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A NOTE ON ENHANCED DISSIPATION OF TIME-DEPENDENT SHEAR FLOWS*

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Abstract. This paper explores the phenomena of enhanced dissipation in solutions to the passive scalar equations subject to time-dependent shear flows. The hypocoercivity functionals with carefully tuned time weights are applied in the analysis. We observe that as long as the critical points of the shear flow vary slowly, one can derive the sharp enhanced dissipation estimates, mirroring the ones obtained for the time-stationary case.

Keywords. Enhanced dissipation; Time-dependent Shear Flows.

AMS subject classifications. 35K10; 35Q35; 35Q92.

1. Introduction

In this paper, we consider the passive scalar equations

$$\partial_t f + V(t,y)\partial_x f = \nu\Delta_\sigma f, \quad f(t=0,x,y) = f_0(x,y). \tag{1.1}$$

Here f denotes the density of the substances, and $(V(t,y),0)$ is a time-dependent shear flow. The Péclet number $\nu > 0$ captures the ratio between the transport and diffusion effects in the process. Here $\Delta_\sigma = \sigma\partial_{xx} + \partial_{yy}, \sigma \in \{0,1\}$. We consider two types of domains: $\mathbb{T}\times\mathbb{R}, \mathbb{T}^2$. The torus $\mathbb{T}$ is normalized such that $\mathbb{T} = [-\pi,\pi]$.

In recent years, much research has been devoted to studying enhanced dissipation and Taylor dispersion phenomena associated with the Equation (1.1) in the regime $0 < \nu \ll 1$. To understand these phenomena, we first identify the relevant time scale of the problem. The standard L^2-energy estimate yields the following energy dissipation equality:

$$\frac{d}{dt}\|f\|_{L^2}^2 = -2\nu\sigma\|\partial_x f\|_{L^2}^2 - 2\nu\|\partial_y f\|_{L^2}^2. \tag{1.2}$$

Hence, at least formally, we expect that the energy (L^2-norm) of the solution decays to half of the original value on a long time scale $\mathcal{O}(\nu^{-1})$. This is called the "heat dissipation time scale". However, a natural question remains: since the fluid transportation can create gradient growth of the density ∇f, which makes the damping effect in (1.2) stronger, can one derive a better decay estimate of the solution to (1.1)? This question was answered by Lord Kelvin in 1887 for a special family of flows $V(t,y) = y$ (Couette flow) [31]. He could explicitly solve the Equation (1.1) and read the exact decay rate through the Fourier transform. To present his observation, we first restrict ourselves to the cylinder $\mathbb{T}\times\mathbb{R}$ or torus $\mathbb{T}^2$ and define the concepts of horizontal average and remainder:

$$\langle f\rangle(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx, \quad f_{\neq}(x,y) = f(x,y) - \langle f\rangle(y).$$

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We observe that the x-average $\langle f\rangle$ of the solution to (1.1) is also a solution to the heat equation. Hence it decays with rate ν. On the other hand, the remainder $f_{\neq}$ still solves the passive scalar Equation (1.1) with $f_{\neq}(t=0,x,y)=f_{0;\neq}(x,y)$ and something nontrivial can be said. Lord Kelvin showed that there exist constants $C,\delta>0$ such that the following estimate holds

$$\|f_{\neq}(t)\|_{L^2}\leq C\|f_{0;\neq}\|_{L^2}e^{-\delta\nu^{1/3}t},\quad \forall t\geq 0. \tag{1.3}$$

One can see that significant decay of the remainder happens on time scale $\mathcal{O}(\nu^{-1/3})$, which is much shorter than the heat dissipation time scale. This phenomenon is called the *enhanced dissipation*.

However, new challenges arise when one considers shear flows different from the Couette flow. In these cases, no direct Fourier analytic proof is available at this point. We focus on two families of shear flows, i.e., strictly monotone shear flows and non-degenerate shear flows. In the paper [5], J. Bedrossian and M. Coti Zelati apply hypocoercivity techniques to show that for *stationary* strictly monotone shear flows $\{(V(y),0)|\inf|V'(y)|\geq c>0, y\in\mathbb{R}\}$, the following estimate is available

$$\|f_{\neq}(t)\|_{L^2}\leq C\|f_{\neq;0}\|_{L^2}e^{-\delta\nu^{1/3}|\log\nu|^{-2}t},\quad \forall t\geq 0.$$

Later on, D. Wei applied resolvent estimate techniques to improve their estimate to (1.3) [34].

When we consider non-constant smooth shear flows on the torus $\mathbb{T}^2$, an important geometrical constraint has to be respected, namely, the shear profile V must have critical points $\mathcal{C}:=\{y_*|\partial_y V(y_*)=0\}$. Nondegenerate shear flows are a family of shear flows such that the second derivative of the shear profile does not vanish at these critical points, i.e., $\min_{y_*\in\mathcal{C}}|\partial_y^2 V(y_*)|\geq c>0$. In the papers [5, 34], it is shown that if the underlying shear flows are *stationary* and non-degenerate, there exist constants $C\geq 1$, $\delta>0$ such that

$$\|f_{\neq}(t)\|_{L^2}\leq C\|f_{0;\neq}\|_{L^2}e^{-\delta\nu^{1/2}t},\quad \forall t\in[0,\infty). \tag{1.4}$$

In the paper [18], it is shown that the enhanced dissipation estimates (1.3), (1.4) are sharp for *stationary* shear flows. In the paper [13, 14, 22], the authors rigorously justify the relation between the enhanced dissipation effect and the mixing effect. In the paper [1], the authors apply Hörmander hypoellipticity technique to derive the estimates (1.3), (1.4) on various domains. For further enhanced dissipation in other flow settings, we refer the interested readers to the papers [16, 21, 26], and the references therein. The enhanced dissipation effects have also found applications in many different areas, ranging from hydrodynamic stability to plasma physics, we refer to the following papers [2–4, 6–12, 15, 17, 19, 20, 23–25, 27–30, 32, 33, 35].

Most of the results we present thus far are centered around *stationary* flows. In this paper, we focus on *time-dependent* shear flows and hope to identify sufficient conditions that guarantee enhanced dissipation and Taylor dispersion. Before stating the main theorems, we provide some further definitions. After applying a Fourier transformation in the x-variable (1.16), we end up with the following k-by-k equation

$$\partial_t\widehat{f}_k(t,y)+V(t,y)ik\widehat{f}_k(t,y)=\nu\partial_y^2\widehat{f}_k-\sigma\nu|k|^2\widehat{f}_k(t,y),\quad \widehat{f}_k(t=0,y)=\widehat{f}_{0;k}(y). \tag{1.5}$$

We will drop the $\widehat{(\cdot)}$ notation later for simplicity. The main statements of our theorems are as follows:

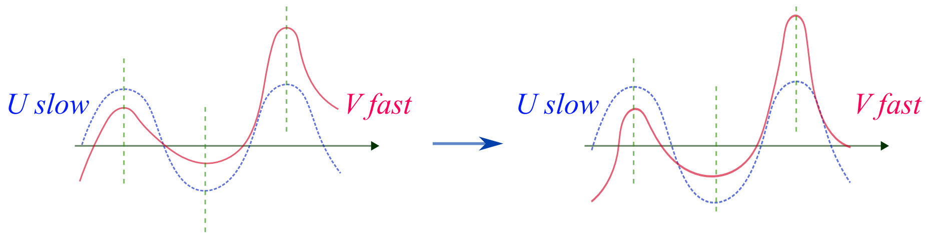

FIG. 1.1. *Relation between U,V. The reference U slowly varies, whereas the actual shear V can change fast. However, the two shears share the same critical points.*

