COMMUN. MATH. SCI.
Vol. 22, No. 6, pp. 1733–1748

LOCALIZATION AND THE LANDSCAPE FUNCTION FOR REGULAR STURM-LIOUVILLE OPERATORS*

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Abstract. We consider the localization in the eigenfunctions of regular Sturm-Liouville operators. After deriving non-asymptotic and asymptotic lower and upper bounds on the localization coefficient of the eigenfunctions, we characterize the landscape function in terms of the first eigenfunction. Several numerical experiments are provided to illustrate the obtained theoretical results.

Keywords. Sturm-Liouville theory; spectral theory.

AMS subject classifications. 34B24; 34B09.

1. Introduction and main results

Localization is an emergent wave phenomenon observed in, e.g., quantum dynamics, electrodynamics and acoustics, where some eigenfunctions attain large values over small subsets of the problem domain and nearly vanish over the rest of the domain [1, 4, 5]. Heilman and Strichartz [8] give an interesting introduction to and examples of localization, while Yamilov *et al.* [11] give a good overview of the different physical settings in which Anderson localization can occur, with special focus on electromagnetic waves in disordered media in dimension three. For other accounts of localization, see, e.g., [3–5] and the references therein. In this work we are concerned with localization in dimension one, specifically with estimating the localization effect in eigenfunctions of regular Sturm-Liouville operators (Theorems 1.1 and 1.2). Next, there is evidence [5] that the pointwise behavior of the so-called landscape function can reveal the subset of the problem domain in which the low-frequency eigenfunctions localize. We shall here characterize the landscape function in terms of low-frequency eigenfunctions of the regular Sturm-Liouville operator (Propositions 1.1 and 1.2).

Let $L>0$, assume $p,w\in C^2([0,L])$ are positive-valued functions satisfying $p^{-1},w^{-1}\in L^\infty((0,L))$, and let $q\in C([0,L])$ be nonnegative-valued. Define the unbounded operator T by

$$Tu=-\frac{1}{w}(pu')'+\frac{q}{w}u,$$

with domain

$$D(T)=\left\{u\in L^2((0,L),w(x)dx): Tu\in L^2((0,L),w(x)dx),\, u(0)=u(L)=0\right\},$$

and recall that T is self-adjoint in $L^2((0,L),w(x)dx)$ with a compact resolvent. We here investigate the "localization" in the solution $\phi_\lambda\in D(T)$ of the regular Sturm-Liouville problem

$$T\phi_\lambda=\lambda\phi_\lambda, \tag{1.1}$$

*Received: January 28, 2023; Accepted (in revised form): February 03, 2024. Communicated by Gang Bao.

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for positive λ. In particular, writing $\|\cdot\|_t$ for $\|\cdot\|_{L^t((0,L))}$, we find *non-asymptotic as well as asymptotic* lower and upper bounds for the 'existence surface' [3, 4], also called the 'localization coefficient',

$$\alpha(\phi_\lambda) = \|\phi_\lambda\|_2^4 / \|\phi_\lambda\|_4^4.$$

The quantity $\alpha(\phi_\lambda)$ is independent of any normalization of ϕ_λ by a scalar factor, and it is a standard measure of the localization of ϕ_λ, with low $\alpha(\phi_\lambda)$ indicating high localization. 'High localization' means that the amplitude of the solution function is relatively high over a small connected sub-interval $I \subset (0,L)$, and relatively low in $(0,L)\setminus I$. Figure 1.1 helps illustrate the concept of localization. Here, we let $L=1$, $q\equiv 0$, $w\equiv 1$, and

$$p(x) = \tanh(40x/L - 10) + 1.1, \quad x \in [0,L], \tag{1.2}$$

making the operator T in (1.1) the Dirichlet Laplacian on $[0,1]$ with a non-trivial metric, $Tu = -(pu')'$. Note that the most localized eigenfunctions correspond to relatively small eigenvalues, and that the localization coefficient seems to approach a constant with increasing eigenvalues. Our results, valid well beyond this single example case, predict both these empirical observations on localization.

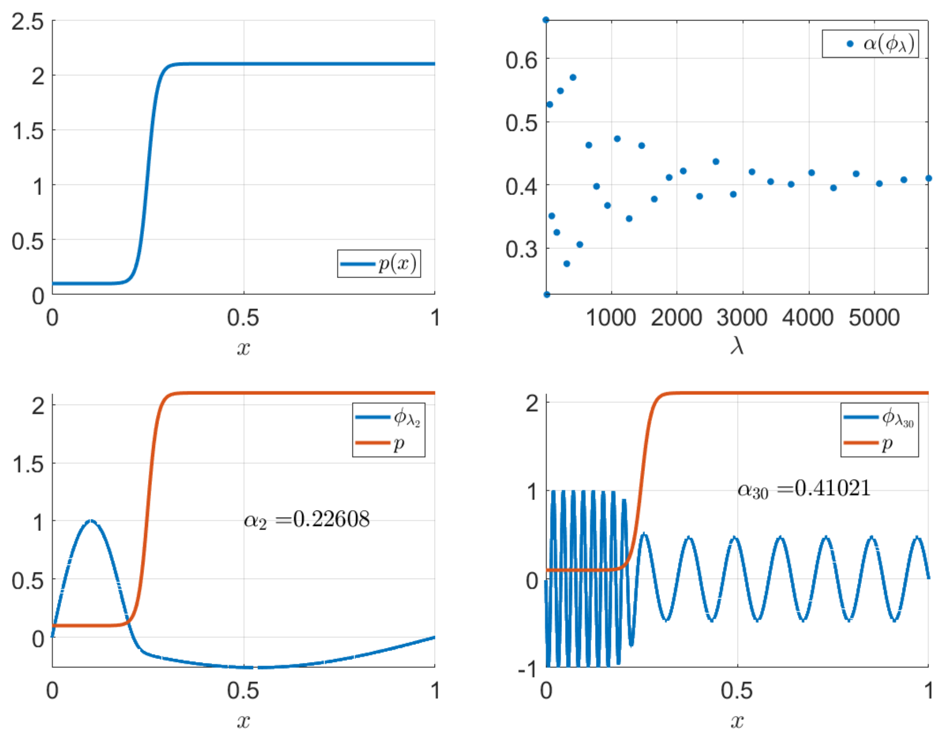

FIG. 1.1. *Localization of eigenfunctions of the Dirichlet Laplacian on* $[0,1]$ *with metric* p.

