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Extremal $\mathcal{N} = (2, 2)$ 2D conformal field theories and constraints of modularity

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We explore the constraints on the spectrum of primary fields implied by modularity of the elliptic genus of $\mathcal{N} = (2, 2)$ 2D CFTs. We show that such constraints have nontrivial implications for the existence of "extremal" $\mathcal{N} = (2, 2)$ conformal field theories. Applications to AdS_3 supergravity and flux compactifications are addressed.

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1. Introduction and summary

In a recent work [47], Witten has revived the study of $2 + 1$ dimensional quantum gravity. In particular, he has defined a notion of pure AdS_3 quantum gravity and investigated its properties in light of the AdS/CFT correspondence. These considerations naturally lead to a notion of an *extremal conformal field theory.* Extremality means that the partition function of the boundary CFT is as close as possible to the Virasoro character of the vacuum. The reason for this is that there are two kinds of excitations in pure gravity: the perturbative excitations and the black holes. The perturbative excitations are identified with Virasoro descendants of the vacuum following [7] while the Virasoro primaries correspond to the BTZ black holes. Since black holes are parametrically heavy, there is a large gap from the vacuum to the first nontrivial Virasoro primary. The present paper addresses similar questions for pure quantum gravity with extended $\mathcal{N} = 2$ supersymmetry. Our main tool will be the elliptic genus of an $\mathcal{N} = 2$ superconformal field theory. As we recall below, this is a weak Jacobi form, and its modular properties impose tight constraints on the partition function. The advantage of this approach is that, unlike the case of [47], we do not have to assume the complete holomorphic factorization of the partition function. The holomorphy and modularity of the elliptic genus holds for any conformal field theory with $\mathcal{N} = 2$ supersymmetry. Thus, we can test the hypothetical existence of a theory of pure AdS_3 supergravity without relying on the additional assumption of holomorphic factorization. We will show that there is some tension between these modular properties and the notion of extremality.

A brief summary of our results is the following:

1. In Section 3.1, we give a definition of an extremal $(2, 2)$ superconformal field theory which, one might expect would constitute a holographic dual to "pure $(2, 2)$ AdS_3 supergravity." In any case, it is a natural generalization of the notion of extremality to $(2, 2)$ supersymmetry. In this paper, we restrict our attention to theories with $c = 0 \bmod 6$ and integral $U(1)_R$ charges for the left- and right-moving $\mathcal{N} = 2$ algebras. Relaxing this assumption is an interesting open problem.

2. In Section 4, we give numerical evidence that only a finite number of "exceptional" examples of extremal $(2, 2)$ theories can exist. Then in Section 5 we give an analytic proof that this is indeed the case. We also present very strong evidence that the extremal elliptic genus only exists for nine values of c, namely

$$6, 12, 18, 24, 30, 42, 48, 66, 78. \tag{1.1}$$

3. In Section 6 we then introduce the notion of a "nearly extremal $(2, 2)$ superconformal theory," whose spectrum only approximates that of pure $(2, 2)$ supergravity. We show that if the degree of approximation is relaxed, then candidate elliptic genera do indeed exist.

4. By quantifying the degree of approximation required to produce candidate elliptic genera we are able to constrain the spectrum as follows. Consider states (in the NSNS-sector) which are right-chiral-primary

and left $\mathcal{N} = 2$ primary with (L_0, J_0) eigenvalue (h, ℓ). In section 6.1, Equation (6.11), we show that for c large any theory with modular elliptic genus must have some such state with

$$h < \frac{c}{24} + \frac{3\ell^2}{2c} - \frac{1}{8} + \mathcal{O}(c^{-1/2}). \tag{1.2}$$

This result is conjectural. It is supported by numerical evidence described in Section 6. Finding a rigorous justification of (6.11) (or a counterexample) is an interesting open problem raised by the present paper.

5. On the other hand, in Section 7, we show that it is possible to construct an elliptic genus which is compatible with the spectrum of an extremal $(2, 2)$ superconformal theory for conformal weights $h \leq \frac{c}{24}$.
6. In Section 9 we comment on a partial generalization of our results to $\mathcal{N} = 4$ theories.

In the remainder of the paper we discuss some implications of the above results. First, in Section 8, we discuss the implications for the existence of pure $(2, 2)$ AdS_3 supergravity. While our results cast some doubt on the existence of such theories, they are not conclusive. It is conceivable that quantum corrections to the cosmic censorship bound for the existence of black holes imply that one should identify a near-extremal rather than an extremal $(2, 2)$ CFT as a holographic dual of pure supergravity. We leave this question for future work. Of course, even when a candidate Jacobi form exists that does not mean a corresponding $(2, 2)$ supergravity necessarily exists. In the analogous $\mathcal{N} = 0$ case, the relevant partition functions can readily be constructed, but it is not known whether the corresponding extremal CFTs exist for general Chern–Simons levels k. Indeed, there is an argument based on the modular differential equation of these partition functions [23, 24] that suggests that the theories are in fact inconsistent for sufficiently large k.

A second motivation for the present work is that constraints on conformal field theory spectra implied by modular invariance might have interesting applications to flux compactifications of string theory and M-theory. This is briefly explained in Section 10. Again, the development of this idea is left to future work.

2. Polar states and the elliptic genus

We will focus on theories with $\mathcal{N} = (2,2)$ two-dimensional superconformal symmetry. It will be convenient to parametrize the (left = right) central charge as $c = 6m$. A simple example of such a theory that the reader might wish to keep in mind is an $\mathcal{N} = (2,2)$ sigma-model based on a Calabi–Yau target space of complex dimension $2m$. In the present paper, we only consider integer values of m, and thus the relevant Calabi–Yau manifolds have even complex dimension.[2] In particular, the smallest nontrivial value of m corresponds to a Calabi–Yau 2-fold, that is a torus T^4 or a K3 surface.

We assume that the Hilbert space of our theory is a direct sum of unitary highest weight representations of the $\mathcal{N} = 2$ algebra. This allows us to define the RR-sector partition function

$$Z_{\mathrm{RR}}(\tau, z; \bar{\tau}, \bar{z}) := \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}} q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\tilde{L}_0 - c/24} e^{2\pi i \bar{z} \tilde{J}_0} e^{i\pi (J_0 - \tilde{J}_0)}, \tag{2.1}$$

which has good modular properties under the SL$(2,\mathbb{Z})$ action $(\tau, z) \to (\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d})$. Here, as usual, $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$, and similarly for $\bar{q}$ and $\bar{y}$.

In these conventions, the elliptic genus of an $\mathcal{N} = (2,2)$ superconformal field theory $\mathcal{C}$ is defined to be

$$\chi(\tau, z; \mathcal{C}) := Z_{\mathrm{RR}}(\tau, z; \bar{\tau}, 0). \tag{2.2}$$

It is holomorphic in (τ, z) by the standard properties of the Witten index. For references on the elliptic genus, see [3, 4, 18, 28–30, 38–42, 44, 45].

$\mathcal{N} = 2$ algebras have the crucial spectral flow isomorphism [43], which allows us to relate the NS- and R-sector partition functions. Recall that spectral flow SF_θ for $\theta \in \frac{1}{2}\mathbb{Z}$ is an isomorphism of $\mathcal{N} = 2$ superconformal algebras which maps eigenvalues

$$L_0 \longrightarrow L_0 + \theta J_0 + \theta^2 m, \tag{2.3}$$

$$J_0 \longrightarrow J_0 + 2\theta m. \tag{2.4}$$

[2]A generalization to half-integer values of m should be possible, but we will not attempt it in the present paper. For $m = 1$, the resulting theory actually has $(4,4)$ supersymmetry, but we will not use this fact.

The spectral flow operators act on $Z = Z_{\mathrm{RR}}$ as:

$$(\mathrm{SF}_\theta \widetilde{\mathrm{SF}}_{\tilde\theta} Z) = e\left(m\theta^2\tau + 2m\theta\left(z+\frac{1}{2}\right)\right) \cdot e\left(m\tilde\theta^2\bar\tau + 2m\tilde\theta\left(\bar z - \frac{1}{2}\right)\right) Z(\tau, z+\tau\theta; \bar\tau, \bar z + \tilde\theta\bar\tau), \tag{2.5}$$

where $e(x) := e^{2\pi i x}$. For simplicity, we restrict our attention to theories with integral spectrum of left- and right-moving $U(1)$ charges $J_0, \tilde J_0$. Again, it should be possible, and would be interesting, to relax this assumption. Spectral-flow invariant theories with integral $U(1)$ charges satisfy

$$Z_{\mathrm{RR}} = (\mathrm{SF}_\theta \widetilde{SF}_{\tilde\theta}) Z_{\mathrm{RR}}, \quad \theta, \tilde\theta \in \mathbb{Z}, \tag{2.6}$$

$$Z_{NSNS} = (\mathrm{SF}_\theta \widetilde{\mathrm{SF}}_{\tilde\theta}) Z_{\mathrm{RR}}, \quad \theta, \tilde\theta \in \mathbb{Z} + \frac{1}{2}. \tag{2.7}$$

As is well known [28], the modularity properties of Z_{RR} together with spectral-flow invariance and unitarity imply that the elliptic genus is a *weak Jacobi form* of index m and weight zero [21]. A weak Jacobi form $\phi(\tau, z)$ of weight w and index $m \in \mathbb{Z}$, with $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, satisfies the transformation laws

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{2\pi i m (cz^2/(c\tau+d))}\phi(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{2.8}$$

$$\phi(\tau, z + \ell\tau + \ell') = e^{-2\pi i m(\ell^2\tau + 2\ell z)}\phi(\tau, z), \quad \ell, \ell' \in \mathbb{Z}, \tag{2.9}$$

and has a Fourier expansion

$$\phi(\tau, z) = \sum_{n\geq 0, \ell\in\mathbb{Z}} c(n, \ell) q^n y^\ell \tag{2.10}$$

with $c(n,\ell) = (-1)^w c(n, -\ell)$. It follows from the spectral-flow identity that $c(n, \ell) = 0$ for $4mn - \ell^2 < -m^2$. Following [21], we denote by $\tilde J_{w,m}$ the vector space of weak Jacobi forms of weight w and index m. A Jacobi form is then a weak Jacobi form whose polar part vanishes (see below).

Suppose we are given an integer $m \in \mathbb{Z}_+$. If $(\ell, n) \in \mathbb{Z}^2$ is a lattice point, we refer to its *polarity* as $p = 4mn - \ell^2$. If $\phi \in \tilde J_{0,m}$, let us define the *polar part* of ϕ, denoted ϕ^-, to be the sum of the terms in the Fourier expansion corresponding to lattice points of negative polarity. By spectral flow, one can always relate the degeneracies to those in the fundamental domain with

$|\ell| \leq m$. If we impose the modular transformation (2.8) with $-\mathbf{1} \in \mathrm{SL}(2,\mathbb{Z})$, which implements charge conjugation, then $c(n,\ell) = c(n,-\ell)$ and therefore the polar coefficients which cannot be related to each other by spectral flow and charge conjugation are $c(n,\ell)$ where (ℓ, n) is valued in the *polar region* $\mathcal{P}$ (of index m), defined to be

$$\mathcal{P}^{(m)} := \{(\ell,n) : 1 \leq \ell \leq m, \quad 0 \leq n, \quad p = 4mn - \ell^2 < 0\}. \tag{2.11}$$

For an example, see Figure 1.

Given any Fourier expansion

$$\psi(\tau,z) = \sum_{\ell,n\in\mathbb{Z}} \hat{\psi}(n,\ell) q^n y^\ell, \tag{2.12}$$

we define its *polar polynomial* (of index m) to be the sum restricted to the polar region $\mathcal{P}^{(m)}$:

$$\mathrm{Pol}(\psi) := \sum_{(\ell,n)\in\mathcal{P}^{(m)}} \hat{\psi}(n,\ell) q^n y^\ell \ . \tag{2.13}$$

Let us moreover denote by V_m the space of polar polynomials, i.e., the vector space generated by the monomials $q^n y^\ell$ with $(\ell,n) \in \mathcal{P}^{(m)}$.

The key mathematical fact we need follows from the theory of "periods of modular forms." The upshot is that one can reconstruct a weak Jacobi

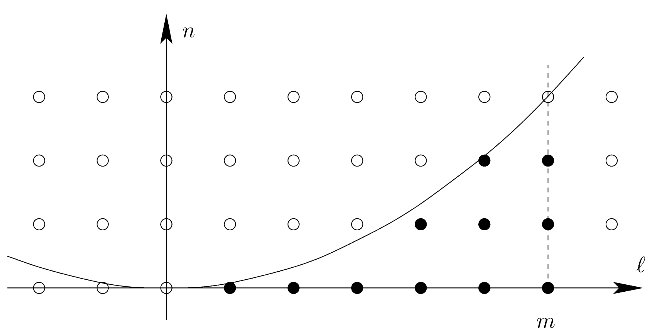

Figure 1: A cartoon showing polar states (represented by "•") in the region $\mathcal{P}^{(m)}$. Spectral flow by $\theta = \frac{1}{2}$ relates these states to particle states in the NS-sector of an $\mathcal{N} = 2$ superconformal field theory which are holographically dual to particle states in AdS_3.

form of weight zero from its polar polynomial. Moreover, there is a sequence

$$0 \longrightarrow \tilde{J}_{0,m} \xrightarrow{\text{Pol}} V_m \xrightarrow{\text{Per}} S_{5/2} \tag{2.14}$$

exact at V_m, where Per is a "period map" to a certain space of vector-valued cusp forms of weight 5/2. A nonzero image in the space of cusp forms means that the polar polynomial cannot be realized by a true weak Jacobi form. For an explanation of these facts in the physics literature, together with references to the mathematical literature, see [15, 33, 34, 37]. The reader interested in these matters should also consult [6].

In the next two sections we will show that there can indeed be nontrivial obstructions simply by computing the dimensions of $\tilde{J}_{0,m}$ and V_m.

Returning to the conformal field theory $\mathcal{C}$, an eigenstate of L_0, J_0 is called a *polar state* if it has negative polarity:

$$p = 4mL_0 - J_0^2 - m^2 = 4m\left(L_0 - \frac{c}{24}\right) - J_0^2 < 0. \tag{2.15}$$

One checks that $4mL_0 - J_0^2$ is spectral-flow invariant, so we can speak of polar states in both the R- and NS-sector. Using the mathematical results explained above, we see that the significance of polar states is that the polar degeneracies of the elliptic genus determine all the other Fourier coefficients of the elliptic genus.

