) dz = 0, \tag{7.7}$$

where $g(z) = e^z p(e^z)$. With new variables, the initial and boundary conditions become

$$u(0,x,y) = (1 - e^x, 0)^+,$$
$$u(t,x,y) \to 1, \ \textit{as} \ \ x \to -\infty, \quad \text{and}, \quad u(t,x,y) \to 0, \ \textit{as} \ x \to +\infty,$$

$$\frac{\partial u(t,x,y)}{\partial y} \to 0, \; as \;\; y \to 0, \quad \text{and,} \quad \frac{\partial u(t,x,y)}{\partial y} \to 0, \; as \; y \to +\infty.$$

Similar to jump-diffusion option pricing model, the infinite domain $(-\infty,+\infty)$ for x should be truncated to be $\Omega_x := (X_l, X_r)$ with a sufficiently small X_l and a sufficiently large X_r, and the infinite domain $(0,\infty)$ for y should be truncated to be $\Omega_y := (0, Y_r)$. On the truncated domain Ω_x, we impose artificial boundary conditions as follows,

$$u(t,X_l,y) \to 1 - e^{r_I t + X_l}, \quad \text{and,} \quad u(t,X_r,y) \to 0.$$

For the Neumann boundary condition on $y=0$ and $y=Y_r$, however, we use the following approximations

$$\frac{\partial u(t,x,y)}{\partial y}\Big|_{x_m,y=0} \approx \frac{u_{m,1}(t)-u_{m,0}(t)}{k_1}, \quad \frac{\partial u(t,x,y)}{\partial y}\Big|_{x_m,y=Y_r} \approx \frac{u_{m,M_y}(t)-u_{m,M_y-1}(t)}{k_{M_y}}.$$

As a consequence, we can discretise the Equation (7.7) on the truncated domain $\Omega := \Omega_x \times \Omega_y$ by using IMEX SBIRK with finite difference methods. It is useful to note that similar to jump-diffusion option pricing model, the remainder $R(t,x,y)$ can be expressed as

$$R(t,x,y) = \mathcal{N}\left(\frac{X_l - x - \mu_{\text{Me}}}{\sigma_{\text{Me}}}\right) - e^{r_I t + x + \mu_{\text{Me}} + \frac{\sigma_{\text{Me}}^2}{2}} \mathcal{N}\left(\frac{X_l - x - \mu_{\text{Me}} - \sigma_{\text{Me}}^2}{\sigma_{\text{Me}}}\right).$$

Now we consider 2-stage IMEX Lobatto IIIC, IMEX Radau IIA, IMEX Gauss methods

		graded time mesh ($\varpi=4$)			graded time mesh ($\varpi=6$)		
	N	RMSE	Order	CPU (s)	RMSE	Order	CPU (s)
IMEX Lobatto IIIC	4	1.0019E-02		12.339	1.4973E-02		12.503
	8	3.6745E-03	1.4471	26.645	6.3782E-03	1.2311	27.790
	16	1.1515E-03	1.6741	56.410	2.2672E-03	1.4923	63.400
IMEX Radau IIA	4	2.2853E-03		12.336	5.6874E-03		12.489
	8	4.0838E-04	2.4844	26.000	9.5675E-04	2.5716	27.287
	16	1.0568E-04	1.9502	56.410	1.1335E-04	3.0773	62.636
IMEX Gauss	4	6.2031E-03		12.224	1.4652E-02		12.558
	8	3.7619E-04	4.0435	25.820	1.3299E-03	3.4617	27.689
	16	6.3597E-05	2.5644	57.649	1.8175E-04	2.8713	62.925
IMEX BDF2	4	4.9547E-02		6.106	8.3204E-02		7.111
	8	1.3007E-02	1.9295	11.992	2.5122E-02	1.7277	14.530
	16	3.2875E-03	1.9842	20.665	6.5796E-03	1.9329	27.838

TABLE 7.5. *The* RMSE *of European put option under Bates model obtained by four IMEX methods with nonuniform space grid ($\delta=0.5$) and graded time mesh. Upper: IMEX Lobatto IIIC; Second: IMEX Radau IIA; Third: IMEX Gauss; Bottom: IMEX BDF2.*

with constant step-size and variable step-size for solving this model. Let the parameters in the Merton's model be [50]

$$\begin{array}{lllll} \sigma = 0.25, & r = 0.03, & \mu_{Me} = -0.5, & \sigma_{Me} = 0.4, & \rho = -0.5, \\ \lambda = 0.2, & T = 0.5, & K = 100, & \kappa = 2, & \theta = 0.04. \end{array}$$