We first derive lower and upper bounds for $\alpha(\phi_\lambda)$ in the non-asymptotic regime, specifically showing that $\alpha(\phi_\lambda)$ can attain relatively low values only at relatively low frequencies (small λ). Then, to complete the picture, we prove the lower and upper bounds for $\alpha(\phi_\lambda)$ in the asymptotic regime as $\lambda \to \infty$.

The treatment of the Sturm-Liouville problem (1.1) when the coefficients are smooth usually starts with the Liouville transformation to the eigenvalue problem for the Schrödinger operator [2]. We work with this transformation in Section 2, but to state our second and third main results we already here define some of the involved quantities. Thus let

$$y(x) = \int_0^x \sqrt{w(s)/p(s)}ds, \quad x \in [0,L],$$

and

$$B = \int_0^L \sqrt{w(s)/p(s)}ds.$$

The function $y:(0,L) \to (0,B)$ is strictly increasing, and has an inverse denoted by $x(y)$. Let

$$f(y) = (w(x(y)))p(x(y)))^{1/4}, \quad y \in [0,B],$$

and

$$Q(y) = f''(y)/f(y) + q(x(y))/w(x(y)), \quad y \in [0,B]. \tag{1.3}$$

Write also

$$a(B,\lambda) = \frac{B\|Q\|_\infty}{2\sqrt{\lambda}}, \tag{1.4}$$

and

$$b(B,\lambda) = \left(\frac{B^3}{12} + \frac{5B}{32\lambda} + \frac{5}{32\lambda^{3/2}}\right)^{1/4} \|Q\|_4/\sqrt{\lambda}. \tag{1.5}$$

Finally, for any real λ, let

$$\Phi_\lambda(y) = \sin(\sqrt{\lambda}y), \quad y \in [0,B].$$

Our first result gives non-asymptotic bounds on $\alpha(\phi_\lambda)$. Let

$$\beta(p,w) = \|w\|_\infty^{-2}\|p^{-1/2}w^{-3/2}\|_\infty^{-1} \quad \text{and} \quad \gamma(p,w) = \|w^{-1}\|_\infty^2\|p^{1/2}w^{3/2}\|_\infty.$$

THEOREM 1.1. *If*

$$a(B,\lambda) < 1 \tag{1.6}$$

and

$$b(B,\lambda) < 1 \tag{1.7}$$

then

$$\beta(p,w)\left(\frac{1-b(B,\lambda)}{1+a(B,\lambda)}\right)^4 \le \frac{\alpha(\phi_\lambda)}{\alpha(\Phi_\lambda)} \le \gamma(p,w)\left(\frac{1+b(B,\lambda)}{1-a(B,\lambda)}\right)^4.$$

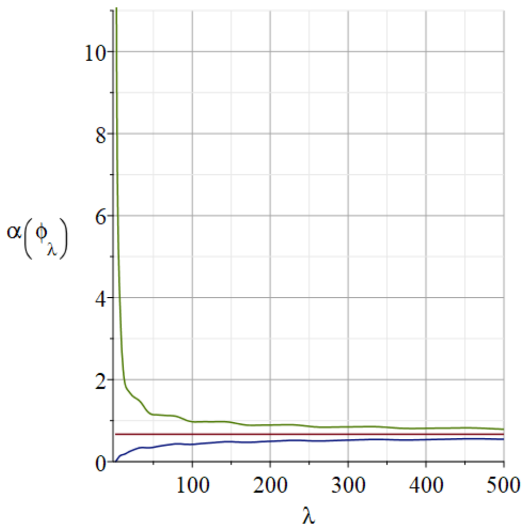

FIG. 1.2. *Lower and upper bounds on $\alpha(\phi_\lambda)$ from Theorem 1.1 for the eigenvalue problem (1.1) after Liouville transformation, with $\beta(p,w)=1$, $\gamma(p,w)=1$, $B=1$, $\|Q\|_\infty=1$, and $\|Q\|_4=1$. The constant value is the asymptotic $2B/3$. For the chosen parameter values, the Assumptions (1.6)–(1.7) are satisfied for $\lambda \gtrapprox 0.74$.*

Figure 1.2 illustrates the bounds on the localization coefficient from Theorem 1.1. Let $\mathrm{BV}([0,B])$ and $\mathrm{AC}([0,B])$ be respectively the space of bounded variation functions, and the space of absolutely continuous functions.

Our second result concerns the asymptotic behavior of $\alpha(\phi_\lambda)$:

THEOREM 1.2. *As $\lambda\to\infty$, we have*

$$\beta(p,w)\frac{2B}{3}+O(\lambda^{-1/2})\le\alpha(\phi_\lambda)\le\gamma(p,w)\frac{2B}{3}+O(\lambda^{-1/2})$$

when $Q\in C([0,B])$,

$$\beta(p,w)\frac{\frac{B^2}{4}-B(\frac{1}{4}+\|Q\|_1B)\lambda^{-1/2}}{\frac{3B}{8}+(\frac{9}{32}+2\|Q\|_1B)\lambda^{-1/2}}+O(\lambda^{-1})\le\alpha(\phi_\lambda)$$
$$\le\gamma(p,w)\frac{\frac{B^2}{4}+B(\frac{1}{4}+\|Q\|_1B)\lambda^{-1/2}}{\frac{3B}{8}-(\frac{9}{32}+2\|Q\|_1B)\lambda^{-1/2}}+O(\lambda^{-1})$$

when $Q\in\mathrm{BV}([0,B])$, and

$$\beta(p,w)\frac{2B}{3}+O(\lambda^{-3/2})\le\alpha(\phi_\lambda)\le\gamma(p,w)\frac{2B}{3}+O(\lambda^{-3/2})$$

when $Q\in C^4([0,B])\cap\mathrm{AC}([0,B])$ and $Q'\in\mathrm{BV}([0,B])$.