2.1. Counting weight zero weak Jacobi forms

Let $\tilde{J}_{\text{ev},*} = \oplus_{w\in 2\mathbb{Z}, m\in\mathbb{Z}} \tilde{J}_{w,m}$ denote the bigraded ring of weak Jacobi forms of even weight. According to [21], Theorem 9.3, $\tilde{J}_{\text{ev},*}$ is a polynomial algebra on four generators of degree

$$(w,m) = (4,0), \quad (6,0), \quad (-2,1), \quad (0,1). \tag{2.16}$$

The first two generators correspond to the Eisenstein series

$$E_4 = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + 6720q^3 + \cdots \tag{2.17}$$

and

$$E_6 = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - 122976q^3 - \cdots, \tag{2.18}$$

where the Fourier coefficients $\sigma_k(n) := \sum_{d|n} d^k$ are defined to be the k-th powers of the positive divisors of n. A generalization of Eisenstein series to Jacobi forms is described in [21]:

(2.19)
$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{c,d\in\mathbb{Z},(c,d)=1} \sum_{\ell\in\mathbb{Z}} (c\tau + d)^{-k} \exp^{\left[2\pi i m\left(\ell^2 \frac{a\tau+b}{c\tau+d} + 2\ell\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right)\right]}.$$

In terms of these generalized Eisenstein series, one can write the remaining two generators in (2.16) as

$$\tilde{\phi}_{-2,1} = \frac{\phi_{10,1}}{\Delta} \in \tilde{J}_{-2,1}, \quad \tilde{\phi}_{0,1} = \frac{\phi_{12,1}}{\Delta} \in \tilde{J}_{0,1}, \tag{2.20}$$

where the first subscript on $\tilde{\phi}$ denotes the weight and the second denotes the index. Here, $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ and

(2.21)
$$\begin{aligned}
\phi_{10,1} &= \frac{1}{144}(E_6E_{4,1} - E_4E_{6,1}) \\
&= (y - 2 + y^{-1})q + (-2y^2 - 16y + 36 - 16y^{-1} - 2y^{-2})q^2 + \cdots, \\
\phi_{12,1} &= \frac{1}{144}(E_4^2E_{4,1} - E_6E_{6,1}) \\
&= (y + 10 + y^{-1})q + (10y^2 - 88y - 132 - 88y^{-1} + 10y^{-2})q^2 + \cdots.
\end{aligned}$$

Thus the two weak Jacobi forms $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ have the series expansion

(2.22)
$$\begin{aligned}
\tilde{\phi}_{-2,1} &= \left(y - 2 + y^{-1}\right) + \left(-2y^2 + 8y - 12 + 8y^{-1} - 2y^{-2}\right) q + \cdots, \\
\tilde{\phi}_{0,1} &= (y + 10 + y^{-1}) + \left(10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2}\right) q + \cdots.
\end{aligned}$$

Much useful information about Jacobi forms can be found in [21].

To summarize, a natural vector space basis of $\tilde{J}_{0,m}$ is given by

$$(\tilde{\phi}_{-2,1})^a(\tilde{\phi}_{0,1})^b E_4^c E_6^d, \tag{2.23}$$

where a, b, c, d are nonnegative integers such that $a + b = m$ and $a = 2c + 3d$. It is straightforward to compute the number of solutions to these constraints and thereby show that

$$j(m) := \dim \tilde{J}_{0,m} = \frac{m^2}{12} + \frac{m}{2} + \left(\delta_{s,0} + \frac{s}{2} - \frac{s^2}{12}\right), \tag{2.24}$$

where $m = 6\rho + s$ with $\rho \geq 0$ and $0 \leq s \leq 5$. Specifically,

$$j(m) = \begin{cases} \frac{m^2}{12} + \frac{m}{2} + 1. & m = 0 \bmod 6, \\ \frac{m^2}{12} + \frac{m}{2} + \frac{5}{12}, & m = 1, 5 \bmod 6, \\ \frac{m^2}{12} + \frac{m}{2} + \frac{2}{3} & m = 2, 4 \bmod 6, \\ \frac{m^2}{12} + \frac{m}{2} + \frac{3}{4} & m = 3 \bmod 6. \end{cases} \tag{2.25}$$

2.2. Counting polar monomials

Let us now compute the dimension of the space V_m and compare it to $j(m)$. In other words, we wish to count the number of integer points in the (ℓ, n) plane bounded (on one side) by the parabola $4mn - \ell^2 = 0$, as shown in Figure 1. We have

$$P(m) := \dim V_m = \sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil . \tag{2.26}$$

Note that we want the *ceiling* function and not the floor function, as we include $n = 0$ up to the largest n with $n < \ell^2/(4m)$ for each $\ell = 1, \ldots, m$.

To compute this, we follow [21] and write our sum as a sum of three terms.

$$\sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil = \sum_{\ell=1}^{m} \frac{\ell^2}{4m} - \sum_{\ell=1}^{m} \left(\left(\frac{\ell^2}{4m} \right)\right) + \frac{1}{2} \sum_{\ell=1}^{m} \left(\left\lceil \frac{\ell^2}{4m} \right\rceil - \left\lfloor \frac{\ell^2}{4m} \right\rfloor \right), \tag{2.27}$$

where

$$((x)) := x - \frac{1}{2}(\lceil x \rceil + \lfloor x \rfloor) = \begin{cases} 0, & x \in \mathbb{Z}, \\ \alpha - \frac{1}{2}, & x = n + \alpha, \quad 0 < \alpha < 1. \end{cases} \tag{2.28}$$

Note that $((x))$ is the sawtooth function. It is periodic of period 1.

Now we evaluate the three terms. The main term comes from the elementary formula

$$\sum_{\ell=1}^{m} \frac{\ell^2}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24}. \tag{2.29}$$

Next, note that the number of integers ℓ with $1 \leq \ell \leq m$ with $\ell^2 = 0 \bmod 4m$ is $\lfloor \frac{b}{2} \rfloor$, where b is the largest integer with $b^2|m$. This follows from the prime factorization of m. Thus, we obtain:

$$\sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil - \sum_{\ell=1}^{m} \left\lfloor \frac{\ell^2}{4m} \right\rfloor = m - \left\lfloor \frac{b}{2} \right\rfloor . \tag{2.30}$$

Finally we come to the most subtle term $\sum_{\ell=1}^{m}((\frac{\ell^2}{4m}))$. The numbers $((\frac{\ell^2}{4m}))$ are, very roughly speaking, randomly distributed between $-1/2$ and $+1/2$, Therefore, the average will go to zero. In fact, they roughly make a random walk, so we expect a quantity on the order of $m^{1/2}$. To be more precise, the discussion of [21, pp. 122–124], shows that

$$\sum_{\ell=1}^{m} \left(\left(\frac{\ell^2}{4m} \right) \right) = -\frac{1}{4} \sum_{d|4m} h'(-d) + \frac{1}{2} \left(\left(\frac{m}{4} \right) \right),$$

where $h'(-d)$ is the class number of a quadratic imaginary field of discriminant $-d$ (with the exception of $d = 3, 4$).

Putting the three terms together, we obtain

$$P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m), \tag{2.31}$$

where $A(m)$ is the arithmetic function

$$A(m) = \frac{1}{4} \sum_{d|4m} h'(-d) - \frac{1}{2} \left\lfloor \frac{b}{2} \right\rfloor - \frac{1}{2} \left(\left(\frac{m}{4} \right) \right) + \frac{1}{24}. \tag{2.32}$$

Very roughly speaking, $A(m)$ grows like $\mathcal{O}(m^{1/2})$, so for large m we have

$$P(m) - j(m) = \frac{m}{8} + \mathcal{O}(m^{1/2}). \tag{2.33}$$

The reader should be warned that we are not using the $\mathcal{O}$ symbol in its precise mathematical sense here, but rather as a heuristic order-of-magnitude for the "average" value of the subleading term to the linear behavior in (2.33).[3]

m	$\dim \tilde{J}_{0,m}$	$\dim V_m$
$m = 0$	1	0
$m = 1$	1	1
$m = 2$	2	2
$m = 3$	3	3
$m = 4$	4	4
$m = 5$	5	6
$m = 6$	7	8
$m = 7$	8	9
$m = 8$	10	11
$m = 9$	12	13
$m = 10$	14	16
$m = 11$	16	18
$m = 12$	19	21

The first few values of $P(m)$ and $j(m)$ are shown in the above table. Note that $P(m) > j(m)$ for $m \geq 5$, and it is straightforward to check with a computer that $(P(m) - j(m) - \frac{m}{8})m^{-1/2}$ is positive and roughly order 1 for values of m out to order several thousand, see Figure 2.

The important conclusion that we draw is that for large m there are on the order of $\frac{m}{8}$ linear constraints on the polar coefficients of the elliptic genus expressing modularity.

Remarks.

1. The action of charge conjugation together with spectral flow defines an action of D_∞ on the (ℓ, n) plane which preserves the space $\mathcal{Q}$ of polar values $-m^2 \leq 4mn - \ell^2 < 0$. A fundamental domain is given by the polar region $\mathcal{P}^{(m)}$, but the quotient $\mathcal{Q}/D_\infty$ has fixed points: for $\ell = -m$ the spectral flow to $\ell = +m$ can be undone by charge conjugation. Therefore, if we compute the *orbifold* Euler character of $\mathcal{Q}/D_\infty$, the

[3]This is a subtle issue which, while fascinating, we believe is a distraction from our main exposition. A theorem of Siegel states that $\lim_{d\to\infty} \frac{\log h'(-d)}{\log d} = \frac{1}{2}$ as d runs through discriminants of quadratic imaginary fields, but $h'(-d)$ itself does not have a simple asymptotic expansion. This follows from its relation to the Dirichlet series $L_d(s)$ at $s = 1$. For a discussion of these and related matters, together with their possible applications to black holes and with references to the math literature, see [35]. For a rigorous discussion of the probability distribution of $h'(-d)$, see [5, 22].

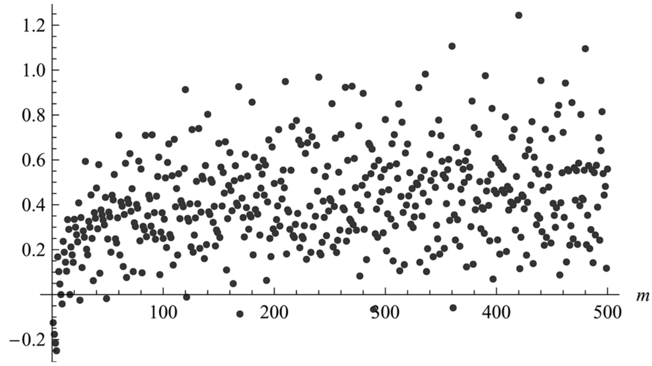

Figure 2: A plot of the first few hundred values of $(P(m) - j(m) - m/8)m^{-1/2}$ shows the quantity remains on the order of 1. The points do not tend to a limiting value — as would be the case for an asymptotic expansion, but are distributed about a mean value and exhibit a considerable amount of scatter. Detailed results on the distribution are available in the math literature, but we will not need these.

line of states (ℓ, h) with $\ell = m$ should be counted with weight $\frac{1}{2}$. There are precisely $m/4$ states on this line and hence $\chi_{\rm orb}(\mathcal{Q}/D_\infty) = P(m) - m/8$, which is a much closer approximation to $j(m)$.

2. Recently, Manschot [34] has reproduced the formula for $P(m) - j(m)$ by directly computing the dimension of the image of the period map Per in (2.14).

3. Extremal $\mathcal{N} = (2, 2)$ conformal field theories

3.1. Definition

In [47], Witten suggested that the holographic dual of pure 2+1 dimensional quantum gravity should be an "extremal conformal field theory." The latter is defined to be a conformal field theory whose modular invariant partition function is "as close as possible" to the Virasoro character of the vacuum. When $c = 24k$ the vacuum character is

$$\chi^{(k)}_{\rm Vac}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n}. \qquad (3.1)$$

The partition function $Z_k(\tau)$ has weight zero. Unlike the elliptic genus case, there is no obstruction to completing an arbitrary polynomial in q^{-1} to a modular function by adding nonpolar terms. Therefore, Witten defines $Z_k(\tau)$ to be the unique modular function with no singularities for $\tau \in \mathbb{H}$ such that the expansion around the cusp at infinity satisfies

$$Z_k(\tau) := \left[q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q^{\leq 0}} + \mathcal{O}(q). \tag{3.2}$$

Following [15], Witten interprets the first Virasoro primary above the vacuum representation to be a state corresponding to the lightest possible BTZ black hole in AdS_3.

Following Witten [47], let us consider "pure $\mathcal{N} = (2, 2)$ supergravity" with negative cosmological constant. This is the hypothetical quantum theory whose classical action is a supersymmetric completion of the Einstein–Hilbert action,

$$I_{\text{sugra}} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(\mathcal{R}(g) + \frac{2}{R^2} + \cdots \right). \tag{3.3}$$

Here, R is the AdS length scale and the ellipses denote contributions of other fields in the $\mathcal{N} = 2$ supergravity multiplet. Specifically, apart from the metric, these fields include real spin-$\frac{3}{2}$ gravitino fields, ψ^i_{L} and ψ^i_{R}, $i = 1, 2$, as well as two abelian gauge fields, a_{L} and a_{R}. In general, if we were interested in $\mathcal{N} = (p, q)$ supergravity theory, the corresponding gauge group would be $\mathrm{SO}(p) \times \mathrm{SO}(q)$. Thus, in the present context of $\mathcal{N} = (2, 2)$ theory, we have $\mathrm{SO}(2) \times \mathrm{SO}(2)$ gauge fields.

In fact, by enlarging the gauge group one can write the entire supergravity action (3.3) as the Chern–Simons action [1, 2]:

$$\begin{aligned} I_{\mathrm{CS}} = \frac{k_{\mathrm{L}}}{4\pi} \int \mathrm{tr} \left(A_{\mathrm{L}} \wedge dA_{\mathrm{L}} + \frac{2}{3} A_{\mathrm{L}} \wedge A_{\mathrm{L}} \wedge A_{\mathrm{L}} \right) \\ - \frac{k_{\mathrm{R}}}{4\pi} \int \mathrm{tr} \left(A_{\mathrm{R}} \wedge dA_{\mathrm{R}} + \frac{2}{3} A_{\mathrm{R}} \wedge A_{\mathrm{R}} \wedge A_{\mathrm{R}} \right), \end{aligned} \tag{3.4}$$

where the gauge fields A_{L} and A_{R} take values in the Lie algebra of the supergroup

$$G = G_{\mathrm{L}} \times G_{\mathrm{R}} = \mathrm{OSp}(2|2)_{\mathrm{L}} \times \mathrm{OSp}(2|2)_{\mathrm{R}}. \tag{3.5}$$

Since the bosonic part of the supergroup $\mathrm{OSp}(2|2)$ is $\mathrm{SO}(2) \times \mathrm{SL}(2,\mathbb{R})$, the gauge group (3.5) contains the classical symmetry[4] group, $\mathrm{SL}(2,\mathbb{R})_{\mathrm{L}} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{R}}$, of the three-dimensional AdS space. In the simple case $k_{\mathrm{L}} = k_{\mathrm{R}}$, which will be of interest to us in the present paper, one finds the following relation between the parameters:

$$k_{\mathrm{L}} = k_{\mathrm{R}} = \frac{R}{16G}. \tag{3.6}$$

Combining this with the Brown–Henneaux formula $c_{\mathrm{L}} = c_{\mathrm{R}} = \frac{3R}{2G}$ and using our expression for the central charge $c_{\mathrm{L}} = c_{\mathrm{R}} = 6m$, we can conveniently write (3.6) as

$$k_{\mathrm{L}} = k_{\mathrm{R}} = \frac{m}{4}. \tag{3.7}$$

Since we take m to be integer, it follows that k_{L} and k_{R} take values in $\frac{1}{4}\mathbb{Z}$. This is consistent with the fact that the bosonic part of our supergroup $\mathrm{OSp}(2|2)$ contains $\mathrm{SL}(2,\mathbb{R})$, which is a double cover of the identity component of $\mathrm{SO}(2,1)$; see Section 2.1 of [47] for further details on the allowed values of k_{L} and k_{R}.

