COMMUN. MATH. SCI.
Vol. 22, No. 6, pp. 1701–1732

THE HYDROSTATIC LIMIT OF THE BERIS-EDWARDS SYSTEM IN DIMENSION TWO*

XINGYU LI†, MARIUS PAICU‡, AND ARGHIR ZARNESCU§

Abstract. We study the scaled anisotropic co-rotational Beris-Edwards system modeling the hydrodynamic motion of nematic liquid crystals in dimension two. We prove the global well-posedness with small analytic data in a thin strip domain. Moreover, we justify the limit to a system involving the hydrostatic Navier-Stokes system with analytic data and prove the convergence.

Keywords. Beris-Edwards system; liquid crystals; Q-tensor; hydrostatic limit.

AMS subject classifications. 35Q30; 76D03.

1. Introduction

The Beris-Edwards system is a widely used model for describing the hydrodynamic motion of nematic liquid crystals. The main characteristic feature of nematic liquid crystals, the local preferred orientation of the rod-like molecules is modeled by the so-called Q-tensors, that in this paper will be assumed to be specific to dimension two. The configuration space of Q-tensors is the set of two-by-two symmetric and traceless matrices, which is

$$\mathcal{S}_0^{(2)} = \{Q \in \mathbb{R}^{2\times 2} : Q = Q^T, \mathrm{tr}Q = 0\}.$$

More details about the modeling are provided in [1].

The Beris-Edwards system couples a dissipative parabolic system for Q-tensor-valued functions, modeling nematic liquid crystal orientation fields, with a forced Navier-Stokes equation for the underlying fluid velocity field u of the molecules. For a general incompressible nematic liquid crystal model within the Q-tensor framework, a result about global well-posedness and decay is provided in [13] and also, more recently, in [18]. The partial regularity has been recently studied by [6] while for the related simplified Ericksen-Leslie system (which was proposed by Ericksen and Leslie in 1960's, see [7, 8] and [9]), it was studied by Lin and Liu in [10] and [11].

In fluid mechanics, a classical reduction of the Navier-Stokes system is obtained assuming that the depth of the domain and the viscosity converge to zero simultaneously, in a related way. It is based on an approximation in geophysical fluid dynamics which assumes that the horizontal scale is large compared to the vertical scale, such that the vertical pressure gradient may be given as the product of density times the gravitational acceleration. In this case, the rescaled system is not isotropic, and we need to study the resulting anisotropic system known as the hydrostatic Navier-Stokes system. Results

*Received: May 04, 2023; Accepted (in revised form): January 30, 2024. Communicated by Chun Liu.

†BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Bizkaia, Spain & Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France (xingyuli92@gmail.com).

‡Université Bordeaux, Institut de Mathématiques de Bordeaux, F-33405 Talence Cedex, France (mpaicu@math.u-bordeaux1.fr).

§ BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Bizkaia, Spain IKERBASQUE, Basque Foundation for Science, Plaza Euskadi 5, 48009 Bilbao, Spain; and Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania (azarnescu@bcamath.org).

and further references about the hydrostatic Navier-Stokes equation can be found in [16], with a hyperbolic version of the system available in [15].

In our case we are interested in studying the similar problem, but in the case of the Beris-Edwards system, aiming to understand the interactions between the flow and fluid near the boundary, in the simplest possible situation, in dimension two and near a flat boundary, assuming two-dimensional Q-tensors. Our results can be interpreted as saying that the isotropic melting condition on Q at the boundary (i.e. zero Dirichlet boundary data) is imposed, through the fluid, also in a thin layer near the boundary, i.e. one cannot have fluid induced turbulent-like behaviour near the boundary.

Thus we study the Beris-Edwards system with small analytic data in a thin strip of $\mathbb{R}^2$. The equations read as follows:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \varepsilon^2 \Delta \mathbf{u} - \varepsilon^4 \nabla\cdot(\nabla Q \odot \nabla Q + (\Delta Q)\cdot Q - Q\cdot \Delta Q) \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{1.2}$$

$$\partial_t Q + \mathbf{u}\cdot\nabla Q + Q\Omega - \Omega Q = \varepsilon^2 \Delta Q - a' Q - c' Q \operatorname{tr}(Q^2) \tag{1.3}$$

where

$$\Omega := \frac{\nabla \mathbf{u} - \nabla^T \mathbf{u}}{2}.$$

The system (1.1)-(1.3), non-dimensionalized in a manner relevant to the study of defect patterns, is the simplified version of the one in [19] (see also [6]). Here $\mathbf{u}$ denotes the fluid velocity field, Q denotes the director field, namely a function taking values in $\mathcal{S}_0^{(2)}$ and P denotes the scalar pressure function which mathematically plays the role of the Lagrange multiplier that guarantees the divergence-free condition of the velocity field $\mathbf{u}$. Moreover, Ω denotes the antisymmetric part of the velocity gradient tensor $\nabla\mathbf{u}$, and $a', c' \in \mathbb{R}$ are constants, $a', c' > 0$.

In [19], the system contains a constant $\xi \in \mathbb{R}$, which is a parameter measuring the ratio between the aligning and tumbing effects that the fluid exerts on the liquid crystal molecules. In this paper, we consider the co-rotational Beris-Edwards system, namely we take $\xi = 0$. Moreover, the relaxation time parameter Γ and the fluid viscosity constant μ are both taken to be one, like in [6].

In this paper, we study the system (1.1)-(1.3) in a thin strip in dimension two, namely in $\mathcal{S}^\varepsilon := \{(x,y)\in\mathbb{R}^2, 0<y<\varepsilon\}$, and we consider the *perturbation of shear flow* case with the scaling as follows:

$$\mathbf{u}(t,x,y) = (u(t,x,y), v(t,x,y)) = \big(U(t,y/\varepsilon) + \varepsilon u^\varepsilon(t,x,y/\varepsilon), \varepsilon^2 v^\varepsilon(t,x,y/\varepsilon)\big) \tag{1.4}$$

and

$$\begin{aligned} P(t,x,y) =& \varepsilon p^\varepsilon(t,x,y/\varepsilon),\; Q_{\alpha\beta}(t,x,y) \\ &= \begin{cases} Q^\varepsilon_{\alpha\beta}(t,x,y/\varepsilon) \text{ if } \alpha,\beta = 1 \text{ or } \alpha,\beta = 2 \\ \varepsilon Q^\varepsilon_{\alpha\beta}(t,x,y/\varepsilon) \text{ if } \alpha = 1, \beta = 2 \text{ or } \alpha = 2, \beta = 1. \end{cases} \end{aligned} \tag{1.5}$$

Furthermore, we assume the non-slip boundary condition on the fluid and isotropic boundary conditions on the director, namely:

$$\mathbf{u}|_{y=0} = \mathbf{u}|_{y=\varepsilon} = 0, \quad Q|_{y=0} = Q|_{y=\varepsilon} = 0.$$

Define $\Delta_\varepsilon := \varepsilon^2\partial_x^2 + \partial_y^2$, and denote the matrix $R^\varepsilon_{ij} := \varepsilon^4(\nabla Q \odot \nabla Q + (\Delta Q)\cdot Q - Q\cdot \Delta Q)_{ij}, 1\le i,j\le 2$. Noting that $\mathrm{tr}\, Q = 0$ and $Q = Q^T$, we have

$$
\begin{aligned}
R^\varepsilon_{11} &= 2\varepsilon^4(\partial_x Q^\varepsilon_{11})^2 + 2\varepsilon^6(\partial_x Q^\varepsilon_{12})^2\\
R^\varepsilon_{12} &= \underbrace{2\varepsilon^3\partial_x Q^\varepsilon_{11}\partial_y Q^\varepsilon_{11} + 2\varepsilon^5\partial_x Q^\varepsilon_{12}\partial_y Q^\varepsilon_{12}}_{R^\varepsilon_{12,1}} \underbrace{-2\varepsilon^3\Delta_\varepsilon Q^\varepsilon_{12}Q^\varepsilon_{11} + 2\varepsilon^3 Q^\varepsilon_{12}\Delta_\varepsilon Q^\varepsilon_{11}}_{R^\varepsilon_{12,2}}\\
R^\varepsilon_{21} &= \underbrace{2\varepsilon^3\partial_x Q^\varepsilon_{11}\partial_y Q^\varepsilon_{11} + 2\varepsilon^5\partial_x Q^\varepsilon_{12}\partial_y Q^\varepsilon_{12}}_{R^\varepsilon_{21,1}} + \underbrace{2\varepsilon^3\Delta_\varepsilon Q^\varepsilon_{12}Q^\varepsilon_{11} - 2\varepsilon^3 Q^\varepsilon_{12}\Delta_\varepsilon Q^\varepsilon_{11}}_{R^\varepsilon_{21,2}}\\
R^\varepsilon_{22} &= 2\varepsilon^2(\partial_y Q^\varepsilon_{11})^2 + 2\varepsilon^4(\partial_y Q^\varepsilon_{12})^2
\end{aligned}
$$

so finally, the scaled system is considered in the strip $\mathcal{S} := \{(x,y)\in\mathbb{R}^2, 0<y<1\}$, and the equations of $u^\varepsilon, v^\varepsilon$ and $Q^\varepsilon_{11}, Q^\varepsilon_{22}$ become

$$
\begin{cases}
\partial_t U + \varepsilon\partial_t u^\varepsilon + \varepsilon(U+\varepsilon u^\varepsilon)\partial_x u^\varepsilon + \varepsilon v^\varepsilon\partial_y(U+\varepsilon u^\varepsilon) + \varepsilon\partial_x p^\varepsilon\\
\quad = \varepsilon^3\partial_x^2 u^\varepsilon + \partial_y^2(U+\varepsilon u^\varepsilon) - \partial_x R^\varepsilon_{11} - \partial_y R^\varepsilon_{21}\\
\varepsilon^2\partial_t v^\varepsilon + \varepsilon^2(U+\varepsilon u^\varepsilon)\partial_x v^\varepsilon + \varepsilon^3 v^\varepsilon\partial_y v^\varepsilon + \partial_y p^\varepsilon = \varepsilon^4\partial_x^2 v^\varepsilon + \varepsilon^2\partial_y^2 v^\varepsilon - \partial_x R^\varepsilon_{12} - \partial_y R^\varepsilon_{22}\\
\partial_x u^\varepsilon + \partial_y v^\varepsilon = 0
\end{cases}
\tag{1.6}
$$

together with the boundary condition

$$
U|_{y=0} = U|_{y=1} = 0, \quad u^\varepsilon|_{y=0} = u^\varepsilon|_{y=\varepsilon} = 0, \quad v^\varepsilon|_{y=0} = v^\varepsilon|_{y=\varepsilon} = 0
$$

and

$$
\begin{cases}
\partial_t Q^\varepsilon_{11} + (U^\varepsilon+\varepsilon u^\varepsilon)\partial_x Q^\varepsilon_{11} + \varepsilon v^\varepsilon\partial_y Q^\varepsilon_{11} + \partial_y(U^\varepsilon+\varepsilon u^\varepsilon)Q^\varepsilon_{12} - \varepsilon^3\partial_x v^\varepsilon Q^\varepsilon_{12}\\
\quad = \varepsilon^2\partial_x^2 Q^\varepsilon_{11} + \partial_y^2 Q^\varepsilon_{11} - a' Q^\varepsilon_{11} - 2c' Q^\varepsilon_{11}((Q^\varepsilon_{11})^2 + 2\varepsilon^2(Q^\varepsilon_{12})^2)\\
\varepsilon\partial_t Q^\varepsilon_{12} + \varepsilon(U^\varepsilon+\varepsilon u^\varepsilon)\partial_x Q^\varepsilon_{12} + \varepsilon^2 v^\varepsilon\partial_y Q^\varepsilon_{12} - \partial_y(U^\varepsilon/\varepsilon + u^\varepsilon)Q^\varepsilon_{11} + \varepsilon^2\partial_x v^\varepsilon Q^\varepsilon_{11}\\
\quad = \varepsilon^3\partial_x^2 Q^\varepsilon_{12} + \varepsilon\partial_y^2 Q^\varepsilon_{12} - a'\varepsilon Q^\varepsilon_{12} - 2c'\varepsilon Q^\varepsilon_{12}((Q^\varepsilon_{11})^2 + 2\varepsilon^2(Q^\varepsilon_{12})^2)
\end{cases}
\tag{1.7}
$$

together with the boundary condition

$$
Q^\varepsilon_{11}|_{y=0} = Q^\varepsilon_{11}|_{y=\varepsilon} = 0, \quad Q^\varepsilon_{12}|_{y=0} = Q^\varepsilon_{12}|_{y=\varepsilon} = 0.
$$

We take U that satisfies $\partial_t U = \partial_y^2 U$. Then[1] we have

$$
U(t,y) = \sum_{m\in\mathbb{N}} c_*(m) e^{-m^2\pi^2 t}\sin(m\pi y), \quad m\in\mathbb{N}. \tag{1.8}
$$

To prove our main result, we need that U (or $\partial_y U$) is uniformly bounded on t,y by some constant $\epsilon > 0$ small enough.

[1]We solve U^ε by separation of variables, using also the boundary conditions. Set $U^\varepsilon(t,y) = T(t)Y(y)$. Then we have $YT' = Y''T$, which is $\frac{T'}{T} = \frac{Y''}{Y} = -\gamma$ as a constant. Then we have $T' + \gamma T = 0$, $Y'' + \gamma Y = 0$. We need to suppose $\gamma > 0$, otherwise we will have $Y(y)\equiv 0$. By solving the ODE satistifed by Y, we have $Y(y) = A\cos\sqrt{\gamma}y + B\sin\sqrt{\gamma}y$. With the boundary condition $Y(0) = Y(1) = 0$, we have $\sqrt{\gamma} = m\pi, m\in\mathbb{Z}$. So $T(t) = T(0)e^{-k^2\pi^2 t}$. We can take $k\ge 0$ without loss of generality.

1.1. Notations and preliminaries. We first introduce the notations that will be used in the paper as follows.

- We write $a \lesssim b$ to mean that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$.
- We write $L^p_T(L^q_h(L^r_v))$ as the space $L^p((0,T);L^q(\mathbb{R}_x;L^r(\mathbb{R}_y)))$.
- We denote $(d_k)_{k\in\mathbb{Z}}$ (resp. $(d_k(t))_{k\in\mathbb{Z}}$) to be a generic element of $l^1(\mathbb{Z})$ so that $\sum_{k\in\mathbb{Z}} d_k = 1$ (resp. $\sum_{k\in\mathbb{Z}} d_k(t) = 1$).
- We denote $|D_x|$ as the Fourier multiplier with symbol $|\xi|$.

Next, we present some basic aspects of the Littlewood-Paley theory (see [2] for more details). For any distribution a with respect to variable x, we define

$$\Delta^h_k(\mathsf{a}) := \mathcal{F}^{-1}(\phi(2^{-k}|\xi|)\hat{\mathsf{a}}), \quad S^h_k(a) := \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\hat{\mathsf{a}}) \tag{1.9}$$

where ϕ, χ are smooth functions that satisfy

$$\operatorname{Supp}\phi \subset \left\{ r\in\mathbb{R}, \frac{3}{4} \le |r| \le \frac{8}{3} \right\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \phi(2^{-j}r) = 1 \quad \text{for all} \quad r>0$$

$$\operatorname{Supp}\chi \subset \left\{ r\in\mathbb{R}, |r| \le \frac{3}{4} \right\} \quad \text{and} \quad \chi(r) + \sum_{j\in\mathbb{Z}} \phi(2^{-j}r) = 1 \quad \text{for all} \quad r>0.$$

We will also need to use Bony's decomposition:

$$fg = T^h_f g + T^h_g f + R^h(f,g) \tag{1.10}$$

where

$$T^f_g := \sum_{k\in\mathbb{Z}} S^h_{k-1} f \Delta^h_k g, \quad R^h(f,g) := \sum_{k\in\mathbb{Z}} \Delta^h_k f \tilde{\Delta}^h_k g$$

with $\tilde{\Delta}^h_k g := \sum_{|k-k'|\le 1} \Delta^h_{k'} g$. Next, we define the functional spaces as follows:

- For any $s\in\mathbb{R}$, we let

$$||u||_{B^s} := \sum_{k\in\mathbb{Z}} 2^{ks} ||\Delta^h_k u||_{L^2}.$$

- For any $p\ge 1, T>0$, we define the *Chemin-Lerner type* space

$$||\mathsf{a}||_{\tilde{L}^p_T(B^s)} := \sum_{k\in\mathbb{Z}} 2^{ks} \left(\int_0^T ||\Delta^h_k \mathsf{a}(t)||^p_{L^2} dt \right)^{\frac{1}{p}}$$

  and the *time weighted Chemin-Lerner type* space

$$||\mathsf{a}||_{\tilde{L}^p_{t,f}(B^s)} := \sum_{k\in\mathbb{Z}} 2^{ks} \left(\int_0^t f(t') ||\Delta^h_k \mathsf{a}(t')||^p_{L^2} dt' \right)^{\frac{1}{p}}.$$

The Chemin-Lerner type space was introduced in [3] to obtain a better description of the regularizing effect of the diffusion equation. Because one cannot use a Gronwall type argument in the framework of Chemin-Lerner space, we need to use the time-weighted Chemin-Lerner norm, which was introduced in [16].