In addition, let $S_{\max} = 4K$, $X_l = -X_r = -\ln(S_{\max}/K)$, and $w_{\max} = 0.5$. The first step approximations u_1 and $U_i^{(0)}$, $i = 1,2$, are obtained by the 2-stage IMEX RK method

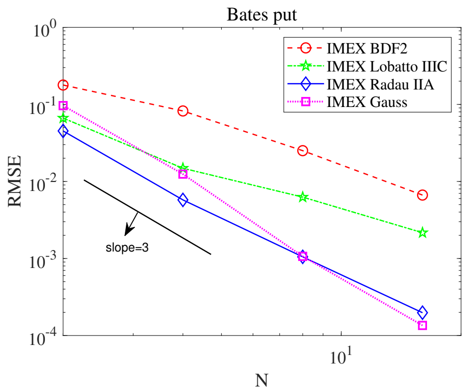


FIG. 7.4. *The RMSE of Bates put option produced by four variable time step-sizes IMEX methods ($\varpi=6$) under nonuniform space grid with $\delta=0.5$, where $N=\{2,4,8,16\}$.*

(7.5). In this example, RMSE of the three points $S=\{90,100,110\}$ with $w=0.04$ is calculated by the following formula

$$\text{RMSE}=\sqrt{\frac{1}{3}\sum_{i=1}^{3}\left(E_i^{N,M_x,M_y}\right)^2},$$

where E_i^{N,M_x,M_y} denotes the error computed at the maturity date T with $M_x\times M_y$ spatial subregions and N time subintervals. To illustrate the time convergence order of the IMEX SBIRK methods, different numbers N of the time steps, together with the same space grid, nonuniform x direction $M_x=513$ with $\delta=0.5$, uniform y direction $M_y=257$, are chosen. The reference value is calculated on a finer grid with $M_x=513$, $M_y=257$, and $N=256$ by the IMEX BDF2 method proposed in [60]. The numerical results are shown in Table 7.5. From Table 7.5 we observe that for this stochastic volatility model, all three variable step-size IMEX SBIRK methods, especially IMEX Radau IIA and IMEX Gauss methods, have higher accuracy than the variable step-size IMEX BDF2 method. Since the IMEX SBIRK methods need to solve higher dimensional algebraic equations than the IMEX BDF2 method, we also present their CPU times for a fair comparison. Then we observe that under almost the same approximation precision requirements, the variable step-size IMEX Lobatto IIIC method requires almost the same CPU time as the variable step-size IMEX BDF2 method, but the variable step-size IMEX Radau IIA and IMEX Gauss methods have less CPU time. These reveal the obvious advantages of high order methods, obtaining higher computational accuracy under fewer computational steps. In addition, we compare the RMSE of the three variable step-size IMEX SBIRK methods and the variable step-size IMEX BDF2 method under a non-uniform time-space grid ($\varpi=6$ and $\delta=0.5$). The numerical results presented in Figure 7.4 illustrate the advantage of the variable-step-size IMEX SBIRK method in terms of the computation accuracy.

8. Concluding remarks

Although many researchers have investigated the stability and error estimates of IMEX multistep methods for PIDEs and their variant because of the importance of these equations in the modeling of finance problems, the topic of error analysis for

IMEX RK methods for such types of equations with practical nonsmooth initial data remain unexplored. In our previous papers [43, 60, 61], the stability and error estimates of variable step-size IMEX multistep methods have been derived. In this paper, we proposed a class of IMEX SBIRK methods for the time discretization of the PIDEs (1.1). This class of methods can achieve arbitrarily high order, requires only the solution of a sparse system of linear equations at each time level and therefore they are extremely effective. The existence and uniqueness of the solution to the implicit algebraic equations were first investigated. The stability of this class of methods was then obtained. To derive the error estimates of this class of methods for solving PIDEs (1.1), higher time regularity results of the solution to abstract PIDEs (2.2) were first obtained based on the lower time regularity results obtained in [60] when the initial data is $u^0 \in V$. The higher time regularity results are found to be crucial in deriving the error estimates of numerical methods for parabolic problems with nonsmooth initial data.

Several numerical experiments for the variable step-size IMEX SBIRK methods, 2-stage IMEX Lobatto IIC, IMEX Radau IIA, IMEX Gauss, for financial models have been implemented. These numerical results suggest that the variable step-size IMEX SBIRK methods are more accurate than the corresponding constant step-size methods and demonstrate the prominent advantages of variable step-size higher order IMEX SBIRK methods compared to the variable step-size IMEX BDF2 method.

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