THEOREM 1.1. *Consider the solution to the Equation (1.5) initiated from the initial data $f_0\in C_c^\infty(\mathbb{T}\times\mathbb{R})$. Assume that on the time interval $[0,T]$, the $C_tC_y^2$ velocity profile $V(t,y)$ satisfies the following constraint*

$$\inf_{t\in[0,T],\ y\in\mathbb{R}}|\partial_y V(t,y)|\geq \mathfrak{c}>0,\quad \|V\|_{L_t^\infty([0,T];W_y^{3,\infty})}<C. \tag{1.6}$$

Then there exists a threshold $\nu_0(V)$ such that for $\nu<\nu_0$, the following estimate holds

$$\|f_k(t)\|_{L^2}\leq e\|f_{0;k}\|_{L^2}\exp\left\{-\delta\nu^{1/3}|k|^{2/3}t\right\},\quad \forall t\in[0,T]. \tag{1.7}$$

Here $\delta>0$ are constants depending only on the parameter $\mathfrak{c}$ and $\|V\|_{L_t^\infty C_y^3}$ (2.11).

The next theorem is stated as follows.

THEOREM 1.2. *Consider the solution to the Equation (1.5) initiated from the smooth initial data $f_0\in C^\infty(\mathbb{T}^2)$. Assume that the shear flow $V(t,y)\in C_{t,y}^2$ satisfies the following structure assumptions on the time interval $[0,T]$:*

(a) *Phase assumption: There exists a nondegenerate reference shear $U\in C_t^1C_y^2$ such that the time-dependent flow $V(t,y)$ and the reference flow $U(t,y)$ share all their nondegenerate critical points $\{y_i(t)\}_{i=1}^N$, where N is a fixed finite number. Moreover,*

$$\begin{aligned}&\partial_y V(t,y)\partial_y U(t,y)\geq 0, && \forall y\in\mathbb{T},\ \forall t\in[0,T],\\ &\|\partial_{ty}U\|_{L_t^\infty([0,T];L_y^\infty)}\leq \nu^{3/4}, && \|V\|_{L_t^\infty([0,T];W_y^{2,\infty})}+\|U\|_{L_t^\infty([0,T];W^{2,\infty})}<C.\end{aligned} \tag{1.8}$$

(b) *Shape assumption: There exist N pairwise disjoint open neighborhoods $\{B_r(y_i(t))\}_{i=1}^N$ with fixed radius $0<r=\mathcal{O}(1)$, and two constants $\mathfrak{C}_0$, $\mathfrak{C}_1>1$ such that the following estimates hold for $Z(t,y)\in\{V(t,y),U(t,y)\}$,*

$$\mathfrak{C}_0^{-1}(y-y_i(t))^2\leq|\partial_y Z|^2\leq \mathfrak{C}_0(y-y_i(t))^2,\quad \mathfrak{C}_0>0,\quad \forall y\in B_r(y_i(t)); \tag{1.9}$$
$$0<\mathfrak{C}_1^{-1}\leq|\partial_y Z|\leq\mathfrak{C}_1,\quad \forall y\notin\cup_{i=1}^N B_r(y_i(t)). \tag{1.10}$$

Then there exists a threshold $\nu_0(U,V)$ such that if $\nu\leq\nu_0$, the following estimate holds

$$\|f_k(t)\|_{L^2}\leq e\|f_k(0)\|_{L^2}\exp\left\{-\delta\nu^{1/2}|k|^{1/2}t\right\},\quad \forall t\in[0,T], \tag{1.11}$$

with δ depending on the functions U,V. In particular, it depends only on the parameters specified in the conditions above.

Remark 1.1. We remark that if we consider the solution $V(t,y)=e^{-\nu t}\sin(y)$ to the heat equation $\partial_t V=\nu\partial_{yy}V$ on the torus, the structure conditions are satisfied for time $t\in[0,\mathcal{O}(\nu^{-1+})]$.

Remark 1.2. We remark that if we consider the solution in our analysis of the time-dependent shear flows, the dynamics of the critical points are crucial. The main theorem encodes the dynamics of the critical points in the reference shear U. The relation between U, V is highlighted in Figure 1.1. The condition $\|\partial_{ty}U\|_\infty\le\nu^{3/4}$ enforces that the critical points of the target shear V cannot move too fast. If this condition is violated, the fluid can trigger mixing and unmixing effects within a short time. Hence, it is not clear whether the enhanced dissipation phenomenon persists.

The hypocoercivity energy functional introduced in [5] is our main tool to prove the main theorems. However, we choose to incorporate time-weights introduced in the papers [35] into our setting. Let us define a parameter and two time weights

$$\epsilon:=\nu|k|^{-1},\quad \psi=\min\{\nu^{1/3}|k|^{2/3}t,1\},\quad \phi=\min\{\nu^{1/2}|k|^{1/2}t,1\}.$$

We observe that the derivatives of the time weights are compactly supported:

$$\psi'(t)=\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t),\quad \phi'(t)=\nu^{1/2}|k|^{1/2}\mathbb{1}_{[0,\nu^{-1/2}|k|^{-1/2}]}(t). \tag{1.12}$$

To prove Theorem 1.1, Theorem 1.2, we invoke the following hypocoercivity functionals

$$\text{Theorem 1.1: } \mathcal{F}[f_k]:=\|f_k\|_2^2+\alpha\psi\epsilon^{2/3}\ \|\partial_y f_k\|_2^2+\beta\psi^2\epsilon^{1/3}\ \Re\langle i\mathrm{sign}(k)f_k,\partial_y f_k\rangle; \tag{1.13}$$

$$\begin{aligned}\text{Theorem 1.2: } \mathcal{G}[f_k]:=&\|f_k\|_2^2+\alpha\phi\epsilon^{1/2}\ \|\partial_y f_k\|_2^2+\beta\phi^2\ \Re\langle i\mathrm{sign}(k)\partial_y U f_k,\partial_y f_k\rangle\\ &+\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f_k\|_2^2.\end{aligned} \tag{1.14}$$

Here, the inner product $\langle\cdot,\cdot\rangle$ is defined in (1.17).

Through detailed analysis, one can derive the following statements. (a) Assume all conditions in Theorem 1.1. There exist parameters $\alpha=\mathcal{O}(1),\beta=\mathcal{O}(1)$ such that the following estimate holds on the time interval $[0,T]$:

$$\mathcal{F}[f_k](t)\le C\mathcal{F}[f_{0;k}]\exp\left\{-\delta\nu^{1/3}|k|^{2/3}t\right\}=C\|f_{0;k}\|_2^2\exp\left\{-\delta\nu^{1/3}|k|^{2/3}t\right\},\quad\forall t\in[0,T]. \tag{1.15a}$$

(b) Assume all conditions in Theorem 1.2. Then there exist parameters $\alpha=\mathcal{O}(1),\beta=\mathcal{O}(1),\gamma=\mathcal{O}(1)$ such that the following estimate holds for $t\in[0,T]$,

$$\mathcal{G}[f_k](t)\le C\mathcal{G}[f_{0;k}]\exp\{-\delta\nu^{1/2}|k|^{1/2}t\}=C\|f_{0;k}\|_2^2\exp\left\{-\delta\nu^{1/2}|k|^{1/2}t\right\},\quad\forall t\in[0,T]. \tag{1.15b}$$

We organize the remaining sections as follows: in Section 2, we prove Theorem 1.1; in Section 3, we prove Theorem 1.2.