REMARK 1.1. A straightforward calculation shows that the localization coefficient of the function Φ_λ is for any positive λ given by

$$\alpha(\Phi_\lambda)=\frac{B^2/4+\cos(\sqrt{\lambda}B)^2/4\lambda-B\cos(\sqrt{\lambda}B)\sin(\sqrt{\lambda}B)/2\sqrt{\lambda}-\cos(\sqrt{\lambda}B)^4/4\lambda}{3B/8+\cos(\sqrt{\lambda}B)^3\sin(\sqrt{\lambda}B)/4\sqrt{\lambda}-5\cos(\sqrt{\lambda}B)\sin(\sqrt{\lambda}B)/8\sqrt{\lambda}}. \quad (1.8)$$

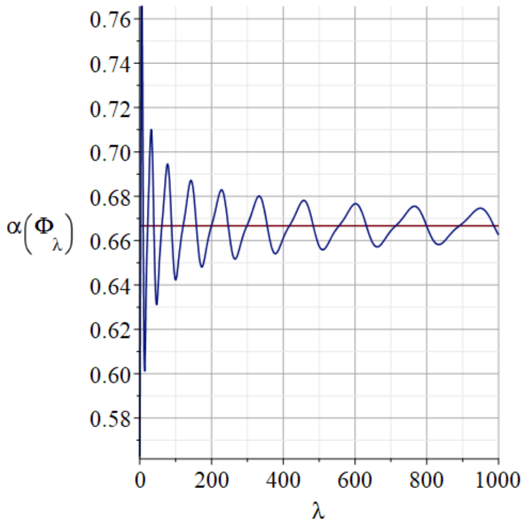

FIG. 1.3. *The localization coefficient* $\alpha(\Phi_\lambda)$ *of the function* $\Phi_\lambda(y)=\sin\sqrt{\lambda}y$, $y\in[0,B]$, *for* $B=1$. *The constant value shown is* $\alpha(\Phi_{\lambda_n})=2/3$, $n\in\boldsymbol{N}_0$, *where the* Φ_{λ_n} *are the eigenfunctions of the Dirichlet Laplacian on the interval* $(0,1)$.

The eigenvalues and eigenfunctions of the Dirichlet Laplacian $-d^2/dy^2$ on $(0,B)$ are given by $\lambda_n=n^2\pi^2/B^2$ and $\Phi_{\lambda_n}(y)=\sin(n\pi y/B)$, respectively, with $n\in\boldsymbol{N}_0:=\boldsymbol{N}\setminus\{0\}=\{1,2,\dots\}$. In particular, $\alpha(\Phi_{\lambda_n})=2B/3$ for $n\in\boldsymbol{N}_0$, that is, all eigenfunctions of the Dirichlet Laplacian on $(0,B)$ have the same localization coefficient. We furthermore readily see that

$$\lim_{\lambda\to\infty}\alpha(\Phi_\lambda)=\frac{2B}{3}, \tag{1.9}$$

and more precisely that, for large λ,

$$\begin{aligned}\alpha(\Phi_\lambda)&=\frac{B^2/4+O(\lambda^{-1/2})}{3B/8+O(\lambda^{-1/2})}=\frac{2B}{3}\frac{1}{1+O(\lambda^{-1/2})}+O(\lambda^{-1/2})\\&=\frac{2B}{3}+O(\lambda^{-1/2}).\end{aligned} \tag{1.10}$$

Figure 1.3 shows $\alpha(\Phi_\lambda)$ as function of λ for the choice $B=1$.

REMARK 1.2. If $p\equiv1$ and $w\equiv1$ then T is the Schrödinger operator $-d^2/dx^2+q(x)$. For this case, in the high-frequency limit ($\lambda\to\infty$) the localization coefficient of ϕ_λ approaches that of Φ_λ (so it approaches the value $2B/3$), that is, **the presence of the potential q becomes insignificant**.

REMARK 1.3. It is readily seen that the lower bound on $\alpha(\phi_\lambda)/\alpha(\Phi_\lambda)$ in Theorem 1.1 is a monotonically increasing function of λ, for any fixed positive B. In view of this, and of the behavior of $\alpha(\Phi_\lambda)$ discussed above, we conclude that if a solution of (1.1) is to exhibit high localization (relatively small value of $\alpha(\phi_\lambda)$) then λ must be relatively small, that is, **localization is a low-frequency phenomenon**.

We next focus on the localization in the eigenfunctions associated to low frequencies (small λ). In [5] the authors have given a simple but efficient way to predict the behavior of first eigenfunctions. Precisely they used the *landscape function* $\ell \in D(T)$, solving

$$T\ell = 1, \quad \ell \in D(T), \tag{1.11}$$

to identify the regions where the solution of (1.1) localizes. This can be observed through the following pointwise key inequality [10]

$$\phi(x) \leq \lambda \ell(x) \|\phi\|_\infty, \quad x \in (0,L).$$

Indeed $\lambda \|\ell\|_\infty \geq 1$ and ϕ can then localize only in the region $\{x \in (0,L);\ \lambda\ell(x) \geq 1\}$.

Our third result thus characterizes the landscape function in terms of the first eigenfunction of the operator T:

PROPOSITION 1.1. *Assume that* $w \equiv 1$, *and let* $k \in \boldsymbol{N}_0$. *If*

$$T^k \widetilde{\ell}_k = 1, \quad \widetilde{\ell}_k \in D(T^k), \quad \ell_k = \widetilde{\ell}_k / \|\widetilde{\ell}_k\|_2,$$

as well as

$$T\phi_1 = \lambda_1 \phi_1, \quad \phi_1 \in D(T), \quad \|\phi_1\|_2 = 1, \quad \phi_1 > 0, \quad \lambda_1 < \lambda_j \text{ for } j = 2,3,\ldots,$$

where $\lambda_j, j \in \boldsymbol{N}_0$ *is the non-decreasing sequence of eigenvalues of* T. *Then*

$$\|\ell_k - \phi_1\|_\infty \leq 2\lambda_1^{1/2} L \|P_1 1\|_2^{-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{k-1/2}, \tag{1.12}$$

where P_1 *is the spectral projection onto the eigenspace associated with* λ_1.

REMARK 1.4. The asymptotic result in (1.12) shows that the convergence is exponentially fast if the *fundamental gap* $\lambda_2 - \lambda_1$ is large enough. When $p = 1$, and q is a weakly convex potential it is known that [9]

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{L^2}.$$

The inequality conjectured by Yau [12] is still an open problem in higher dimensions. It turns out that this spectral gap also determines the rate at which positive solutions of the heat equation tend to their projections onto the first eigenspace.