We need an anisotropic Bernstein type lemma:[2]

LEMMA 1.1. *Let $\mathcal{B}_h$ be a ball of $\mathbb{R}_h$ and $\mathcal{C}_h$ a ring of $\mathbb{R}_h$. For any $1\le p_2\le p_1\le\infty$ and $1\le q\le\infty$, we have the following results:*

(1) *If Supp $\hat{\mathsf{a}}\subset 2^k\mathcal{B}_h$, then* $||\partial_x^\alpha \mathsf{a}||_{L_h^{p_1}(L_v^q)}\lesssim 2^{k\left(|\alpha|+\frac{1}{p_2}-\frac{1}{p_1}\right)}||a||_{L_h^{p_2}(L_v^q)}$.

(2) *If Supp $\hat{\mathsf{a}}\subset 2^k\mathcal{C}_h$, then* $||\mathsf{a}||_{L_h^{p_1}(L_v^q)}\lesssim 2^{-kN}||\partial_x^N a||_{L_h^{p_1}(L_v^q)}$.

From now on, for any constant $a>0$, we define the function

$$u_\psi(t,x,y):=\mathcal{F}^{-1}_{\xi\to x}(e^{\psi(t,\xi)}\hat{u}(t,\xi,y)) \tag{1.11}$$

where the phase function ψ is defined by

$$\psi(t,\xi):=(a-\lambda\eta(t))|\xi|$$

recall U defined in (1.8). We define the function $\eta(t)$ as follows:

$$\eta(0)=0,\quad \eta'(t)=\delta e^{-\mathcal{R}t}+\sum_{m>0}|c_*(m)|e^{-m^2\pi^2t}$$

here $\mathcal{R}$ is half of the constant of Poincaré inequality for functions in $H_0^1(0,1)$. The smallness of $\delta,\eta(t),\eta'(t),\partial_yU$ is necessary to prove the global well-posedness of the anisotropic system.

For the choices of λ,δ, let C be a constant large enough (that satisfies (2.23)), and $\lambda=C^2,\delta=\frac{c\epsilon}{2C^2}$, where c is a positive universal constant, determined by (2.9), and $\epsilon>0$ is small enough. Notice that for any $t\ge0$,

$$\eta'(t)<\delta+\sum_{m>0}|c_*(m)|,\quad \eta(t)<\frac{\delta}{\mathcal{R}}+\sum_{m>0}\frac{|c_*(m)|}{m^2\pi^2}$$

moreover, recall that

$$|\partial_yU|=|\sum_{m\in\mathbb{N}}c_*(m)m\pi e^{-m^2\pi^2t}\cos(m\pi y)|\le\sum_{m\in\mathbb{N}}m\pi|c_*(m)|$$

thus we need to suppose that $\sum_{m\in\mathbb{N}}m|c_*(m)|$ is small enough. Then $\eta(t),\eta'(t),\partial_yU$ are small enough, and $\eta(t)=o(\frac{1}{C^2})<\frac{a}{\lambda}$, which assures the positiveness of ψ.

1.2. Main result and strategies. We first explain the main strategies and ideas of the proof.

Similarly as in the Prandtl equation and in the hydrostatic Navier-Stokes equation (see [14, 16] for more information), because of the nonlinear term $v\partial_yu$, we have one derivative loss in the x variable in the process of energy estimates. So we need to work with analytic data to solve this problem. For the scaled anisotropic system, we have the following existence theorem for fixed $\varepsilon>0$.

THEOREM 1.1. *Let $a>0$. There exists a constant $c_0>0$ small enough, such that if U defined in (1.8) satisfies $\sum_{m>0}m|c_*(m)|<c_0$, and the initial data $(u_0^\varepsilon,v_0^\varepsilon,(Q_{11})_0^\varepsilon,(Q_{12})_0^\varepsilon)$ satisfies*

$$\begin{aligned}&||e^{a|D_x|}(\varepsilon u_0^\varepsilon,\varepsilon^2v_0^\varepsilon)||_{B^{\frac12}}+||e^{a|D_x|}((Q_{11})_0^\varepsilon,\varepsilon(Q_{12})_0^\varepsilon)||_{B^{\frac12}}\\&\quad+||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0^\varepsilon,\varepsilon(Q_{12})_0^\varepsilon)||_{B^{\frac12}}\le c_0\end{aligned} \tag{1.12}$$

[2]More details can be found in [4, 12].

then the rescaled system (1.6)-(1.7) has a unique global solution $(u^\varepsilon, v^\varepsilon, (Q_{11})^\varepsilon, (Q_{12})^\varepsilon)$ *and there exists a constant* $C>0$*, such that for any* $t>0$*, there holds*

$$\begin{aligned}
&||e^{\mathcal{R}t'}(\varepsilon u^\varepsilon, \varepsilon^2 v^\varepsilon)_\psi||_{\tilde{L}^\infty_t(B^{\frac12})} + ||e^{\mathcal{R}t'}(Q^\varepsilon_{11}, \varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^\infty_t(B^{\frac12})}\\
&\quad + ||e^{\mathcal{R}t'}(\varepsilon^2\partial_x, \varepsilon\partial_y)(Q^\varepsilon_{11}, \varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^\infty_t(B^{\frac12})}\\
&\quad + ||e^{\mathcal{R}t'}\partial_y(U^\varepsilon + \varepsilon u^\varepsilon, \varepsilon^2 v^\varepsilon)_\psi||_{\tilde{L}^2_t(B^{\frac12})} + ||e^{\mathcal{R}t'}\partial_x(\varepsilon u^\varepsilon, \varepsilon^2 v^\varepsilon)_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\quad + ||e^{\mathcal{R}t'}\partial_y(Q^\varepsilon_{11}, \varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})} + ||e^{\mathcal{R}t'}\varepsilon\partial_x(Q^\varepsilon_{11}, \varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\quad + \varepsilon||e^{\mathcal{R}t'}\Delta_\varepsilon(Q^\varepsilon_{11}, \varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\le C||e^{a|D_x|}(\varepsilon u^\varepsilon_0, \varepsilon^2 v^\varepsilon_0)||_{B^{\frac12}} + C||e^{a|D_x|}((Q_{11})^\varepsilon_0, \varepsilon(Q_{12})^\varepsilon_0)||_{B^{\frac12}}\\
&\quad + C||e^{a|D_x|}(\varepsilon^2\partial_x, \varepsilon\partial_y)((Q_{11})^\varepsilon_0, \varepsilon(Q_{12})^\varepsilon_0)||_{B^{\frac12}}
\end{aligned} \tag{1.13}$$

where $(u^\varepsilon_\psi, v^\varepsilon_\psi, (Q_{11})^\varepsilon_\psi, (Q_{12})^\varepsilon_\psi)$ *are defined as in (1.11).*

The key point in the proof of the theorem is that we are able to control the lower frequency part u by the higher frequency part $\partial_y u$ because the domain is a thin strip and we can use on it the Poincaré inequality. The reason we choose Besov space instead of Sobolev is that we can get the optimal rate, which is just the constant of Poincaré inequality. We remind that here the smallness condition of the rescaled system (1.6)-(1.7) is independent of ε. For the initial system (1.1)-(1.3), the analyticity radius of the initial data a should be a/ε, and the small constant c_0 in (1.12) should be $c_0/\varepsilon^{\frac12}$ instead.

We next explain why we need to consider our fluid part to be a perturbation of a shear flow. The term $Q\Omega - \Omega Q$, when considered in the anistropic Besov space of the strip, will have a $1/\varepsilon$ part, which would not allow the convergence in the limit $\varepsilon\to 0$. To solve this problem, we consider $U^\varepsilon(t, y/\varepsilon) + \varepsilon u^\varepsilon(t, x, y/\varepsilon)$ instead of $\varepsilon u^\varepsilon(t, x, y/\varepsilon)$. For the U^ε term, the $1/\varepsilon$ part will vanish because U^ε does not depend on x.

Moreover, unlike in the case of the Navier-Stokes equation in [16], we also need to consider the second order frequency part $\Delta_\varepsilon Q$. This is because $\Delta_\varepsilon Q$ appears in the equation of $\mathbf{u}$, and $\partial_y^2 Q$ cannot be directly estimated, as we do not have the boundary condition for $\partial_y Q$. This problem can be solved by multiplying the equation for Q with $\Delta_\varepsilon Q$ where we have this term and all the other terms can be directly controlled.

We next consider the hydrostatic approximation with small analytic data. Letting $\varepsilon\to 0$ the limit of (1.1), (1.2) formally is:

$$\begin{cases} \partial_t u + U\partial_x u + v\partial_y U + \partial_x p = \partial_y^2 u\\ \partial_y p = 0\\ \partial_x u + \partial_y v = 0\end{cases} \tag{1.14}$$

with the boundary condition

$$U|_{y=0} = U|_{y=1} = 0, \quad u|_{y=0} = u|_{y=1} = 0, \quad v|_{y=0} = v|_{y=1} = 0$$

and the function U satisfies $\partial_t U = \partial_y^2 U$. Similarly as the scaled anisotropic system, we obtain that $U(t,y)$ also has the similar form (1.8).

Next, recall the boundary condition $v|_{y=0} = v|_{y=1} = 0$. Integrating the equation $\partial_x u + \partial_y v = 0$ (over $[0,1]$ with respect to the y variable) we get $\partial_x \int_0^1 u(t,x,y)dy = 0$.

Because of the assumption on the space in which the solution is, we have $u(t,x,y)\to 0$ as $|x|\to\infty$. So finally we have the compatibility condition

$$\int_0^1 u(t,x,y)dy=0$$

and (1.3) becomes

$$\begin{cases}\partial_t Q_{11}+U\partial_x Q_{11}+\partial_y U Q_{12}=\partial_y^2 Q_{11}-a'Q_{11}-2c'Q_{11}^3\\ \partial_y U Q_{11}=0.\end{cases}\tag{1.15}$$

We need to suppose that $\partial_y U$ is not identically 0. Then we deduce from (1.15) that $Q_{11}=Q_{12}=0$.

Remark 1.1. For more complicated 3D hydrostatic limit system $\varepsilon\to 0$ of $Q_{ij}(1\le i,j\le 3)(Q=Q^T,\operatorname{tr}Q=0)$, there exist non-trivial values of Q. In fact, after direct calculation, we have $Q_{12}^2=(2Q_{11}+Q_{22})(Q_{11}+2Q_{22})$, and the estimate of the related system is more declicate. We leave it as a future work.

The result about global well-posedness of the hydrostatic system is then as follows:

Theorem 1.2. *Let $a>0$ and let $U(t,y)$ be defined in (1.8) with $\partial_y U\not\equiv 0$. Suppose that the initial data u_0 satisfies the compatibility condition $\int_0^1 u_0 dy=0$.*

(i) *Assume $\sum_{m>0}\frac{|c_*(m)|}{m}<c_1$ for some c_1 small enough. Then the system (1.14) has a unique global solution u that satisfies for any $s>0$,*

$$||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}_t^\infty(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}_t^2(B^s)}\le ||e^{a|D_x|}u_0||_{B^s}.\tag{1.16}$$

(ii) *Assume $\sum_{m>0}m|c_*(m)|<c_1$ for some c_1 small enough. Then the system (1.14) has a unique global solution u, and for any $s>0$, there exists a constant $C>0$, such that*

$$\begin{aligned}&||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}_t^\infty(B^s)}+||e^{\mathcal{R}t'}\partial_y^2 u_\phi||_{\tilde{L}_t^2(B^s)}\\ \le& C(||e^{a|D_x|}u_0||_{B^s}+||e^{a|D_x|}u_0||_{B^{s+1}}+||e^{a|D_x|}\partial_y u_0||_{B^s})\\ &||e^{\mathcal{R}t'}(\partial_t u)_\phi||_{\tilde{L}_t^2(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}_t^\infty(B^s)}\\ \le& C(||e^{a|D_x|}\partial_y u_0||_{B^s}+||e^{a|D_x|}u_0||_{B^{s+1}})\end{aligned}\tag{1.17}$$

where u_ϕ is given by (3.1).

Remark 1.2. It is worth pointing out that in order to prove the global well-posedness of the hydrostatic case, we do not need the smallness of the initial data u_0 as [16], because in our case the function η does not include the norm u,v as in the anisotropic case. However, the smallness of u_0 is still required to prove the convergence.

Finally, we study the convergence of the scaled anisotropic system to the limit hydrostatic system. For the vanishing viscosity of the analytical solutions of Navier-Stokes system in the half space, the local-in-time convergence was studied in [17], and for the scaled anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes system, the global-in-time convergence was studied in [16]. Following the strategy of [16], we have a theorem about the global-in-time convergence, as follows:

THEOREM 1.3. *Let $a>0$. Suppose there exists $c_0>0$, such that U satisfies $\sum_{m>0} m|c_*(m)|<c_0$ and $(u_0^\varepsilon, v_0^\varepsilon, (Q_{11})_0^\varepsilon, (Q_{12})_0^\varepsilon)$ satisfy (1.12). Moreover, suppose that u_0 satisfies $e^{a|D_x|}u_0 \in B^{\frac{1}{2}} \cap B^{\frac{5}{2}}, e^{a|D_x|}\partial_y u_0 \in B^{\frac{3}{2}}$, $||e^{a|D_x|}u_0||_{B^{\frac{1}{2}}} < c_0$, and the compatibility condition $\int_0^1 u_0 dy = 0$ is satisfied. The function v_0 is determined by $\partial_x u_0 + \partial_y u_0 = 0$ and $v_0 = 0$ for $y=0$ or 1. Define $w_1 := u^\varepsilon - u, w_2 := v^\varepsilon - v$. Then there exist constants $C, M>0$, such that*

$$\begin{aligned}
&||(\varepsilon w_\Theta^1, \varepsilon^2 w_\Theta^2)||_{\tilde{L}_t^\infty(B^{\frac{1}{2}})} + ||(Q_{11}^\varepsilon, \varepsilon Q_{12}^\varepsilon)_\Theta||_{\tilde{L}_t^\infty(B^{\frac{1}{2}})} + ||(\varepsilon^2\partial_x, \varepsilon\partial_y)(Q_{11}^\varepsilon, \varepsilon Q_{12}^\varepsilon)_\Theta||_{\tilde{L}_t^\infty(B^{\frac{1}{2}})} \\
&\quad + ||\partial_y(V^\varepsilon + \varepsilon w^1, \varepsilon^2 w^2)_\Theta||_{\tilde{L}_t^2(B^{\frac{1}{2}})} + ||e^{\mathcal{R}t'}\partial_x(\varepsilon u^\varepsilon, \varepsilon^2 v^\varepsilon)_\psi||_{\tilde{L}_t^2(B^{\frac{1}{2}})} \\
&\quad + ||\partial_y(Q_{11}^\varepsilon, \varepsilon Q_{12}^\varepsilon)_\Theta||_{\tilde{L}_t^2(B^{\frac{1}{2}})} + ||\varepsilon\partial_x(Q_{11}^\varepsilon, \varepsilon Q_{12}^\varepsilon)_\Theta||_{\tilde{L}_t^2(B^{\frac{1}{2}})} + \varepsilon||\Delta_\varepsilon(Q_{11}^\varepsilon, \varepsilon Q_{12}^\varepsilon)_\psi||_{\tilde{L}_t^2(B^{\frac{1}{2}})} \\
&\le C||e^{a|D_x|}(\varepsilon(u_0^\varepsilon - u_0), \varepsilon^2(v_0^\varepsilon - v_0))||_{B^{\frac{1}{2}}} + C||e^{a|D_x|}((Q_{11})_0, \varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}} \\
&\quad + C||e^{a|D_x|}(\varepsilon^2\partial_x, \varepsilon\partial_y)((Q_{11})_0, \varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}} + CM\varepsilon
\end{aligned} \tag{1.18}$$

where u_Θ is determined by (3.19) and a similar definition holds for Q_Θ.

To prove this theorem, we still need to control the difference between the scaled anisotropic and hydrostatic systems, and we use energy estimates with exponential weights in the Fourier variable. The weights still depend on time as before, but the loss of the analyticity in x variable needs to be considered for both systems.

1.3. Outline of the paper. In Section 2 we prove the global wellposedness of the scaled anisotropic system. In Section 3, we prove the global wellposedness of the hydrostatic system and the convergence from anisotropic system to hydrostatic system.