Notations: We define the Fourier transform in the x variable,

$$\widehat{f}_k(y)=\frac{1}{2\pi}\int_{\mathbb{T}}f(x,y)e^{-ikx}dx. \tag{1.16}$$

For two complex-valued functions f,g, we define the inner product

$$\langle f,g\rangle=\int_D f\overline{g}dy. \tag{1.17}$$

Here D is the domain of interest. Furthermore, we introduce the L^p-norms ($p\in[1,\infty)$)

$$\|f\|_p=\|f\|_{L^p}=\left(\int|f|^p dy\right)^{1/p},\quad p\in[1,\infty).$$

We also recall the standard extension of this definition to the $p=\infty$ case. We further recall the standard definition for Sobolev norms of functions $f(y),g(t,y)$:

$$\|f\|_{W_y^{m,p}}=\left(\sum_{k=0}^m\|\partial_y^m f\|_{L^p}^p\right)^{1/p},\quad p\in[1,\infty];\qquad \|g\|_{L_t^qW_y^{m,p}}=\|\|g\|_{W_y^{m,p}}\|_{L_t^q},\quad p,q\in[1,\infty].$$

We will also use classical notations $H^1=W^{1,2}$ and H_0^1 (the H^1 functions with zero trace on the boundary). We use the notation $A\approx B$ ($A,B>0$) if there exists a constant $C>0$ such that $\frac{1}{C}B\le A\le CB$. Similarly, we use the notation $A\lesssim B$ ($A\gtrsim B$) if there exists a constant C such that $A\le CB$ ($A\ge B/C$). Throughout the paper, the constant C can depend on the norm $\|V\|_{L_t^\infty W_y^{3,\infty}},\|U\|_{L_t^\infty W_y^{3,\infty}}$, but it will never depend on $\nu,|k|$. The meaning of the notation C can change from line to line.

2. Enhanced dissipation: strictly monotone shear flows

In this section, we prove the estimate (1.7) for the hypoelliptic passive scalar equation $(1.5)_{\sigma=0}$. The proof of the $\sigma=1$ case is similar and simpler. Throughout the remaining part of the paper, we adopt the following notation

$$f(t,y):=\widehat{f}_k(t,y).$$

Without loss of generality, we assume that

$$\partial_y V>0,\quad k\ge1. \tag{2.1}$$

Let us start with a simple observation.

Lemma 2.1. *Assume the relation*

$$\alpha>\beta^2. \tag{2.2}$$

Then, the following relations hold

$$\frac{1}{2}(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2)\le\mathcal{F}[f]\le\frac{3}{2}(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2),\quad\forall t\in[0,T]. \tag{2.3}$$

Proof. To prove the estimate, we recall the definition of $\mathcal{F}$ (1.13), and estimate it using Hölder inequality, Young's inequality,

$$\mathcal{F}[f]\le\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2+\beta\psi^2\epsilon^{1/3}\|f\|_2\|\partial_y f\|_2\le\left(1+\frac{\beta^2}{2\alpha}\psi^3\right)\|f\|_2^2+\frac{3\alpha}{2}\epsilon^{2/3}\psi\|\partial_y f\|_2^2$$

Similarly, we have the following lower bound,

$$\mathcal{F}[f]\ge\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2-\beta\epsilon^{1/3}\psi^2\|f\|_2\|\partial_y f\|_2\ge\left(1-\frac{\beta^2}{2\alpha}\psi^3\right)\|f\|_2^2+\frac{\alpha}{2}\epsilon^{2/3}\psi\|\partial_y f\|_2^2$$

Since $\alpha>\beta^2$, we obtain that

$$\frac{1}{2}\|f\|_2^2+\frac{1}{2}\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\le\mathcal{F}[f]\le\frac{3}{2}\|f\|_2^2+\frac{3}{2}\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2,\quad\forall t\in[0,T].$$

This concludes the proof of the lemma. ☐

By taking the time derivative of the hypocoercivity functional, (1.13), we end up with the following decomposition:

$$\frac{d}{dt}\mathcal{F}[f]=\frac{d}{dt}\|f\|_2^2+\alpha\epsilon^{2/3}\frac{d}{dt}\left(\psi\|\partial_y f\|_2^2\right)+\beta\epsilon^{1/3}\frac{d}{dt}\left(\psi^2\Re\langle if,\partial_y f\rangle\right)=:T_{L^2}+T_\alpha+T_\beta. \quad (2.4)$$

Through standard energy estimates, we observe that

$$T_{L^2}=-2\nu\int|\partial_y f|^2dy-2\Re i\int Vf\overline{f}dy=-2\nu\|\partial_y f\|_2^2. \quad (2.5)$$

The estimates for the T_α, T_β terms are trickier, and we collect them in the following technical lemmas whose proofs will be postponed to the end of this section.

LEMMA 2.2 (α-estimate). *For any constant $B>0$, the following estimate holds on the interval $[0,T]$:*

$$\begin{aligned}T_\alpha\le&\alpha\epsilon^{2/3}\psi'\|\partial_y f\|_2^2-2\alpha\psi\epsilon^{2/3}\nu\|\partial_{yy}f\|_2^2\\&+\frac{\beta}{B}\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}f\right\|_2^2+\frac{B\alpha^2}{\beta}\|\partial_y V\|_\infty\nu\|\partial_y f\|_2^2.\end{aligned} \quad (2.6)$$

LEMMA 2.3 (β-estimate). *The following estimate holds*

$$\begin{aligned}T_\beta\le&\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)\\&+\frac{\beta^2}{\alpha}\psi^3\nu\|\partial_y f\|_2^2+\alpha\psi\epsilon^{2/3}\nu\|\partial_{yy}f\|_2^2-\beta\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}|f|\right\|_2^2.\end{aligned} \quad (2.7)$$

We are ready to prove Theorem 1.1 with these estimates.

Proof. **(Proof of Theorem 1.1.)** If $T\le 2\nu^{-1/3}|k|^{-2/3}$, then standard L^2-energy estimate yields (1.7). Hence, we assume $T>2\nu^{-1/3}|k|^{-2/3}$ without loss of generality. We distinguish between two time intervals, i.e.,

$$\mathcal{I}_1=[0,\nu^{-1/3}|k|^{-2/3}],\quad \mathcal{I}_2=[\nu^{-1/3}|k|^{-2/3},T].$$

We organize the proof in three steps.