Our fourth and final result is as follows:

PROPOSITION 1.2. *Assume that* $w \equiv 1$, *and let* λ_j, $j \in \boldsymbol{N}_0$, *be the non-decreasing sequence of eigenvalues of* T. *Let* P_j *be the spectral projection onto the eigenspace associated with* λ_j. *Let* k, $n_0 \in \boldsymbol{N}_0$, *and* $t \in (\lambda_{n_0+1}^{-1}, \lambda_{n_0}^{-1})$. *If*

$$(tT)^k \ell_{k,t} = 1, \quad \ell_{k,t} \in D(T^k),$$

then

$$\left\| \ell_{k,t} - \sum_{j=1}^{n_0} \frac{1}{(t\lambda_j)^k} P_j 1 \right\|_\infty \leq \frac{L}{t^{1/2}} \frac{1}{(t\lambda_{n_0+1})^{k-1/2}}. \tag{1.13}$$

REMARK 1.5. The value of t fixes the number of the eigenfunctions covered by the generalized landscape function $\ell_{t,k}$. Notice that $t \in (\lambda_{n_0+1}^{-1}, \lambda_{n_0}^{-1})$ is equivalent to

$$\frac{1}{t\lambda_{n_0+1}} < 1 < \frac{1}{t\lambda_{n_0}},$$

which implies that the contribution of the eigenfunctions $P_j 1$, $j > n_0$ in the localization of $\ell_{k,t}$ is exponentially small for large k while the contribution of $P_j 1$ for $1 \le j \le n_0$ can be exponentially large if in addition $P_j 1$ is not zero. These observations are confirmed in Section 6 by several numerical tests. Finally the results of Propositions 1.1 and 1.2 are still valid for w non-constant and sufficiently smooth ($\|\cdot\|_2$ should be substituted by $\|\cdot\|_{L^2((0,L);wdx)}$).

Theorem 1.1 is proved in Section 2 using a Volterra integral equation representation of solutions of (1.1) given by Fulton [6], while Theorem 1.2 is proved in Section 3 via the asymptotic expansions of ϕ_λ, as $\lambda \to \infty$, given in Fulton and Pruess [7]. Finally, we prove Propositions 1.1 and 1.2 in Sections 4 and 5 respectively using the power method.

2. Proof of Theorem 1.1 (non-asymptotic bounds on $\alpha(\phi_\lambda)$)

Using the Liouville transformation

$$\begin{aligned} y(x) &= \int_0^x \sqrt{w(s)/p(s)}\,ds \text{ for } x \in [0,L]; \quad B = y(L); \\ f(y) &= (w(x(y))p(x(y)))^{1/4}, \quad y \in [0,B]; \\ v_\lambda(y) &= \phi_\lambda(x(y)) f(y), \quad y \in [0,B]; \end{aligned} \tag{2.1}$$

and

$$Q(y) = f''(y)/f(y) + q(x(y))/w(x(y)), \quad y \in [0,B],$$

we recast the problem (1.1) in the Liouville normal form [7, pp. 303–304]

$$\begin{cases} -v_\lambda'' + Q(y) v_\lambda = \lambda v_\lambda, & y \in (0,B), \\ v_\lambda(0) = v_\lambda(B) = 0. \end{cases} \tag{2.2}$$

It follows from our assumptions on p, w and q that $Q \in C([0,B])$, hence $Q \in L^t([0,B])$ for $t \in [1,\infty]$. Now

$$\int_0^L \phi_\lambda(x)^2 dx = \int_0^B \frac{v_\lambda(y)^2}{w(x(y))} dy \in \left[\|w\|_\infty^{-1}, \|w^{-1}\|_\infty\right] \times \int_0^B v_\lambda(y)^2 dy$$

and

$$\begin{aligned} \int_0^L \phi_\lambda(x)^4 dx &= \int_0^B \frac{v_\lambda(y)^4}{p(x(y))^{1/2} w(x(y))^{3/2}} dy \\ &\in \left[\|p^{1/2} w^{3/2}\|_\infty^{-1}, \|p^{-1/2} w^{-3/2}\|_\infty\right] \times \int_0^B v_\lambda(y)^4 dy, \end{aligned}$$

so it remains to examine $\|v_\lambda\|_2^2$ and $\|v_\lambda\|_4^4$. To this end we recall from Fulton and Pruess [7, p. 308] that a solution of the ODE in (2.2), normalized such that $v_\lambda(0) = 0$ and $v_\lambda'(0) = (w(0)p(0))^{-1/4} \neq 0$, satisfies the associated Volterra integral equation

$$\left(\mathrm{Id} - \frac{1}{\sqrt{\lambda}} K_Q\right) v_\lambda(y) = \frac{v_\lambda'(0)}{\sqrt{\lambda}} \Phi_\lambda(y), \quad y \in [0,B], \tag{2.3}$$

where for any $u \in C^2([0,B])$ we have

$$K_Q u(y) = \int_0^y Q(z) \sin(\sqrt{\lambda}(y-z)) u(z)\, dz, \quad y \in (0,B).$$

Now write $\|K_Q\|_t$ for the operator norm of K_Q as a mapping from $L^t((0,B))$ to $L^t((0,B))$.

LEMMA 2.1. *For every positive λ we have*

$$\|K_Q\|_2^2 \le \frac{B^2}{4}\|Q\|_\infty^2$$

and

$$\|K_Q\|_4^4 \le \left(\frac{B^3}{12}+\frac{5B}{32\lambda}+\frac{5}{32\lambda^{3/2}}\right)\|Q\|_4^4.$$

Proof. The estimates follow readily from applying Hölder's inequality. We have

$$\begin{aligned}\|K_Q u\|_2^2 &\le \|Q\|_\infty^2\|u\|_2^2\int_0^B \|\sin(\sqrt{\lambda}(y-\cdot))\|_{L^2((0,y))}^2 dy\\ &= \|Q\|_\infty^2\|u\|_2^2\int_0^B\left(\frac{y}{2}-\frac{\cos\sqrt{\lambda}y\sin\sqrt{\lambda}y}{2\sqrt{\lambda}}\right)dy\\ &= \frac{\|Q\|_\infty^2}{4}\|u\|_2^2\left(B^2-\frac{\sin^2\sqrt{\lambda}B}{\lambda}\right)\\ &\le \frac{\|Q\|_\infty^2 B^2}{4}\|u\|_2^2, \quad u\in L^2((0,B)),\end{aligned}$$

as well as

$$\begin{aligned}\|K_Q u\|_4^4 \le& \|Q\|_4^4\|u\|_4^4\int_0^B \|\sin(\sqrt{\lambda}(y-\cdot))\|_{L^2((0,y))}^4 dy\\ =&\|Q\|_4^4\|u\|_4^4\left(\frac{B^3}{12}+\frac{B\cos(\sqrt{\lambda}B)^2}{4\lambda}-\frac{\sin\sqrt{\lambda}B\cos(\sqrt{\lambda}B)^3}{16\lambda^{3/2}}\right.\\ &\left.-\frac{3B}{32\lambda}-\frac{3\cos\sqrt{\lambda}B\sin\sqrt{\lambda}B}{32\lambda^{3/2}}\right)\\ \le&\|Q\|_4^4\left(\frac{B^3}{12}+\frac{5B}{32\lambda}+\frac{5}{32\lambda^{3/2}}\right)\|u\|_4^4, \quad u\in L^4([0,B]),\end{aligned}$$