2. Global well-posedness

We define the maximal time T^* by

$$\begin{aligned}
T^* := \sup\{t>0, \varepsilon||u_\psi||_{B^{\frac{3}{2}}} + ||\partial_y u_\psi||_{B^{\frac{1}{2}}} + \varepsilon||(Q_{11}, \varepsilon Q_{12})_\psi||_{B^{\frac{3}{2}}} \\
+ ||\partial_y(Q_{11}, \varepsilon Q_{12})_\psi||_{B^{\frac{1}{2}}} \le \delta e^{-\mathcal{R}t}\}.
\end{aligned} \tag{2.1}$$

Note that $u_\psi, v_\psi, (Q_{11})_\psi, (Q_{12})_\psi$ satisfy the equations

$$\begin{cases}
\varepsilon\partial_t u_\psi + \lambda\varepsilon\eta'(t)|D_x|u_\psi + \varepsilon([U+\varepsilon u]\partial_x u)_\psi + \varepsilon(v\partial_y[U+\varepsilon u])_\psi + \varepsilon\partial_x p_\psi \\
= \varepsilon^3\partial_x^2 u_\psi + \varepsilon\partial_y^2 u_\psi - \partial_x(R_{11}^\varepsilon)_\psi - \partial_y(R_{21}^\varepsilon)_\psi \\
\varepsilon^2\partial_t v_\psi + \varepsilon^2\lambda\eta'(t)|D_x|v_\psi + \varepsilon^2([U+\varepsilon u]\partial_x v)_\psi + \varepsilon^3(v\partial_y v)_\psi + \partial_y p_\psi \\
= \varepsilon^4\partial_x^2 v_\psi + \varepsilon^2\partial_y^2 v_\psi - \partial_x(R_{12}^\varepsilon)_\psi - \partial_y(R_{22}^\varepsilon)_\psi \\
\partial_x u_\psi + \partial_y v_\psi = 0
\end{cases} \tag{2.2}$$

and

$$\begin{cases}
\partial_t(Q_{11})_\psi + \lambda\eta'(t)|D_x|(Q_{11})_\psi \\
\quad + ([U+\varepsilon u]\partial_x Q_{11} + \varepsilon v\partial_y Q_{11})_\psi + (\partial_y[U+\varepsilon u]Q_{12} - \varepsilon^3\partial_x v Q_{12})_\psi \\
= \varepsilon^2\partial_x^2(Q_{11})_\psi + \partial_y^2(Q_{11})_\psi - (a'Q_{11} + c'Q_{11}(2Q_{11}^2 + 2\varepsilon^2 Q_{12}^2))_\psi \\
\varepsilon\partial_t(Q_{12})_\psi + \varepsilon\lambda\eta'(t)|D_x|(Q_{12})_\psi \\
\quad + (\varepsilon[U+\varepsilon u]\partial_x Q_{12} + \varepsilon^2 v\partial_y Q_{12})_\psi - (\partial_y[U/\varepsilon + u]Q_{11} - \varepsilon^2\partial_x v Q_{11})_\psi \\
= \varepsilon^3\partial_x^2(Q_{12})_\psi + \varepsilon\partial_y^2(Q_{12})_\psi - \varepsilon\left[a'Q_{12} + c'Q_{12}(2Q_{11}^2 + 2\varepsilon^2 Q_{12}^2)\right]_\psi.
\end{cases} \tag{2.3}$$

We apply the dyadic operator Δ_k^h. Notice that $\partial_x u_\psi + \partial_y v_\psi = 0$, so

$$(\partial_x \Delta_k^h p_\psi, \Delta_k^h u_\psi)_{L^2} + (\partial_y \Delta_k^h p_\psi, \Delta_k^h v_\psi)_{L^2} = -(\Delta_k^h p_\psi, \partial_x \Delta_k^h u_\psi + \partial_y \Delta_k^h v_\psi)_{L^2} = 0.$$

Next, observing that $\partial_x U = 0$, we have from integration by parts,

$$\begin{aligned}(\Delta_k^h(\partial_x uU)_\psi, \Delta_k^h u_\psi)_{L^2} &= (\Delta_k^h(\partial_x uU)_\psi, \Delta_k^h u_\psi)_{L^2} + (\Delta_k^h(u\partial_x U)_\psi, \Delta_k^h u_\psi)_{L^2} \\ &= -(\Delta_k^h(uU)_\psi, \Delta_k^h \partial_x u_\psi)_{L^2} \end{aligned} \tag{2.4}$$

and similarly,

$$\begin{aligned}(\Delta_k^h(\partial_x vU)_\psi, \Delta_k^h v_\psi)_{L^2} &= (\Delta_k^h(\partial_x Q_{11} U)_\psi, \Delta_k^h (Q_{11})_\psi)_{L^2} \\ &= (\Delta_k^h(\partial_x Q_{12} U)_\psi, \Delta_k^h (Q_{12})_\psi)_{L^2} = 0. \end{aligned} \tag{2.5}$$

Moreover,

$$(\Delta_k^h(\partial_y U Q_{11})_\psi, \Delta_k^h(Q_{12})_\psi)_{L^2} = (\Delta_k^h(\partial_y U Q_{12})_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2}. \tag{2.6}$$

Consider the Equations (2.2), (2.3), multiply respectively by $\varepsilon\Delta_k^h u_\psi, \varepsilon^2\Delta_k^h v_\psi, \Delta_k^h(Q_{11})_\psi, \varepsilon\Delta_k^h(Q_{12})_\psi$ and take L^2 scalar product, to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\Delta_k^h(\varepsilon u)_\psi||_{L^2}^2 \\ &\quad + ||\Delta_k^h(\varepsilon^2 v_\psi)||_{L^2}^2) + \lambda\varepsilon^2\eta'(t)(|D_x|\Delta_k^h u_\psi, |\Delta_k^h u_\psi)_{L^2} + \lambda\eta'(t)(|D_x|\Delta_k^h(\varepsilon^2 v_\psi), |\Delta_k^h(\varepsilon^2 v_\psi))_{L^2} \\ &\quad + \varepsilon^4(||\partial_x\Delta_k^h u_\psi||_{L^2}^2 + ||\partial_x\Delta_k^h(\varepsilon v_\psi)||_{L^2}^2) + (||\partial_y\Delta_k^h(\varepsilon u)_\psi||_{L^2}^2 + ||\partial_y\Delta_k^h(\varepsilon^2 v_\psi)||_{L^2}^2) \\ &= -\varepsilon^2(\Delta_k^h(vU)_\psi, \Delta_k^h\partial_y u_\psi)_{L^2} - \varepsilon^3(\Delta_k^h(u\partial_x u)_\psi, \Delta_k^h u_\psi)_{L^2} - \varepsilon^3(\Delta_k^h(v\partial_y u)_\psi, \Delta_k^h u_\psi)_{L^2} \\ &\quad - \varepsilon^5(\Delta_k^h(u\partial_x v)_\psi, \Delta_k^h v_\psi)_{L^2} - \varepsilon^5(\Delta_k^h(v\partial_y v)_\psi, \Delta_k^h v_\psi)_{L^2} + \varepsilon(\Delta_k^h(R_{11}^\varepsilon)_\psi, \Delta_k^h\partial_x u_\psi)_{L^2} \\ &\quad + \varepsilon(\Delta_k^h(R_{21}^\varepsilon)_\psi, \Delta_k^h\partial_y u_\psi)_{L^2} + \varepsilon^2(\Delta_k^h(R_{12}^\varepsilon)_\psi, \Delta_k^h\partial_x v_\psi)_{L^2} + \varepsilon^2(\Delta_k^h(R_{22}^\varepsilon)_\psi, \Delta_k^h\partial_y v_\psi)_{L^2} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\Delta_k^h(Q_{11})_\psi||_{L^2}^2 + ||\Delta_k^h(\varepsilon(Q_{12})_\psi)||_{L^2}^2) \\ &\quad + a'(||\Delta_k^h(Q_{11})_\psi||_{L^2}^2 + ||\Delta_k^h(\varepsilon Q_{12})_\psi||_{L^2}^2) + \lambda\eta'(t)(|D_x|\Delta_k^h(Q_{11})_\psi, |\Delta_k^h(Q_{11})_\psi)_{L^2} \\ &\quad + \lambda\eta'(t)(|D_x|\Delta_k^h(\varepsilon(Q_{12})_\psi), |\Delta_k^h(\varepsilon(Q_{12})_\psi))_{L^2} + \varepsilon^2(||\partial_x\Delta_k^h(Q_{11})_\psi||_{L^2}^2 + ||\partial_x\Delta_k^h(\varepsilon(Q_{12})_\psi)||_{L^2}^2) \\ &\quad + ||\partial_y\Delta_k^h(Q_{11})_\psi||_{L^2}^2 + ||\partial_y\Delta_k^h(\varepsilon(Q_{12})_\psi)||_{L^2}^2 \\ &= -\varepsilon(\Delta_k^h(u\partial_x(Q_{11}))_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2} - \varepsilon(\Delta_k^h(v\partial_y(Q_{11}))_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2} \\ &\quad - \varepsilon^3(\Delta_k^h(u\partial_x(Q_{12}))_\psi, \Delta_k^h(Q_{12})_\psi)_{L^2} - \varepsilon^3(\Delta_k^h(v\partial_y(Q_{12}))_\psi, \Delta_k^h(Q_{12})_\psi)_{L^2} \\ &\quad - (\Delta_k^h(\partial_y u Q_{11} - \varepsilon^2\partial_x v Q_{11})_\psi, \Delta_k^h(\varepsilon Q_{12})_\psi)_{L^2} \\ &\quad - \varepsilon(\Delta_k^h(\partial_y u Q_{12} - \varepsilon^2\partial_x v Q_{12})_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2} - 2c'(\Delta_k^h[Q_{11}(Q_{11}^2 + \varepsilon^2 Q_{12}^2)], \Delta_k^h(Q_{11})_\psi)_{L^2} \\ &\quad - 2c'\varepsilon^2(\Delta_k^h[Q_{12}(Q_{11}^2 + \varepsilon^2 Q_{12}^2)], \Delta_k^h(Q_{12})_\psi)_{L^2}. \end{aligned} \tag{2.8}$$

REMARK 2.1. Recall that the scaled model is in the strip $\mathcal{S}$. Because u, v, Q_{11}, Q_{12} all equal to 0 on the boundary of y variable, then according to Poincaré inequality, there exists a universal constant $\mathcal{R} > 0$, such that

$$2\mathcal{R}||\Delta_k^h(u,v,Q_{11},Q_{12})_\psi||_{L^2}^2 \le ||\partial_y\Delta_k^h(u,v,Q_{11},Q_{12})_\psi||_{L^2}^2.$$

Moreover, for the equation of Q_{11} and Q_{12}, we need the assumption $a' > 0$.

Recall from (2) of Lemma 1.1 that there exists $c>0$, such that

$$c2^{2k}||\Delta_k^h(u_\psi,\varepsilon v_\psi)||_{L^2}^2 \leq ||\Delta_k^h\partial_x(u_\psi,\varepsilon v_\psi)||_{L^2}^2 \tag{2.9}$$

multiply (2.7) by $e^{2\mathcal{R}t}$ and integrate over $[0,t]$, to get

$$\begin{aligned}
&\frac{1}{2}(||e^{\mathcal{R}t'}\Delta_k^h(\varepsilon u)_\psi||_{L_t^\infty(L^2)}^2+||e^{\mathcal{R}t'}\Delta_k^h(\varepsilon^2 v_\psi)||_{L_t^\infty(L^2)}^2)\\
&\quad+\lambda 2^k\int_0^t \eta'(t')(||e^{\mathcal{R}t'}\Delta_k^h(\varepsilon u)_\psi||_{L^2}^2+||e^{\mathcal{R}t'}\Delta_k^h\varepsilon^2 v_\psi||_{L^2}^2)dt'\\
&\quad+\frac{1}{2}\int_0^t e^{2\mathcal{R}t'}(||\Delta_k^h\partial_y(\varepsilon u_\psi)||_{L^2}^2+||\Delta_k^h\partial_y(\varepsilon^2 v_\psi)||_{L^2}^2)dt'\\
&\quad+\frac{1}{2}\int_0^t e^{2\mathcal{R}t'}\cdot c\varepsilon^4 2^{2k}(||\Delta_k^h u_\psi||_{L^2}^2+||\Delta_k^h(\varepsilon v_\psi)||_{L^2}^2)dt'\\
&\leq ||e^{a|D_x|}\Delta_k^h(\varepsilon u_0)||_{L^2}^2+||e^{a|D_x|}\Delta_k^h(\varepsilon^2 v_0)||_{L^2}^2+\underbrace{\varepsilon^3\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(u\partial_x u)_\psi,e^{\mathcal{R}t'}\Delta_k^h u_\psi)_{L^2}|dt'}_{A_1}\\
&\quad+\underbrace{\varepsilon^3\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(v\partial_y u)_\psi,e^{\mathcal{R}t'}\Delta_k^h u_\psi)_{L^2}|dt'}_{A_2}+\underbrace{\varepsilon^5\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(u\partial_x v)_\psi,e^{\mathcal{R}t'}\Delta_k^h v_\psi)_{L^2}|dt'}_{A_3}\\
&\quad+\underbrace{\varepsilon^5\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(v\partial_y v)_\psi,e^{\mathcal{R}t'}\Delta_k^h v_\psi)_{L^2}|dt'}_{A_4}+\underbrace{\varepsilon\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(R_{11}^\varepsilon)_\psi,e^{\mathcal{R}t'}\Delta_k^h\partial_x u_\psi)_{L^2}|dt'}_{A_5}\\
&\quad+\underbrace{\varepsilon\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(R_{21}^\varepsilon)_\psi,e^{\mathcal{R}t'}\Delta_k^h\partial_y u_\psi)_{L^2}|dt'}_{A_6}+\underbrace{\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(R_{12}^\varepsilon)_\psi,e^{\mathcal{R}t'}\Delta_k^h\partial_x v_\psi)_{L^2}|dt'}_{A_7}\\
&\quad+\underbrace{\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(R_{22}^\varepsilon)_\psi,e^{\mathcal{R}t'}\Delta_k^h\partial_y v_\psi)_{L^2}|dt'}_{A_8}+\underbrace{\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(v\partial_y U)_\psi,e^{\mathcal{R}t'}\Delta_k^h u_\psi)_{L^2}|dt'}_{A_9}
\end{aligned} \tag{2.10}$$

similarly for the Equation (2.8), we have

$$\begin{aligned}
&\frac{1}{2}||e^{\mathcal{R}'t'}\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi||_{L_t^\infty(L^2)}^2+a'\int_0^t||e^{\mathcal{R}'t'}\Delta_k^h(Q_{11},\varepsilon(Q_{12})_\psi)(t')||_{L^2}^2dt'\\
&\quad+\lambda 2^k\int_0^t\eta'(t')||e^{\mathcal{R}'t'}\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi||_{L^2}^2dt'\\
&\quad+\frac{1}{2}\int_0^t e^{2\mathcal{R}'t'}(||\Delta_k^h\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{L^2}^2+c\varepsilon^2 2^{2k}||\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi||_{L^2}^2)dt'\\
&\leq ||e^{a|D_x|}\Delta_k^h(Q_{11},\varepsilon Q_{12})_0||_{L^2}^2\\
&\quad+\underbrace{\varepsilon\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(u\partial_x(Q_{11},\varepsilon Q_{12}))_\psi,e^{\mathcal{R}t'}\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{B_1}\\
&\quad+\underbrace{\varepsilon\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(v\partial_y(Q_{11},\varepsilon Q_{12}))_\psi,e^{\mathcal{R}t'}\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{B_2}
\end{aligned}$$

$$+\underbrace{\varepsilon\int_0^t |\Big(e^{\mathcal{R}t'}\Delta_k^h\left[(\varepsilon^2\partial_x v-\partial_y u)Q_{11}\right]_\psi, e^{\mathcal{R}t'}\Delta_k^h(Q_{12})_\psi\Big)_{L^2}|dt'}_{B_3}$$

$$+\underbrace{\varepsilon\int_0^t |\Big(e^{\mathcal{R}t'}\Delta_k^h[(\partial_y u-\varepsilon^2\partial_x v)Q_{12}]_\psi, e^{\mathcal{R}'t'}\Delta_k^h(Q_{11})_\psi\Big)_{L^2}|dt'}_{B_4}$$

$$+\underbrace{2c'\int_0^t |(e^{\mathcal{R}t'}\Delta_k^h[(Q_{11}^2+\varepsilon^2 Q_{12}^2)(Q_{11},\varepsilon Q_{12})]_\psi, e^{\mathcal{R}'t'}\Delta_k^h(Q_{11},\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{B_5} \quad (2.11)$$

from Lemma 3.1-3.3 of [16], we have

$$A_1+A_2+A_3+A_4\lesssim \varepsilon^3 d_k^2 2^{-k}||e^{\mathcal{R}t'}(u,\varepsilon v)_\psi||^2_{\tilde{L}^2_{t,\eta'(t)}(B^1)}$$

and

$$B_1\lesssim \varepsilon d_k^2 2^{-k}||e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi||^2_{\tilde{L}^2_{t,\eta'(t)}(B^1)}$$

recalling that $v(t,x,y)=-\int_0^y \partial_x u(t,x,y')dy'$ and

$$\partial_y U=\sum_{m>0} m\pi c_m e^{-m^2\pi^2 t}\cos(m\pi y)\lesssim \eta'(t)$$

we have

$$A_9\lesssim \varepsilon^2 d_k^2 2^{-k}||e^{\mathcal{R}t'}u_\psi||^2_{\tilde{L}^2_{t,\eta'(t)}(B^1)}$$

next, noticing the fact that $||Q^3||_{\mathcal{B}^{1/2}}\lesssim ||Q||^3_{\mathcal{B}^{1/2}}$, we use Cauchy-Schwartz inequality to obtain that

$$B_5\lesssim d_k^2 2^{-k}||e^{\mathcal{R}'t'}(Q_{11},\varepsilon Q_{12})_\psi||^2_{L^2(B^{1/2})}||(Q_{11},\varepsilon Q_{12})_\psi||^2_{L^2(B^{1/2})}$$

for the rest of the terms, we need to calculate terms A and B seperately.

2.0.1. Estimates of B_2, B_3, B_4. We first show the following lemmas.

LEMMA 2.1. *For any $s\in(0,\frac{1}{2}]$ and any $t\le T^*$, we have*

$$\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(w\tilde{w})_\psi, e^{\mathcal{R}t'}\Delta_k^h(\partial_y u)_\psi)_{L^2}|dt' \\ \lesssim d_k^2 2^{-2ks}||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}.$$

Proof. By using Bony's decomposition, we have $w\tilde{w}=T_w^h\tilde{w}+T_{\tilde{w}}^h w+R^h(w,\tilde{w})$.