Step # 1: Energy bounds. Combining the estimates (2.5), (2.6), (2.7), we obtain that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}[f]\le&\alpha\epsilon^{2/3}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\|\partial_y f\|_2^2\\&+\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)\\&-2\nu\|\partial_y f\|_2^2-2\alpha\psi\epsilon^{2/3}\nu\|\partial_{yy}f\|_2^2\\&+\frac{\beta}{2}\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}f\right\|_2^2+\frac{2\alpha^2}{\beta}\nu\|\partial_y V\|_\infty\|\partial_y f\|_2^2\\&+\alpha\psi\epsilon^{2/3}\nu\|\partial_{yy}f\|_2^2+\frac{\beta^2}{\alpha}\psi^3\nu\|\partial_y f\|_2^2-\beta\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}|f|\right\|_2^2\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)\\
&\quad -\nu\left(2-\alpha\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)-\frac{2\alpha^2}{\beta}\|\partial_y V\|_\infty-\frac{\beta^2}{\alpha}\psi^3\right)\|\partial_y f\|_2^2\\
&\quad -\frac{1}{2}\beta\epsilon^{1/3}\psi^2|k|\left\|\sqrt{|\partial_y V|}f\right\|_2^2.
\end{aligned}
$$

Now we choose the α,β as follows:

$$
\alpha=\beta=\frac{1}{2(1+\|\partial_y V\|_\infty)}. \tag{2.8}
$$

Then we check that the condition (2.2) and the following hold for all $t\in[0,T]$,

$$
\begin{aligned}
&2-\alpha\mathbb{1}_{[0,\nu^{-1/3}]}(t)-\frac{2\alpha^2}{\beta}\|\partial_y V\|_\infty-\frac{\beta^2}{\alpha}\psi^3\\
\ge& 2-\frac{1}{2(1+\|\partial_y V\|_\infty)}-\frac{\|\partial_y V\|_\infty}{1+\|\partial_y V\|_\infty}-\frac{1}{2(1+\|\partial_y V\|_\infty)}\ge 1.
\end{aligned}
$$

As a result, we have (2.3) and the following,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}[f]\le& \frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)\\
&-\nu\|\partial_y f\|_2^2-\frac{\beta\epsilon^{1/3}|k|}{2}\psi^2\left\|\sqrt{|\partial_y V|}f\right\|_2^2.
\end{aligned} \tag{2.9}
$$

Step # 2: Initial time layer estimate. Thanks to the estimate (2.9) and the equivalence (2.3), we have that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}[f](t)\le 2\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathcal{F}[f](t)=\frac{\sqrt{2}}{(1+\|\partial_y V\|_\infty)^{1/2}}\nu^{1/3}|k|^{2/3}\mathcal{F}[f](t),\\
&\mathcal{F}[f](t=0)=\|f_{0;k}\|_2^2.
\end{aligned}
$$

By solving this differential inequality, we have that

$$
\mathcal{F}[f](t)\le \exp\left\{\frac{\sqrt{2}}{(1+\|\partial_y V\|_\infty)^{1/2}}\right\}\|f_{0;k}\|_2^2,\quad \forall t\in[0,\nu^{-1/3}|k|^{-2/3}]. \tag{2.10}
$$

Step # 3: Long time estimate. Now, we focus on the long time interval $\mathcal{I}_2$. On this interval, we have that $\psi\equiv 1$. The estimate (2.9), together with the lower bound on $|\partial_y V|$ (1.6), the choice of β (2.8) yields that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}[f](t)\le& -\nu\|\partial_y f\|_2^2-\frac{\nu^{1/3}|k|^{2/3}}{4(1+\|\partial_y V\|_\infty)}\left\|\sqrt{|\partial_y V|}f\right\|_2^2\\
\le& -\frac{\nu^{1/3}|k|^{2/3}}{4(1+\mathfrak{c})(1+\|\partial_y V\|_\infty)}(\|f\|_2^2+\alpha\epsilon^{2/3}\|\partial_y f\|_2^2)\\
\le& -\frac{\nu^{1/3}|k|^{2/3}}{6(1+\mathfrak{c})(1+\|\partial_y V\|_\infty)}\mathcal{F}[f](t).
\end{aligned}
$$

In the last line, we invoked the equivalence (2.3). Hence, for all $t\in[\nu^{-1/3}|k|^{-2/3},T]$

$$\begin{aligned}&\mathcal{F}[f](t)\le\mathcal{F}[f](t=\nu^{-1/3}|k|^{-2/3})\exp\left\{-\delta\nu^{1/3}|k|^{2/3}(t-\nu^{-1/3}|k|^{-2/3})\right\},\\&\delta:=\frac{1}{6(1+\mathfrak{c})(1+\|\partial_yV\|_\infty)}.\end{aligned}\tag{2.11}$$

Thanks to the relation (2.10), we have that

$$\mathcal{F}[f_k](t)\le e^2\|f_{0;k}\|_2^2\exp\left\{-\delta\nu^{1/3}|k|^{2/3}t\right\},\quad\forall t\in[\nu^{-1/3}|k|^{-2/3},T].$$

This concludes the proof of (1.15a) and Theorem 1.1. □

Finally, we collect the proofs of the technical lemmas.

Proof. **(Proof of Lemma 2.2.)** We recall the definition of T_α (2.4). Invoking the Equation (1.5) and integration by parts yields that

$$\begin{aligned}T_\alpha=&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2+\alpha\psi\epsilon^{2/3}\frac{d}{dt}\|\partial_yf\|_2^2\\=&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2+2\alpha\psi\epsilon^{2/3}\Re\int\partial_y(\nu\partial_y^2f-ikVf)\overline{\partial_yf}dy\\=&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2+2\alpha\psi\epsilon^{2/3}\Re\int\left(\nu\partial_y^3f-ik\partial_yVf-ikV\partial_yf\right)\overline{\partial_yf}dy\\=&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2+2\alpha\psi\epsilon^{2/3}\left(\nu\Re\int\partial_y^3f\overline{\partial_yf}dy-\Re\int ik\partial_yVf\overline{\partial_yf}dy-\Re\int ikV\partial_yf\overline{\partial_yf}dy\right)\\=&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2-2\alpha\psi\epsilon^{2/3}\left(\nu\|\partial_y^2f\|_2^2+\Re\int ik\partial_yVf\overline{\partial_yf}dy\right)\\\le&\alpha\psi'\epsilon^{2/3}\|\partial_yf\|_2^2-2\alpha\psi\epsilon^{2/3}\nu\|\partial_y^2f\|_2^2+2\alpha\psi\epsilon^{2/3}|k|\|\partial_yV\|_\infty^{1/2}\|\sqrt{|\partial_yV|}f\|_2\|\partial_yf\|_2.\end{aligned}$$

An application of Young's inequality yields (2.6). □

Proof. **(Proof of Lemma 2.3.)** The estimate of the T_β term in (2.4) is technical. Hence, we further decompose it into three terms:

$$\begin{aligned}T_\beta=&2\beta\psi\psi'\epsilon^{1/3}\Re\langle if,\partial_yf\rangle+\beta\psi^2\epsilon^{1/3}\Re\int i\partial_tf\overline{\partial_yf}dy+\beta\psi^2\epsilon^{1/3}\Re\int if\overline{\partial_{yt}f}dy\\=:&T_{\beta;1}+T_{\beta;2}+T_{\beta;3}.\end{aligned}\tag{2.12}$$

We estimate these terms one by one. To begin with, we have the following bound for the $T_{\beta;1}$:

$$\begin{aligned}|T_{\beta;1}|\le&2\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\sqrt{\psi}\|f\|_2(\sqrt{\alpha}\epsilon^{1/3}\sqrt{\psi}\|\partial_yf\|_2)\\\le&\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_yf\|_2^2\right).\end{aligned}\tag{2.13}$$