∎

We have from (2.3) and from Lemma 2.1 that, for all positive λ,

$$\begin{aligned}\|v_\lambda\|_2 &\ge \lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_2-\lambda^{-1/2}\|K_Q\|_2\|v_\lambda\|_2\\ &\ge\lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_2-\lambda^{-1/2}B\|Q\|_\infty\|v_\lambda\|_2/2,\end{aligned}$$

$$\begin{aligned}\|v_\lambda\|_2 &\le \lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_2+\lambda^{-1/2}\|K_Q\|_2\|v_\lambda\|_2\\ &\le\lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_2+\lambda^{-1/2}B\|Q\|_\infty\|v_\lambda\|_2/2,\end{aligned}$$

$$\begin{aligned}\|v_\lambda\|_4 &\ge \lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_4-\lambda^{-1/2}\|K_Q\|_4\|v_\lambda\|_4\\ &\ge\lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_4-\lambda^{-1/2}\left(\frac{B^3}{12}+\frac{5B}{32\lambda}+\frac{5}{32\lambda^{3/2}}\right)^{1/4}\|Q\|_4\|v_\lambda\|_4,\end{aligned}$$

and

$$\begin{aligned}\|v_\lambda\|_4 &\le \lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_4 + \lambda^{-1/2}\|K_Q\|_4\|v_\lambda\|_4\\ &\le \lambda^{-1/2}|v_\lambda'(0)|\|\Phi_\lambda\|_4 + \lambda^{-1/2}\left(\frac{B^3}{12}+\frac{5B}{32\lambda}+\frac{5}{32\lambda^{3/2}}\right)^{1/4}\|Q\|_4\|v_\lambda\|_4.\end{aligned}$$

Specifically, for $B\|Q\|_\infty/2 < \sqrt{\lambda}$ (Assumption (1.6)) and

$$\left(\frac{B^3}{12}+\frac{5B}{32\lambda}+\frac{5}{32\lambda^{3/2}}\right)^{1/4}\|Q\|_4 < \sqrt{\lambda}$$

(Assumption (1.7)), we have

$$\left(\frac{1-b(B,\lambda)}{1+a(B,\lambda)}\right)^4 \le \frac{\alpha(v_\lambda)}{\alpha(\Phi_\lambda)} \le \left(\frac{1+b(B,\lambda)}{1-a(B,\lambda)}\right)^4.$$

3. Proof of Theorem 1.2 (asymptotic bounds on $\alpha(\phi_\lambda)$)

The first part of Theorem 1.2 follows from the fact that

$$\frac{1\mp b(B,\lambda)}{1\pm a(B,\lambda)} = 1+O(\lambda^{-1/2}), \quad \lambda\to\infty,$$

together with (1.10) and the estimates in Theorem 1.1.

Next, if $Q\in \mathrm{BV}([0,B])$ then we can use the asymptotic expansion of v_λ from (2.2) given by Equation $(3.3)_{2N}$ of Fulton and Pruess [7] with $N=1$, and get

$$v_\lambda(y)/v_\lambda'(0) = \lambda^{-1/2}\sin(\lambda^{1/2}y) - \frac{1}{2\lambda}\int_0^y Q(s)ds\cdot\cos(\lambda^{1/2}y) + O(\lambda^{-3/2}), \quad y\in[0,B],$$

where the remainder $O(\lambda^{-3/2})$ is uniform in $y\in[0,B]$. This, in turn, implies

$$\begin{aligned}\frac{\|v_\lambda\|_2^4}{v_\lambda'(0)^4} =& \lambda^{-2}\frac{B^2}{4}\\ &-\lambda^{-5/2}B\left(\frac{1}{4}\sin(2B\lambda^{1/2}) + \int_0^B\int_0^y Q(s)ds\,\sin(\lambda^{1/2}y)\cos(\lambda^{1/2}y)dy\right) + O(\lambda^{-3})\end{aligned}$$

and

$$\begin{aligned}\frac{\|v_\lambda\|_4^4}{v_\lambda'(0)^4} =& \lambda^{-2}\frac{3B}{8} + \lambda^{-5/2}\left(\frac{\sin(4B\lambda^{1/2}) - 8\sin(2B\lambda^{1/2})}{32}\right.\\ &\left. -2\int_0^B\int_0^y Q(s)ds\,\sin^3(\lambda^{1/2}y)\cos(\lambda^{1/2}y)dy\right) + O(\lambda^{-3})\end{aligned}$$

as $\lambda\to\infty$. Thus $c_- \le \lambda^2\|v_\lambda\|_2^4/v_\lambda'(0)^4 \le c_+$ with

$$c_\pm = \frac{B^2}{4} \pm \lambda^{-1/2}B\left(\frac{1}{4} + B\|Q\|_1\right) + O(\lambda^{-1}), \quad \lambda\to\infty,$$

and $d_- \le \lambda^2\|v_\lambda\|_4^4/v_\lambda'(0)^4 \le d_+$ with

$$d_\pm = \frac{3B}{8} \pm \lambda^{-1/2}\left(\frac{9}{32} + 2B\|Q\|_1\right) + O(\lambda^{-1}), \quad \lambda\to\infty.$$