The equivalence of $\mathcal{N} = (2,2)$ supergravity and Chern–Simons theory based on the supergroup (3.5) is valid not only classically, but to all orders in perturbation theory, as long as the perturbative expansion starts with a nondegenerate classical solution. This way of formulating perturbative $\mathcal{N} = (2,2)$ supergravity will be useful to us in what follows, in particular, in Section 8 where we discuss quantum corrections.

The $\mathcal{N} = (2,2)$ case is similar to the $\mathcal{N} = 0$ case of Chern–Simons gravity. There are no local degrees of freedom, but the Chern–Simons theory does give rise to "edge states." These are $\mathcal{N} = 2$ descendants of the vacuum representation, that is, the irreducible highest weight representation defined by $(h = 0, q = 0)$.

The natural generalization of Witten's proposal to $(2,2)$ supergravity in $2+1$ dimensions is that the holographic dual should be an "extremal $(2,2)$ superconformal field theory," where we define the latter to be a theory whose partition function is "as close as possible" to the vacuum character of the

[4] This symmetry group is the gauge group of the analogous formulation of $\mathcal{N} = 0$ gravity theory.

$\mathcal{N}=2$ algebra. The vacuum character of the $\mathcal{N}=2$ algebra is [8]

$$\begin{aligned}\chi^{(m)}_{\text{vac}}(\tau,z) &:= \text{Tr}_{V_{0,0}} q^{L_0-c/24} e^{2\pi i(z+1/2)J_0} \\ &= q^{-m/4}(1-q)\prod_{n=1}^{\infty}\frac{(1-yq^{n+1/2})(1-y^{-1}q^{n+1/2})}{(1-q^n)^2}.\end{aligned} \tag{3.8}$$

We have shifted z by 1/2 relative to the standard definition for later convenience. The expression in (3.8) is neither spectral-flow invariant, nor modular invariant, and hence more terms must certainly be added to get a physical partition function.

In [10] the near horizon geometry of the D1D5 system was investigated and it was observed that the cosmic censorship bound for the BTZ black hole, which requires $r_\pm \geq 0$ for the two roots of the lapse function, can be translated into the holographic conformal field theory as the bounds

$$4m\left(L_0-\frac{c}{24}\right)-J_0^2\geq 0 \quad \text{and} \quad 4m\left(\tilde{L}_0-\frac{c}{24}\right)-\tilde{J}_0^2\geq 0. \tag{3.9}$$

In [15], the connection of these inequalities to the conditions on polarity of terms in the partition function was pointed out. We will assume here that for general $\mathcal{N}=(2,2)$ supergravity the cosmic censorship bound continues to be (3.9). That is, black hole states must have $p,\tilde{p}\geq 0$, where p and $\tilde{p}$ refer to the polarity of the left- and right-moving states (i.e., $p=4mn-\ell^2$). In a theory of "pure supergravity," we would certainly want to require that all states with $p<0$ and $\tilde{p}<0$ are $\mathcal{N}=2$ descendents of the vacuum (or their spectral-flow images). These considerations, then, motivate our definition of an $\mathcal{N}=(2,2)$ extremal conformal field theory to be:

Definition. An $\mathcal{N}=(2,2)$ *extremal conformal field theory of level* m ("$\mathcal{N}=2$ ECFT" for short) is a hypothetical theory whose partition function is of the form:

$$\begin{aligned}Z_{\text{NSNS}}(\tau,z;\bar{\tau},\bar{z}) &:= \text{Tr}_{\mathcal{H}_{\text{NSNS}}} q^{L_0-c/24}e^{2\pi i z J_0}\bar{q}^{\tilde{L}_0-c/24}e^{2\pi i\bar{z}\tilde{J}_0}e^{i\pi(J_0-\tilde{J}_0)} \\ &= \sum_{s,\bar{s}\in\mathbb{Z}} \text{SF}_s\chi^{(m)}_{\text{vac}}(\tau,z)\text{SF}_{\bar{s}}\bar{\chi}^{(m)}_{\text{vac}}(\bar{\tau},\bar{z}) \\ &\quad+\sum_{s\in\mathbb{Z}}\text{SF}_s\chi^{(m)}_{\text{vac}}(\tau,z)\bar{f}(\bar{\tau},\bar{z})+\sum_{\bar{s}\in\mathbb{Z}} f(\tau,z)\text{SF}_{\bar{s}}\bar{\chi}^{(m)}_{\text{vac}}(\bar{\tau},\bar{z}) \\ &\quad+\sum_{p,\tilde{p}\geq 0} a(n,\ell;\tilde{n},\tilde{\ell})q^n y^\ell\bar{q}^{\tilde{n}}\bar{y}^{\tilde{\ell}}.\end{aligned} \tag{3.10}$$

Here the coefficients $a(n, \ell; \tilde{n}, \tilde{\ell})$ are integers, and the sum over nonpolar states in the last line means that *both* the left and right polarity of the state is nonnegative. The functions $f(\tau, z)$ and $\bar{f}(\bar{\tau}, \bar{z})$ describe the contribution of terms with nonnegative polarity with respect to the left and right polarity, respectively. We need to include such terms since states with either $p \geq 0$ or $\tilde{p} \geq 0$ are not polar and are allowed by the extremality condition.

Using spectral flow (2.2), we can compute $Z_{\mathrm{RR}}(\tau, z; \bar{\tau}, \bar{z})$ for an $\mathcal{N} = 2$ ECFT from (3.10). The elliptic genus is then obtained upon setting $\bar{z} = 0$. In this limit, only those terms that have $\bar{q}^0$ contribute. All of these terms have negative polarity, with the exception of the $\bar{q}^0 \bar{y}^0$ term that has polarity zero. Thus the elliptic genus of an $\mathcal{N} = 2$ ECFT of level m is of the form

$$(2(-1)^m + u) \sum_{\theta \in \mathbb{Z}+1/2} \mathrm{SF}_\theta \chi^{(m)}_{\mathrm{vac}} + \mathrm{Nonpolar}, \tag{3.11}$$

where u is the coefficient of the $\bar{q}^0 \bar{y}^0$ term coming from $\bar{f}(\bar{\tau}, \bar{z})$. The factor $2(-1)^m$ is the limit $\bar{z} \to 0$ of the first term in (3.10), as we will see momentarily. Using (5.21), below one can determine the constant to be $u = 12m - 2$. For convenience, we drop the overall constant factor from the right-movers and define

$$\chi^{(m)}_{\mathrm{ext}}(\tau, z) := \sum_{\theta \in \mathbb{Z}+1/2} \mathrm{SF}_\theta \chi^{(m)}_{\mathrm{vac}} + \mathrm{Nonpolar}. \tag{3.12}$$

We will call a weak Jacobi form that satisfies (3.12) an *extremal elliptic genus*. Because the only unknown terms in (3.12) are nonpolar terms, we can compute the polar polynomial of such an extremal elliptic genus. We will give an explicit formula for it in Section 3.2. Then, in Section 4, we investigate whether such a polar polynomial is consistent with modularity.

3.2. The extremal polar polynomial

Let us compute the polar polynomial of a would-be extremal elliptic genus. We begin by demonstrating the following useful fact:

$$\mathrm{Pol}\left(\sum_{\theta \in \mathbb{Z}+1/2} \mathrm{SF}_\theta \chi^{(m)}_{\mathrm{vac}} \right) = \mathrm{Pol}(\mathrm{SF}_{1/2} \chi^{(m)}_{\mathrm{vac}}) \ . \tag{3.13}$$

Indeed, if we apply the spectral flow by $\theta = l + \frac{1}{2}$ to the vacuum character (3.8), we obtain an expression of the form

$$(3.14)\qquad (-1)^m q^{l(l+1)m} y^{(2l+1)m} (1-q) \prod_{n=1}^{\infty} \frac{(1-yq^{n+l+1})(1-y^{-1}q^{n-l})}{(1-q^n)^2}.$$

We wish to show that this expression contains no polar terms in the fundamental domain (2.11) for $l \neq 0$. Without loss of generality, we can assume $l > 0$. Note that it is not true that (3.14) has no polar terms. In fact, already the first term $q^{l(l+1)m} y^{(2l+1)m}$ is polar for every l; it has polarity $p = -m^2$. However, it does not belong to the polar region $\mathcal{P}^{(m)}$ since the power of y is not in the allowed range $1 \leq \ell \leq m$.

On the other hand, there are terms in (3.14) with $1 \leq \ell \leq m$ but, as we show momentarily, these terms are not polar. We can simplify the problem a little bit and omit the denominator in (3.14) and the factor $(1-q)$ which can only increase the polarity. Then, our goal is to show that

$$(3.15)\qquad q^{l(l+1)m} y^{(2l+1)m} \prod_{n=1}^{\infty} (1-yq^{n+l+1})(1-y^{-1}q^{n-l})$$

has no polar terms in the range $1 \leq \ell \leq m$. From the above discussion, we already know that the term $q^{l(l+1)m} y^{(2l+1)m}$ is polar. We can combine it with the terms from factors $(1-yq^{n+l+1})$ and $(1-y^{-1}q^{n-l})$ for various n to bring the power of y to the desired range. Since l is assumed to be positive, it is easy to see that the terms coming from factors $(1-yq^{n+l+1})$ can be ignored, while from $\prod_{n=1}^{\infty}(1-y^{-1}q^{n-l})$ we need to collect at least $2lm$ factors of y^{-1} to bring the overall power of y to the desired range. The most economical way to do this (which yields the minimal increase in polarity) is to collect the factors in the infinite product with the smallest powers of q. These are the terms with $n = 1, \ldots, 2lm$:

$$(3.16)\qquad q^{l(l+1)m} y^{(2l+1)m} \prod_{n=1}^{2lm} y^{-1} q^{n-l} = q^{(2lm-l+2)lm} y^m.$$

The resulting term has polarity $p = 4(2lm - l + 2)lm^2 - m^2$ which satisfies $p > 0$ for any $l, m \geq 1$. It is easy to see that including other factors from the infinite product in (3.15) only increases the polarity further.

Having proven (3.13), we now define

$$(3.17)\qquad p_{\text{ext}}^{(m)} := (-1)^m \,\text{PolSF}_{1/2} \chi_{\text{vac}}^{(m)}.$$

On the other hand, setting $l = 0$ in (3.14), one finds

$$(-1)^m \mathrm{SF}_{1/2}\chi^{(m)}_{\mathrm{vac}} = (1-q)y^m \prod_{n=1}^{\infty} \frac{(1-yq^{n+1})(1-y^{-1}q^n)}{(1-q^n)^2}. \tag{3.18}$$

The Fourier expansion of (3.18) begins:

$$y^m + q(y^m - y^{m-1}) + q^2(-2y^{-1+m} + 3y^m - y^{1+m}) + \cdots. \tag{3.19}$$

The first few polar polynomials follow easily from (3.19) since the polar terms for index m have $n \le \lfloor \frac{m}{4} \rfloor$. In this way, we find that the first few polar polynomials are:

$$p^1_{\mathrm{ext}} = y, \tag{3.20}$$
$$p^2_{\mathrm{ext}} = y^2, \tag{3.21}$$
$$p^3_{\mathrm{ext}} = y^3, \tag{3.22}$$
$$p^4_{\mathrm{ext}} = y^4, \tag{3.23}$$
$$p^5_{\mathrm{ext}} = (1+q)y^5, \tag{3.24}$$
$$p^6_{\mathrm{ext}} = (1+q)y^6 - qy^5, \tag{3.25}$$
$$p^7_{\mathrm{ext}} = (1+q)y^7 - qy^6, \tag{3.26}$$
$$p^8_{\mathrm{ext}} = (1+q)y^8 - qy^7, \tag{3.27}$$
$$p^9_{\mathrm{ext}} = (1+q+3q^2)y^9 - qy^8, \tag{3.28}$$
$$p^{10}_{\mathrm{ext}} = (1+q+3q^2)y^{10} - (q+2q^2)y^9, \tag{3.29}$$
$$p^{11}_{\mathrm{ext}} = (1+q+3q^2)y^{11} - (q+2q^2)y^{10}, \tag{3.30}$$
$$p^{12}_{\mathrm{ext}} = (1+q+3q^2)y^{12} - (q+2q^2)y^{11}. \tag{3.31}$$

4. Experimental search for the extremal elliptic genus

Since $P(m) > j(m)$ for $m \ge 5$, and since Equation (3.18) does not have any obvious modular properties, it is far from obvious that (3.13) is the polar polynomial of a true weak Jacobi form. In this section, we describe numerical results suggesting that in fact, for all but finitely many m, it is not in the image of Pol applied to $\tilde{J}_{0,m}$. We will find that there are actually some "exceptional" cases where it is in the image for $m \ge 5$. In Section 5, we will show analytically that there can only be a finite number of such exceptional cases. That might seem to obviate the need for the present section, but the methods we employ here will prove very useful when we come to describe nearly extremal theories in Section 6.

Choose a basis ϕ_i, $i = 1, \ldots, j(m)$, for $\tilde{J}_{0,m}$. We are searching for real numbers x_i such that

$$\sum_{i=1}^{j(m)} x_i \mathrm{Pol}(\phi_i) = p_{\mathrm{ext}}^{(m)}. \tag{4.1}$$

A useful way of trying to solve this equation is the following. We choose a polarity-ordered basis of monomials $q^n y^\ell$ for V_m, that is the basis monomials $q^{n(a)} y^{l(a)}$, where $a = 1, \ldots, \dim V_m = P(m)$ so that polarity increases as a increases, and terms with the same polarity are ordered in increasing powers of y. For example, for $a = 1$, the most polar term is y^m. A polarity-ordered basis for V_5 would be

$$y^5, y^4, y^3, qy^5, y^2, y^1 \tag{4.2}$$

with $a = 1, \ldots, 6$. The polarity-ordered basis will be very useful for our discussion of β-extremal $\mathcal{N} = 2$ conformal field theories in Section 6.