Step 1: Estimate of $\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T_w^h\tilde{w})_\psi, e^{\mathcal{R}t'}\Delta_k^h(\partial_y u)_\psi)_{L^2}dt'$. We have

$$\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T_w^h\tilde{w})_\psi, e^{\mathcal{R}t'}\Delta_k^h(\partial_y u)_\psi)_{L^2}dt' \\ \lesssim \sum_{|k'-k|\le 4}\int_0^t ||e^{\mathcal{R}t'}S_{k'-1}^h w_\psi(t')||_{L^\infty}||e^{\mathcal{R}t'}\Delta_{k'}^h\tilde{w}_\psi(t')||_{L^2}||\Delta_k^h(\partial_y u)_\psi(t')||_{L^2}dt'$$

$$
\begin{aligned}
&\lesssim 2^{-\frac{k}{2}} d_k(t) \sum_{|k'-k|\le 4} \int_0^t ||e^{\mathcal{R}t'} S^h_{k'-1} w_\psi(t')||_{L^\infty} ||e^{\mathcal{R}t'} \Delta^h_{k'} \tilde{w}_\psi(t')||_{L^2} ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} dt' \\
&\lesssim 2^{-\frac{k}{2}} d_k(t) \sum_{|k'-k|\le 4} \left(\int_0^t ||e^{\mathcal{R}t'} S^h_{k'-1} w_\psi(t')||^2_{L^\infty} ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^t ||e^{\mathcal{R}t'} \Delta^h_{k'} \tilde{w}_\psi(t')||^2_{L^2} ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\left(\int_0^t ||e^{\mathcal{R}t'} S^h_{k'-1} w_\psi(t')||^2_{L^\infty} ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{l\le k'-2} 2^{\frac{l}{2}} \left(\int_0^t ||e^{\mathcal{R}t'} \Delta^h_l w_\psi(t')||^2_{L^2} ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{l\le k'-2} d_l 2^{l(\frac{1}{2}-s)} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} \lesssim 2^{k'(\frac{1}{2}-s)} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}
\end{aligned}
\tag{2.12}
$$

so we obtain that

$$
\begin{aligned}
&\int_0^t (e^{\mathcal{R}t'} \Delta^h_k (T^h_w \tilde{w})_\psi, e^{\mathcal{R}t'} \Delta^h_k (\partial_y u)_\psi)_{L^2} dt' \\
&\lesssim \sum_{|k'-k|\le 4} d_k d_{k'} 2^{-2ks} 2^{(k-k')(2s-\frac{1}{2})} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'} \tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} \\
&\lesssim d_k^2 2^{-2ks} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'} \tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}.
\end{aligned}
$$

Step 2: Estimate of $\int_0^t (e^{\mathcal{R}t'} \Delta^h_k (T^h_{\tilde{w}} w)_\psi, e^{\mathcal{R}t'} \Delta^h_k (\partial_y u)_\psi)_{L^2} dt'$. Similarly as Step 1, just exchange w and $\tilde{w}$.

Step 3: Estimate of $\int_0^t (e^{\mathcal{R}t'} \Delta^h_k (R^h(w,\tilde{w}))_\psi, e^{\mathcal{R}t'} \Delta^h_k (\partial_y u)_\psi)_{L^2} dt'$. We have

$$
\begin{aligned}
&\int_0^t (e^{\mathcal{R}t'} \Delta^h_k (R^h(w,\tilde{w}))_\psi, e^{\mathcal{R}t'} \Delta^h_k (\partial_y u)_\psi)_{L^2} dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\ge k-3} \int_0^t ||e^{\mathcal{R}t'} \tilde{\Delta}^h_{k'} w_\psi(t')||_{L^2_h(L^\infty_v)} ||e^{\mathcal{R}t'} \Delta^h_{k'} \tilde{w}_\psi(t')||_{L^2} ||\Delta^h_k \partial_y u_\psi(t')||_{L^2} dt' \\
&\lesssim \sum_{k'\ge k-3} \int_0^t ||e^{\mathcal{R}t'} \tilde{\Delta}^h_{k'} w_\psi(t')||_{L^2} ||e^{\mathcal{R}t'} \Delta^h_{k'} \tilde{w}_\psi(t')||_{L^2} ||\partial_y u_\psi||_{B^{\frac{1}{2}}} dt' \\
&\lesssim \sum_{k'\ge k-3} \left(\int_0^t ||e^{\mathcal{R}t'} \Delta^h_{k'} \tilde{w}_\psi(t')||^2_{L^2} ||\partial_y u_\psi||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \cdot \left(\int_0^t ||e^{\mathcal{R}t'} \tilde{\Delta}^h_{k'} w_\psi(t')||^2_{L^2} ||\partial_y u_\psi||_{B^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2ks} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'} \tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} \left(\sum_{k'\ge k-3} d_{k'} 2^{(k-k')s} \right)^2 \\
&\lesssim d_k^2 2^{-2ks} ||e^{\mathcal{R}t'} w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'} \tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}
\end{aligned}
$$

and the lemma is proved after gathering three terms. ∎

LEMMA 2.2. *For any $s\in(0,\frac{1}{2}]$ and any $t\le T^*$, we have*

$$\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(\partial_y uw)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}|dt'$$
$$\lesssim d_k^2 2^{-2ks}||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}.$$

Proof. By Bony's decomposition. we have

$$\partial_y uw = T^h_{\partial_y u}w + T^h_w\partial_y u + R^h(\partial_y u, w).$$

We recall from (3.15) of [16] that

$$||\Delta_k^h u_\psi(t)||_{L^\infty}\lesssim d_j(t)||\partial_y u_\psi(t)||_{B^{\frac{1}{2}}} \tag{2.13}$$

because of Bernstein lemma 1.1 and Poincaré inequality.

Step 1: Estimate of $\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T^h_{\partial_y u}w)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt'$. Notice that

$$\begin{aligned}
&\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T^h_{\partial_y u}w)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}\int_0^t ||S^h_{k'-1}\partial_y u_\psi(t')||_{L^\infty}||e^{\mathcal{R}t'}\Delta^h_{k'}w_\psi(t')||_{L^2}||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}\int_0^t ||\partial_y u_\psi(t)||_{B^{\frac{1}{2}}}||e^{\mathcal{R}t'}\Delta^h_{k'}w_\psi(t')||_{L^2}||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}\left(\int_0^t ||\partial_y u_\psi(t)||_{B^{\frac{1}{2}}}||e^{\mathcal{R}t'}\Delta^h_{k'}\tilde{w}_\psi(t')||^2_{L^2}dt'\right)^{\frac{1}{2}}\\
&\qquad\cdot\left(\int_0^t ||\partial_y u_\psi(t)||_{B^{\frac{1}{2}}}||e^{\mathcal{R}t'}\Delta^h_{k}w_\psi(t')||^2_{L^2}dt'\right)^{\frac{1}{2}}\\
&\lesssim d_k 2^{-2ks}||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}\left(\sum_{|k'-k|\le 4} d_{k'}2^{(k-k')s}\right)\\
&\lesssim d_k^2 2^{-2ks}||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}.
\end{aligned}$$

Step 2: Estimate of $\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T^h_w\partial_y u)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt'$. We have

$$\begin{aligned}
&\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T^h_w\partial_y u)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}\int_0^t ||S^h_{k'-1}w_\psi(t')||_{L^\infty}||e^{\mathcal{R}t'}\Delta^h_{k'}\partial_y u_\psi(t')||_{L^2}||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}2^{-\frac{k'}{2}}\int_0^t d_{k'}(t)||S^h_{k'-1}w_\psi(t')||_{L^\infty}||e^{\mathcal{R}t'}\partial_y u_\psi(t')||_{B^{\frac{1}{2}}}||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt'\\
&\lesssim \sum_{|k'-k|\le 4}d_{k'}2^{-\frac{k'}{2}}\left(\int_0^t ||S^h_{k'-1}w_\psi(t')||^2_{L^\infty}||e^{\mathcal{R}t'}\partial_y u_\psi(t')||_{B^{\frac{1}{2}}}dt'\right)^{\frac{1}{2}}\\
&\qquad\cdot\left(\int_0^t ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}}||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||^2_{L^2}dt'\right)^{\frac{1}{2}}.
\end{aligned}$$

We use (2.12) again to obtain that

$$
\begin{aligned}
&\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(T_w^h\partial_y u)_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt' \\
&\lesssim \sum_{|k'-k|\le 4} d_k d_{k'} 2^{-2ks} 2^{(k-k')(s-\frac{1}{2})} ||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} \\
&\lesssim d_k^2 2^{-2ks} ||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}.
\end{aligned}
$$

Step 3: Estimate of $\int_0^t (e^{\mathcal{R}t'}\Delta_k^h(R^h(\partial_y u, w))_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt'$. We have

$$
\begin{aligned}
&\int_0^t (\Delta_k^h(e^{\mathcal{R}t'}R^h(\partial_y u, w))_\psi, e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi)_{L^2}dt' \\
&\lesssim 2^{\frac{k}{2}} \int_0^t ||\tilde{\Delta}_{k'}^h\partial_y u_\psi(t')||_{L^2_h(L^\infty_v)} ||e^{\mathcal{R}t'}\Delta_{k'}^h w_\psi(t')||_{L^2} ||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\ge k-3} 2^{-\frac{k'}{2}} \int_0^t ||\partial_y u_\psi(t')||_{B^{\frac{1}{2}}} ||e^{\mathcal{R}t'}\Delta_{k'}^h w_\psi(t')||_{L^2} ||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{k'\ge k-3} 2^{-\frac{k'}{2}} \left(\int_0^t ||e^{\mathcal{R}t'}\Delta_k^h\tilde{w}_\psi(t')||_{L^2}^2 ||\partial_y u_\psi||_{B^{\frac{1}{2}}}dt'\right)^{\frac{1}{2}} \\
&\qquad\qquad \cdot \left(\int_0^t ||e^{\mathcal{R}t'}\Delta_{k'}^h w_\psi(t')||_{L^2}^2 ||\partial_y u_\psi||_{B^{\frac{1}{2}}}dt'\right)^{\frac{1}{2}} \\
&\lesssim d_k 2^{-2ks} ||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} \left(\sum_{k'\ge k-3} d_{k'} 2^{(k-k')(s+\frac{1}{2})}\right) \\
&\lesssim d_k^2 2^{-2ks} ||e^{\mathcal{R}t'}w_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)} ||e^{\mathcal{R}t'}\tilde{w}_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^s)}
\end{aligned}
$$

so we finish the proof after gathering all the terms. ∎

REMARK 2.2. Lemma 2.1 and Lemma 2.2 give the estimates on the terms A_6 and the B_3 term with $\partial_y u$ part. But since for these terms, there is no ∂_x or v term included, the Besov norm becomes $B^{\frac{1}{2}}$ instead of B^1, which makes the function $\eta(t)$ a bit different from the one in [16].

PROPOSITION 2.1.

$$
B_2 \lesssim \frac{1}{\varepsilon} d_k^2 2^{-k} ||\partial_y(Q_{11}, \varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}. \tag{2.14}
$$

Proof. From Poincaré inequality and $\partial_y v = -\partial_x u$, we have

$$
\begin{aligned}
&\int_0^t (\Delta_k^h(T^h_{\partial_y Q_{11}} v)_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2}dt' \\
&\lesssim \sum_{|k'-k|\le 4} \int_0^t ||S_{k'-1}^h\partial_y(Q_{11})_\psi(t')||_{L^\infty} ||\Delta_{k'}^h v_\psi(t')||_{L^2} ||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt' \\
&\lesssim \sum_{|k'-k|\le 4} \int_0^t ||\partial_y(Q_{11})_\psi(t')||_{B^{\frac{1}{2}}} ||\Delta_{k'}^h v_\psi(t')||_{L^2} ||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'
\end{aligned}
$$

$$\lesssim \sum_{|k'-k|\le 4}\left(\int_0^t ||\Delta_{k'}^h(\partial_x u)_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h\partial_y(Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim d_k^2 2^{-k}||\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\partial_y(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}.$$

Next,

$$\int_0^t (\Delta_k^h(T_v^h\partial_y Q_{11})_\psi,\Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$
$$\lesssim \sum_{|k'-k|\le 4}\int_0^t ||S_{k'-1}^h v_\psi(t')||_{L^\infty}|||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim \sum_{|k'-k|\le 4}2^{-\frac{k'}{2}}\int_0^t ||S_{k'-1}^h v_\psi(t')||_{L^\infty}||(\partial_y Q_{11})_\psi(t')||_{B^{\frac12}}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim \sum_{|k'-k|\le 4}2^{-\frac{k'}{2}}\left(\int_0^t ||S_{k'-1}^h v_\psi(t')||_{L^\infty}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}.$$

Notice that for any $s\in(0,\frac12]$,

$$\left(\int_0^t ||S_{k'-1}^h v_\psi(t')||_{L^\infty}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim \sum_{l\le k'-2}2^{\frac{l}{2}}\left(\int_0^t ||\Delta_l^h v_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\lesssim \sum_{l\le k'-2}2^{\frac{l}{2}}\left(\int_0^t ||\Delta_l^h\partial_x u_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim \sum_{l\le k'-2}d_l 2^{l(\frac12-s)}||\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^s)}\lesssim d_k 2^{k(\frac12-s)}||\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^s)}$$

hence

$$\int_0^t (\Delta_k^h(T_v^h\partial_y Q_{11})_\psi,\Delta_k^h(Q_{11})_\psi)_{L^2}dt'\lesssim d_k^2 2^{-k}||\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||(\partial_y Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}$$

For the third term, we have

$$\int_0^t (\Delta_k^h(R^h(v,\partial_y Q_{11}))_\psi,\Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$
$$\lesssim 2^{\frac{k}{2}}\sum_{k'\ge k-3}\int_0^t ||\Delta_{k'}^h v_\psi(t')||_{L^2}||\tilde\Delta_{k'}^h(\partial_y Q_{11})_\psi(t')||_{L^2}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim 2^{\frac{k}{2}}\sum_{k'\ge k-3}2^{-\frac{k'}{2}}\int_0^t ||(\partial_y Q_{11})_\psi(t')||_{B^{\frac12}}||\Delta_{k'}^h v_\psi(t')||_{L^2}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim 2^{\frac{k}{2}}\sum_{k'\ge k-3}2^{-\frac{k'}{2}}\int_0^t ||(\partial_y Q_{11})_\psi(t')||_{B^{\frac12}}||\Delta_{k'}^h\partial_x u_\psi(t')||_{L^2}||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim 2^{\frac{k}{2}}\sum_{k'\ge k-3}2^{-\frac{k'}{2}}\left(\int_0^t ||\Delta_{k'}^h(\partial_y Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h\partial_x u_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$

$$\lesssim d_k 2^{-k}||\partial_y(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}||\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\left(\sum_{k'\geq k-3} d_{k'}2^{(k-k')}\right)$$

$$\lesssim d_k^2 2^{-k}||\partial_y(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}||\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}$$

and the proposition is proved. □

PROPOSITION 2.2.

$$B_3 \lesssim d_k^2 2^{-k}\left(||e^{\mathcal{R}t'}(\varepsilon\partial_x u)_\psi||^2_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} + ||e^{\mathcal{R}t'}(\partial_y(Q_{11},\varepsilon Q_{12})_\psi||^2_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\right) \tag{2.15}$$

Proof. The terms related to $\partial_y u$ can be estimated by using Lemma 2.2. Next, recall from (3.18) of [16] that because $v(t,x,y) = -\int_0^y \partial_x u(t,x,y')dy'$, we have

$$||\Delta_k^h v_\psi(t)||_{L^\infty} \lesssim 2^{\frac{3k}{2}}||\Delta_k^h u_\psi(t)||_{L^2}. \tag{2.16}$$

Notice that

$$\varepsilon^3\int_0^t (\Delta_k^h(T^h_{\partial_x v}Q_{12})_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$

$$\lesssim \varepsilon^3 \sum_{|k'-k|\leq 4}\int_0^t ||S^h_{k'-1}\partial_x v_\psi(t')||_{L^\infty}||\Delta^h_{k'}(Q_{12})_\psi(t')||_{L^2}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$

$$\lesssim \varepsilon \sum_{|k'-k|\leq 4} 2^{-\frac{3k'}{2}}\int_0^t ||S^h_{k'-1}\partial_x v_\psi(t')||_{L^\infty}||\Delta^h_{k'}(\varepsilon^2\partial_x Q_{12})_\psi(t')||_{B^{\frac{1}{2}}}||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}dt'$$

$$\lesssim \sum_{|k'-k|\leq 4} 2^{-\frac{3k'}{2}}\varepsilon\left(\int_0^t ||S^h_{k'-1}\partial_x v_\psi(t')||^2_{L^\infty}\eta'(t)dt'\right)^{\frac{1}{2}}\cdot\left(\int_0^t ||\Delta_k^h\partial_y(Q_{11})_\psi(t')||^2_{L^2}\eta'(t)dt'\right)^{\frac{1}{2}}$$

and we deduce from (2.16) that

$$\left(\int_0^t ||S^h_{k'-1}\varepsilon\partial_x v_\psi(t')||^2_{L^\infty}\eta'(t)dt'\right)^{\frac{1}{2}} \lesssim \sum_{l\leq k'-2} 2^{\frac{3l}{2}}\left(\int_0^t ||\Delta_l^h\varepsilon\partial_x u_\psi(t')||^2_{L^2}\eta'(t)dt'\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{l\leq k'-2} d_l 2^l||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} \lesssim d_{k'}2^{k'}||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}$$

so

$$\varepsilon^3\int_0^t (\Delta_k^h(T^h_{\partial_x v}Q_{12})_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$
$$\lesssim d_k^2 2^{-k}||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}||\partial_y(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}.$$

Similarly, we have

$$\varepsilon^3\int_0^t (\Delta_k^h(T^h_{Q_{12}}\partial_x v)_\psi, \Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$

$$\lesssim \varepsilon^3 \sum_{|k'-k|\leq 4}\int_0^t ||S^h_{k'-1}(Q_{12})_\psi(t')||_{L^\infty}|||\Delta^h_{k'}(\partial_x v)_\psi(t')||_{L^2}||\Delta_k^h(Q_{11})_\psi(t')||_{L^2}dt'$$

$$\lesssim \sum_{|k'-k|\leq 4}\int_0^t ||(\varepsilon^2 Q_{12})_\psi(t')||_{B^{\frac{3}{2}}}||(\varepsilon\Delta^h_{k'}\partial_x u)_\psi(t')||_{L^2}||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}dt'$$

$$\lesssim \sum_{|k'-k|\leq 4}\left(\int_0^t ||(\varepsilon\partial_x u)_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim d_k^2 2^{-k}||\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\partial_y(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}$$

and

$$\varepsilon^3\int_0^t(\Delta_k^h(R^h(\partial_x v,Q_{12}))_\psi,\Delta_k^h(Q_{11})_\psi)_{L^2}dt'$$
$$\lesssim \varepsilon^3 2^{\frac k2}\sum_{k'\geq k-3}\int_0^t ||\Delta_{k'}^h\partial_x v_\psi(t')||_{L^2}||\tilde\Delta_{k'}^h(Q_{12})_\psi(t')||_{L^2}||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim 2^{\frac k2}\sum_{k'\geq k-3}2^{-\frac{k'}{2}}\int_0^t ||(\varepsilon\partial_x u)_\psi(t')||_{L^2}||\Delta_{k'}^h(\varepsilon^2\partial_x Q_{12})_\psi(t')||_{B^{\frac12}}||\Delta_k^h(\partial_y Q_{11})_\psi(t')||_{L^2}dt'$$
$$\lesssim 2^{\frac k2}\sum_{k'\geq k-3}2^{-\frac{k'}{2}}\left(\int_0^t ||\Delta_{k'}^h\varepsilon\partial_x u_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h\partial_y(Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim d_k 2^{-k}||\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\partial_y(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\left(\sum_{k'\geq k-3}d_{k'}2^{(k-k')}\right)$$
$$\lesssim d_k^2 2^{-k}||\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\partial_y(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}$$

which together with Lemma 2.2 provides the result. □

REMARK 2.3. The term B_4 can be estimated similarly as B_3 by exchanging the order of Q_{11} and Q_{12}.