Finally, if $Q\in \mathrm{AC}([0,B])$ and $Q'\in \mathrm{BV}([0,B])$ then we can use the asymptotic expansion of v_λ given by [7, Equation $(3.3)_{2N+1}$] with $N=1$, to get

$$\begin{aligned} v_\lambda(y)/v_\lambda'(0) = &\lambda^{-1/2}\sin(\lambda^{1/2}y) - \lambda^{-1}\frac{1}{2}\cos(\lambda^{1/2}y)\int_0^y Q(s)ds \\ &+\lambda^{-3/2}\frac{1}{4}\sin(\lambda^{1/2}y)\left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds + Q(0)+Q(y)\right)+O(\lambda^{-2}), \end{aligned} \tag{3.1}$$

where the remainder $O(\lambda^{-2})$ is uniform in y. This, in turn, implies

$$\begin{aligned} \frac{\|v_\lambda\|_2^4}{v_\lambda'(0)^4} = &\lambda^{-2}\frac{B^2}{4} \\ &-\lambda^{-5/2}B\left(\frac{\sin(2B\lambda^{1/2})}{4}+\int_0^B \sin(\lambda^{1/2}y)\cos(\lambda^{1/2}y)\int_0^y Q(s)ds dy\right) \\ &+\lambda^{-3}B\left(\frac{1}{4}\int_0^B \cos^2(\lambda^{1/2}y)\left(\int_0^y Q(s)ds\right)^2 dy\right. \\ &\left.+\frac{1}{2}\int_0^B \sin^2(\lambda^{1/2}y)\left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds + Q(0)+Q(y)\right)dy\right)+O(\lambda^{-7/2}) \end{aligned}$$

and

$$\begin{aligned} \frac{\|v_\lambda\|_4^4}{v_\lambda'(0)^4} = &\lambda^{-2}\frac{3B}{8} \\ &+\lambda^{-5/2}\left(\frac{\sin(4B\lambda^{1/2})-8\sin(2B\lambda^{1/2})}{32}-2\int_0^B \sin^3(\lambda^{1/2}y)\cos(\lambda^{1/2}y)\int_0^y Q(s)ds dy\right) \\ &+\lambda^{-3}\left(\frac{3}{2}\int_0^B \sin^2(\lambda^{1/2}y)\cos^2(\lambda^{1/2}y))\left(\int_0^y Q(s)ds\right)^2 dy\right. \\ &\left.+\int_0^B \sin^4(\lambda^{1/2}y)\left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds + Q(0)+Q(y)\right)dy\right)+O(\lambda^{-7/2}). \end{aligned}$$

Now each eigenvalue λ is a zero of $\lambda\mapsto v_\lambda(B)$ [7, p. 319, Case 4], and in light of (3.1) we therefore have

$$\sin(\lambda^{1/2}B) = \lambda^{-1/2}\frac{1}{2}\int_0^B Q(s)ds\cdot\cos(\lambda^{1/2}B)+O(\lambda^{-3/2}),$$

$$\sin^2(\lambda^{1/2}B) = \lambda^{-1}\frac{1}{4}\left(\int_0^B Q(s)ds\right)^2\cos^2(\lambda^{1/2}B)+O(\lambda^{-2}), \quad \cos^2(\lambda^{1/2}B)=1+O(\lambda^{-1}),$$

$$\sin(2\lambda^{1/2}B) = \lambda^{-1/2}\int_0^B Q(s)ds\cdot\cos^2(\lambda^{1/2}B)+O(\lambda^{-3/2}),$$

and

$$\sin(4\lambda^{1/2}B) = \lambda^{-1/2}2\int_0^B Q(s)ds\cdot\cos(2\lambda^{1/2}B)\cos^2(\lambda^{1/2}B)+O(\lambda^{-3/2}).$$

Also, using integration by parts, we find for any $\phi,\psi\in C^4([0,B])$ with $\phi(0)=0$ that

$$\int_0^B \sin(\lambda^{1/2}y)\cos(\lambda^{1/2}y)\phi(y)dy$$
$$=\lambda^{-1/2}\left(\frac{\sin^2(\lambda^{1/2}B)}{2}-\frac{1}{4}\right)\phi(B)+O(\lambda^{-1})=-\lambda^{-1/2}\frac{1}{4}\phi(B)+O(\lambda^{-1}),$$

$$\int_0^B \cos^2(\lambda^{1/2}y)\phi(y)dy$$
$$=\frac{1}{2}\int_0^B \phi(y)dy+\lambda^{-1/2}\frac{\sin(2\lambda^{1/2}B)}{4}\phi(B)+O(\lambda^{-1})=\frac{1}{2}\int_0^B \phi(y)dy+O(\lambda^{-1}),$$

$$\int_0^B \sin^2(\lambda^{1/2}y)\psi(y)dy$$
$$=\frac{1}{2}\int_0^B \psi(y)dy-\lambda^{-1/2}\frac{\sin(2\lambda^{1/2}B)}{4}\psi(B)+O(\lambda^{-1})=\frac{1}{2}\int_0^B \psi(y)dy+O(\lambda^{-1}),$$

$$\int_0^B \sin^3(\lambda^{1/2}y)\cos(\lambda^{1/2}y)\phi(y)dy$$
$$=\lambda^{-1/2}\left(\frac{\sin^4(\lambda^{1/2}B)}{4}-\frac{3}{32}\right)\phi(B)+O(\lambda^{-1})=-\lambda^{-1/2}\frac{3}{32}\phi(B)+O(\lambda^{-1}),$$

$$\int_0^B \sin^2(\lambda^{1/2}y)\cos^2(\lambda^{1/2}y)\phi(y)dy$$
$$=\frac{1}{8}\int_0^B \phi(y)dy+\lambda^{-1/2}\left(\frac{\sin(2\lambda^{1/2}B)-4\sin(\lambda^{1/2}B)\cos^3(\lambda^{1/2}B)}{16}\phi(B)\right.$$
$$\left.-\frac{\cos^4(\lambda^{1/2}B)}{2}\phi'(B)+\frac{3}{16}\phi'(B)+\frac{5}{16}\phi'(0)\right)+O(\lambda^{-1})$$
$$=\frac{1}{8}\int_0^B \phi(y)dy+\lambda^{-1/2}\frac{5}{16}(\phi'(0)-\phi'(B))+O(\lambda^{-1}),$$

and

$$\int_0^B \sin^4(\lambda^{1/2}y)\psi(y)dy$$
$$=\frac{3}{8}\int_0^B \psi(y)dy-\lambda^{-1/2}\left(\frac{\sin^3(\lambda^{1/2}B)\cos(\lambda^{1/2}B)}{4}+\frac{3\sin(2\lambda^{1/2}B)}{16}\right)\psi(B)+O(\lambda^{-1})$$
$$=\frac{3}{8}\int_0^B \psi(y)dy+O(\lambda^{-1}).$$

Using these expansions, we get

$$\lambda^2\frac{\|v_\lambda\|_2^4}{v_\lambda'(0)^4}=\frac{B^2}{4}+\lambda^{-1}\frac{B}{4}\left[\frac{1}{2}\int_0^B\left(\int_0^y Q(s)ds\right)^2dy\right.$$
$$\left.+\int_0^B\left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds+Q(0)+Q(y)\right)dy\right]+O(\lambda^{-3/2})$$

and

$$\lambda^2 \frac{\|v_\lambda\|_4^4}{v'_\lambda(0)^4} = \frac{3B}{8} + \lambda^{-1}\frac{3}{8}\left[\frac{1}{2}\int_0^B \left(\int_0^y Q(s)ds\right)^2 dy \right.$$
$$\left. + \int_0^B \left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds + Q(0) + Q(y)\right) dy\right] + O(\lambda^{-3/2}).$$