Next we compute the term $T_{\beta;2}$ using the Equation (1.5) and the assumption $\partial_yV>0$ (2.1):

$$\begin{aligned}T_{\beta;2}=&\beta\psi^2\epsilon^{1/3}\Re\int i(\nu\partial_{yy}f-ikVf)\overline{\partial_yf}dy\\=&\beta\psi^2\epsilon^{1/3}\left(\nu\Re\int i\partial_{yy}f\overline{\partial_yf}dy+k\Re\int V\partial_y\left(\frac{|f|^2}{2}\right)dy\right)\end{aligned}$$

$$=\beta\psi^2\epsilon^{1/3}\left(\nu\Re\int i\partial_{yy}f\overline{\partial_y f}dy-\frac{k}{2}\Re\int|f|^2\partial_y V dy\right). \tag{2.14}$$

Finally, we focus on the $T_{\beta;3}$ term in (2.12). Recalling that $0\le\partial_y V\in\mathbb{R}$, we have that

$$\begin{aligned}T_{\beta;3}=&\beta\psi^2\epsilon^{1/3}\Re\int i f\overline{\left(\nu\partial_y^3 f-ik\partial_y V f-ikV\partial_y f\right)}dy\\=&\beta\psi^2\epsilon^{1/3}\left(-\nu\Re\int i\partial_y f\overline{\partial_y^2 f}dy-k\Re\int f\overline{\partial_y V f}dy-k\Re\int V\partial_y\left(\frac{|f|^2}{2}\right)dy\right)\\=&-\beta\psi^2\epsilon^{1/3}\nu\Re\int i\partial_y f\overline{\partial_y^2 f}dy-\frac{\beta}{2}\psi^2\epsilon^{1/3}k\Re\int|f|^2\partial_y V dy.\end{aligned} \tag{2.15}$$

Combining the estimates (2.13), (2.14), (2.15), we have that

$$\begin{aligned}T_\beta\le&\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)\\&-2\beta\psi^2\epsilon^{1/3}\nu\Re\int i\partial_y f\overline{\partial_y^2 f}dy-\beta\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}|f|\right\|_2^2\\\le&\frac{\beta}{\sqrt{\alpha}}\nu^{1/3}|k|^{2/3}\mathbb{1}_{[0,\nu^{-1/3}|k|^{-2/3}]}(t)\left(\|f\|_2^2+\alpha\epsilon^{2/3}\psi\|\partial_y f\|_2^2\right)+\frac{\beta^2}{\alpha}\psi^3\nu\|\partial_y f\|_2^2\\&+\alpha\psi\epsilon^{2/3}\nu\|\partial_{yy}f\|_2^2-\beta\psi^2\epsilon^{1/3}|k|\left\|\sqrt{|\partial_y V|}|f|\right\|_2^2.\end{aligned}$$

□

3. Enhanced dissipation: nondegenerate shear flows

In this section, we prove the estimate (1.11) for the hypoelliptic passive scalar equation $(1.5)_{\sigma=0}$. Without loss of generality, we assume that $k\ge1$. Let us start with a lemma.

LEMMA 3.1. *Consider the flow $V(t,y)$ and the reference flow $U(t,y)$ as in Theorem 1.2. There exists a constant $C_*(\mathfrak{C}_0,\mathfrak{C}_1)>1$ such that the following estimate holds*

$$C_*^{-1}|\partial_y U(t,y)|\le|\partial_y V(t,y)|\le C_*|\partial_y U(t,y)|,\quad\forall y\in\mathbb{T},\quad\forall t\in[0,T]. \tag{3.1}$$

Proof. We distinguish between two cases: (a) $y\in B_r(y_i(t))$; b) $y\in(\cup_{i=1}^N B_r(y_i(t)))^c$. If $y\in B_r(y_i(t))$, by (1.9),

$$|\partial_y V(t,y)|\le\mathfrak{C}_0^{1/2}|y-y_i(t)|\le\mathfrak{C}_0|\partial_y U(t,y)|,\quad|\partial_y U(t,y)|\le\mathfrak{C}_0^{1/2}|y-y_i(t)|\le\mathfrak{C}_0|\partial_y V(t,y)|.$$

In case (b), since $|\partial_y V|,|\partial_y U|\in[\mathfrak{C}_1^{-1},\mathfrak{C}_1]$, the relation (3.1) is direct. □

LEMMA 3.2. *Assume the relation*

$$\beta^2\le\alpha\gamma. \tag{3.2}$$

Then, the following equivalence relation concerning the functional $\mathcal{G}$ (1.14) holds

$$\begin{aligned}&\|f\|_2^2+\frac{1}{2}\left(\alpha\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2\right)\\\le&\mathcal{G}[f]\le\|f\|_2^2+\frac{3}{2}\left(\alpha\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2\right).\end{aligned} \tag{3.3}$$

Proof. We recall the definition of $\mathcal{G}$ (1.14), and estimate $\mathcal{G}[f]$ using Hölder inequality and Young's inequality,

$$\begin{aligned}\mathcal{G}[f] &\leq \|f\|_2^2+\alpha\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\beta\phi^2\|\partial_y U f\|_2\|\partial_y f\|_2+\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2\\ &\leq \|f\|_2^2+\frac{3\alpha}{2}\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\left(\gamma+\frac{\beta^2}{2\alpha}\right)\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2.\end{aligned}$$

Similarly, we have the lower bound,

$$\mathcal{G}[f]\geq \|f\|_2^2+\frac{\alpha}{2}\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\left(\gamma-\frac{\beta^2}{2\alpha}\right)\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2.$$

Since (3.2) implies that $\frac{\beta^2}{2\alpha}\leq\frac{\gamma}{2}$, we obtain that

$$\begin{aligned}&\|f\|_2^2+\frac{1}{2}\alpha\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\frac{1}{2}\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2\\ \leq&\mathcal{G}[f(t)]\leq \|f\|_2^2+\frac{3}{2}\alpha\phi\epsilon^{1/2}\|\partial_y f\|_2^2+\frac{3}{2}\gamma\phi^3\epsilon^{-1/2}\|\partial_y U f\|_2^2.\end{aligned}$$

This concludes the proof of the lemma. ∎

By taking the time derivative of the hypocoercivity functional, (1.13), we end up with the following decomposition:

$$\begin{aligned}&\frac{d}{dt}\mathcal{G}[f(t)]\\ =&\frac{d}{dt}\|f\|_2^2+\alpha\epsilon^{1/2}\frac{d}{dt}\left(\phi\|\partial_y f\|_2^2\right)+\beta\frac{d}{dt}\left(\phi^2\Re\langle i\partial_y U f,\partial_y f\rangle\right)+\gamma\epsilon^{-1/2}\frac{d}{dt}\left(\phi^3\|\partial_y U f\|_2^2\right)\\ =:&\mathbb{T}_{L^2}+\mathbb{T}_\alpha+\mathbb{T}_\beta+\mathbb{T}_\gamma.\end{aligned}\tag{3.4}$$

The estimates for the $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, and $\mathbb{T}_\gamma$ terms are tricky, and we collect them in the following technical lemmas whose proofs will be postponed to the end of this section.

LEMMA 3.3 (α-estimate). *The following estimate holds on the interval $[0,T]$:*

$$\mathbb{T}_\alpha\leq\alpha\nu\left(1+\frac{4\alpha}{\beta}C_*^3\right)\|\partial_y f\|_2^2-2\alpha\phi\epsilon^{1/2}\nu\|\partial_y^2 f\|_2^2+\frac{\beta\phi^2}{4C_*}|k|\|\partial_y U f\|_2^2.\tag{3.5}$$

Here, the constant C_* is defined in (3.1).