2.0.2. Estimates of A_5, A_6, A_7, A_8. We prove the following propositions:

PROPOSITION 2.3. *We have*

$$A_5+A_8\lesssim \varepsilon d_k^2 2^{-k}||e^{\mathcal{R}'t'}(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||e^{\mathcal{R}'t'}\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}. \tag{2.17}$$

Proof. Notice that

$$\varepsilon^3\int_0^t(\Delta_k^h(T^h_{\partial_x Q_{11}}\partial_x Q_{11})_\psi,\Delta_k^h\partial_x u_\psi)_{L^2}dt'$$
$$\lesssim \varepsilon^3\sum_{|k'-k|\leq 4}\int_0^t ||S_{k'-1}^h\partial_x(Q_{11})_\psi(t')||_{L^\infty}||\Delta_{k'}^h(\partial_x Q_{11})_\psi(t')||_{L^2}||\Delta_k^h\partial_x u_\psi(t')||_{L^2}dt'$$
$$\lesssim \sum_{|k'-k|\leq 4}\int_0^t ||(\varepsilon\partial_x Q_{11})_\psi(t')||_{B^{\frac12}}||\Delta_{k'}^h(\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2}||\Delta_k^h\varepsilon\partial_x u_\psi(t')||_{L^2}dt'$$
$$\lesssim \sum_{|k'-k|\leq 4}\left(\int_0^t ||\Delta_{k'}^h(\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}\left(\int_0^t ||\Delta_k^h\varepsilon\partial_x u_\psi(t')||_{L^2}^2\eta'(t)dt'\right)^{\frac12}$$
$$\lesssim d_k^2 2^{-k}||\varepsilon\partial_x(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}$$

and

$$\varepsilon^3\int_0^t(\Delta_k^h(R^h(Q_{11},\partial_y Q_{12}))_\psi,\Delta_k^h\partial_x^2 u_\psi)_{L^2}dt'$$

$$
\begin{aligned}
&\lesssim \varepsilon^3 2^{\frac{k}{2}} \sum_{k'\geq k-3} \int_0^t ||\Delta_{k'}^h (\partial_x Q_{11})_\psi(t')||_{L^2} ||\tilde{\Delta}_{k'}^h (\partial_x Q_{11})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x u_\psi(t')||_{L^2} dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{|k'-k|\leq 4} 2^{-\frac{k'}{2}} \int_0^t ||(\varepsilon\partial_x Q_{11})_\psi(t')||_{B^{\frac12}} ||\Delta_{k'}^h (\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2} ||\Delta_k^h \varepsilon\partial_x u_\psi(t')||_{L^2} dt' \\
&\lesssim 2^{\frac{k}{2}} \sum_{|k'-k|\leq 4} 2^{\frac{-k'}{2}} \left(\int_0^t ||\Delta_{k'}^h (\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2}^2 \eta'(t) dt'\right)^{\frac12} \\
&\qquad \cdot \left(\int_0^t ||\Delta_k^h \varepsilon\partial_x u_\psi(t')||_{L^2}^2 \eta'(t) dt'\right)^{\frac12} \\
&\lesssim d_k 2^{-k} ||\partial_x (Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})} \left(\sum_{k'\geq k-3} d_{k'} 2^{2(k-k')}\right) \\
&\lesssim d_k^2 2^{-k} ||\partial_x (Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}
\end{aligned}
$$

so we have proved that

$$
A_5 \lesssim \varepsilon d_k^2 2^{-k} ||e^{\mathcal{R}'t'} \varepsilon\partial_x (Q_{11}, \varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})} ||e^{\mathcal{R}'t'} \varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}.
$$

The estimate of A_8 is similar, just exchanging $\varepsilon\partial_x$ to ∂_y and using that $\partial_y v = -\partial_x u$. □

PROPOSITION 2.4.

$$
A_7 \lesssim \varepsilon d_k^2 2^{-k} ||(\varepsilon\partial_x, \partial_y)(Q_{11}, \varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})} ||(\varepsilon\partial_x u, \varepsilon^2 \partial_x v)_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}. \tag{2.18}
$$

Proof. We write

$$
\Delta_\varepsilon Q_{12} Q_{11} - \Delta_\varepsilon Q_{11} Q_{12} = \varepsilon^2 \partial_x^2 Q_{12} Q_{11} - \varepsilon^2 \partial_x^2 Q_{11} Q_{12} + \partial_y^2 Q_{12} Q_{11} - \partial_y^2 Q_{11} Q_{12}
$$

from integration by parts,

$$
\begin{aligned}
&(\Delta_k^h (\partial_y^2 Q_{12} Q_{11})_\psi, \Delta_k^h \partial_x v_\psi)_{L^2} + (\Delta_k^h (\partial_y Q_{12} \partial_y Q_{11})_\psi, \Delta_k^h \partial_x v_\psi)_{L^2} \\
&= -(\Delta_k^h (\partial_y Q_{12} Q_{11})_\psi, \Delta_k^h \partial_x \partial_y v_\psi)_{L^2}
\end{aligned}
$$

so we have (recall that $\partial_y v = -\partial_x u$)

$$
\begin{aligned}
&(\Delta_k^h (\partial_y^2 Q_{12} Q_{11} - \partial_y^2 Q_{11} Q_{12})_\psi, \Delta_k^h \partial_x v_\psi)_{L^2} \\
&= (\Delta_k^h (\partial_y Q_{12} Q_{11})_\psi, \Delta_k^h \partial_x^2 u_\psi)_{L^2} - (\Delta_k^h (\partial_y Q_{11} Q_{12})_\psi, \Delta_k^h \partial_x^2 u_\psi)_{L^2}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\varepsilon^3 \int_0^t (\Delta_k^h (T^h_{\partial_y Q_{12}} Q_{11})_\psi, \Delta_k^h \partial_x^2 u_\psi)_{L^2} dt' \\
&\lesssim \varepsilon^3 \sum_{|k'-k|\leq 4} \int_0^t ||S^h_{k'-1} \partial_y (Q_{12})_\psi(t')||_{L^\infty} ||\Delta_{k'}^h (Q_{11})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x^2 u_\psi(t')||_{L^2} dt' \\
&\lesssim \varepsilon^2 \sum_{|k'-k|\leq 4} \int_0^t ||\partial_y (\varepsilon Q_{12})_\psi(t')||_{B^{\frac12}} ||\Delta_{k'}^h (Q_{11})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x^2 u_\psi(t')||_{L^2} dt'
\end{aligned}
$$

$$\lesssim \sum_{|k'-k|\leq 4} \left(\int_0^t ||\Delta_{k'}^h(\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}} \left(\int_0^t ||\Delta_k^h \varepsilon\partial_x u_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}}$$
$$\lesssim d_k^2 2^{-k} ||\varepsilon\partial_x(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}$$

and similarly,

$$\varepsilon^3 \int_0^t (\Delta_k^h(T^h_{Q_{11}}\partial_y Q_{12})_\psi, \Delta_k^h\partial_x^2 u_\psi)_{L^2} dt'$$
$$\lesssim \varepsilon^3 \sum_{|k'-k|\leq 4} \int_0^t ||S^h_{k'-1}(Q_{11})_\psi(t')||_{L^\infty} ||\Delta_{k'}^h(\partial_y Q_{12})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x^2 u_\psi(t')||_{L^2} dt'$$
$$\lesssim \varepsilon^2 \sum_{|k'-k|\leq 4} 2^{-k'} \int_0^t ||(\varepsilon Q_{11})_\psi(t')||_{B^{\frac{3}{2}}} ||\Delta_{k'}^h(\partial_y Q_{12})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x^2 u_\psi(t')||_{L^2} dt'$$
$$\lesssim \sum_{|k'-k|\leq 4} \left(\int_0^t ||\Delta_{k'}^h(\partial_y Q_{12})_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}} \left(\int_0^t ||\Delta_k^h \varepsilon\partial_x u_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}}$$
$$\lesssim d_k^2 2^{-k} ||\partial_y(Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}$$

moreover,

$$\varepsilon^2 \int_0^t (\Delta_k^h(R^h(Q_{11},\partial_y Q_{12}))_\psi, \Delta_k^h\partial_x^2 u_\psi)_{L^2} dt'$$
$$\lesssim \varepsilon^2 2^{\frac{k}{2}} \sum_{k'\geq k-3} \int_0^t ||\Delta_{k'}^h(Q_{11})_\psi(t')||_{L^2} ||\tilde{\Delta}_{k'}^h(\partial_y Q_{12})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x^2 u_\psi(t')||_{L^2} dt'$$
$$\lesssim \varepsilon^2 2^{\frac{3k}{2}} \sum_{|k'-k|\leq 4} 2^{-\frac{3k'}{2}} \int_0^t ||(\partial_y Q_{12})_\psi(t')||_{B^{\frac{1}{2}}} ||\Delta_{k'}^h(\partial_x Q_{11})_\psi(t')||_{L^2} ||\Delta_k^h \partial_x u_\psi(t')||_{L^2} dt'$$
$$\lesssim 2^{\frac{3k}{2}} \sum_{|k'-k|\leq 4} 2^{\frac{-3k'}{2}} \left(\int_0^t ||\Delta_{k'}^h(\varepsilon\partial_x Q_{11})_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}}$$
$$\cdot \left(\int_0^t ||\Delta_k^h \varepsilon\partial_x u_\psi(t')||_{L^2}^2 \eta'(t)dt'\right)^{\frac{1}{2}}$$
$$\lesssim d_k 2^{-k} ||\partial_x(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} \left(\sum_{k'\geq k-3} d_{k'} 2^{2(k-k')}\right)$$
$$\lesssim d_k^2 2^{-k} ||\partial_x(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})},$$

so after combining these three terms, we have

$$\varepsilon^3 \int_0^t (\Delta_k^h(\partial_y Q_{12} Q_{11})_\psi, \Delta_k^h\partial_x^2 u_\psi)_{L^2} dt'$$
$$\lesssim d_k^2 2^{-k} \left(||\varepsilon\partial_x(Q_{11})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})} + ||\partial_y(\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\right) ||\varepsilon\partial_x u_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})},$$

next,

$$(\Delta_k^h(\partial_x^2 Q_{12} Q_{11} - \partial_x^2 Q_{11} Q_{12})_\psi, \Delta_k^h \partial_x v_\psi)_{L^2}$$

$$=(\Delta_k^h(\partial_x Q_{12}Q_{11})_\psi,\Delta_k^h\partial_x^2 v_\psi)_{L^2}-(\Delta_k^h(\partial_x Q_{11}Q_{12})_\psi,\Delta_k^h\partial_x^2 v_\psi)_{L^2}.$$

Similarly as the proof above (just change $\partial_y Q_{12}$ to $\varepsilon\partial_x Q_{12}$ and $\partial_x u$ to $\varepsilon\partial_x v$), we have

$$\begin{aligned}&\varepsilon^5\int_0^t(\Delta_k^h(\partial_x Q_{12}Q_{11})_\psi,\Delta_k^h\partial_x^2 v_\psi)_{L^2}dt'\\ &\lesssim d_k^2 2^{-k}\left(||\varepsilon\partial_x(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}+||\partial_x(\varepsilon^2 Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\right)||\varepsilon^2\partial_x v_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\end{aligned}$$

and then

$$\begin{aligned}&\varepsilon(\Delta_k^h(R^\varepsilon_{12,2})_\psi,\Delta_k^h\partial_x v_\psi)_{L^2}\\ &\lesssim d_k^2 2^{-k}||\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\varepsilon^2\partial_x v_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\\ &\quad+d_k^2 2^{-k}||(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\varepsilon\partial_x u_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\end{aligned}$$

similarly, we have

$$\varepsilon(\Delta_k^h(R^\varepsilon_{12,1})_\psi,\Delta_k^h\partial_x v_\psi)_{L^2}\lesssim d_k^2 2^{-k}||\varepsilon^2\partial_x v_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}$$

and the proposition is proved by gathering the estimates above. ◻

PROPOSITION 2.5.

$$\begin{aligned}A_6&\lesssim\varepsilon^3 d_k^2 2^{-k}||\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\\ &\lesssim\varepsilon^3 d_k^2 2^{-k}||(Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||(\Delta_\varepsilon Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}\\ &\quad+\varepsilon^3 d_k^2 2^{-k}||(Q_{12})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}||(\Delta_\varepsilon Q_{11})_\psi||_{\tilde L^2_{t,\eta'(t)}(B^{\frac12})}.\end{aligned}\tag{2.19}$$

Proof. The proof is similar as that of Lemma 2.1. We skip it for simplicity. ◻

From the estimate of A_6, we notice that there exists an extra term $\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})$ (more precisely, it is $\partial_y^2(Q_{11},\varepsilon Q_{12})$), which could be estimated from the calculations above. To solve this problem, we come back to the Equation (1.7). It has the term $\Delta_\varepsilon Q$ on the right, so we apply Δ_k^h to it and take the L^2 inner product with $-\varepsilon^2\Delta_k^h(\Delta_\varepsilon(Q_{11},\varepsilon Q_{22}))_\psi$ to provide the bound related to $\Delta_\varepsilon Q$. The reason that we multiply by an extra ε^2 is to prove the convergence to the hydrostatic system, to be done later. We have

$$\begin{aligned}&\frac12\frac{d}{dt}||(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi||_{L^2}^2\\ &\quad+a'||(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi||_{L^2}^2+||\varepsilon\Delta_k^h(\Delta_\varepsilon Q_{11},\varepsilon\Delta_\varepsilon Q_{12})_\psi||_{L^2}^2\\ &\quad+\lambda\eta'(t')(|D_x|\Delta_k^h(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{22})_\psi,\Delta_k^h(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{22})_\psi)_{L^2}\\ &=\varepsilon^2(\Delta_k^h((U+\varepsilon u)\partial_x Q_{11})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}+\varepsilon^4(\Delta_k^h((U+\varepsilon u)\partial_x Q_{12})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}\\ &\quad+\varepsilon^3(\Delta_k^h((v\partial_y Q_{11})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}+\varepsilon^5(\Delta_k^h((v\partial_y Q_{12})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}\\ &\quad+\varepsilon^2(\Delta_k^h((\partial_y(U+\varepsilon u)Q_{12})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}\\ &\quad+\varepsilon^2(\Delta_k^h((\partial_y(U+\varepsilon u)Q_{11})_\psi,e^{\mathcal R't'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}\\ &\quad-\varepsilon^5(\Delta_k^h((\partial_x vQ_{12})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}-\varepsilon^5(\Delta_k^h((\partial_x vQ_{11})_\psi,\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+2c\varepsilon^2(\Delta_k^h(Q_{11}^3+2\varepsilon^2Q_{11}Q_{12}^2)_\psi,\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}\\
&+2c\varepsilon^4(\Delta_k^h(2\varepsilon^2Q_{12}^3+Q_{12}Q_{11}^2)_\psi,\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}
\end{aligned}
\tag{2.20}
$$

multiply (2.20) by $e^{2\mathcal{R}t}$ and integrate over $[0,t]$, to get

$$
\begin{aligned}
&\frac{1}{2}||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi||^2_{L^\infty_t(L^2)}+a'\int_0^t||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi(t')||^2_{L^2}dt'\\
&\quad+\lambda 2^k\int_0^t\eta'(t)||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi(t')||^2_{L^2}dt'\\
&\quad+\int_0^t||e^{\mathcal{R}t'}\varepsilon\Delta_k^h(\Delta_\varepsilon Q_{11},\varepsilon\Delta_\varepsilon Q_{12})_\psi(t')||^2_{L^2}dt'\\
&\le||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h((Q_{11})_0,\varepsilon(Q_{12})_0)||^2_{L^2}\\
&\quad+\underbrace{\varepsilon^2\int_0^t|e^{\mathcal{R}t'}(\Delta_k^h((U+\varepsilon u)\partial_xQ_{11})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}|dt'}_{C_1}\\
&\quad+\underbrace{\varepsilon^4\int_0^t|e^{\mathcal{R}t'}(\Delta_k^h((U+\varepsilon u)\partial_xQ_{12})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{C_2}\\
&\quad+\underbrace{\varepsilon^3\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h((v\partial_yQ_{11})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}|dt'}_{C_3}\\
&\quad+\underbrace{\varepsilon^5\int_0^t|e^{\mathcal{R}t'}(\Delta_k^h((v\partial_yQ_{12})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{C_4}\\
&\quad+\underbrace{\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h((\partial_y(U+\varepsilon u)Q_{12})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}|dt'}_{C_5}\\
&\quad+\underbrace{\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h((\partial_y(U+\varepsilon u)Q_{11})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{C_6}\\
&\quad+\underbrace{\varepsilon^5\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h((\partial_xvQ_{12})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}|dt'}_{C_7}\\
&\quad+\underbrace{\varepsilon^5\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h((\partial_xvQ_{11})_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{C_8}\\
&+\underbrace{2c\varepsilon^2\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(Q_{11}^3+2\varepsilon^2Q_{11}Q_{12}^2)_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11})_\psi)_{L^2}|dt'}_{C_9}\\
&\quad+\underbrace{2c'\varepsilon^4\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(2\varepsilon^2Q_{12}^3+Q_{12}Q_{11}^2)_\psi,e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{12})_\psi)_{L^2}|dt'}_{C_{10}}.
\end{aligned}
\tag{2.21}
$$