Having chosen these two bases, we can define a matrix N_{ia} of dimensions $j(m) \times P(m)$ from the expansion

$$\mathrm{Pol}(\phi_i) = \sum_{a=1}^{P(m)} N_{ia} q^{n(a)} y^{\ell(a)}. \tag{4.3}$$

Similarly, we can define coefficients d_a by

$$p_{\mathrm{ext}}^{(m)} = \sum_{a=1}^{P(m)} d_a q^{n(a)} y^{\ell(a)}. \tag{4.4}$$

Thus, we are trying to solve the linear equations

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \quad a = 1, \ldots, P(m). \tag{4.5}$$

It should be stressed that even if we can find a solution x_i to (4.5), we are far from establishing the existence of an $\mathcal{N} = 2$ extremal theory. If a solution exists, then the next test we should apply is to see whether the resulting form $\sum x_i \phi_i$ has *integral* Fourier coefficients. Integrality is clearly a necessary condition for any candidate elliptic genus since it arises in conformal field theory from the trace on a Hilbert space.

Using a computer (and the explicit basis (2.23) above), we have examined equation (4.5) for $1 \leq m \leq 36$. We have found that there is a solution x_i in rational numbers for $1 \leq m \leq 5$ and for $m = 7, 8, 11, 13$, but there is no solution for $m = 6, 9, 10$ and $14 \leq m \leq 36$.[5] Moreover, remarkably, for those values of m which give a solution, the Fourier coefficients we have explicitly evaluated turn out to be integral.

The simplest example is the case $m = 1$, in which case $\chi^{(1)}_{\rm ext} = \tilde{\phi}_{0,1}$. The next simplest case, $m = 2$, yields

$$\chi^{(2)}_{\rm ext} = \frac{1}{6}(\tilde{\phi}_{0,1})^2 + \frac{5}{6}(\tilde{\phi}_{-2,1})^2 E_4. \tag{4.6}$$

Although it is not obvious, one can prove that the Fourier coefficients are all integral. Indeed, the claim that this expression has integer Fourier coefficients is equivalent to the statement

$$(\tilde{\phi}_{0,1})^2 + 5(\tilde{\phi}_{-2,1})^2 E_4 = 0 \bmod 6. \tag{4.7}$$

In order to prove this, it is convenient to note (see (2.17) and (2.18)) that

$$E_4 = 1 \bmod 6, \quad E_6 = 1 \bmod 6.$$

Moreover, from (2.21), it also follows that $\phi_{10,1} = \phi_{12,1}$ mod 6, which in turn implies $\tilde{\phi}_{-2,1} = \tilde{\phi}_{0,1}$ mod 6, cf. (2.22). Substituting this into $(\tilde{\phi}_{0,1})^2 + 5(\tilde{\phi}_{-2,1})^2$ and using the fact that $\tilde{\phi}_{0,1}$ and $\tilde{\phi}_{-2,1}$ have integer Fourier coefficients, we therefore demonstrate (4.7).

When we use the basis (2.23), the solutions x_i are rational numbers with increasingly large denominators as m increases. For example, already the next case, $m = 3$, looks like

$$\chi^{(3)}_{\rm ext} = \frac{1}{48}(\tilde{\phi}_{0,1})^3 + \frac{7}{16}\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2 E_4 + \frac{13}{24}(\tilde{\phi}_{-2,1})^3 E_6. \tag{4.8}$$

Even though the coefficients x_i of every monomial $(\tilde{\phi}_{-2,1})^a(\tilde{\phi}_{0,1})^b E_4^c E_6^d$ are rational numbers, the Fourier coefficients $c(n, \ell)$ are integers. In order to show this, as in the previous example, we express this as the following

[5]The arguments of Section 5 demonstrate that there can only be finitely many solutions. Using the constraints of that section, it is easy to check that there are no further solutions up to $m \leq 400$. This suggests that the above list is in fact complete.

statement

$$(\tilde{\phi}_{0,1})^3 + 21\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2 E_4 + 26(\tilde{\phi}_{-2,1})^3 E_6 = 0 \bmod 48. \tag{4.9}$$

Then, using (2.17), we note that $E_4 = 1 \bmod 48$, so we can ignore E_4 in this computation. It is not true, however, that $E_6 = 1 \bmod 48$. Instead, from (2.18) we find that $E_6^2 = 1 \bmod 48$. According to (2.20) and (2.21), this implies the following identity:

$$\tilde{\phi}_{-2,1} = \tilde{\phi}_{0,1} E_6 \bmod 48,$$

which, after substituting in the LHS of (4.9), proves the desired result.

Using the basis of weak Jacobi forms described in Section 7 below, one can check that for the "miraculous" values $m = 5, 7, 8, 11, 13$ the solution does indeed have the property that all the Fourier coefficients $c(n, \ell)$ are integers.

5. The extremal elliptic genus does not exist for m sufficiently large

In this section, we give an analytic proof that there is no weak Jacobi form in $\tilde{J}_{0,m}$ satisfying (3.12) for m sufficiently large. Since this section is rather long and technical, let us summarize the main idea here. Using the spectral-flow symmetry, one can determine the NS-sector character (without an insertion of y^{J_0} or $(-1)^F$) from the elliptic genus. This character is a modular form for a congruence subgroup Γ_θ of the modular group. It is therefore highly constrained, and as in the case discussed in [47], determined by the coefficients of the negative powers of q, which in turn are fixed by the polar terms of the original elliptic genus. On the other hand, given the full NS-sector character, we can also determine from it, by a suitable modular transformation, the R-sector character (without an insertion of $(-1)^F$), and thus, in particular, its leading term in the q-expansion. This coefficient is however also directly determined by the extremal hypothesis and a sum rule (5.20) for Fourier coefficients. The two ways of evaluating the same coefficient lead to a nontrivial constraint on m, Equation (5.19). Using properties of modular forms, one can show that this constraint is violated for sufficiently large m. The argument must be broken up into cases: m odd, $m = 2 \bmod 4$ and $m = 0 \bmod 4$, of which the last case is technically the most difficult. In this section, we will give the main line of argument, whereas the technical details can be found in Appendix A.

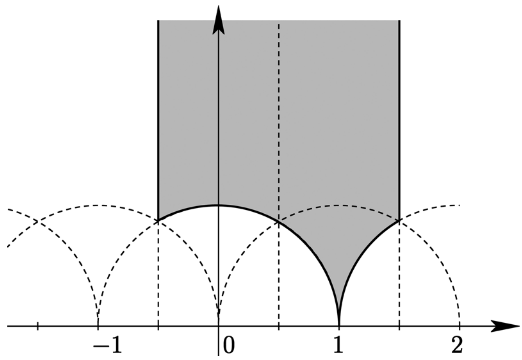

Figure 3: The fundamental domain $\mathcal{F}_\theta$ of the genus zero subgroup Γ_θ of Γ.

5.1. NS-sector elliptic genus

Suppose $\chi(\tau, z)$ is the elliptic genus of a CFT with $\chi \in \tilde{J}_{0,m}$. By spectral flow we define the "NS-sector elliptic genus" to be[6]

$$\chi_{\mathrm{NS}}(\tau, z) := e\left[m\left(\frac{\tau}{4} + z + \frac{1}{2}\right)\right] \chi\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right). \tag{5.1}$$

Using the transformation properties of a Jacobi form, it follows easily that

$$\begin{aligned} \chi_{\mathrm{NS}}(-1/\tau, z/\tau) &= (-1)^m e\left(\frac{mz^2}{\tau}\right) \chi_{\mathrm{NS}}(\tau, z), \\ \chi_{\mathrm{NS}}(\tau + 2, z) &= (-1)^m \chi_{\mathrm{NS}}(\tau, z). \end{aligned} \tag{5.2}$$

If we put $z = 0$, we thus obtain simple transformation laws for $\chi_{\mathrm{NS}}(\tau) := \chi_{\mathrm{NS}}(\tau, 0)$ under the congruence subgroup $\Gamma_\theta = \langle T^2, S\rangle$. (In this section, we consider the modular group to be $\mathrm{PSL}(2, \mathbb{Z})$.) For m even we have a strict modular function and for m odd we have a function with multiplier system given by -1 on the two generators.

To begin, let us sketch a few mathematical facts. The group Γ_θ is a genus zero subgroup of Γ. It has modular domain $\mathcal{F}_\theta = \mathcal{F} \cup T \cdot \mathcal{F} \cup TS \cdot \mathcal{F}$ shown in Figure 3. Note there are two cusps, equivalent to $\tau = i\infty$ and $\tau = 1$.

Since $\mathbb{H}/\Gamma_\theta$ is genus zero, the function field has a generator $\hat{K}(\tau)$ which can be uniquely specified (up to an additive and multiplicative constant) by

[6]Note that unlike the NS vacuum character (3.8), $\chi_{\mathrm{NS}}(\tau, z)$ does not involve the shift of z by 1/2.

demanding that $\hat{K}$ takes $i\infty$ to ∞.[7] An explicit choice is:

$$\hat{K}(\tau) := \frac{\vartheta_3^{12}}{\eta^{12}}(\tau) = \frac{\Delta^2(\tau)}{\Delta(2\tau)\Delta(\tau/2)} = q^{-1/2} + 24 + 276q^{1/2} + \cdots . \tag{5.3}$$

The expansion of $\hat{K}$ around the cusp at $\tau = 1$ is obtained by writing $\tau = 1 - \frac{1}{\tau_r}$ and observing that

$$\hat{K}(\tau) := \tilde{K}(\tau_r) = -\frac{\vartheta_2^{12}}{\eta^{12}}(\tau_r) = -2^{12}q_r + \cdots , \tag{5.4}$$

where $q_r = e(\tau_r)$.

In order to work with the case of m odd, it will be useful to consider the index two subgroup[8] $\tilde{\Gamma}_\theta := \langle T^4, ST^2\rangle$ such that $\Gamma_\theta = \tilde{\Gamma}_\theta \cup S \cdot \tilde{\Gamma}_\theta$. This is again a genus zero subgroup, and its Hauptmodul is the NS-sector character of $\tilde{\phi}_{0,1}$ (i.e., the elliptic genus for K3 divided by two). Using the definition (5.1) with $m = 1$ and $\chi = \tilde{\phi}_{0,1}$ and putting $z = 0$, one finds

$$\kappa(\tau) := \left(\frac{2\vartheta_4}{\vartheta_2}\right)^2 - \left(\frac{2\vartheta_2}{\vartheta_4}\right)^2 = q^{-1/4}(1 - 20q^{1/2} + \cdots). \tag{5.5}$$

This function satisfies $\kappa|_S = -\kappa$ and $\kappa|_{T^2} = -\kappa$, and is thus odd under the Deck transformation $\mathbb{H}/\tilde{\Gamma}_\theta \to \mathbb{H}/\Gamma_\theta$. Indeed,

$$\kappa^2(\tau) = \hat{K}(\tau) - 64, \tag{5.6}$$

giving the explicit double cover. Near the Ramond cusp, κ has the expansion

$$\kappa(1 - 1/\tau_r) := \tilde{\kappa}(\tau_r) = -4i\left[\left(\frac{\vartheta_3}{\vartheta_4}\right)^2 + \left(\frac{\vartheta_4}{\vartheta_3}\right)^2\right](\tau_r) = -8i(1 + 32q_r + \mathcal{O}(q_r^2)). \tag{5.7}$$

Now, χ_{NS} has no singularities for $\tau \in \mathbb{H}$, and, moreover, using again the transformation laws of a Jacobi form

$$\chi_{\mathrm{NS}}(1 - 1/\tau_r) = e\left(-\frac{m}{4}\right)\chi\left(\tau_r, \frac{1}{2}\right) = e\left(-\frac{m}{4}\right)\sum_{n,\ell} c(n,\ell)(-1)^\ell q_r^n. \tag{5.8}$$

[7] Such a function for a genus zero congruence subgroup is often referred to as a "Hauptmodul."

[8] To prove the subgroup is index 2 note that for all $n \in \mathbb{Z}$, T^{4n}, ST^{4n+2}, $T^{4n+2}S$ and $ST^{4n}S$ are in $\tilde{\Gamma}_\theta$. Then use induction on the length of the word in S, T^2. Recall that in this section modular transformations are regarded as elements of $\mathrm{PSL}(2,\mathbb{Z})$.

By unitarity, the sum is over $n \geq 0$ and hence $\chi_{\mathrm{NS}}(\tau)$ must be a polynomial in $\kappa(\tau)$. This polynomial will be even for m even and odd for m odd. Moreover, the polynomial is fixed by the coefficients of the nonpositive powers of q. Those coefficients in turn are related to the polar contributions to $\chi(\tau, z)$. To demonstrate the relationship, note that

$$\chi_{\mathrm{NS}}(\tau) = \sum_{n,\ell} c(n,\ell)\,(-1)^{m+\ell}\, q^{m/4} + n + \ell/2. \tag{5.9}$$

Now write

$$4mn - \ell^2 = 4m\left(\frac{m}{4} + n + \frac{\ell}{2}\right) - (m+\ell)^2. \tag{5.10}$$

The nonpolar terms in $\chi(\tau, z)$ have $4mn - \ell^2 \geq 0$ and therefore from (5.10) contribute only nonnegative powers of q in (5.9). In fact, they always contribute positive powers with precisely one exception: when $4mn - \ell^2 = 0$ and $\ell = -m$. In that case, $n = m/4$. Note that this cannot happen if $m \neq 0 \bmod 4$ because n is integral.

5.2. A nontrivial constraint

In this subsection, we assume $m \neq 0 \bmod 4$. We return to a discussion of the case $m = 0 \bmod 4$ in Section 5.3 below.

Our conclusion thus far is that for $m \neq 0 \bmod 4$, $\chi_{\mathrm{NS}}(\tau)$ is a modular function for $\tilde{\Gamma}_\theta$ such that

$$\chi_{\mathrm{NS}}(\tau) = \sum_{\theta \in \mathbb{Z}} \mathrm{SF}_\theta \chi^{(m)}_{\mathrm{vac}}(\tau, z)|_{z=1/2} + \mathcal{O}(q^{1/4}). \tag{5.11}$$

One easily finds that only $\theta = 0$ can contribute to negative powers of q and hence we can simplify this equation to

$$\chi_{\mathrm{NS}}(\tau) = q^{-m/4}(1-q)\prod_{n=1}^{\infty} \frac{(1+q^{n+1/2})^2}{(1-q^n)^2} + \mathcal{O}(q^{1/4}). \tag{5.12}$$

This has expansion

$$q^{-m/4}(1 + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 6q^3 + \cdots). \tag{5.13}$$

While the expression on the RHS of (5.12) is not modular, it can be written as

$$\chi_{\mathrm{NS}}(\tau) = q^{-m/4}\frac{1-q^{1/2}}{1+q^{1/2}}q^{1/8}\frac{\vartheta_3}{\eta^3} + \mathcal{O}(q^{1/4}). \tag{5.14}$$

Now we can write an explicit formula for $\chi_{\mathrm{NS}}(\tau)$. Define expansion coefficients:

$$q^{-m/4}q^{1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{\vartheta_3}{\eta^3} = \sum_{\alpha=-m/4}^{\infty}\tilde{h}(\alpha)q^{\alpha}. \tag{5.15}$$

Note that $\tilde{h}(\alpha)$ is only nonzero for $\alpha \in \frac{1}{2}\mathbb{Z}$, for m even and $\frac{1}{4}+\frac{1}{2}\mathbb{Z}$ for m odd. For $\alpha \in \frac{1}{4}\mathbb{Z}_+$, let $\wp_\alpha$ be the unique polynomial of degree 4α such that

$$\wp_\alpha(\kappa) = q^{-\alpha} + \mathcal{O}(q^{1/4}), \quad \alpha \in \frac{1}{4}\mathbb{Z}_+. \tag{5.16}$$

Then for $m \neq 0 \bmod 4$,

$$\chi_{\mathrm{NS}} = \sum_{\alpha=-m/4}^{0}\tilde{h}(\alpha)\wp_{-\alpha}(\kappa). \tag{5.17}$$

On the other hand, if we expand around the cusp $\tau = 1$, then, by (5.8),

$$\sum_{\alpha=-m/4}^{0}\tilde{h}(\alpha)\wp_{-\alpha}(\tilde{\kappa}(\tau_r)) = e^{-i\pi m/2}\sum_{n,\ell\in\mathbb{Z}}c(n,\ell)(-1)^{\ell}\,q_r^{n}. \tag{5.18}$$

In particular, if we take $\tau_r \to i\infty$, then we arrive at the key constraint

$$L := \sum_{\alpha=-m/4}^{0}\tilde{h}(\alpha)\wp_{-\alpha}(-8i) = e^{-i\pi m/2}\sum_{\ell}c(0,\ell)(-1)^{\ell}. \tag{5.19}$$

The argument for the nonexistence of the extremal elliptic genus is based on showing that, for large m, the left-hand side and right-hand side of (5.19) have different growth rates. As we shall see momentarily, the right-hand side is always an affine linear function of m, while the left-hand side grows exponentially for $m = 2 \bmod 4$; for m odd, the left-hand side grows also linearly in m, but the coefficient is different.