Note that the factors multiplying λ^0 and λ^{-1} in $\|v_\lambda\|_2^4$ are proportional to those in $\|v_\lambda\|_4^4$, with the proportionality constant $2B/3$. We thus have

$$\alpha(v_\lambda) = \frac{B^2/4 + \lambda^{-1}s + O(\lambda^{-3/2})}{(3/2B)(B^2/4 + \lambda^{-1}s + O(\lambda^{-3/2}))} = \frac{2B}{3} + O(\lambda^{-3/2}),$$

where

$$s = \frac{B}{4}\left[\frac{1}{2}\int_0^B \left(\int_0^y Q(s)ds\right)^2 dy + \int_0^B \left(\int_0^y Q(s)\int_0^s Q(\tau)d\tau ds + Q(0) + Q(y)\right) dy\right].$$

4. Proof of Proposition 1.1

By construction T is self-adjoint and diagonalizable. Hence it can be written in the following form

$$T = \sum_{j=1}^{\infty} \lambda_j P_j, \tag{4.1}$$

where $(\lambda_j)_{j\in \boldsymbol{N}_0}$ is the strictly increasing sequence of eigenvalues of T, and P_j are the orthogonal projections onto the eigenspaces associated to λ_j, $j \in \boldsymbol{N}_0$.

Since $\ell_k \in D(T)$, it has the following expansion

$$\ell_k = \tilde{\ell}_k / \|\tilde{\ell}_k\|_2, \quad \tilde{\ell}_k = \sum_{j=1}^{\infty} \lambda_j^{-k} P_j 1.$$

Straightforward computations give

$$\|\lambda_1^k \tilde{\ell}_k - P_1 1\|_2 \le L^{1/2}\left(\frac{\lambda_1}{\lambda_2}\right)^k, \quad \text{and } \|P_1 1\|_2 \le \|\lambda_1^k \tilde{\ell}_k\|_2. \tag{4.2}$$

We then deduce

$$\|\lambda_1^k \tilde{\ell}_k\|_2 - \|P_1 1\|_2 \le L^{1/2}\left(\frac{\lambda_1}{\lambda_2}\right)^k. \tag{4.3}$$

Similarly, since $T\tilde{\ell}_k \in L^2((0,L))$, we have

$$\|\lambda_1^k T^{1/2}\tilde{\ell}_k - T^{1/2} P_1 1\|_2 \le (\lambda_1 L)^{1/2}\left(\frac{\lambda_1}{\lambda_2}\right)^{k-1/2}. \tag{4.4}$$

Recall that $\phi_1 > 0$, which implies $\|P_1 1\|_2 > 0$ and $\phi_1 = P_1 1/\|P_1 1\|_2$. Now combining inequalities (4.2), (4.3) and (4.4), we get

$$\|T^{1/2}\ell_k - T^{1/2}\phi_1\|_2 \le \|\lambda_1^k T^{1/2}\tilde{\ell}_k - T^{1/2}P_1 1\|_2 \|P_1 1\|_2^{-1} + \lambda_1^{1/2}\left(\|\lambda_1^k \tilde{\ell}_k\|_2 - \|P_1 1\|_2\right)\|P_1 1\|_2^{-1}$$
$$\le 2(\lambda_1 L)^{1/2}\left(\frac{\lambda_1}{\lambda_2}\right)^{k-1/2} \|P_1 1\|_2^{-1}. \tag{4.5}$$

On the other hand, we have

$$|\varphi(x)| \leq \int_0^x |\varphi'(s)| ds \leq L^{1/2} \|\varphi'\|_2, \quad \forall x \in (0,L),$$

for all $\varphi \in C_0^\infty((0,L))$.

Since $C_0^\infty((0,L))$ is dense in $D(T^{1/2})$, we get

$$\|\varphi\|_\infty \leq L^{1/2} \|T^{1/2}\varphi\|_2, \quad \forall \varphi \in D(T^{1/2}). \tag{4.6}$$

Combining inequalities (4.6) and (4.5), we obtain the desired result.

5. Proof of Proposition 1.2

Using the spectral expansion (4.1), we get

$$\ell_{k,t} = \sum_{j=1}^{\infty} \frac{1}{(t\lambda_j)^k} P_j 1.$$

Hence

$$T^{1/2}\left(\ell_{k,t} - \sum_{j=1}^{n_0} \frac{1}{(t\lambda_j)^k} P_j 1\right) = \sum_{j=n_0+1}^{\infty} \frac{1}{(t\lambda_j)^k} T^{1/2} P_j 1.$$

Therefore

$$\left\| T^{1/2}\left(\ell_{k,t} - \sum_{j=1}^{n_0} \frac{1}{(t\lambda_j)^k} P_j 1\right)\right\|_2 \leq \frac{L^{1/2}}{t^{1/2}} \frac{1}{(t\lambda_{n_0+1})^{k-1/2}}. \tag{5.1}$$

Applying again the Sobolev inequality (4.6), we recover the final estimate.

6. The landscape function: numerical tests

We start by illustrating the consequences of Proposition 1.1. For Figure 6.1 we use $L=1$ and

$$p(x) = \tanh(40x/L - 10) + 1.1, \quad q(x) = 0, \quad x \in [0,L],$$

as in (1.2) in Section 1), while Figure 6.2 shows the graphs of

$$p(x) = \tanh(40x/L - 20) + 1.1, \quad q(x) = 2 + \sin(2\pi x), \quad x \in [0,L],$$

used, with $L=1$, for the results of Figure 6.3. Finally, Figure 6.4 shows the functions

$$p(x) = \tanh(40x/L - 10) + 1.1, \quad q(x) = 2 + \sin(2\pi x), \quad x \in [0,L],$$

used, with $L=5$, for the results of Figure 6.5.