Similarly as the estimate of B_1 and recalling that $U\lesssim\eta'(t)$, we have

$$
C_1+C_2\lesssim\varepsilon^2d_k^22^{-k}||e^{\mathcal{R}t'}\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}||e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}.
$$

Similarly as the estimate of B_2, we get

$$C_3+C_4\lesssim\varepsilon^3 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\partial_x u_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\|e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}.$$

Similarly as the estimate of B_3, and recalling that $\partial_y U\lesssim\eta'(t)$, we have

$$\begin{aligned}C_5+C_6\lesssim&\varepsilon^2 d_k^2 2^{-k}(\|e^{\mathcal{R}t'}(Q_{12})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\|e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})})\\&+\varepsilon^2 d_k^2 2^{-k}(\|e^{\mathcal{R}t'}(Q_{11})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\|e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{12})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})})\end{aligned}$$

and

$$C_7+C_8\lesssim\varepsilon^3 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\partial_x u_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\|e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}.$$

Then we get

$$\begin{aligned}C_9+C_{10}\lesssim&\varepsilon^2 d_k^2 2^{-k}\big(\|e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\|(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\\&+\|e^{\mathcal{R}t'}(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\|(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\big)\end{aligned}$$

so finally we have that there exists a universal constant $C>0$, such that

$$\begin{aligned}&\frac{1}{2}\|e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi\|^2_{L^\infty_t(L^2)}\\&\quad+a'\int_0^t\|e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)\Delta_k^h(Q_{11},\varepsilon Q_{22})_\psi(t')\|^2_{L^2}dt'\\&\quad+\lambda 2^k\int_0^t\eta'(t)\|e^{\mathcal{R}t'}\Delta_k^h(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{22})_\psi(t')\|^2_{L^2}dt'\\&\quad+\frac{\varepsilon^2}{2}\int_0^t\|e^{\mathcal{R}t'}\Delta_k^h(\Delta_\varepsilon Q_{11},\varepsilon\Delta_\varepsilon Q_{12})_\psi(t')\|^2_{L^2}dt'\\&\le\|e^{a|D_x|}\Delta_k^h(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0,\varepsilon(Q_{12})_0)\|^2_{L^2}\\&\quad+C\varepsilon^2 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\partial_x(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}+C\varepsilon^4 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\partial_x u_\psi\|^2_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}\\&\quad+Cd_k^2 2^{-k}\|e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac{1}{2}})}+C\varepsilon^2 d_k^2 2^{-k}\|e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi\|^4_{\tilde{L}^2_t(B^{\frac{1}{2}})}\\&\quad+C\varepsilon^2 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\|\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\\&\quad+C\varepsilon^2 d_k^2 2^{-k}\|e^{\mathcal{R}t'}\partial_y(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}\|\partial_y(Q_{11},\varepsilon Q_{12})_\psi\|^2_{\tilde{L}^2_t(B^{\frac{1}{2}})}.\end{aligned}\tag{2.22}$$

Now we come back to prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Multiply the inequalities above 2^k, take square root and sum up over $\mathbb{Z}$, and recall that $a>0$. Then there exists a universal constant $C>0$, such that (recall c defined in (2.9))

$$\begin{aligned}&\|e^{\mathcal{R}t'}(\varepsilon u,\varepsilon^2 v)_\psi\|_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}+\|e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi\|_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}\\&\quad+\|e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{12})_\psi\|_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}\\&\quad+\sqrt{\lambda}\|e^{\mathcal{R}t'}(\varepsilon u,\varepsilon^2 v)_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^1)}+\sqrt{\lambda}\|e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi\|_{\tilde{L}^2_{t,\eta'(t)}(B^1)}\end{aligned}$$

$$
\begin{aligned}
&+c||e^{\mathcal{R}t'}\partial_y(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_t(B^{\frac12})}+c||e^{\mathcal{R}t'}\partial_x(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&+c||e^{\mathcal{R}t'}\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}+c||e^{\mathcal{R}t'}\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&+\sqrt{\lambda}||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^1)}+c\varepsilon||e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
\le&||e^{a|D_x|}(\varepsilon u_0,\varepsilon^2 v_0)||_{B^{\frac12}}+||e^{a|D_x|}((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac12}}\\
&+||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac12}}\\
&+C\varepsilon^{\frac32}||e^{\mathcal{R}t'}(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^1)}+C\varepsilon^{\frac12}||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^1)}\\
&+C||e^{\mathcal{R}t'}\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}+C||e^{\mathcal{R}t'}\partial_x(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}\\
&+C||e^{\mathcal{R}t'}\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}+C\varepsilon||e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_{t,\eta'(t)}(B^{\frac12})}\\
&+C||e^{\mathcal{R}t'}\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}||\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&+C||e^{\mathcal{R}t'}\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}||\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})} \qquad (2.23)
\end{aligned}
$$

recall that on $(0,T^*]$, we have $||\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}+||\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}<\delta$. Choosing $\lambda=C^2$ and $\delta=\frac{c\epsilon}{2C^2}$, where $\epsilon>0$ is small enough, then $C\delta=\frac{c\epsilon}{2C}<\frac{c}{2}$. Thus we have

$$
\begin{aligned}
&||e^{\mathcal{R}t'}(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^\infty_t(B^{\frac12})}+||e^{\mathcal{R}t'}(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^\infty_t(B^{\frac12})}\\
&\quad+||e^{\mathcal{R}t'}(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^\infty_t(B^{\frac12})}+\frac{c}{2}||e^{\mathcal{R}t'}(\varepsilon\partial_x,\partial_y)(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\quad+\frac{c}{2}||e^{\mathcal{R}t'}(\varepsilon\partial_x,\partial_y)(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}+\frac{c}{2}\varepsilon||e^{\mathcal{R}t'}\Delta_\varepsilon(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\le||e^{a|D_x|}(\varepsilon u_0,\varepsilon^2 v_0)||_{B^{\frac12}}+||e^{a|D_x|}((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac12}}\\
&\quad+||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac12}}:=c_0 \qquad (2.24)
\end{aligned}
$$

so for $t\le T^*$, we have

$$
\begin{aligned}
&||\partial_y(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_t(B^{\frac12})}+||\partial_x(\varepsilon u,\varepsilon^2 v)_\psi||_{\tilde{L}^2_t(B^{\frac12})}\\
&\qquad+||\partial_y(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}+||\varepsilon\partial_x(Q_{11},\varepsilon Q_{12})_\psi||_{\tilde{L}^2_t(B^{\frac12})}\le\frac{2c_0}{c}e^{-\mathcal{R}t}.
\end{aligned}
$$

Recall the definition of T^*. If we choose c_0 small enough, such that $c_0\le\frac{c\delta}{4}$, then we deduce that T^* equals to ∞. This finishes the proof of Theorem 1.1. □

3. Global well-posedness of the hydrostatic system and the convergence

In this section, we study the global well-posedness of the hydrostatic approximate Equations (1.14) with small analytic data and justify the limit from the scaled anisotropic system to the hydrostatic system.

3.1. Global solutions of the hydrostatic approximate. Recall that for the hydrostatic limit Equation (1.15), we should skip the ordinary case that $\partial_y U=0$. Then (1.15) becomes $Q_{11}=Q_{12}=0$. So we only need to focus on the Equation (1.14). Recall that U has the expression given in (1.8) and we suppose that $\sum_{m\ge0}m|c_*(m)|<\mathsf{c}^*$. Define

$$
u_\phi(t,x,y)=\mathcal{F}^{-1}_{\xi\to x}(e^{\phi(t,\xi)}\hat{u}(t,\xi,y))\quad\text{with}\quad\phi(t,\xi):=(a-\lambda\theta(t))|\xi| \qquad (3.1)
$$

where $\theta(t)$ denotes the evolution of the analytic band of u, determined by

$$\theta'(t)=\sum_{m>0} m|c_*(m)|e^{-m^2\pi^2 t} \quad \text{with} \quad \theta(0)=0. \tag{3.2}$$

Notice that $\theta(t)\leq\sum_{m>0}\frac{|c_*(m)|}{m\pi^2}$ for any $t>0$. We first prove the following proposition.

Proposition 3.1. *For any $s>0$, there exists some constant $\mathsf{c}^*>0$, such that if $\sum_{m>0}\frac{|c_*(m)|}{m}<\mathsf{c}^*$, then*

$$||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^\infty_t(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^s)}\leq||e^{a|D_x|}u_0||_{B^s}. \tag{3.3}$$

Proof. Recall that $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$. Using (1.14), we deduce that u_ϕ satisfies

$$\partial_t u_\phi+\lambda\theta'(t)|D_x|u_\psi+(U\partial_x u)_\phi+(v\partial_y U)_\phi-\partial_y^2 u_\phi+\partial_x p_\phi=0. \tag{3.4}$$

Applying Δ_k^h to (3.4) and taking the L^2 inner product with $\Delta_k^h u_\phi$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\Delta_k^h u_\phi||_{L^2}^2+\lambda\theta'(|D_x|\Delta_k^h u_\phi,\Delta_k^h u_\phi)_{L^2}+||\Delta_k^h\partial_y u_\phi||_{L^2}^2\\ &=-(\Delta_k^h(U\partial_x u)_\phi,\Delta_k^h u_\phi)_{L^2}-(\Delta_k^h(v\partial_y U)_\phi,\Delta_k^h u_\phi)_{L^2}-(\Delta_k^h(\partial_x p_\phi,\Delta_k^h u_\phi)_{L^2}.\end{aligned} \tag{3.5}$$

Similarly as the anisotropic case, we obtain that $(\Delta_k^h(U\partial_x u)_\phi,\Delta_k^h u_\phi)_{L^2}=0$, and

$$\begin{aligned}(\Delta_k^h(\partial_x p_\phi,\Delta_k^h u_\phi)_{L^2}&=-(\Delta_k^h(p_\phi,\Delta_k^h\partial_x u_\phi)_{L^2}=-(\Delta_k^h(p_\phi,\Delta_k^h\partial_y v_\phi)_{L^2}\\ &=-(\Delta_k^h(\partial_y p_\phi,\Delta_k^h v_\phi)_{L^2}=0.\end{aligned}$$

Next, recalling that $\partial_y U\lesssim\theta'(t)$, we can directly deduce that for any $s>0$,

$$-(\Delta_k^h(v\partial_y U)_\phi,\Delta_k^h u_\phi)_{L^2}\lesssim d_k^2 2^{-2ks}||u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})} \tag{3.6}$$

multiplying (3.5) by $e^{2\mathcal{R}t}$ and integrating over $[0,t]$, we obtain that there exists a constant $C>0$ such that

$$\begin{aligned}&\frac{1}{2}||e^{\mathcal{R}t'}\Delta_k^h u_\phi||^2_{L^\infty_t(L^2)}+\lambda 2^k\int_0^t\theta'(t')||e^{\mathcal{R}t'}\Delta_k^h u_\phi(t')||^2_{L^2}dt'+\frac{1}{2}||e^{\mathcal{R}t'}\Delta_k^h\partial_y u_\phi||^2_{L^2_t(L^2)}\\ &\leq||e^{a|D_x|}\Delta_k^h u_0||^2_{L^2}+Cd_k^2 2^{-2ks}||e^{\mathcal{R}t'}u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}\end{aligned} \tag{3.7}$$

for any $s>0$, we multiply (3.7) by 2^{2ks}, take square root and sum up over $\mathbb{Z}$ to obtain that for any $t\leq T^*$,

$$\begin{aligned}&||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^\infty_t(B^s)}+\sqrt{\lambda}||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^s)}\\ &\leq||e^{a|D_x|}u_0||_{B^s}+C||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}\end{aligned} \tag{3.8}$$

finally, if $\mathsf{c}^*<\frac{a\pi^2}{C^2}$, then we can choose a suitable λ such that $\lambda>C^2$, and we have proved the proposition. ∎

Remark 3.1. Here we just need $\theta(t)$ to be small enough instead of $\theta'(t)$, which allows the condition on U to be weaker.

Next, we prove

PROPOSITION 3.2. *For any $s>0$, there exists some constant $\mathsf{c}^*>0$, such that if $\sum_{m>0} m|c_*(m)|<\mathsf{c}^*$, then there exists a constant $C>0$ such that*

$$\begin{aligned}&||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^\infty_t(B^s)}+||e^{\mathcal{R}t'}\partial_y^2 u_\phi||_{\tilde{L}^2_t(B^s)}\\ \le& C(||e^{a|D_x|}u_0||_{B^s}+||e^{a|D_x|}u_0||_{B^{s+1}}+||e^{a|D_x|}\partial_y u_0||_{B^s}).\end{aligned} \tag{3.9}$$

Proof. We come back to the Equation (1.14), and we follow the idea of Lemma 3.2 of [16]. Applying ∂_y to both sides, we have

$$\partial_t\partial_y u+U\partial_x\partial_y u+v\partial_y^2 U-\partial_y^3 u+\partial_x\partial_y p=0.$$

Recall that $(\Delta_k^h(U\partial_x\partial_y u)_\phi,\Delta_k^h\partial_y u_\phi)_{L^2}=0$. Similarly as before, we obtain that for any $0<t<T^*$,

$$\begin{aligned}&\frac{1}{2}||e^{\mathcal{R}t'}\Delta_k^h\partial_y u_\phi||^2_{L^\infty_t(L^2)}+\lambda 2^k\int_0^t\theta'(t)||e^{\mathcal{R}t'}\Delta_k^h\partial_y u_\phi(t')||^2_{L^2}dt'+\frac{1}{2}||e^{\mathcal{R}t'}\Delta_k^h\partial_y^2 u_\phi||^2_{L^2_t(L^2)}\\ \le&\frac{1}{2}||e^{a|D_x|}\Delta_k^h\partial_y u_0||^2_{L^2}+\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h(v\partial_y^2 U)_\psi,e^{\mathcal{R}t'}\Delta_k^h\partial_y u_\phi|)_{L^2}|dt'\\ &+\int_0^t|(e^{\mathcal{R}t'}\Delta_k^h\partial_x p_\phi,e^{\mathcal{R}t'}\Delta_k^h\partial_y^2 u_\phi|)_{L^2}|dt'.\end{aligned} \tag{3.10}$$

From integration by parts, we have

$$(\Delta_k^h(v\partial_y^2 U)_\phi,\Delta_k^h\partial_y u_\phi)_{L^2}=-(\Delta_k^h(v\partial_y U)_\phi,\Delta_k^h\partial_y^2 u_\phi)_{L^2}+(\Delta_k^h(\partial_x u\partial_y U)_\phi,\Delta_k^h\partial_y u_\phi)_{L^2}.$$