Let us first establish the growth property of the right-hand side. By the ansatz for pure supergravity, we know that the only nonzero polar coefficients $c(0,\ell)$ occur for $\ell = \pm m$ and are given by 1. The coefficient $c(0,0)$ is *not* polar. Fortunately, Gritsenko [25] has proven a useful identity for the Fourier coefficients of weak Jacobi forms of index m:[9]

$$m \sum_{\ell} c(0,\ell) = 6 \sum_{\ell} \ell^2 c(0,\ell). \tag{5.20}$$

Using (5.20) and (3.18), we can solve for $c(0,0)$ to get $c(0,0) = 12m - 2$, and therefore

$$\sum_{\ell} c(0,\ell)(-1)^{\ell} = 12m - 2 + 2(-1)^m = \begin{cases} 12m, & m \text{ even}, \\ 12m-4, & m \text{ odd}. \end{cases} \tag{5.21}$$

In particular, the right-hand side of (5.19) grows linearly with m.

Now let us turn to the left-hand side of (5.19). Observe that this is the q^0 term in the q-expansion of

$$\left(\sum_{\alpha \geq -m/4} \tilde{h}(\alpha) q^{\alpha} \right) \left(\sum_{n \geq 0} q^{n/4} \wp_{n/4}(-8i) \right). \tag{5.22}$$

On the other hand, using the fact that κ is a Hauptmodul, one can show that[10]

$$\sum_{n=0}^{\infty} q^{n/4} \wp_{n/4}(z) = \frac{4q \frac{d}{dq} \kappa}{z - \kappa}. \tag{5.23}$$

[9]The proof is very simple: $\exp[-8\pi^2 m G_2(\tau) z^2] \chi(\tau, z)$ transforms as a weight zero modular form. Therefore, the coefficients of z^{2n} in the Taylor series around $z = 0$ transform like forms of weight $2n$. In particular the coefficient of z^2 must vanish, since there are no modular forms of weight 2.

[10]Write $\wp_\alpha(z) = \oint_C \frac{\wp_\alpha(\ell)}{\ell - z} \frac{d\ell}{2\pi i}$, where the contour is on a large circle C in the ℓ plane. Now make the change of variables $\ell \to \ell(r) := r^{-1} - 20r + \cdots$ so that $\ell(q^{1/4}) = \kappa$. This gives a one-one map of C to a small circle C' around the origin. Using $\wp_\alpha(\ell(r)) = r^{-4\alpha} + \mathcal{O}(r)$, and taking the circle to be small, we see $\wp_\alpha(z) = -\oint_{C'} \frac{\ell'(r) r^{-4\alpha}}{\ell(r) - z} \frac{dr}{2\pi i}$. It is now straightforward to sum the series and apply Cauchy's theorem to arrive at (5.23). We thank Terry Gannon for pointing out this crucial identity to us.

In order to apply this to our problem, we use the identities[11]

$$
\begin{aligned}
24q\frac{d}{dq}\log\vartheta_4 &= E_2 - (\vartheta_2^4 + \vartheta_3^4),\\
24q\frac{d}{dq}\log\vartheta_3 &= E_2 + (\vartheta_2^4 - \vartheta_4^4),\\
24q\frac{d}{dq}\log\vartheta_2 &= E_2 + (\vartheta_3^4 + \vartheta_4^4),
\end{aligned} \tag{5.24}
$$

to compute $4q\frac{d}{dq}\kappa = -4\vartheta_3^8/(\vartheta_2^2\vartheta_4^2)$. Using the "abstruse identity" $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$, it follows that

$$\sum_{n=0}^{\infty} q^{n/4}\wp_{n/4}(-8i) = (\vartheta_4^2 - i\vartheta_2^2)^2. \tag{5.25}$$

Thus, we need to estimate the large m behavior of

$$L := \left[q^{-m/4+1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{\vartheta_3}{\eta^3}(\vartheta_4^2 - i\vartheta_2^2)^2\right]_{q^0}. \tag{5.26}$$

We estimate the growth behavior of L in Appendix A, and it turns out to be quite different for even and odd.

For m odd, $e^{i\pi m/2}L$ is positive and is bounded below by

$$e^{i\pi m/2}L \geq 4\pi m - 8\pi\sqrt{m - \frac{5}{2}} - 6\pi. \tag{5.27}$$

Since $4\pi > 12$, this will asymptotically (i.e., for $m \geq 2000$) grow more quickly than (5.21). We have checked that among the first 2000 terms,

---

[11]To prove these identities note that $(24q\frac{d}{dq} - E_2)\vartheta_2$ must be a weight 5/2 modular form for $\Gamma(2)$ and hence is a polynomial of degree 5 in $\vartheta_2, \vartheta_3, \vartheta_4$. Moreover, the q expansion has only coefficients $q^{1/8+n}$ with n integer. Together with the transformation property under $\tau \to \tau + 1$, this fixes it to be of the form $\vartheta_2(a(\vartheta_3^4 + \vartheta_4^4) + b\vartheta_3\vartheta_4(\vartheta_3^2 + \vartheta_4^2) + c\vartheta_3^2\vartheta_4^2)$ for some constants a, b, c. Now, matching the first three coefficients of the q expansion on the left- and right-hand sides, we find $a = 1, b = c = 0$. The other two equations now follow by modular transformations. These identities also have nice interpretations in terms of massless free fermions on a two-dimensional torus. One can compute the expectation value of their energy either by differentiating their partition function or by evaluating the energy–momentum tensor using the fermion two-point function. Requiring that these two methods produce the same answer implies these identities [17].

the two numbers only agree for $m = 1, 3, 5, 7, 11, 13, 19, 31, 41$. For $m = 1, 3, 5, 7, 11, 13$, there exists indeed a sugra elliptic genus, while for $m = 19, 31, 41$ there does not, as we have verified explicitly. (Note that the fact that the two numbers agree does not imply that there must exist a sugra elliptic genus.)

For $m = 2 \bmod 4$, L turns out to grow exponentially, so that (5.19) cannot be satisfied for m large enough. For details of the calculation, see again Appendix A.

5.3. A constraint for $m = 0 \bmod 4$

We now turn to the case $m = 0 \bmod 4$. As we have pointed out above, in this case nonpolar terms contribute to the constant term of χ_{NS}. We thus need to make the more general ansatz

(5.28)
$$\chi_{\mathrm{NS}}(\tau, z) = q^{-m/4+1/8} \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})} \frac{\vartheta_3(\tau, z)}{\eta^3} + d + \mathcal{O}(q^{1/2}).$$

Instead of (5.17), we obtain

$$\chi_{\mathrm{NS}} = \sum_{\alpha=-m/4}^{0} \tilde{h}(\alpha) \wp_{-\alpha}(\kappa) + d. \tag{5.29}$$

The argument of Section 5.2 can then be used to fix the value of d:

$$d = 12m - \left[q^{-m/4+1/8} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3(\tau)}{\eta^3} (\vartheta_4^4 - \vartheta_2^4) \right]_{q^0}. \tag{5.30}$$

We obtain an additional constraint on the theory in the following way. Let

$$\hat{D} := \left(y \frac{d}{dy} \right)^2 - \frac{m}{6} E_2. \tag{5.31}$$

Then $\hat{\chi}_{\mathrm{NS}}(\tau) := \hat{D}(\chi_{\mathrm{NS}}(\tau, z))|_{z=0}$ is a weight 2 weakly holomorphic modular form for Γ_θ which moreover satisfies

$$\hat{\chi}_{\mathrm{NS}}(1 - 1/\tau_r) = \tau_r^2 \hat{D}(\chi(\tau, z))|_{z=1/2}. \tag{5.32}$$

The $q_r \to 0$ limit of the coefficient of τ_r^2 of the right-hand side of (5.32) is

$$\sum_\ell c(0,\ell)(-1)^\ell \ell^2 - \frac{m}{6} \sum c(0,\ell)(-1)^\ell = 2m^2 - \frac{m}{6} 12m = 0. \tag{5.33}$$

On the other hand, weakly holomorphic modular forms of weight 2 for Γ_θ are of the form

$$(\vartheta_2^4 - \vartheta_4^4) \times L(\hat{K}), \tag{5.34}$$

where $L(\hat{K})$ is a Laurent series in $\hat{K}$. By examining the Ramond cusp, we see that $L(\hat{K})$ must be a polynomial in $\hat{K}$. Define polynomials $P_a(\hat{K}) = q^{-a/2} + \mathcal{O}(q^{1/2})$ for $a \geq 0$ and

$$\tilde{P}_a(\hat{K})(\vartheta_2^4 - \vartheta_4^4) = \begin{cases} 1 + \mathcal{O}(q^{1/2}), & a = 0, \\ aq^{-a/2} + \mathcal{O}(q^{1/2}), & a > 0. \end{cases} \tag{5.35}$$

Using (5.24), we find

$$2q\frac{d}{dq}\hat{K} = \hat{K}(\vartheta_2^4 - \vartheta_4^4), \tag{5.36}$$

from which we deduce

$$\tilde{P}_a(z) = \begin{cases} -1, & a = 0, \\ -zP_a'(z), & a > 0. \end{cases} \tag{5.37}$$

Define expansion coefficients

$$\hat{\chi}_{\mathrm{NS}}(\tau) = \sum_{\alpha=-m/4} (-2\alpha)x(\alpha)q^\alpha + X(0). \tag{5.38}$$

If the extremal elliptic genus exists, then

$$\hat{\chi}_{\mathrm{NS}}(\tau) = \sum_{\alpha<0} x(\alpha)\tilde{P}_{-\alpha}(\hat{K})(\vartheta_2^4 - \vartheta_4^4) - X(0)(\vartheta_2^4 - \vartheta_4^4). \tag{5.39}$$

Evaluating at the Ramond cusp, we have

$$\tau_r^2 \left(X(0)(\vartheta_4^4 + \vartheta_3^4) - \sum_{\alpha<0} x(\alpha)\tilde{P}_{-\alpha}(\tilde{K})(\vartheta_4^4 + \vartheta_3^4) \right), \tag{5.40}$$

and evaluating at $q_r \to 0$ the coefficient of τ_r^2 becomes simply $2X(0)$ since $\tilde{P}_\alpha(0) = 0$ for $\alpha > 0$. Therefore, $X(0) = 0$.

On the other hand, we can deduce the coefficient $X(0)$ directly from the q^0 term of $\hat{D}\chi_{\rm NS}$. Expressing $\chi_{\rm NS}$ by (5.28) and (5.30) and then using

$$(y\partial_y)^2 \frac{1}{(1+yq^{1/2})(1+y^{-1}q^{1/2})}\bigg|_{y=1} = -\frac{2q^{1/2}}{(1+q^{1/2})^4}, \tag{5.41}$$

$$y\partial_y \vartheta_3|_{y=1} = 0, \tag{5.42}$$

$$(y\partial_y)^2\vartheta_3|_{y=1} = 2q\partial_q\vartheta_3, \tag{5.43}$$

and (5.24), we obtain the constraint

$$\begin{aligned} 0 = \left[\hat{D}\chi_{\rm NS}\right]_{q^0} &= \left[(y\partial_y)^2\chi_{\rm NS} - \frac{m}{6}E_2\,\chi_{\rm NS}\right]_{q^0} \\ &= -2m^2 + \left[q^{-m/4+1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{-2q^{1/2}}{(1+q^{1/2})^2}\frac{\vartheta_3}{\eta^3}\right]_{q^0} \\ &\quad - (4m-2)\left[q^{-m/4+1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{q\partial_q\vartheta_3}{\eta^3}\right]_{q^0} \\ &= -2m^2 - R_1 - (4m-2)\,R_2, \end{aligned} \tag{5.44}$$

where R_1 and R_2 are defined as

$$R_1 = \left[2\,q^{1/2}\,\frac{(1-q^{1/2})^4}{(1-q)^3}\frac{\vartheta_3}{\eta^3}\right]_{q^{m/4-1/8}} \tag{5.45}$$

$$R_2 = \left[\frac{(1-q^{1/2})^2}{1-q}\frac{q\partial_q\vartheta_3}{\eta^3}\right]_{q^{m/4-1/8}}. \tag{5.46}$$

In Appendix A, we show that for large enough m both R_1 and R_2 are positive. It is then clear that (5.44) cannot be satisfied.

5.4. What are the exceptional values of m?

The results of the previous subsections establish rigorously that there are at most a finite number of values of m for which a candidate extremal elliptic genus can exist. The results of Section 4 suggest that there are in fact precisely nine such values namely $1 \leq m \leq 5$, and $m = 7, 8, 11, 13$. Although we do not have a rigorous proof, we strongly believe this list to be complete.

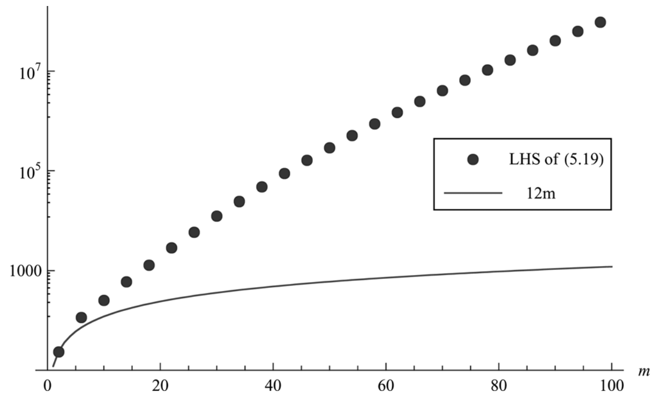

Figure 4: A semi-log plot of the left-hand side and right-hand side of the constraint (5.19) for $m = 2 \bmod 4$. The left-side grows exponentially with m, as shown on the log plot.