LEMMA 3.4 (β-estimate). *The following estimate holds*

$$\begin{aligned}\mathbb{T}_\beta\leq&\left(\frac{1}{4}+4\beta C_*\right)\nu\|\partial_y f\|_2^2+2\alpha\phi\epsilon^{1/2}\nu\|\partial_{yy} f\|_2^2-\frac{3}{4}\frac{\beta\phi^2|k|}{C_*}\|\partial_y U f\|_2^2\\ &+\left(\frac{\beta\phi^2}{|k|^{1/2}}+\frac{\beta\phi}{2\alpha}\|\partial_{yy}U\|_\infty^2\right)\beta\phi^2\nu^{1/2}|k|^{1/2}\|f\|_2^2+\left(\frac{3\beta^2}{4\alpha\gamma}\right)\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2.\end{aligned}\tag{3.6}$$

Here the constant C_* is defined in (3.1).

REMARK 3.1. The phase assumption $\partial_y V(t,y)\partial_y U(t,y)\geq 0$ and the shape assumption (1.10) play a major role in Lemma 3.4. They guarantee the existence of a dissipation term of the form $\sim -\phi^2|k|\|\partial_y U f\|_2^2$. For details, we refer the readers to (3.10).

Lemma 3.5 (γ-estimate). *The following estimate holds on the interval* $[0,T]$

$$\mathbb{T}_\gamma \le \left(\frac{3\gamma C_*}{\beta}+\frac{1}{4}\right)\frac{\beta|k|\phi^2\|\partial_y U f\|_2^2}{C_*}$$
$$+\left(\frac{4C_*\gamma^2\phi^2}{\beta^2|k|^{1/2}}+\frac{4\gamma}{\beta}\phi\|\partial_{yy}U\|_\infty^2\right)\beta\phi^2\nu^{1/2}|k|^{1/2}\|f\|_2^2-\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2. \quad (3.7)$$

Here the C_* is defined in (3.1).

These estimates allow us to prove Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** If $T\le 2\nu^{-1/2}|k|^{-1/2}$, then standard L^2-energy estimate yields (1.11). Hence, we assume $T>2\nu^{-1/2}|k|^{-1/2}$ without loss of generality. We distinguish between two time intervals, i.e.,

$$\mathcal{I}_1=[0,\nu^{-1/2}|k|^{-1/2}],\quad \mathcal{I}_2=[\nu^{-1/2}|k|^{-1/2},T].$$

We organize the proof into three steps. In step # 1, we choose the α,β,γ parameters and derive the energy dissipation relation. In step # 2, we estimate the functional $\mathcal{G}$ in the time interval $\mathcal{I}_1$. In step # 3, we estimate the functional $\mathcal{G}$ in the time interval $\mathcal{I}_2$ and conclude the proof.

Step # 1: Energy bounds. Combining the estimates (2.5), (3.5), (3.6), (3.7), we obtain that

$$\frac{d}{dt}\mathcal{G}[f(t)]\le -\left(\frac{7}{4}-\alpha-\frac{4\alpha^2}{\beta}C_*^3-4\beta C_*\right)\nu\|\partial_y f\|_2^2-\left(\frac{1}{4}-\frac{3\gamma C_*}{\beta}\right)\frac{\beta|k|\phi^2}{C_*}\|\partial_y U f\|_2^2$$
$$+\left(\frac{\beta\phi^2}{|k|^{1/2}}+\frac{\beta\phi}{2\alpha}\|\partial_{yy}U\|_\infty^2+\frac{4C_*\gamma^2\phi^2}{\beta^2|k|^{1/2}}+\frac{4\gamma}{\beta}\phi\|\partial_{yy}U\|_\infty^2\right)\beta\phi^2\nu^{1/2}|k|^{1/2}\|f\|_2^2$$
$$-\gamma\left(1-\frac{3\beta^2}{4\alpha\gamma}\right)\phi^3\nu\epsilon^{-1/2}\|\partial_y U\partial_y f\|_2^2.$$

We choose α, γ in terms of $\beta(\le 1)$ as follows

$$\alpha=\frac{\beta^{1/2}}{4C_*^{3/2}},\qquad \gamma=4\beta^{3/2}C_*^{3/2}.$$

The resulting differential inequality is

$$\frac{d}{dt}\mathcal{G}[f(t)]$$
$$\le -\left(\frac{5}{4}-4\beta C_*\right)\underbrace{\epsilon|k|}_{=\nu}\|\partial_y f\|_2^2-\left(\frac{1}{4}-12\beta^{1/2}C_*^{5/2}\right)\frac{\beta|k|\phi^2}{C_*}\|\partial_y U f\|_2^2$$
$$+\underbrace{\left(\beta+2\beta^{1/2}C_*^{3/2}+64C_*^4\beta+16\beta^{1/2}C_*^{3/2}\right)}_{\le 83\beta^{1/2}C_*^4}\max\{1,\|\partial_{yy}U\|_\infty^2\}\beta\phi^2\underbrace{\epsilon^{1/2}|k|}_{=\nu^{1/2}|k|^{1/2}}\|f\|_2^2$$
$$-\frac{\gamma}{4}\phi^3\nu\epsilon^{-1/2}\|\partial_y U\partial_y f\|_2^2.$$

Now we invoke the spectral inequality (A.1) to obtain that

$$\frac{d}{dt}\mathcal{G}[f(t)]\le -\left(\frac{5}{4}-4\beta C_*-83\beta^{3/2}C_*^4\max\{1,\|\partial_{yy}U\|_\infty^2\}\right)\nu\|\partial_y f\|_2^2$$

$$-\left(\frac{1}{4C_*}-12\beta^{1/2}C_*^{3/2}-83\beta^{1/2}C_*^4\mathfrak{C}_{\text{spec}}\max\{1,\|\partial_{yy}U\|_\infty^2\}\right)\beta|k|\phi^2\|\partial_yUf\|_2^2.$$

Hence we can choose

$$\beta=\beta(C_*,\mathfrak{C}_{\text{spec}},\|\partial_{yy}U\|_\infty)<1$$

small enough, invoke the spectral inequality (A.1) and the equivalence relation (3.3) to obtain that

$$\begin{aligned}\frac{d}{dt}\mathcal{G}[f(t)]\le&-\frac{1}{2}\epsilon|k|\|\partial_yf\|_2^2-\frac{\beta}{8C_*}|k|\phi^2\|\partial_yUf\|_2^2\\ \le&-\frac{\beta\phi^2}{16\mathfrak{C}_{\text{spec}}C_*}\epsilon^{1/2}|k|\|f\|_2^2-\frac{1}{4}\nu^{1/2}|k|^{1/2}\epsilon^{1/2}\phi\|\partial_yf\|_2^2\\ &-\frac{\beta}{16C_*}\nu^{1/2}|k|^{1/2}\epsilon^{-1/2}\phi^3\|\partial_yUf\|_2^2\\ \le&-\delta(\beta,\mathfrak{C}_{\text{spec}}^{-1},C_*^{-1})\nu^{1/2}|k|^{1/2}\mathcal{G}[f].\end{aligned}\tag{3.8}$$

Finally, we observe that the parameter δ depends only on three parameters $C_*,\mathfrak{C}_{\text{spec}}$ and $\|\partial_{yy}U\|_\infty$.