Notice that $\partial_y U\lesssim\theta'(t)\le\mathsf{c}^*$, so

$$\begin{aligned}&(\Delta_k^h(v\partial_y U)_\phi,\Delta_k^h\partial_y^2 u_\phi)_{L^2}\\ \lesssim& d_k^2 2^{-2ks}||\partial_y^2 u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^s)}+d_k^2 2^{-2ks}||u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^{s+1})}\\ \lesssim&\mathsf{c}^* d_k^2 2^{-2ks}||\partial_y^2 u_\phi||^2_{\tilde{L}^2_t(B^s)}+\mathsf{c}^* d_k^2 2^{-2ks}||u_\phi||^2_{\tilde{L}^2_t(B^{s+1})}\end{aligned} \tag{3.11}$$

and similarly from Poincaré inequality,

$$\begin{aligned}&(\Delta_k^h(\partial_x u\partial_y U)_\phi,\Delta_k^h\partial_y u_\phi)_{L^2}\\ \lesssim& d_k^2 2^{-2ks}||\partial_y u_\phi||_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}||u_\phi||_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}\\ \lesssim&\frac{1}{\mathcal{R}}d_k^2 2^{-2ks}||\partial_y u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac{1}{2}})}\end{aligned} \tag{3.12}$$

next, for the $\partial_x p_\phi$ term, we integrate (1.14) for $y\in[0,1]$ to get that

$$\partial_x p=\partial_y u(t,x,1)-\partial_y u(t,x,0)-2\partial_x\int_0^1 U(t,y)u(t,x,y)dy \tag{3.13}$$

so we have

$$\begin{aligned}&(e^{\mathcal{R}t}\Delta_k^h\partial_x p_\phi(t),e^{\mathcal{R}t}\Delta_k^h\partial_y^2 u_\phi(t))_{L^2}\\ =&\int_{\mathbb{R}}e^{\mathcal{R}t}\Delta_k^h\partial_x p_\phi\cdot e^{\mathcal{R}t}(\Delta_k^h\partial_y u_\phi(t,x,1)-\Delta_k^h\partial_y u_\phi(t,x,0))dx\end{aligned}$$

$$
\begin{aligned}
&=\int_{\mathbb{R}}(e^{\mathcal{R}t}(\Delta_k^h\partial_y u_\phi(t,x,1)-\Delta_k^h\partial_y u_\phi(t,x,0)))^2dx\\
&\quad+2\int_{\mathbb{R}}\left(e^{\mathcal{R}t}\int_0^1\Delta_k^h\partial_x(Uu)_\phi dy\right)\cdot e^{\mathcal{R}t}(\Delta_k^h\partial_y u_\phi(t,x,1)-\Delta_k^h\partial_y u_\phi(t,x,0))dx\\
&\lesssim||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\phi(t)||^2_{L^\infty_v(L^2_h)}+||e^{\mathcal{R}t}\Delta_k^h\partial_x(Uu)_\phi(t)||^2_{L^1_v(L^2_h)}
\end{aligned}
$$

notice that for any $s>0$,

$$
\begin{aligned}
||e^{\mathcal{R}t}\Delta_k^h\partial_x(Uu)_\phi(t)||_{L^2_t(L^1_v(L^2_h))}&\leq||e^{\mathcal{R}t}\Delta_k^h\partial_x(Uu)_\phi(t)||_{L^2_t(L^2)}\\
&\lesssim d_k2^{-ks}||U||_{L^2_tL^\infty_y}||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^\infty_t(B^{s+1})}\\
&\lesssim \mathsf{c}^*d_k2^{-ks}||e^{\mathcal{R}t'}u_\phi||_{\tilde{L}^\infty_t(B^{s+1})}
\end{aligned}
$$

and similarly as the proof of Proposition 4.2 of [16], we have

$$
||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\phi||^2_{L^\infty_v(L^2_h)}\leq 2||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\phi||_{L^2}||e^{\mathcal{R}t}\Delta_k^h\partial_y^2 u_\phi||_{L^2}
$$

combining the estimates together, we have

$$
\begin{aligned}
&\int_0^t|(e^{\mathcal{R}t}\Delta_k^h\partial_x p_\phi(t),e^{\mathcal{R}t}\Delta_k^h\partial_y^2u_\phi(t))_{L^2}|dt'\\
&\leq\frac{1}{4}||e^{\mathcal{R}t}\Delta_k^h\partial_y^2u_\phi||^2_{L^2_t(L^2)}+C||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\phi||^2_{L^2_t(L^2)}+C(\mathsf{c}^*)^2d_k^22^{-2ks}||e^{\mathcal{R}t'}u_\phi||^2_{\tilde{L}^\infty_t(B^{s+1})}\\
&\leq\frac{1}{4}||e^{\mathcal{R}t}\Delta_k^h\partial_y^2u_\phi||^2_{L^2_t(L^2)}+Cd_k^22^{-2ks}(||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\phi||^2_{\tilde{L}^2_t(B^s)}+(\mathsf{c}^*)^2||e^{\mathcal{R}t'}u_\phi||^2_{\tilde{L}^\infty_t(B^{s+1})}).
\end{aligned}
\tag{3.14}
$$

Take the square root and sum up in $\mathbb{Z}$, so from (3.11) to (3.14), we have

$$
\begin{aligned}
&||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^\infty_t(B^s)}+\sqrt{\lambda}||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac12})}+||e^{\mathcal{R}t'}\partial_y^2 u_\phi||_{\tilde{L}^2_t(B^s)}\\
&\leq||e^{a|D_x|}\partial_y u_0||_{B^s}+C\mathsf{c}^*(||e^{\mathcal{R}t}\partial_y^2u_\phi||_{\tilde{L}^2_t(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^{s+1})})\\
&\quad+C(||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^s)}+\mathsf{c}^*||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^\infty_t(B^{s+1})}+\frac{1}{\mathcal{R}}||e^{\mathcal{R}t'}\partial_y u_\phi||^2_{\tilde{L}^2_{t,\theta'(t)}(B^{s+\frac12})})
\end{aligned}
$$

together with the Proposition 3.1 and Poincaré inequality, if c^* is small enough, we choose $\lambda\gg\frac{C^2}{\mathcal{R}^2}$ and the proposition is proved. □

Now we come back to finish proving Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** The first two inequalities have been proved in the two previous propositions. The main structure of proving the third one is similar to Theorem 1.2 of [16]. For the Equation (1.14), we apply the operator Δ_k^h and take the L^2 inner product with $e^{2\mathcal{R}t}\Delta_k^h(\partial_t u)_\phi$ to obtain that

$$
\begin{aligned}
||e^{\mathcal{R}t}\Delta_k^h(\partial_t u)_\phi||^2_{L^2}=&e^{2\mathcal{R}t}(\Delta_k^h\partial_y^2u_\phi,\Delta_k^h(\partial_y u)_\phi)_{L^2}\\
&-e^{2\mathcal{R}t}(\Delta_k^h(U\partial_x u)_\phi,\Delta_k^h(\partial_y u)_\phi)_{L^2}-e^{2\mathcal{R}t}(\Delta_k^h(v\partial_y U)_\phi,\Delta_k^h(\partial_y u)_\phi)_{L^2}.
\end{aligned}
\tag{3.15}
$$

From integration by parts we get

$$
(e^{2\mathcal{R}t}\Delta_k^h\partial_y^2u_\psi,\Delta_k^h(\partial_y u)_\psi)_{L^2}=-\frac{1}{2}\frac{d}{dt}||e^{\mathcal{R}t}\Delta_h^h\partial_y u_\psi||^2_{L^2}-\theta'(t)2^k||e^{\mathcal{R}t}\Delta_k^h\partial_y u_\psi||^2_{L^2}. \tag{3.16}
$$

For an arbitary $s>0$, we multiply (3.15) by 2^{2ks}, use (3.16) and sum over $\mathbb{Z}$, to get

$$\begin{aligned}&||e^{\mathcal{R}t'}(\partial_t u)_\psi||_{\tilde{L}^2_t(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\psi||_{\tilde{L}^\infty_t(B^s)}\\ \leq& C(||e^{a|D_x|}\partial_y u_0||_{B^s}+||e^{\mathcal{R}t'}(U\partial_x u)_\psi||_{\tilde{L}^2_t(B^s)}+||e^{\mathcal{R}t'}(v\partial_y U)_\psi||_{\tilde{L}^2_t(B^s)}).\end{aligned}\tag{3.17}$$

Recalling that U and $\partial_y U$ are uniformly bounded by c^*, by Poincare inequality, we have

$$||e^{\mathcal{R}t'}(U\partial_x u)_\phi||_{\tilde{L}^2_t(B^s)},||e^{\mathcal{R}t'}(v\partial_y U)_\phi||_{\tilde{L}^2_t(B^s)}\lesssim ||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^{s+1})}$$

so finally we obtain

$$||e^{\mathcal{R}t'}(\partial_t u)_\phi||_{\tilde{L}^2_t(B^s)}+||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^\infty_t(B^s)}\leq C(||e^{a|D_x|}\partial_y u_0||_{B^s}+||e^{a|D_x|}u_0||_{\tilde{L}^2_t(B^{s+1})})$$

and Theorem 1.2 is thus proved. □

REMARK 3.2. Notice that in this subsection, the evolution equation $\theta(t)$ depends only on U. In fact, in next subsection, we need to add another term $||\partial_y u_\phi(t')||_{B^{\frac{1}{2}}}$ to prove the convergence. Notice that

$$\int_0^t ||\partial_y u_\phi(t')||_{B^{\frac{1}{2}}}dt' \lesssim ||e^{\mathcal{R}t'}\partial_y u_\phi||_{\tilde{L}^2_t(B^{\frac{1}{2}})}\lesssim ||e^{a|D_x|}u_0||_{B^{\frac{1}{2}}}$$

so if we also suppose $||e^{a|D_x|}u_0||_{B^{\frac{1}{2}}}$ is small enough, then $||\partial_y u_\phi(t')||_{B^{\frac{1}{2}}}$ will also appear in $\theta(t)$. Comparing with the anistropic case, we can similarly define the evolution of the analytic band of $u^\varepsilon - u$ to prove the convergence. The details can be seen later.

3.2. Convergence to the hydrostatic system. Define $w^1_\varepsilon := u^\varepsilon - u, w^2_\varepsilon := v^\varepsilon - v$. Then they satisfy the equations:

$$\begin{cases}\varepsilon\partial_t w^1_\varepsilon - \varepsilon^3\partial_x^2 w^1_\varepsilon - \varepsilon\partial_y^2 w^1_\varepsilon + \varepsilon\partial_x q_\varepsilon + \partial_x R_{11} + \partial_y R_{21}\\ = \varepsilon^3\partial_x^2 u_\varepsilon - \varepsilon[(U+\varepsilon u^\varepsilon)\partial_x u^\varepsilon - U\partial_x u] - \varepsilon[v^\varepsilon\partial_y(U+\varepsilon u^\varepsilon) - v\partial_y U]\\ \varepsilon^2\partial_t w^2_\varepsilon - \varepsilon^4\partial_x^2 w^2_\varepsilon - \varepsilon^2\partial_y^2 w^2_\varepsilon + \partial_y q^\varepsilon + \partial_x R_{12} + \partial_y R_{22}\\ = -\varepsilon^2\partial_t v + \varepsilon^4\partial_x^2 v + \varepsilon^2\partial_y^2 v - \varepsilon^2(U+\varepsilon u^\varepsilon)\partial_x v^\varepsilon - \varepsilon^3 v^\varepsilon\partial_y v^\varepsilon\\ \partial_x w^1_\varepsilon + \partial_y w^2_\varepsilon = 0.\end{cases}\tag{3.18}$$

For a function u, we define

$$u_\Theta(t,x,y) := \mathcal{F}^{-1}_{\xi\to x}(e^{\Theta(t,\xi)}\hat{u}(t,\xi,y)),\quad \Theta(t,\xi) := (a-\mu\zeta(t))|\xi|\tag{3.19}$$

where $\mu\geq\lambda$ will be determined later, and $\zeta(t)$ is given by

$$\begin{aligned}&\zeta(0)=0,\\ &\zeta'(t) = ||(\varepsilon\partial_x,\partial_y)u^\varepsilon_\psi(t')||_{B^{\frac{1}{2}}} + ||(\varepsilon\partial_x,\partial_y)(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\psi(t')||_{B^{\frac{1}{2}}} + \sum_{m>0} m|c_m|e^{-m^2\pi^2 t} + ||u_\phi(t')||_{B^{\frac{1}{2}}}\end{aligned}$$

out of Theorems 1.1 and 1.2, we deduce that

$$\begin{aligned}&||\varepsilon u_\Theta||_{\tilde{L}^\infty(\mathbb{R}^+,B^{\frac{1}{2}})} + ||u_\Theta||_{\tilde{L}^\infty(\mathbb{R}^+,B^{\frac{1}{2}}\cap B^{\frac{5}{2}})}\\ &\quad + ||\partial_y u_\Theta||_{\tilde{L}^2(\mathbb{R}^+,B^{\frac{1}{2}}\cap B^{\frac{5}{2}})} + ||(\partial_t u)_\Theta||_{\tilde{L}^2(\mathbb{R}^+,B^{\frac{3}{2}})} \leq M\end{aligned}\tag{3.20}$$

where $M>1$ is a constant independent of ε. For convenience, we define

$$\mathcal{S}^1 := \varepsilon^3\partial_x^2 u_\varepsilon - \varepsilon[(U+\varepsilon u^\varepsilon)\partial_x u^\varepsilon - U\partial_x u] - \varepsilon[v^\varepsilon\partial_y(U+\varepsilon u^\varepsilon) - v\partial_y U]$$

and

$$\mathcal{S}^2 := -\varepsilon^2\partial_t v + \varepsilon^4\partial_x^2 v + \varepsilon^2\partial_y^2 v - \varepsilon^2(U+\varepsilon u^\varepsilon)\partial_x v^\varepsilon - \varepsilon^3 v^\varepsilon\partial_y v^\varepsilon.$$

We prove two propositions about $\mathcal{S}^1$ and $\mathcal{S}^2$:

PROPOSITION 3.3.

$$\begin{aligned}\int_0^t |(\Delta_k^h\mathcal{S}_\Theta^1, \Delta_k^h w_\Theta^1)_{L^2}|dt' \lesssim{}& \varepsilon d_k^2 2^{-k}||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac12}_{\tilde{L}^\infty_t(B^{\frac32})}||\partial_y(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac12})}\\ &+\varepsilon^3 d_k^2 2^{-k}||\partial_y u_\Theta||_{\tilde{L}^2_t(B^{\frac32})}||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac32})} + d_k^2 2^{-k}||(\varepsilon w^1)_\Theta||^2_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}\\ &+\varepsilon d_k^2 2^{-k}(||u_\Theta||_{\tilde{L}^\infty_t(B^{\frac12})}||\partial_y u_\Theta||_{\tilde{L}^2_t(B^{\frac32})} + ||u_\Theta||_{\tilde{L}^\infty_t(B^{\frac32})}||\partial_y u_\Theta||_{\tilde{L}^2_t(B^{\frac12})})||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac12})}\end{aligned} \tag{3.21}$$

Proof. From integration by parts and Poincare inequality, we have

$$\varepsilon^4\int_0^t |(\Delta_k^h\partial_x^2 u_\Theta, \Delta_k^h w_\Theta^1)_{L^2}|dt' \lesssim \varepsilon^3 d_k^2 2^{-k}||\partial_y u_\Theta||_{\tilde{L}^2_t(B^{\frac32})}||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac32})} \tag{3.22}$$

we next write

$$(U+\varepsilon u^\varepsilon)\partial_x u^\varepsilon - U\partial_x u = U\partial_x w^1 + u^\varepsilon\partial_x w^1 + \varepsilon w^1\partial_x u + \varepsilon u\partial_x u$$

and

$$v^\varepsilon\partial_y(U+\varepsilon u^\varepsilon) - v\partial_y U = w^2\partial_y U + \varepsilon w^2\partial_y u + \varepsilon v^\varepsilon\partial_y w^1 + \varepsilon v\partial_y u$$

similarly as in Lemma 3.1 of [16] and recalling that $U,\partial_y U\lesssim\zeta'(t)$, we have

$$\varepsilon^2\int_0^t |(\Delta_k^h[(U+\varepsilon u^\varepsilon)\partial_x w^1 + \partial_y U w^2]_\Theta, \Delta_k^h w_\Theta^1)_{L^2}|dt' \lesssim d_k^2 2^{-k}||(\varepsilon w^1)_\Theta||^2_{\tilde{L}^2_{t,\zeta'(t)}(B^1)} \tag{3.23}$$

analogously to (5.11) of [16] we get

$$\begin{aligned}&\varepsilon\int_0^t |(\Delta_k^h(\varepsilon w^1\partial_x u)_\Theta, \Delta_k^h(\varepsilon w^1)_\Theta)_{L^2}|dt'\\ &\lesssim \varepsilon d_k^2 2^{-k}(||(\varepsilon w^1)_\Theta||^2_{\tilde{L}^2_{t,\zeta'(t)}(B^1)} + ||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac12}_{\tilde{L}^\infty_t(B^{\frac32})}||\partial_y(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac12})})\end{aligned} \tag{3.24}$$

recall that $v^\varepsilon = w^2 + v$. Similarly as in (5.13) of [16], we obtain

$$\begin{aligned}&\varepsilon\int_0^t |(\Delta_k^h[v^\varepsilon\partial_y(\varepsilon w^1)]_\Theta, \Delta_k^h(\varepsilon w^1)_\Theta)_{L^2}|dt'\\ &\lesssim \varepsilon d_k^2 2^{-k}||(\varepsilon w^1)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac12}_{\tilde{L}^\infty_t(B^{\frac32})}||\partial_y(\varepsilon w^1)_\Theta||_{\tilde{L}^2_t(B^{\frac12})}\end{aligned} \tag{3.25}$$

finally, similarly as (5.14) of [16], we have

$$\varepsilon\int_0^t |(\Delta_k^h(\varepsilon w^2\partial_y u)_\Theta, \Delta_k^h(\varepsilon w^1)_\Theta)_{L^2}|dt' \lesssim \varepsilon d_k^2 2^{-k}||(\varepsilon w^1)_\Theta||^2_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}. \tag{3.26}$$