Next, in Figure 6.6 we illustrate the validity of the upper bound on $\|\ell_{k,t} - \sum_{j=1}^{n_0} (t\lambda_j)^{-k} P_j 1\|_\infty$, as given in Proposition 1.2. Since the constants $(t\lambda_j)^{-1}$, $j = 1,\ldots,n_0$, are greater than 1, the numerical error present in the above L^∞-norm can grow exponentially with k. To avoid this numerical instability, in Figure 6.6 we plot the equivalent quantity $\|\sum_{j=n_0+1}^{\infty} (t\lambda_j)^{-k} (t\lambda_j)^{-k} P_j 1\|_\infty$, with the series truncated at $j=20$.

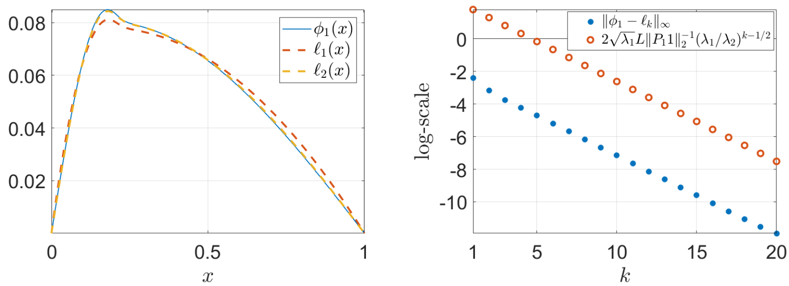

FIG. 6.1. *Left: the functions ℓ_k approach the first eigenvector ϕ_1 pointwise as k increases. Right: actual value vs. upper bound on $\|\phi_1 - \ell_k\|_\infty$, see Proposition 1.1.*

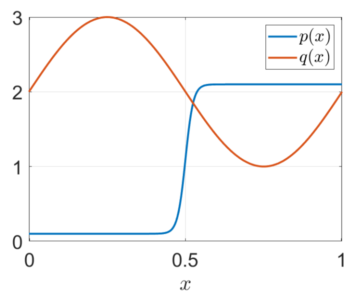

FIG. 6.2. *The functions $p(x)$ and $q(x)$ used for the results of Figure 6.3.*

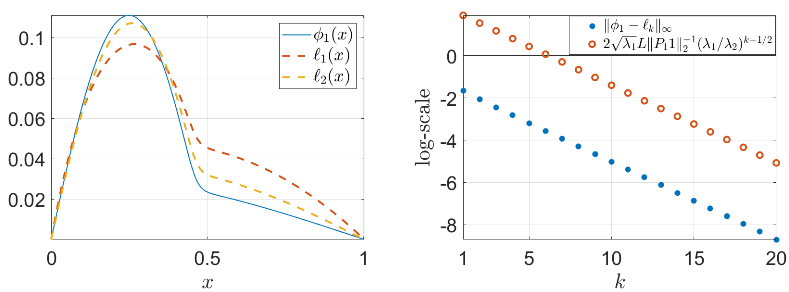

FIG. 6.3. *Left: the functions ℓ_k approach the first eigenvector ϕ_1 pointwise as k increases. Right: actual value vs. upper bound on $\|\phi_1 - \ell_k\|_\infty$, see Proposition 1.1.*

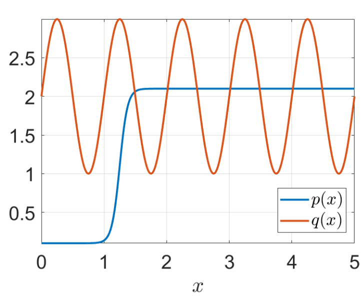

FIG. 6.4. *The functions $p(x)$ and $q(x)$ used for the results of Figure 6.5.*

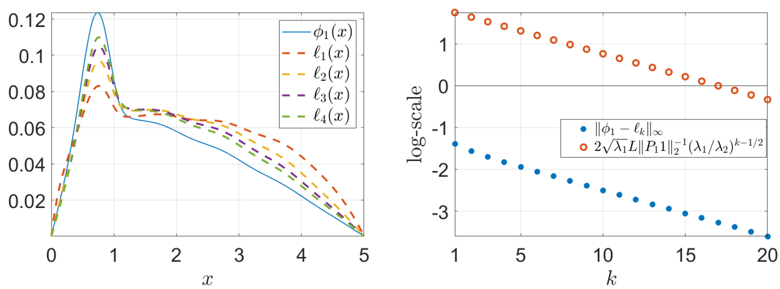

FIG. 6.5. *Left: the functions ℓ_k approach the first eigenvector ϕ_1 pointwise as k increases. Right: actual value vs. upper bound on $\|\phi_1 - \ell_k\|_\infty$, see Proposition 1.1.*

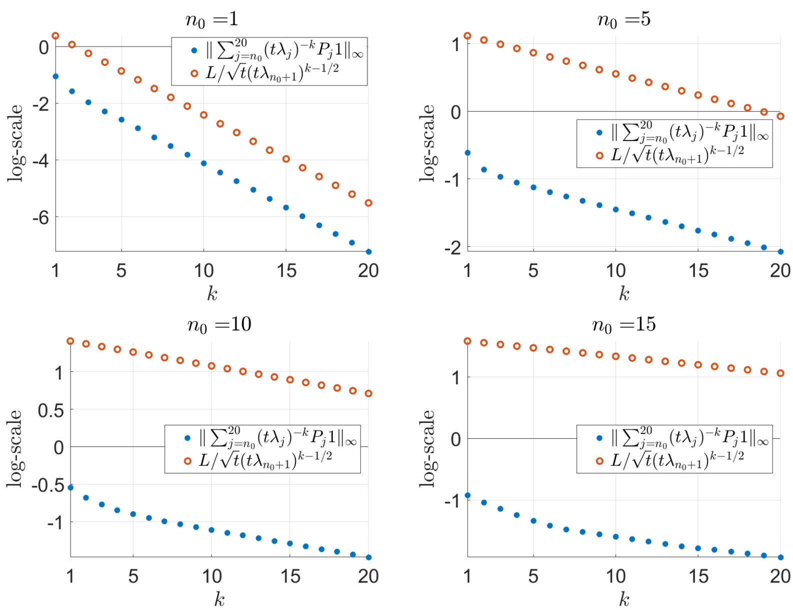

FIG. 6.6. *Illustration of the upper bound of Proposition 1.2 for different values of n_0.*

Acknowledgments. This work was funded by the Villum Foundation research grant no. 25893. M. Karamehmedović was additionally funded by Villum Foundation research grant VIL58857 and by the project 20FUN02/f10 POLight from the European Metrology Programme for Innovation and Research co-financed by the Participating States and from the European Union's Horizon 2020 research and innovation programme.

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