As we have mentioned, for m odd, we have studied the first 2000 terms and the only possibilities are the values mentioned above. For $m \sim 2000$, we are well within the regime for which our asymptotic bounds are valid. For m even, we have also examined the constraints numerically and it appears that $m \geq 36$ is well within the range of validity of our bounds, see Figures 4 and 5.

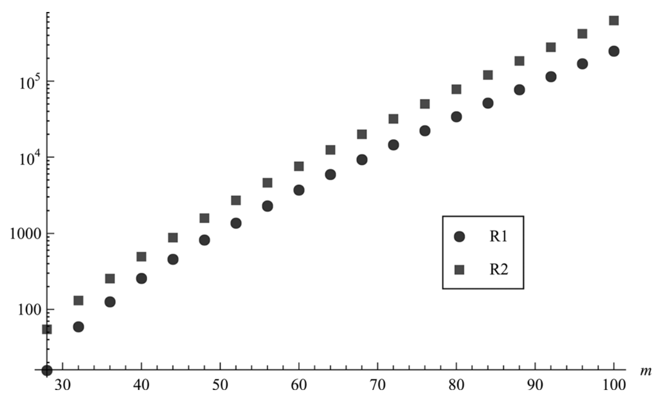

Figure 5: This figure shows a semi-log plot of the quantities R_1 and R_2 appearing in the constraint (5.44). The constraint is violated when they are positive. The plot shows that starting with $m = 28$ they are indeed positive and even exponentially growing.

6. Near-extremal $\mathcal{N} = 2$ conformal field theories

In Section 5, we showed that $\mathcal{N} = 2$ ECFTs, as we have defined them, at best exist only for a finite number of exceptional values of m. One might object that our definition is too narrow and that we should simply modify the definition of an extremal theory.

In this section, we consider one way of modifying the notion of an extremal theory, by demanding only that some "significant" fraction of the polar degeneracies $c(n,\ell)$ coincide with those predicted from the vacuum character.

Returning to the system of equations (4.5), for fixed m, define $k(m)$ to be the largest integer such that

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \quad a = 1, \dots, k(m) \tag{6.1}$$

admits a solution x_i for which the elliptic genus $\sum x_i \phi_i$ has an integral Fourier expansion. We would like to show that we can choose $k(m)$ to be "close" to $P(m)$.

Turning again to a numerical analysis, we studied the truncation of (6.1) to the first $j(m)$ equations: $1 \leq a \leq j(m)$, where we ordered the polar terms via their polarity. We found that in all cases $1 \leq m \leq 36$ there is indeed a solution x_i in rational numbers. Moreover, for all m *except* $m = 17$ the Fourier expansion coefficients are integral — in so far as we have tested them. This indicates that $k(m) = j(m) + \mathcal{O}(1)$.[12] We conjecture that this is the case in general, and in Section 6.1, assuming this conjecture to be true, we derive an interesting constraint on the spectrum of $\mathcal{N} = 2$ CFTs.

For the analysis in Section 6.1, it turns out to be more convenient to define a "β-extremal $\mathcal{N} = 2$ CFT" by imposing the less restrictive condition of only requiring that polar degeneracies are predicted from the vacuum character in the β-truncated polar region:

$$\mathcal{P}_\beta := \{(\ell, n) : 1 \leq \ell \leq m, n \geq 0, 4mn - \ell^2 \leq -\beta\}. \tag{6.2}$$

We know that for suitable β candidate elliptic genera exist. For example, if we take $\beta = m^2$, then we can always construct a candidate elliptic genus. We get a better approximation to an extremal theory by lowering the value

[12] Note that at least for the exceptional solutions, $m = 7, 8, 11, 13$ we have $k(m) > j(m)$.

of β. Therefore, let $P_\beta(m)$ be the number of independent polar monomials of polarity $\le -\beta$, and let β_* be the *smallest* integer β such that

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \quad a = 1, \ldots, P_\beta(m), \tag{6.3}$$

admits a solution x_i for which $\sum x_i \phi_i$ has *integral* coefficients in its Fourier expansion. According to our conjecture, $P_{\beta_*}(m) \cong j(m)$. We would therefore like to estimate the value of β for which $P_\beta(m) = j(m) + \mathcal{O}(m^{1/2})$ for large m. The computation follows closely the analysis of Section 2.2.

We now have

$$P_\beta(m) = \sum_{r=r_0}^{m} \left\lceil \frac{r^2-\beta}{4m} \right\rceil, \tag{6.4}$$

where $r_0 := \lceil \sqrt{\beta} \rceil$. As before, we write this as a sum of three terms,

$$P_\beta(m) = \sum_{r=r_0}^{m} \frac{r^2-\beta}{4m} - \sum_{r=r_0}^{m} \left(\left(\frac{r^2-\beta}{4m}\right)\right) + \frac{1}{2} \sum_{r=r_0}^{m} \left(\left\lceil \frac{r^2-\beta}{4m} \right\rceil - \left\lfloor \frac{r^2-\beta}{4m} \right\rfloor \right). \tag{6.5}$$

The first term is

$$\sum_{r=r_0}^{m} \frac{r^2-\beta}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24} - \frac{r_0(2r_0-1)(r_0-1)}{24m} - \beta \frac{(m-r_0+1)}{4m}. \tag{6.6}$$

Denote the number of integers r such that $r_0 \le r \le m$ with $r^2 = \beta \bmod 4m$ by $\nu(m,\beta)$. Unlike the case $\beta = 0$, we cannot write down an exact formula, but it is clear that asymptotically $\nu(m,\beta) \sim m^{1/2}$. The second term is

$$\sum_{r=r_0}^{m} \left\lceil \frac{r^2-\beta}{4m} \right\rceil - \sum_{r=r_0}^{m} \left\lfloor \frac{r^2-\beta}{4m} \right\rfloor = m + 1 - r_0 - \nu(m,\beta). \tag{6.7}$$

For the third term, we again use the argument that the numbers $((\frac{r^2-\beta}{4m}))$ are randomly distributed. We thus have a random walk between $-1/2$ and $+1/2$ and the sum is expected to be of order $m^{1/2}$.

To conclude, note that for $\beta = \alpha m$ with α a constant $0 < \alpha < 1$, we have $r_0 \sim m^{1/2}$, so the large m asymptotics are

$$P_\beta(m) = \frac{m^2}{12} + \left(\frac{5}{8} - \frac{\alpha}{4}\right) m + \mathcal{O}(m^{1/2}). \tag{6.8}$$

Comparing to (2.25), we see that for large m, the reduction of polarity to obtain the truncated supergravity elliptic genus is given by $\beta = \frac{1}{2}m + \mathcal{O}(m^{1/2})$.

As in Equation (2.33) above, the symbol $\mathcal{O}(m^{1/2})$ is to be understood heuristically. It would be worthwhile being more rigorous about this point.

6.1. A constraint on the spectrum of $\mathcal{N} = 2$ theories with integral $U(1)$ charges

In the previous sections, we have found strong evidence that we must have $P_{\beta_*}(m) \cong j(m)$, and hence by (6.8)

$$\beta_* \geq \frac{m}{2} + \mathcal{O}(m^{1/2}) \tag{6.9}$$

for large m.

Now a monomial $q^n y^\ell$ of polarity β corresponds by spectral flow to a state in the NS-sector that contributes as $q^{h-m/4} y^\ell$ with

$$h = \frac{m}{4} + \frac{\ell^2}{4m} - \frac{\beta}{4m}. \tag{6.10}$$

Therefore, if we accept (6.9), then we can obtain an interesting constraint on the spectrum of a $(2,2)$ AdS_3 supergravity with a holographically dual CFT. It must contain at least one state which is a left-moving $\mathcal{N} = 2$ primary (not necessarily chiral primary) tensored with a right-moving chiral primary such that

$$h < \frac{m}{4} + \frac{\ell^2}{4m} - \frac{1}{8} + \mathcal{O}(m^{-1/2}). \tag{6.11}$$

It would be interesting and useful to sharpen this bound. However, we will show in Section 7 that it is possible to construct elliptic genera, which, after spectral flow, do match the spectrum of the vacuum character for all conformal weights with $h \leq \frac{m}{4}$. There is no contradiction between this result and (6.11) because under $1/2$ unit of spectral flow, $0 \leq |\ell| \leq 2m$ and hence $\frac{\ell^2}{4m}$ could be as large as m, and thus the bound can be as large as $\frac{5m}{4} - \frac{1}{8} + \mathcal{O}(m^{-1/2})$.

7. Construction of nearly extremal elliptic genera

In this section, we consider an alternative basis for the weak Jacobi forms which has a "triangular" nature, allowing us to replace the polar region $\mathcal{P}^{(m)}$

by an alternative region S. We will see that for large m, S "approximates" $\mathcal{P}^{(m)}$. In the next section, we discuss the possible physical significance of this fact.

It is shown in [25] that there is an *integral* basis of the ring of weak Jacobi forms of weight zero with integral coefficients

$$\tilde{J}^{\mathbb{Z}}_{0,*} = \mathbb{Z}[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}]/I, \tag{7.1}$$

where I is the ideal generated by the relation

$$\phi_{0,1}\phi_{0,3} = 4\phi_{0,4} + \phi_{0,2}^2. \tag{7.2}$$

The generators are elliptic genera of Calabi–Yau manifolds, and explicit formulae are given in [25]. In the basis (2.23), they can be expressed as[13]

$$\phi_{0,1} = \tilde{\phi}_{0,1}, \tag{7.3}$$

$$\phi_{0,2} = \frac{1}{24}\tilde{\phi}_{0,1}^2 - \frac{1}{24}\tilde{\phi}_{-2,1}^2 E_4, \tag{7.4}$$

$$\phi_{0,3} = \frac{1}{432}\tilde{\phi}_{0,1}^3 - \frac{1}{144}\tilde{\phi}_{0,1}\tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{216}\tilde{\phi}_{-2,1}^3 E_6, \tag{7.5}$$

$$\phi_{0,4} = \frac{1}{6912}\tilde{\phi}_{0,1}^4 - \frac{1}{1152}\tilde{\phi}_{0,1}^2\tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{864}\tilde{\phi}_{0,1}\tilde{\phi}_{-2,1}^3 E_6 - \frac{1}{2304}\tilde{\phi}_{-2,1}^4 E_4^2. \tag{7.6}$$

To make the triangular nature of this basis manifest, it is useful to consider the NS-sector images of the generators,

$$\hat{\phi}_{0,m} = (-1)^m q^{m/4} y^m \phi_{0,m}\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right). \tag{7.7}$$

We now consider ordering the q, y expansion by the leading power of q and, for each power of q, by the *largest* positive power of y. (Recall that $\chi_{\mathrm{NS}}(\tau, z)$ is an even function of z, so the positive powers of y determine the negative powers of y.) With this ordering of terms, we have

$$\begin{aligned}
\hat{\phi}_{0,1} &= q^{-1/4} + \mathcal{O}(q^{1/4}),\\
\hat{\phi}_{0,2} &= (y + y^{-1}) + \mathcal{O}(q^{1/2}),\\
\hat{\phi}_{0,3} &= q^{1/4}(y - y^{-1})^2 + \mathcal{O}(q^{3/4}),\\
\hat{\phi}_{0,4} &= 1 + \mathcal{O}(q^{1/2}).
\end{aligned} \tag{7.8}$$

[13] We have redefined $\phi_{0,4}$ in [25] by a factor of -1.

By (7.1), an overcomplete linear basis of $\tilde{J}_{0,m}$ is given by

$$(\hat{\phi}_{0,1})^i(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,3})^k(\hat{\phi}_{0,4})^l \tag{7.9}$$

with $i + 2j + 3k + 4l = m$, $i, j, k, l \geq 0$. In order to obtain a set of linearly independent basis vectors, we distinguish the monomials in (7.9) according to whether $i > k$ or $i \leq k$ and then use identity (7.2) to eliminate $\hat{\phi}_{0,3}$ or $\hat{\phi}_{0,1}$, respectively. The result is that there exists a vector space basis for $\tilde{J}_{0,m}$ which is a disjoint union of two sets $A \amalg B$ with

$$A := \{(\hat{\phi}_{0,1})^i(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,4})^l| \quad i > 0, j \geq 0, l \geq 0 \quad i + 2j + 4l = m\}, \tag{7.10}$$

$$B := \{(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,3})^k(\hat{\phi}_{0,4})^l|, \quad j \geq 0, k \geq 0, l \geq 0 \quad 2j + 3k + 4l = m\}. \tag{7.11}$$

A tedious but elementary counting argument shows that

$$|A| = \begin{cases} \dfrac{m^2}{16} + \dfrac{3m}{8} - \dfrac{s^2}{16} + \dfrac{s}{8} + \dfrac{1}{2}, & m = s \bmod 4, \quad s = 1, 3, \\ \dfrac{m^2}{16} + \dfrac{m}{4} - \dfrac{s^2}{16} + \dfrac{s}{4}, & m = s \bmod 4, \quad s = 0, 2, \end{cases} \tag{7.12}$$

and $|A| + |B| = j(m)$.

Now note that the leading expression in the q, y expansion of an element in the set A is $q^{-i/4}y^j$, while that in the set B is $q^{k/4}y^{j+2k}$. It thus follows that *an* (NS-*sector*) *Jacobi form of weight zero and index* m *with integral Fourier coefficients is uniquely determined by the coefficients of* $q^n y^\ell$ *where* (ℓ, n) *run over the set:*

$$S = S_A \amalg S_B \tag{7.13}$$

where

$$S_A = \left\{ (\ell, n) | n < 0, 0 \leq \ell, n + \frac{m}{4} \geq \frac{\ell}{2} \right\} \tag{7.14}$$

and

$$S_B = \left\{ (\ell, n) | 0 \leq n, 8n \leq \ell, n + \frac{m}{4} \geq \frac{\ell}{2} \right\}. \tag{7.15}$$

In both S_A and S_B, the (ℓ, n) are in the lattice $(\ell, n) \in \mathbb{Z} \times \frac{1}{4}\mathbb{Z}$, subject to the quantization condition

$$\left(n + \frac{m}{4}\right) - \frac{\ell}{2} = 0 \bmod 1. \tag{7.16}$$

(This quantization is equivalent to the statement that in the Ramond sector the elliptic genus has a Fourier expansion in q, y with integral powers of q, y.) The regions S_A and S_B in the (ℓ, h) plane are triangles and their union is a triangle. The full region S can serve as a surrogate for the polar region $\mathcal{P}^{(m)}$, as explained in Figure 6.

Recall that n, the power of q in the NS-sector character, is related to h as $n = h - \frac{m}{4}$. It then follows from (7.14) that S_A contains all possible points with $h < m/4$ that occur in the NS vacuum character (3.8). Thus it *is* possible to construct a weak Jacobi form with integral coefficients whose q-expansion agrees with that of an extremal theory for all NS-sector Virasoro weights up to $h = m/4$ (for m even) and $h = (m-1)/4$ (for m odd). This fits in very nicely with the bound (6.11), which puts an upper bound on the range of h where all states can be descendants of the vacuum.