A difference from [16] is that in here there is an extra term $\varepsilon(u\partial_x u+v\partial_y u)$. Notice that for any $s>0$, from the law of product in anisotropic Besov space (see [5] for the proof) and Poincare inequality, we have

$$\begin{aligned}&||(u\partial_x u)_\Theta||_{\tilde L^2_t(B^s)}+||(v\partial_y u)_\Theta||_{\tilde L^2_t(B^s)}\\ \lesssim&||u_\Theta||_{\tilde L^\infty_t(B^{\frac12})}||\partial_y u_\Theta||_{\tilde L^2_t(B^{s+1})}+||u_\Theta||_{\tilde L^\infty_t(B^{s+1})}||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac12})}\end{aligned}\tag{3.27}$$

and the proposition is proved after combining (3.22)-(3.27). ∎

Proposition 3.4.

$$\begin{aligned}&\int_0^t|(\Delta_k^h\mathcal S^2_\Theta,\Delta_k^h w^2_\Theta)_{L^2}|dt'\\ \lesssim&\varepsilon d_k^2 2^{-k}||(\varepsilon w^1,\varepsilon^2 w^2)_\Theta||^2_{\tilde L^2_{t,\zeta'(t)}(B^1)}+\varepsilon^2 d_k^2 2^{-k}||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}||\partial_y u_\Theta||_{\tilde L^2_t(B^2)}\\ &+\varepsilon^2 d_k^2 2^{-k}(||(\partial_t u)_\Theta||_{\tilde L^2_t(B^{\frac32})}+||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac32})})||\partial_y w^2_\Theta||_{\tilde L^2_t(B^{\frac12})}\\ &+\varepsilon^4 d_k^2 2^{-k}||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac52})}||w^2_\Theta||_{\tilde L^2_t(B^{\frac32})}\\ &+\varepsilon d_k^2 2^{-k}||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}||u^\varepsilon_\Theta||^{\frac12}_{L^\infty_t(B^{\frac12})}||\partial_y u_\Theta||_{\tilde L^2_t(B^2)}\\ &+\varepsilon d_k^2 2^{-k}||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac12}_{L^\infty_t(B^{\frac32})}(||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac32})}+||\partial_y(\varepsilon^2 w^2)_\Theta||_{\tilde L^2_t(B^{\frac12})}).\end{aligned}\tag{3.28}$$

Proof. Similarly as (5.15) of [16], we deduce from Poincaré inequality that

$$\begin{aligned}&\int_0^t|\Delta_k^h(-\varepsilon^2\partial_t v+\varepsilon^4\partial_x^2 v+\varepsilon^2\partial_y^2 v)_\Theta,\Delta_k^h w^2_\Theta|_{L^2}dt\\ \lesssim&\varepsilon^2 d_k^2 2^{-k}\left(||(\partial_t u)_\Theta||_{\tilde L^2_t(B^{\frac32})}+||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac32})}\right)||\partial_y w^2_\Theta||_{\tilde L^2_t(B^{\frac12})}\\ &+\varepsilon^4 d_k^2 2^{-k}||\partial_y u_\Theta||_{\tilde L^2_t(B^{\frac52})}||w^2_\Theta||_{\tilde L^2_t(B^{\frac32})}.\end{aligned}\tag{3.29}$$

We decompose $v^\varepsilon=v+w^2$. Recall that $U\lesssim\zeta'(t)$, so we have

$$\int_0^t|\Delta_k^h(U\partial_x w^2)_\Theta,\Delta_k^h w^2_\Theta|_{L^2}dt'=0$$

and

$$\varepsilon^2\int_0^t|\Delta_k^h(U\partial_x v)_\Theta,\Delta_k^h\varepsilon^2 w^2_\Theta|_{L^2}dt'\lesssim\varepsilon^2 d_k^2 2^{-k}||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}||\partial_y u_\Theta||_{\tilde L^2_t(B^2)}.$$

Next, similarly as (5.16) and (5.17) of [16], we have

$$\begin{aligned}&\varepsilon^3\int_0^t|\Delta_k^h(u^\varepsilon\partial_x v^\varepsilon)_\Theta,\Delta_k^h\varepsilon^2 w^2_\Theta|_{L^2}dt'\\ \lesssim&\varepsilon d_k^2 2^{-k}||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}(||\varepsilon^2 w^2_\Theta||_{\tilde L^2_{t,\zeta'(t)}(B^1)}+||u^\varepsilon_\Theta||^{\frac12}_{L^\infty_t(B^{\frac12})}||\partial_y u_\Theta||_{\tilde L^2_t(B^2)}).\end{aligned}\tag{3.30}$$

Note that $v^\varepsilon\partial_y v^\varepsilon=v\partial_y w^2+w^2\partial_y w^2+v\partial_y v+w^2\partial_y v$. Recall from Lemma 3.3 of [16] that

$$\varepsilon^3\int_0^t|(\Delta_k^h(w^2\partial_y w^2)_\Theta,\Delta_k^h\varepsilon^2 w^2_\Theta)_{L^2}|dt'\lesssim\varepsilon d_k^2||(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||^2_{\tilde L^2_{t,\zeta'(t)}(B^1)}.$$

Next, recall that $\partial_y v = -\partial_x u$. Similarly as (3.24), we have

$$\begin{aligned}&\varepsilon^3\int_0^t |(\Delta_k^h(w^2\partial_x u)_\Theta, \Delta_k^h \varepsilon^2 w_\Theta^2)_{L^2}|dt' \\ \lesssim& \varepsilon d_k^2 2^{-k}||\varepsilon^2 w_\Theta^2||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}(||\varepsilon^2 w_\Theta^2||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)} + ||u_\Theta||^{\frac{1}{2}}_{L_t^\infty(B^{\frac{3}{2}})}||\partial_y(\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_t(B^{\frac{1}{2}})}) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}&\varepsilon^3\int_0^t |(\Delta_k^h(v\partial_y w^2)_\Theta, \varepsilon^2\Delta_k^h w_\Theta^2)_{L^2}|dt' \\ \lesssim& \varepsilon d_k^2 2^{-k}||(\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac{1}{2}}_{\tilde{L}_t^\infty(B^{\frac{3}{2}})}||\partial_y(\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_t(B^{\frac{1}{2}})} \end{aligned} \tag{3.32}$$

finally, again using the product law of anistropic Besov norms, we have

$$\begin{aligned}&\varepsilon^3\int_0^t |(\Delta_k^h(v\partial_y v)_\Theta, \Delta_k^h\varepsilon^2 w_\Theta^2)_{L^2}|dt' \\ \lesssim& \varepsilon^3 d_k^2 2^{-k}||(\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}||u_\Theta||^{\frac{1}{2}}_{\tilde{L}_t^\infty(B^{\frac{3}{2}})}||\partial_y u_\Theta||_{\tilde{L}^2_t(B^{\frac{3}{2}})} \end{aligned} \tag{3.33}$$

and the propostion is proved after combining (3.29) to (3.33). ∎

Finally, we come back to proving Theorem 1.3.

Proof. **(Proof of Theorem 1.3.)** Recall that in the hydrostatic limit case we assume that $Q_{11} = Q_{12} = 0$ and we still drop 'ε' in Q_{11}^ε and Q_{12}^ε, for simplicity, as the anisotropic case. By using a similar derivation of the scaled anisotropic system and recalling that the terms R_{ij} can be estimated similarly as in the Propositions 2.3-2.5, we obtain that

$$\begin{aligned}&||\Delta_k^h(\varepsilon w_\Theta^1, \varepsilon^2 w_\Theta^2)||^2_{L_t^\infty(L^2)} + ||\Delta_k^h(Q_{11}, \varepsilon Q_{12})_\Theta)||^2_{L_t^\infty(L^2)} \\ &\quad + ||\Delta_k^h(\varepsilon^2\partial_x, \varepsilon\partial_y)(Q_{11}, \varepsilon Q_{12})_\Theta)||^2_{L_t^\infty(L^2)} \\ &\quad + \mu 2^k\int_0^t \zeta'(t')||\Delta_k^h((\varepsilon w^1)_\Theta, (\varepsilon^2 w^2)_\Theta))||^2_{L^2}dt' + \int_0^t ||\Delta_k^h\Delta_\varepsilon((Q_{11}, \varepsilon Q_{12})_\Theta))||^2_{L^2}dt' \\ &\quad + \int_0^t ||\Delta_k^h\partial_y((\varepsilon w^1)_\Theta, (\varepsilon^2 w^2)_\Theta))||^2_{L^2} + \varepsilon^2 2^{2k}||\Delta_k^h(\varepsilon w^1)_\Theta, (\varepsilon^2 w^2)_\Theta))||^2_{L^2}dt' \\ &\quad + \mu 2^k\int_0^t \zeta'(t')||\Delta_k^h(Q_{11}, \varepsilon Q_{12})_\Theta)||^2_{L^2}dt' \\ &\quad + \mu 2^k\int_0^t \zeta'(t')||\Delta_k^h(\varepsilon^2\partial_x, \varepsilon\partial_y)(Q_{11}, \varepsilon Q_{12})_\Theta)||^2_{L^2}dt' \\ &\quad + \int_0^t ||\Delta_k^h\partial_y((Q_{11}, \varepsilon Q_{12})_\Theta))||^2_{L^2} + \varepsilon^2 2^{2k}||\Delta_k^h((Q_{11}, \varepsilon Q_{12})_\Theta))||^2_{L^2}dt' \\ \leq& ||e^{a|D_x|}\Delta_k^h(\varepsilon(u_0^\varepsilon - u_0), \varepsilon^2(v_0^\varepsilon - v_0))||^2_{L^2} \\ &\quad + \int_0^t |(\Delta_k^h\mathcal{S}_\Theta^1, \Delta_k^h w_\Theta^1)_{L^2}|dt' + \int_0^t |(\Delta_k^h\mathcal{S}_\Theta^2, \Delta_k^h w_\Theta^2)_{L^2}|dt' \end{aligned} \tag{3.34}$$

we deduce from Propsitions 3.3, 3.4 and $M > 1$ that

$$\int_0^t |(\Delta_k^h\mathcal{S}_\Theta^1, \Delta_k^h w_\Theta^1)_{L^2}|dt' + \int_0^t |(\Delta_k^h\mathcal{S}_\Theta^2, \Delta_k^h w_\Theta^2)_{L^2}|dt'$$

$$\lesssim d_k^2 2^{-k}(||(\varepsilon w^1,\varepsilon^2 w^2)_\Theta||^2_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}+M\varepsilon||(\varepsilon\partial_x,\partial_y)(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^2_t(B^{\frac{1}{2}})})$$
$$+d_k^2 2^{-k}M^{\frac{1}{2}}\varepsilon||\partial_y(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^2_t(B^{\frac{1}{2}})}||(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}$$
$$+d_k^2 2^{-k}M^{\frac{3}{2}}\varepsilon||\varepsilon^2 w^2_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)} \tag{3.35}$$

so from Cauchy-Schwartz inequality, we have

$$||(\varepsilon w^1)_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}+\mu||(\varepsilon w^1)_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}$$
$$+||(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\Theta||_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}+||(\varepsilon^2\partial_x,\varepsilon\partial_y)(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\Theta||_{\tilde{L}^\infty_t(B^{\frac{1}{2}})}$$
$$+||\partial_y(\varepsilon w^1,\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_t(B^{\frac{1}{2}})}+||\partial_x(\varepsilon u^\varepsilon,\varepsilon^2 v^\varepsilon)_\psi||_{\tilde{L}^2_t(B^{\frac{1}{2}})}$$
$$+||\partial_y(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\Theta||_{\tilde{L}^2_t(B^{\frac{1}{2}})}+||\varepsilon\partial_x(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\Theta||_{\tilde{L}^2_t(B^{\frac{1}{2}})}+\varepsilon||\Delta_\varepsilon(Q^\varepsilon_{11},\varepsilon Q^\varepsilon_{12})_\psi||_{\tilde{L}^2_t(B^{\frac{1}{2}})}$$
$$\le C||e^{a|D_x|}(\varepsilon(u^\varepsilon_0-u_0),\varepsilon^2(v^\varepsilon_0-v_0))||_{B^{\frac{1}{2}}}+C||e^{a|D_x|}((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}}$$
$$+C||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}}+C(||(\varepsilon w^1,\varepsilon^2 w^2)_\Theta||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}$$
$$+M^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}||(\varepsilon\partial_x,\partial_y)(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||^{\frac{1}{2}}_{\tilde{L}^2_t(B^{\frac{1}{2}})}+M^{\frac{3}{4}}\varepsilon^{\frac{1}{2}}||\varepsilon^2 w^2_\Theta||^{\frac{1}{2}}_{\tilde{L}^2_{t,\eta'(t)}(B^1)}$$
$$+M^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}||\partial_y(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||^{\frac{1}{2}}_{\tilde{L}^2_t(B^{\frac{1}{2}})}||(\varepsilon w^1_\Theta,\varepsilon^2 w^2_\Theta)||^{\frac{1}{2}}_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}$$
$$\le C||e^{a|D_x|}(\varepsilon(u^\varepsilon_0-u_0),\varepsilon^2(v^\varepsilon_0-v_0))||_{B^{\frac{1}{2}}}+C||e^{a|D_x|}((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}}$$
$$+C||e^{a|D_x|}(\varepsilon^2\partial_x,\varepsilon\partial_y)((Q_{11})_0,\varepsilon(Q_{12})_0)||_{B^{\frac{1}{2}}}+CM(\varepsilon+||(\varepsilon w^1)_\Theta,\varepsilon^2 w^2_\Theta)||_{\tilde{L}^2_{t,\zeta'(t)}(B^1)}) \tag{3.36}$$

and (3.35) is proved by choosing $\mu\ge C^2M^2$. □

Acknowledgments. X. Li and A. Zarnescu have been partially supported by the Basque Government through the BERC 2022- 2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa CEX2021-001142 and through project PID2020-114189RB-I00 funded by Agencia Estatal de Investigación (PID2020-114189RB-I00 / AEI / 10.13039/501100011033). X. Li has been partially supported by the LTC-Transmath project funded by Fundacíon Euskampus. A. Zarnescu has been also partially supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P4-PCE-2021-0921, within PNCDI III. M. Paicu has been supported by Université de Bordeaux.

REFERENCES

[1] J.M. Ball and A. Zarnescu, *Orientability and energy minimization in liquid crystal models*, Arch. Ration. Mech. Anal., 202(2):493–535, 2011. 1
[2] H. Bahouri, J-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011. 1.1
[3] J. Chemin and N. Lerner, *Flot de champs de vecteurs non Lipschitziens et equations de Navier-Stokes*, J. Differ. Equ., 121(2):314–328, 1995. 1.1
[4] J. Chemin and P. Zhang, *On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations*, Commun. Math. Phys., 272(2):529–566, 2007. 2
[5] J. Chemin, M. Paicu, and P. Zhang, *Global large solutions to 3-D inhomogeneous Navier–Stokes system with one slow variable*, J. Differ. Equ., 256(1):223–252, 2014. 3.2
[6] H. Du, H. Wu, and C. Wang, *Suitable weak solutions for the co-rotational Beris-Edwards system in dimension three*, Arch. Ration. Mech. Anal., 238(2):749–803, 2020. 1, 1
[7] J. Ericksen, *Conservation laws for liquid crystals*, Trans. Soc. Rheol., 5(1):23–34, 1961. 1
[8] J. Ericksen, *Continuum theory of nematic liquid crystals*, Res. Mech., 21(4):381–392, 1987. 1

[9] F. Leslie, *Some constitutive equations for liquid crystals*, Arch. Ration. Mech. Anal., 28:265–283, 1968. 1

[10] F. Lin and C. Liu, *Non parabolic dissipative systems modeling the flow of liquid crystals*, Commun. Pure Appl. Math., 48(5):501–537, 1995. 1

[11] F. Lin and C. Liu, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Discret. Contin. Dyn. Syst., 2(1):1–22, 1996. 1

[12] M. Paicu, *Equation anisotrope de Navier-Stokes dans des espaces critiques*, Rev. Mat. Iberoam., 21(1):179–235, 2005. 2

[13] M. Paicu and A. Zarnescu, *Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system*, SIAM J. Math. Anal., 43(5):2009–2049, 2011. 1

[14] M. Paicu and P. Zhang, *Global existence and decay of solutions to Prandtl system with small analytic data*, Arch. Ration. Mech. Anal., 241:403–446, 2021. 1.2

[15] M. Paicu and P. Zhang, *Global hydrostatic approximation of the hyperbolic Navier-Stokes system with small Gevrey class 2 data*, Sci. China Math., 65(6):1109–1146, 2022. 1

[16] M. Paicu, P. Zhang, and Z. Zhang, *On the hydrostatic approximation of the Navier-Stokes equations in a thin strip*, Adv. Math., 372:107293, 2020. 1, 1.1, 1.2, 1.2, 1.2, 2, 2.0.1, 2.2, 2.0.1, 3.1, 3.1, 3.1, 3.2, 3.2, 3.2, 3.2, 3.2, 3.2, 3.2, 3.2

[17] M. Sammartino and R. Caflisch, *Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations*, Commun. Math. Phys., 192:433–461, 1998. 1.2

[18] M. Schonbek and Y. Shibata, *Global well-posedness and decay of a Q-tensor model of incompressible nematic liquid crystal in* $\mathbb{R}^N$, J. Differ. Equ., 266(6):3034–3065, 2019. 1

[19] H. Wu, X. Xu, and A. Zarnescu, *Dynamics and flow effects in the Beris-Edwards* , Arch. Ration. Mech. Anal., 231:1217–1267, 2019. 1