Step # 2: Initial time layer estimate. This step is similar to the argument in the strictly monotone shear case. Thanks to the energy dissipation relation (3.8), we obtain that

$$\mathcal{G}[f_k](t)\le\|f_{0;k}\|_{L^2}^2,\quad\forall t\in[0,\nu^{-1/2}|k|^{-1/2}].$$

Step # 3: Long time estimate. Assume $t\ge\nu^{-1/2}|k|^{-1/2}$. Thanks to the energy dissipation relation (3.8), we obtain

$$\frac{d}{dt}\mathcal{G}[f]\le-\delta\nu^{1/2}|k|^{1/2}\mathcal{G}[f].$$

Hence, we obtain that

$$\begin{aligned}\mathcal{G}[f(t)]\le&\mathcal{G}[f(\nu^{-1/2}|k|^{-1/2})]e^{-\delta\nu^{1/2}|k|^{1/2}\left(t-\nu^{-1/2}|k|^{-1/2}\right)}\le e\mathcal{G}[f(0)]e^{-\delta\nu^{1/2}|k|^{1/2}t}\\ =&e\|f(0)\|_2^2e^{-\delta\nu^{1/2}|k|^{1/2}t}.\end{aligned}$$

Now, the results from Step 2 and 3 yields (1.15b). □

We conclude the section by providing the details of the proofs of Lemmas 3.3, 3.4, and 3.5.

Proof. **(Proof of Lemma 3.3.)** We recall the definition of $\mathbb{T}_\alpha$ (3.4). Invoking the Equation (1.5) and integration by parts yields that

$$\begin{aligned}\mathbb{T}_\alpha=&\alpha\phi'\epsilon^{1/2}\|\partial_yf\|_2^2+\alpha\phi\epsilon^{1/2}\frac{d}{dt}\|\partial_yf\|_2^2\\ =&\alpha\phi'\epsilon^{1/2}\|\partial_yf\|_2^2+2\alpha\phi\epsilon^{1/2}\Re\int\partial_y(\nu\partial_y^2f-ikVf)\overline{\partial_yf}dy\\ =&\alpha\phi'\epsilon^{1/2}\|\partial_yf\|_2^2-2\alpha\phi\epsilon^{1/2}\left(\nu\|\partial_y^2f\|_2^2+\Re\int ik\partial_yVf\overline{\partial_yf}dy\right).\end{aligned}$$

Now we apply Hölder inequality, the expression (1.12), and the equivalence relation (3.1) to obtain that

$$\begin{aligned}\mathbb{T}_\alpha \leq&\alpha\nu\|\partial_y f\|_2^2-2\alpha\phi\epsilon^{1/2}\nu\|\partial_y^2 f\|_2^2+2\alpha\phi\epsilon^{1/2}|k|\|\partial_y V f\|_2\|\partial_y f\|_2\\ \leq&\alpha\nu\|\partial_y f\|_2^2-2\alpha\phi\epsilon^{1/2}\nu\|\partial_y^2 f\|_2^2+\frac{4\alpha^2}{\beta}C_*^3\nu\|\partial_y f\|_2^2+\frac{\beta\phi^2|k|}{4C_*}\|\partial_y U f\|_2^2.\end{aligned}$$

This is (3.5). ∎

Proof. **(Proof of Lemma 3.4.)** The estimate of the $\mathbb{T}_\beta$ term in (2.4) is technical. We further decompose it into four terms and estimate them one by one:

$$\begin{aligned}\mathbb{T}_\beta =&2\beta\phi\phi'\Re\langle i\partial_y U f,\partial_y f\rangle+\beta\phi^2\Re\int i\partial_{ty}U f\overline{\partial_y f}dy+\beta\phi^2\Re\int i\partial_y U\partial_t f\overline{\partial_y f}dy\\ &+\beta\phi^2\Re\int i\partial_y U f\overline{\partial_{yt}f}dy\\ =:&\mathbb{T}_{\beta;1}+\mathbb{T}_{\beta;2}+\mathbb{T}_{\beta;3}+\mathbb{T}_{\beta;4}.\end{aligned}\tag{3.9}$$

To begin with, we apply the expression (1.12), the Hölder and Young's inequalities to derive the following bound for the $\mathbb{T}_{\beta;1}$ term,

$$\mathbb{T}_{\beta;1}\leq 2\beta\phi\nu^{1/2}|k|^{1/2}\|\partial_y U f\|_2\|\partial_y f\|_2\leq\frac{\beta\phi^2|k|}{4C_*}\|\partial_y U f\|_2^2+4\beta C_*\nu\|\partial_y f\|_2^2.$$

Next we estimate the term $\mathbb{T}_{\beta;2}$ using the assumption (1.8),

$$\mathbb{T}_{\beta;2}\leq\beta\phi^2\|\partial_{ty}U\|_\infty\|f\|_2\|\partial_y f\|_2\leq\beta^2\phi^4\nu^{1/2}\|f\|_2^2+\frac{1}{4}\nu\|\partial_y f\|_2^2.$$

We estimate the $\mathbb{T}_{\beta;3}$-term in (3.9) as follows

$$\begin{aligned}\mathbb{T}_{\beta;3}=&\beta\phi^2\Re\int i\partial_y U(\nu\partial_{yy}f-iVkf)\overline{\partial_y f}dy\\ =&\beta\phi^2\left(\nu\Re\int i\partial_y U\partial_{yy}f\overline{\partial_y f}dy+k\Re\int\partial_y UVf\overline{\partial_y f}dy\right)\\ \leq&\beta\phi^2\nu\|\partial_y U\partial_y f\|_2\|\partial_{yy}f\|_2+\beta\phi^2 k\Re\int\partial_y UVf\overline{\partial_y f}dy\\ \leq&\alpha\phi\epsilon^{1/2}\nu\|\partial_{yy}f\|_2^2+\left(\frac{\beta^2}{4\alpha\gamma}\right)\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2+\beta\phi^2 k\Re\int\partial_y UVf\overline{\partial_y f}dy.\end{aligned}$$

Finally we estimate the term $\mathbb{T}_{\beta;4}$ in (3.9)

$$\begin{aligned}\mathbb{T}_{\beta;4}=&\beta\phi^2\Re\int i\partial_y U f\overline{\left(\nu\partial_y^3 f-ik\partial_y V f-ikV\partial_y f\right)}dy\\ =&\beta\phi^2\left(-\nu\Re\int i(\partial_{yy}U f+\partial_y U\partial_y f)\overline{\partial_{yy}f}dy-k\Re\int(\partial_y U\partial_y V)|f|^2dy\right.\\ &\left.-k\Re\int\partial_y UVf\overline{\partial_y f}dy\right).\end{aligned}$$

Here, we observe that the Assumption (1.8) guarantees that the second term on the right-hand side is negative ($k\geq 1$). Now, we invoke the Assumption (1.8) and the equivalence relation (3.1) to obtain that

$$\mathbb{T}_{\beta;4}\leq\beta\phi^2\nu\|\partial_{yy}U\|_\infty\|f\|_2\|\partial_{yy}f\|_2+\beta\phi^2\nu\|\partial_y U\partial_y f\|_2\|\partial_{yy}f\|_2-\frac{\beta\phi^2|k|}{C_*}\Re\int|\partial_y U|^2|f|^2dy$$

$$
\begin{aligned}
&-\beta\phi^2 k\Re\int \partial_y U V f\overline{\partial_y f}dy\\
\le&\alpha\phi\epsilon^{1/2}\nu\|\partial_{yy}f\|_2^2+\frac{\beta^2}{2\alpha}\phi^3\epsilon^{-1/2}\nu\|\partial_{yy}U\|_\infty^2\|f\|_2^2+\left(\frac{\beta^2}{2\alpha\gamma}\right)\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2\\
&-\frac{\beta\phi^2|k|}{C_*}\|\partial_y U f\|_2^2-\beta\phi^2 k\Re\int \partial_y U V f\overline{\partial_y f}dy. \qquad (3.10)
\end{aligned}
$$