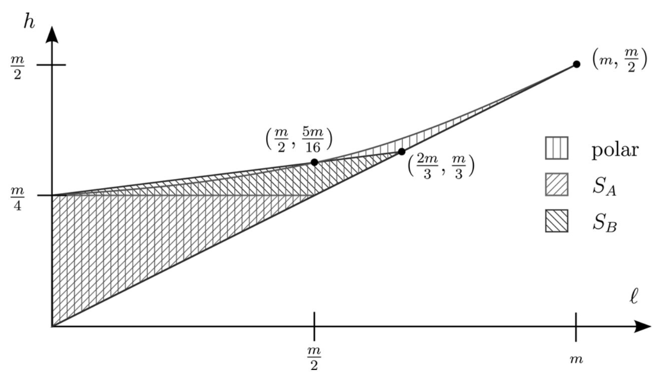

Figure 6: A comparison of the polar region $\mathcal{P}^{(m)}$ and the region S. The NS-sector polar region is bounded by $\ell \geq 0, h \geq \ell/2, h \leq \frac{m}{4} + \frac{\ell^2}{4m}$. The region S is the triangular region, $\ell \geq 0, h \geq \frac{\ell}{2}, h - \frac{m}{4} \leq \frac{\ell}{8}$, which itself is a union of two triangular regions S_A and S_B, where S_A is the subregion of S with $h < \frac{m}{4}$. The polar region contains S_A, while S_B is an "approximation" to the remainder.

8. Discussion: quantum corrections to the cosmic censorship bound

If the pure $\mathcal{N} = (2, 2)$ supergravity is a consistent quantum theory, its Hilbert space should be spanned by states which can be identified as excitations of the supergravity fields. One class of such states are perturbative and normalizable excitations of the supergravity fields in AdS_3, which generate the vacuum representation in the boundary CFT [7]. It is expected that these are the only states up to the cosmic censorship bound. We define this bound to be the boundary of the region in the space of energy and charges in which states corresponding semi-classically to black hole solutions can exist. In the classical limit, the cosmic censorship bound is the condition on mass and charges of a black hole such that there is a regular horizon.

It turns out that the classical cosmic censorship bound is exactly equal to the upper bound of the polar part of the CFT spectrum [15]. This was the motivation for the definition of $\mathcal{N} = (2, 2)$ extremal conformal field theory in Section 3.1. On the other hand, in Section 5, we proved that such a conformal field theory does not exist for sufficiently large m. This result, however, does not immediately rule out the conjectured existence of pure $\mathcal{N} = (2, 2)$ supergravity since the cosmic censorship bound might receive quantum corrections. That is, there might be quantum corrections to the relation between the values of the mass and charges of those quantum states whose semi-classical manifestation are black holes. There are two potential sources for such corrections, and we will discuss each of them below.

As far as perturbative effects are concerned, the pure supergravity theory can be treated as the Chern–Simons gauge theory with the gauge group (3.5). Since the classical equations of motion of the Chern–Simons theory imply vanishing of the gauge field strength and since any perturbative corrections to the equations of motion can be expressed as a polynomial of the field strength and its covariant derivatives, black hole solutions are not corrected to all orders in the perturbative (i.e., $1/m$) expansion. However, values of the mass and charge of a given black hole solution can receive corrections since computing them requires knowing the action as well as the equations of motion. In particular, the "level" m, whose inverse appears in front of the action, can be corrected. The leading discrepancy between the dimension of the space of polar polynomials, $P(m)$, and the dimension of the space of weak Jacobi forms, $j(m)$,

$$P(m) - j(m) = \frac{m}{8} + \mathcal{O}(m^{1/2}), \tag{8.1}$$

can be explained if m is shifted by an appropriate constant by quantum effects. Such a shift is known to occur in perturbative Chern–Simons gauge theory [46], where the level k is shifted at one loop by the dual Coxeter number of the gauge group, $C_2(G)$. For the supergroup OSp(2|2), we have $C_2 = -2$, so that in the present case both k_{L} and k_{R} are shifted as[14]

$$k_{\mathrm{L}} \to k_{\mathrm{L}} - 2. \tag{8.2}$$

Combining this with Equation (3.7), we can express this as the shift of m,

$$m \to m - 8, \tag{8.3}$$

which, unfortunately, does not account for the difference in (8.1). Furthermore, it seems difficult to attribute subleading terms in $(P(m) - j(m))$ to higher order perturbative effects since subleading terms in $P(m)$ contain the arithmetic function $A(m)$, which does not have a nice $1/m$ expansion (see footnote 2).

There is another source of corrections which are nonperturbative in nature. To see this, we note that conformal weights h for states counted by the elliptic genus are integers, as required by modular invariance. This granularity, which is smeared out in any perturbative analysis, gives rise to an intrinsic ambiguity in the cosmic censorship bound of $O(1)$ in h. Since the boundary of the polar region in the (L_0, J_0) plane has a length of order m, it is possible that the discrepancy of $P(m)$ and $j(m)$ mentioned above is entirely attributed to this granularity. For example, the bound on h for a new primary state found in (6.11) is within $O(1)$ of the cosmic censorship bound.

It is possible that a combination of these two effects resolves the apparent contradiction between the conjectured existence of pure $\mathcal{N} = (2, 2)$ supergravity and the properties of the elliptic genus we found in this paper.

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[14]One way to think about this shift is as follows. The supergroup OSp(2|2) is the superconformal group of AdS_2, and its dual Coxeter number, C_2, can be thought of as the beta-function of the world-sheet sigma-model defining AdS_2 space-time. If instead of AdS_2 we consider a positive curvature space, that is a 2-sphere S^2, the contribution to the beta-function of the world-sheet theory should have opposite sign and, hence, the opposite shift of k. In particular, for S^2, which has the isometry group SU(2), the shift $k \to k + 2$ is familiar in the study of SU(2) Chern–Simons theory [46]. In the case of OSp(2|2) Chern–Simons theory, the shift should have opposite sign, therefore justifying (8.2).

Given the close resemblance of the region $S_A \cup S_B$ identified in Section 7 with the polar region, it is natural to ask whether the boundary of that region might in fact constitute the quantum-corrected cosmic censorship bound. This seems unlikely to us. Along the line $h = \frac{\ell}{8} + \frac{m}{4}$, $0 \leq \ell \leq \frac{2m}{3}$, the polarity becomes as great as $p = \frac{m^2}{16}$. It seems unlikely that quantum corrections will modify the mass and charge in such a way as to change a semi-classical black hole state with such a polarity to a descendent of the vacuum.

9. Extremal $\mathcal{N} = 4$ theories

The analysis for the case of the pure $\mathcal{N} = (2,2)$ supergravity theories is somewhat inconclusive since we cannot rule out that there are quantum corrections to the classical supergravity ansatz. The situation is sharper for the case with $\mathcal{N} = (4,4)$ superconformal symmetry since the possible quantum corrections of these theories are well constrained [11]. Therefore, in this section, we shall begin to address whether modular invariance allows for a pure $\mathcal{N} = (4,4)$ supergravity theory. Unfortunately, our results are somewhat incomplete.

Following the earlier definition, we define an extremal $\mathcal{N} = (4,4)$ theory to be a theory whose partition function is of the form (3.10), where $\chi^{(m)}_{\text{vac}}$ is now the vacuum character of the $\mathcal{N} = 4$ algebra [19, 20]:

$$\chi^{(m)}_{\text{vac}} = q^{-m/4} \prod_{n=1}^{\infty} \frac{(1 - yq^{n-1/2})^2 (1 - y^{-1} q^{n-1/2})^2}{(1 - q^n)} \chi(q, y), \tag{9.1}$$

with

$$\chi(q, y) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n-1})} \times \sum_{j \in \mathbb{Z}} q^{(m+1)j^2 + j} \left(\frac{y^{2(m+1)j}}{(1 - yq^{j+1/2})^2} - \frac{y^{-2(m+1)j-2}}{(1 - y^{-1} q^{j+1/2})^2} \right). \tag{9.2}$$

As in the case of the $\mathcal{N} = 2$ vacuum character, we have evaluated this expression at $z + \frac{1}{2}$. To get rid of the negative powers of q in the denominator, we can rewrite it as two separate sums over positive j,

$$\chi(q,y) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-y^2q^n)(1-y^{-2}q^{n-1})} \times \left[\sum_{j\geq 0} q^{(m+1)j^2+j} \left(\frac{y^{2(m+1)j}}{(1-yq^{j+1/2})^2} - \frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j+1/2})^2} \right) + \sum_{j\geq 1} q^{(m+1)j^2+j-1} \left(\frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j-1/2})^2} - \frac{y^{2(m+1)j}}{(1-yq^{j-1/2})^2} \right) \right]. \tag{9.3}$$

It is straightforward to read off the polar polynomial from this expression.

Using the same methods as in Section 4, we have analyzed whether this polar polynomial can be completed to a weak Jacobi form. We have performed the analysis for $1 \leq m \leq 20$, and we have found that the only cases where this is possible are $m = 1, 2, 3, 4, 5$. (Note that for $1 \leq m \leq 4$ this is automatic since $P(m) = j(m)$.) Thus, apart from a few low-level exceptions, we expect that the pure $\mathcal{N} = (4, 4)$ sugra ansatz is incompatible with modular invariance. It might be possible to prove this assertion by suitably modifying the methods of Section 5, but the expressions appear to be challenging and we have not attempted to do so.

An important loophole in our argument is the possibility that there are zero modes making the elliptic genus vanish. This might happen when there is an extension of the chiral algebra and m is odd. In order to demonstrate this, write the character expansion of the RR-sector partition function as

$$Z_{\mathrm{RR}} = \sum_{1\leq \ell,\tilde{\ell}\leq m} c_{\ell\tilde{\ell}} \chi_\ell \overline{\chi_{\tilde{\ell}}} + c_{00}\chi_0\overline{\chi_0} + \sum_{1\leq \ell\leq m} c_{\ell 0}\chi_\ell\overline{\chi_0} + \sum_{1\leq\tilde{\ell}\leq m} c_{0\tilde{\ell}}\chi_0\overline{\chi_{\tilde{\ell}}} + \cdots \tag{9.4}$$

Here χ_ℓ denote the characters of the unitary massless representations, with $0 \leq \ell \leq m$ denoting twice the spin of the highest weight vector and $+\cdots$ refers to terms with a massive representation on the left or the right. The reason for separating out the $\ell = 0$ spin as special is that its highest weight vector is not a polar state, whereas the highest weight vectors of all the other massless representations are polar states. An extremal theory must have an expansion of the form

$$Z_{\mathrm{RR}} = \chi_m\overline{\chi_{\tilde{m}}} + c_{00}\chi_0\overline{\chi_0} + \sum_{1\leq \ell\leq m} c_{\ell 0}\chi_\ell\overline{\chi_0} + \sum_{1\leq\tilde{\ell}\leq m} c_{0\tilde{\ell}}\chi_0\overline{\chi_{\tilde{\ell}}} + \cdots \tag{9.5}$$

since χ_m is the spectral-flow image of the NS vacuum. Now, the elliptic genus of χ_ℓ is $(-1)^\ell(\ell+1)$, while that of the massive representations is zero. Thus, if the elliptic genus vanishes then, comparing the coefficient of the left-moving vacuum character χ_m, we see that

$$c_{m0} = (-1)^{m+1}(m+1). \tag{9.6}$$

Note that a nonvanishing coefficient c_{m0} implies that the right-moving chiral algebra is enhanced, as claimed. Also, since c_{m0} is a positive integer, this can only happen when m is odd. Moreover, by comparing the coefficients of the other left-moving characters, we find the constraints $c_{\ell 0} = 0$ for $1 \leq \ell \leq m-1$ and $\sum_{\tilde{\ell}=0}^{m} c_{0\tilde{\ell}}(-1)^{\tilde{\ell}}(\tilde{\ell}+1) = 0$. Since our no-go theorem would apply if either the holomorphic or anti-holomorphic elliptic genus is nonvanishing, we might as well assume the anti-holomorphic elliptic genus also vanishes. In this case, we find that $c_{0\ell} = 0$ for $1 \leq \ell \leq m-1$ and hence $c_{00} = (m+1)^2$, so that $Z_{\mathrm{RR}} = |\chi_m + (m+1)\chi_0|^2 + \cdots$. Thus, for extremal theories of this type our arguments fail, and further investigation is necessary.

It should be noted that a vanishing elliptic genus does indeed occur in some important examples. One example arises in $\mathrm{AdS}_3 \times S^3 \times T^4$ compactifications [31]. A second example is in the MSW conformal field theory with $(0,4)$ supersymmetry, which is dual to an $\mathrm{AdS}_3 \times S^2 \times X$ compactification, where X is Calabi–Yau [32,36]. In all these cases, there is an extended chiral algebra due to singleton modes. In such a case, one must take derivatives with respect to $\bar{z}$ and set $\bar{z} = 0$ [9,31]. The resulting modular object is a nonholomorphic generalization of a Jacobi theta function [12,14]. A similar phenomenon happens in the analog of the elliptic genus for the *large* $\mathcal{N} = 4$ superconformal algebra [26]. Of course, the examples we have just cited are not extremal theories. However, these examples do suggest that it would be useful to extend the investigation of extremal theories to the cases of vanishing elliptic genera, or $(0,4)$ supersymmetry, or large $\mathcal{N} = 4$ supersymmetry.

10. Applications to flux compactifications

Flux compactifications of M-theory and string theory have been a very popular subject of investigation in recent years [13,16]. Unfortunately, these compactifications are in general very complicated and it is difficult to be sure that they are valid solutions of string theory within a controlled approximation scheme. The demonstration of holographically dual conformal field theories would definitively settle such difficulties, at least for flux compactifications to anti-de Sitter spacetimes. The considerations and techniques of this paper might put interesting constraints on the allowed spectra of some classes of flux compactifications, namely compactifications to AdS_3 with a holographically dual $(2,2)$ conformal field theory. One could imagine, for example, flux compactifications of M-theory on a suitable Calabi–Yau 4-fold, where one includes $M5$ instanton effects, in order to exclude no-scale compactifications.

The compactifications of greatest interest are those with a small cosmological constant and a large gap from the ground state to the Kaluza–Klein scale. These simple aspects of the spectrum already have implications for the conformal field theory. If the cosmological constant is small, then the Brown–Henneaux central charge $c = \frac{3}{2}RM_{\rm pl}^{(3)}$ is large. This implies that the level

$$m = \frac{RM_{\rm pl}^{(3)}}{4} \tag{10.1}$$

is large.