Combining the estimates, we have

$$
\begin{aligned}
\mathbb{T}_\beta\le&\left(\frac14+4\beta C_*\right)\nu\|\partial_y f\|_2^2-\frac34\frac{\beta\phi^2|k|}{C_*}\|\partial_y U f\|_2^2\\
&+\left(\frac{\beta\phi^2}{|k|^{1/2}}+\frac{\beta}{2\alpha}\phi\|\partial_{yy}U\|_\infty^2\right)\beta\phi^2\nu^{1/2}|k|^{1/2}\|f\|_2^2\\
&+2\alpha\phi\epsilon^{1/2}\nu\|\partial_{yy}f\|_2^2+\left(\frac{3\beta^2}{4\alpha\gamma}\right)\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2.
\end{aligned}
$$

This is the estimate (3.6). ☐

Proof. **(Proof of Lemma 3.5.)** Combining the Equation (1.5), the smallness Assumption (1.8), and integration by parts yields the following bound

$$
\begin{aligned}
\mathbb{T}_\gamma\le&3\gamma\phi^2|k|\|\partial_y U f\|_2^2\\
&+2\gamma\phi^3\epsilon^{-1/2}\left(\int|\partial_{ty}U||\partial_y U||f|^2dy+\Re\int|\partial_y U|^2(\nu\partial_{yy}f-iVkf)\overline{f}dy\right)\\
\le&3\gamma\phi^2|k|\|\partial_y U f\|_2^2\\
&+2\gamma\phi^3\epsilon^{-1/2}\left(\nu^{3/4}\|f\|_2\|\partial_y U f\|_2-2\nu\Re\int\partial_y U\partial_y f\ \overline{\partial_{yy}U f}dy-\nu\|\partial_y U\partial_y f\|_2^2\right)\\
\le&\left(\frac{3\gamma C_*}{\beta}+\frac14\right)\frac{\beta\phi^2|k|}{C_*}\|\partial_y U f\|_2^2+\left(\frac{4C_*\gamma^2}{\beta^2|k|^{1/2}}\phi^2+\frac{4\gamma\phi}{\beta}\|\partial_{yy}U\|_\infty^2\|\right)\beta\phi^2\nu^{1/2}|k|^{1/2}\|f\|_2^2\\
&-\gamma\phi^3\epsilon^{-1/2}\nu\|\partial_y U\partial_y f\|_2^2.
\end{aligned}
$$

This is (3.7). ☐

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Appendix. Technical lemmas. The proof makes use of several spectral inequalities. We present them below.

LEMMA A.1. *Consider the domain, i.e., $y\in\mathbb{T}$. Assume that $U(t,y)$ has N nondegenerate critical points $\{y_i(t)\}_{i=1}^N$ for $t\in[0,T]$. Moreover, there exist N open neighbourhoods $B_r(y_i(t)), i=1,\cdots,N$, such that*

$$
\begin{aligned}
&|\partial_y U(t,y)|^2\ge\mathfrak{C}_0^{-1}(y-y_i(t))^2,\quad\forall t\in[0,T],\quad\forall y\in B_r(y_i(t)),\quad\forall y_i(t)\in\{y\big|\partial_y U(t,y)=0\},\\
&|\partial_y U(t,y)|\in[\mathfrak{C}_1^{-1},\mathfrak{C}_1],\quad\forall y\in(\cup_{i=1}^N B_r(y_i(t)))^c.
\end{aligned}
$$

Then for ν small enough depending on the shear U, there exists a constant $\mathfrak{C}_{Spec}\ge1$ such that the following estimate holds ($\epsilon=\nu/|k|$)

$$
\epsilon^{1/2}\|f\|_{L^2(\mathbb{T})}^2\le\epsilon\|\partial_y f\|_{L^2(\mathbb{T})}^2+\mathfrak{C}_{Spec}\|\partial_y U(t,\cdot)f\|_{L^2(\mathbb{T})}^2. \qquad (A.1)
$$

Proof. The proof of the theorem is stated in the paper [5]. For the sake of completeness, we provide a different proof here. We can apply a partition of unity $\{\chi_i\}_{i=0}^N$ to decompose the function $f=f(\chi_0+\sum_{i=1}^n\chi_i)$, where $\{\chi_i\}_{i\neq 0}$ are supported near the critical points $y_i(t)$ and χ_0 is supported away from the critical points. Moreover, $\sum_{i=0}^n\|\partial_y\chi_i\|_\infty\le C$ and the supports of $\{\chi_i\}_{i\neq 0}$ are pairwise disjoint. Now we use the integration by parts formula

$$\begin{aligned}\epsilon^{1/2}\int_{\mathbb{R}}|f_i|^2dy=&\frac{1}{2}\epsilon^{1/2}\left|\int_{\mathbb{R}}|f_i|^2\frac{d^2}{dy^2}(y-y_i)^2dy\right|=\epsilon^{1/2}\left|\int_{\mathbb{R}}\partial_y|f_i|^2(y-y_i)dy\right|\\ \le&2\mathfrak{C}_0\epsilon^{1/2}\left|\Re\int_{\mathbb{R}}\overline{f_i}\partial_yf_i|\partial_yU|dy\right|\le\frac{1}{2}\epsilon\|\partial_yf_i\|_{L^2(\mathbb{R})}^2+C(\mathfrak{C}_0)\|\partial_yUf_i\|_{L^2(\mathbb{R})}^2,\quad i\neq 0.\end{aligned}$$

Since the supports of the cutoff functions $\chi_i, i\neq 0$ are disjoint, we have that

$$\epsilon^{1/2}\int_{\mathbb{T}}|f(1-\chi_0)|^2dy\le\epsilon\|\partial_y(f(1-\chi_0))\|_{L^2}^2+C(\mathfrak{C}_0)\|\partial_yUf(1-\chi_0)\|_{L^2}^2.$$

We further observe that, since the $|\partial_yU|\ge c>0$ on the support of χ_0,

$$\epsilon^{1/2}\|f\chi_0\|_{L^2}^2\le C\||\partial_yU|f\chi_0\|_{L^2}^2.$$

Combining the above estimates, we have that

$$\begin{aligned}\epsilon^{1/2}\|f\|_{L^2}^2\le&2\epsilon^{1/2}\|f\chi_0\|_{L^2}^2+2\epsilon^{1/2}\|f(1-\chi_0)\|_{L^2}^2\le\epsilon\|\partial_y(f(1-\chi_0))\|_{L^2}^2+C(\mathfrak{C}_0)\,\||\partial_yU|f\|_{L^2}^2\\ \le&\epsilon\|\partial_yf\|_{L^2}^2+C(\mathfrak{C}_0)\,\||\partial_yU|f\|_{L^2}^2+\epsilon\|\partial_y\chi_0\|_{L^\infty}^2\,\|f\|_{L^2}^2.\end{aligned}$$

We can take the ν small enough so that the left-hand side absorbs the last term. This concludes the proof of the lemma. ∎

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