Now let us consider the spectrum of the theory. The supergravity multiplet corresponds to the super-Virasoro descendants. Next, if V_8 is the volume of the Calabi–Yau 4-fold in 11-dimensional Planck units, then

$$[V_8(M_{\rm pl}^{(11)})^8]M_{\rm pl}^{(11)} = M_{\rm pl}^{(3)}, \tag{10.2}$$

and therefore, $M_{\rm pl}^{(11)} \sim M_{\rm pl}^{(3)}$ unless V_8 is unnaturally large, and hence in AdS units, the KK scale is of order m. Thus, we naturally expect a large gap to the primary fields corresponding to the KK modes.

In addition to the supergravity multiplet and the KK modes, there will typically be other primary fields, for example, the moduli fields, many of which might have acquired masses in the compactification scheme. Our conjectured bound (6.11) might possibly put constraints on the masses which the moduli acquire.

It would clearly be of interest to make these considerations more precise, and moreover to extend them to theories with holographic duals with only $(1,1)$ supersymmetry. Indeed, one does not expect generic flux compactifications to lead to $\mathcal{N} = 2$ supersymmetry since there is no candidate isometry for the $U(1)$ current algebra. Given the $\mathcal{N} = 1$ supersymmetry, one can still form a holomorphic elliptic genus, but the existence of the Hauptmodul $\hat{K}$ for Γ_θ (see Equation (5.3) above) shows that the techniques of this paper cannot be used to exclude compactifications just based on the polar polynomial of the elliptic genus. Further work is needed to see whether modularity, combined with other ideas, puts any interesting constraints on the landscape of three-dimensional AdS compactifications.[15]

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A. Growth properties

A.1. Analysis of the constraint for m odd

For m odd, we have

$$L = \left[-2iq^{-m/4+1/8} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3(\tau)}{\eta^3} \vartheta_2^2 \vartheta_4^2 \right]_{q^0}, \tag{A.1}$$

where L was defined in (5.26). We can simplify this significantly using the triple product identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$. Next, shifting $\tau \to \tau + 1$ (which cannot change the q^0 coefficient) we obtain

$$L = 4e^{-i\pi m/2} \left[q^{-m/4+1/8} \frac{1+q^{1/2}}{1-q^{1/2}} \vartheta_2 \vartheta_3 \right]_{q^0}. \tag{A.2}$$

Now use the usual sum formula for ϑ_2 and ϑ_3 to obtain

(A.3)
$$\vartheta_2\vartheta_3 = \sum_{r,s\in\mathbb{Z}} q^{(r-1/2)^2/2+s^2/2} = \sum_{r,s\in\mathbb{Z}} q^{(2r-1)^2/8+(2s)^2/8} = \sum_{n\in\mathbb{N}_0} B(n) q^{n/8},$$

where $B(n)$ is the number of ways of writing n as a sum of an even and an odd integer squared, i.e., $n = (2r-1)^2 + (2s)^2$ with both r and s integer. We also observe that the series expansion of the other factor is

$$\frac{1+q^{1/2}}{1-q^{1/2}} = 1 + 2\sum_{\ell=1}^{\infty} q^{\ell/2}. \tag{A.4}$$

Thus the exact result for (5.26) is

$$L = 4e^{-i\pi m/2} \left[B(2m-1) + 2 \sum_{\ell=1}^{(2m-1)/4} B(2m-1-4\ell) \right]. \tag{A.5}$$

The dominant contribution comes from the second term. This sum is precisely equal to all combinations of an odd and an even integer whose square sum up to a number less or equal to $2m-5$. Now draw a rectangular lattice whose unit cell is a square with length 2, where we shift the lattice by one unit in the x^1-direction say, so that the centers of the cells are at $(x^1, x^2) = (2r-1, 2s)$. Consider the area of all those unit cells for

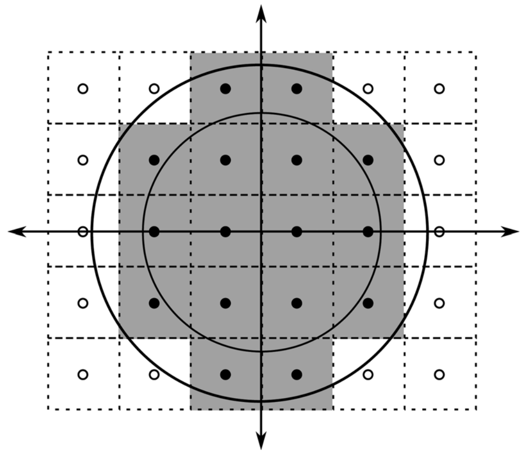

Figure 7: The grey area is given by those boxes whose centers lie within the outer circle of radius $\sqrt{2m-5}$. The inner circle has radius $\sqrt{2m-5} - \sqrt{2}$ and is completely contained in the grey area.

which the corresponding center point $(2r-1, 2s)$ has the property that $(2r-1)^2 + (2s)^2 \leq 2m-5$. It follows from elementary geometry that this area is bigger than the area of the disk with radius $\sqrt{2m-5} - \sqrt{2}$ (see Figure 7). Since each unit cell has area 4, it follows that

(A.6)
$$\sum_{\ell=1}^{(2m-1)/4} B(2m-1-4\ell) \geq \frac{1}{4}\pi\left(\sqrt{2m-5} - \sqrt{2}\right)^2 = \frac{\pi}{2}m - \pi\sqrt{m - \frac{5}{2}} - \frac{3}{4}\pi.$$

Thus it follows that $e^{i\pi m/2}L$, which is positive, is bounded below by

$$e^{i\pi m/2}L \geq 4\pi m - 8\pi\sqrt{m - \frac{5}{2}} - 6\pi. \tag{A.7}$$

A.2. Analysis of the constraint for $m = 2 \bmod 4$

In the case of m odd, we saw that L only grew linearly. Since the original expression contained exponentially growing function such as η^{-3}, this means that there had to occur cancellations. We will now show that for $m =$

2 mod 4, such cancellations do not occur, i.e., that

$$L = \left[q^{-m/4+1/8} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3}{\eta^3} (\vartheta_4^4 - \vartheta_2^4) \right]_{q^0} \tag{A.8}$$

grows exponentially with m. To this end, use (5.24) to write

$$\left[q^{-1/2+1/8} \frac{(1-q^{1/2})^2}{1-q} \left(-24 \frac{q\partial_q \vartheta_3}{\eta^3} + \frac{E_2 \vartheta_3}{\eta^3} \right) \right]_{q^N} , \tag{A.9}$$

where $N = m/4 - 1/2$. The following form of E_2 will be useful:

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k) q^k, \tag{A.10}$$

where $\sigma_1(k)$ is the divisor function.

Let us first consider the second term of (A.9). We will show that this is negative and grows exponentially fast with N. We introduce the expansion coefficients of ϑ_3/η^3,

$$\frac{\vartheta_3}{\eta^3} = q^{-1/8} \sum_{n \geq 0} (F_1(n) q^n + F_2(n) q^{n+1/2}). \tag{A.11}$$

From these, we obtain the discrete derivative $(1-q^{1/2})^2 \vartheta_3/\eta^3$,

$$q^{-1/2+1/8} (1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n \geq 0} (K(n) q^n + K'(n) q^{n-1/2}) \tag{A.12}$$

with $K(n) = F_2(n) - 2F_1(n) + F_2(n-1)$, and, including E_2,

$$q^{-1/2+1/8} E_2 (1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n \geq 0} (\hat{K}(n) q^n + \hat{K}'(n) q^{n-1/2}) \tag{A.13}$$

with

$$\hat{K}(n) = K(n) - 24 \sum_{s=1}^{n} \sigma_1(s) K(n-s). \tag{A.14}$$

Finally, the desired second term of (A.9) is $\sum^N \hat{K}(n)$. It will therefore suffice to show that $\hat{K}(n)$ grows exponentially and is negative for large n.

To examine the large n behavior, we begin with the Rademacher expansions for $F_1(n)$ and $F_2(n)$. These are summarized in Appendix B with the result that

$$\begin{aligned} F_1(n) &= (8n)^{-5/4} e^{\pi\sqrt{2n}} \Bigg(1 - \frac{15+\pi^2}{8\sqrt{2}\pi} n^{-1/2} + \frac{105+10\pi^2+\pi^4}{256\pi^2} n^{-1} \\ &\quad + \mathcal{O}(n^{-3/2}) \Bigg), \\ F_2(n) &= (8n)^{-5/4} e^{\pi\sqrt{2n}} \Bigg(1 + \frac{3(\pi^2-5)}{8\sqrt{2}\pi} n^{-1/2} + \frac{3(35-10\pi^2+3\pi^4)}{256\pi^2} n^{-1} \\ &\quad + \mathcal{O}(n^{-3/2}) \Bigg). \end{aligned}$$

From this, we compute the discrete derivative

$$K(n) = \pi^2 (8n)^{-9/4} e^{\pi\sqrt{2n}} (1 + O(n^{-1/2})). \tag{A.15}$$

Note the exponential growth with n. Now write

$$\hat{K}(n) = K(n) - 24K(n-1) - 24S \tag{A.16}$$

with $S := \sum_{s=2}^n \sigma(s) K(n-s)$. It is straightforward to see that the sum S is positive definite for large n: first note that because of (A.15), $K(n)$ is negative for at most finitely many n. Since $K(n)$ grows exponentially and $\sigma(s)$ only grows like $\sigma(s) \sim e^\gamma s \ln\ln s$, where γ is the Euler–Mascheroni constant [27], it follows that the first terms of the sum dominate the (potentially negative) terms at its tail. The first two terms on the RHS of (A.16) clearly grow like $-23\pi^2(8n)^{-9/4}e^{\pi\sqrt{2n}}$, hence $\hat{K}(n)$ is negative and exponentially growing for large n. Therefore the same is true for $\sum^N \hat{K}(n)$.

In the analysis of the case $m = 0 \bmod 4$ below, we will show that the first term of (A.9) is negative, so that there can be no cancellations between the two. We thus conclude that (A.9) grows exponentially.

A.3. Analysis of the constraint for $m = 0 \bmod 4$

Define

$$R_1 = \left[2q^{1/2} \frac{(1-q^{1/2})^4}{(1-q)^3} \frac{\vartheta_3}{\eta^3} \right]_{q^{m/4-1/8}} \tag{A.17}$$

$$R_2 = \left[\frac{(1-q^{1/2})^2}{1-q} \frac{q\partial_q \vartheta_3}{\eta^3} \right]_{q^{m/4-1/8}}. \tag{A.18}$$

We shall show that for large enough m, both R_1 and R_2 are positive. Consider first R_2. Note that the only negative coefficients that can appear are due to the factor $(1-q^{1/2})^2$. It will suffice to show that the coefficients

$$\left[\frac{(1-q^{1/2})^2}{(1-q)^3(1-q^2)^3} q\partial_q \vartheta_3 \right]_{q^N} \tag{A.19}$$

are positive for N large enough. We have dropped the factor of $(1-q)^{-1}$ and included only the first two factors of η^3, which will turn out to be sufficient to ensure positivity. Defining

$$\frac{1}{(1-q)^3(1-q^2)^3} = \sum_{n=0}^{\infty} b(n) q^n, \tag{A.20}$$

it is straightforward to calculate

$$b(n) = \begin{cases} \frac{1}{1920}(2+n)(4+n)(6+n)(8+n)(5+2n), & n \text{ even}, \\ \frac{1}{1920}(1+n)(3+n)(5+n)(7+n)(13+2n), & n \text{ odd}. \end{cases} \tag{A.21}$$

Note in particular that

$$b(n) = \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \mathcal{O}(n^2). \tag{A.22}$$

We now want to calculate the coefficients $p(N)$ of

$$\frac{1}{(1-q)^3(1-q^2)^3} q\partial_q \vartheta_3 = \sum_{N \in 1/2\mathbb{N}} p(N) q^N. \tag{A.23}$$

We need to distinguish the cases $N \in \mathbb{N}$ and $N \in \mathbb{N} + \frac{1}{2}$:

$$N \in \mathbb{N} : p(N,K) = \sum_{s=0}^{K} b(N - 2s^2)4s^2, \tag{A.24}$$

$$N \in \mathbb{N} + \frac{1}{2} : p(N,K) = \sum_{s=0}^{K} b(N - (2s+1)^2/2)(2s+1)^2. \tag{A.25}$$

In principle, the upper bound K is given by the requirement that the argument of b be nonnegative, and its explicit expression will involve some floor function of a square root of N. For the moment, we will leave K as an auxiliary integer parameter. One can then evaluate the sums explicitly to obtain polynomials in both N and K, again distinguishing the cases N odd and N even. As the resulting expressions are rather lengthy, we refrain from writing them down explicitly. To determine the Nth coefficient of (A.19), we then need to evaluate

$$p(N,K_1) - 2p(N-1/2,K_2) + p(N-1,K_3). \tag{A.26}$$

In principle, we would now have to determine the exact values of K_i, which are complicated step functions of $N^{1/2}$. For our purposes, however, it is enough to know their leading behavior. In particular, we know that $K_i = \sqrt{\frac{N}{2}} - \epsilon_i$, where $0 \leq \epsilon_i < 2$, so that ϵ_i is of order 1. We then obtain for (A.26) the expression

$$\frac{N^{9/2}}{1890\sqrt{2}} + \mathcal{O}(N^{7/2}). \tag{A.27}$$

Note that this holds for all $N \in \frac{1}{2}\mathbb{N}$. (Hence, our estimates can also be applied to the analysis of Section A.2.) This shows that the leading term has a positive coefficient and that it is independent of the ϵ_i, which only appear in the subleading terms. This then shows that (A.19) has positive coefficients for N large enough.

Note that for low values of N, the coefficients of (A.19) can still be negative. To complete the argument, we thus have to show that after convolution with the remaining factors in (A.18) the potentially negative coefficients for $N < N_0$ cannot render negative the coefficients at arbitrarily large N. To see this, note that it follows from the Rademacher expansion that, for any set of positive integers $a_1, \ldots, a_k$, the Fourier coefficients of

$$(1-q)^{a_1}(1-q^2)^{a_2}\cdots(1-q^k)^{a_k}\eta^{-3} \tag{A.28}$$

will have the asymptotic behavior $\sim n^p e^{\pi\sqrt{2n}}$. For example, in Appendix B, we show that for the case of interest, $(1-q)^3(1-q^2)^3\eta^{-3}$, the leading asymptotics is given by

$$\frac{\pi^6}{8\sqrt{2}} n^{-9/2} e^{\pi\sqrt{2n}}. \tag{A.29}$$

We approximate the convolution sum as the integral

$$\int^N ds s^{9/2}(N-s)^{-9/2} e^{\pi\sqrt{2(N-s)}}. \tag{A.30}$$

The position of the saddle point of this integral grows as

$$s_0 \sim N^{1/2}. \tag{A.31}$$

This means that for N large enough, the contribution of the negative coefficients around $s \sim 1$ will be negligible, so that the total coefficient is positive.

Turning to R_1, we need to consider

$$(1-q)^{-3}(1-q^2)^{-3}(1-q^3)^{-3}(1-q^4)^{-3} = \sum_{n=0}^{\infty} \tilde{b}(n) q^n. \tag{A.32}$$

A straightforward, but somewhat tedious calculation then gives expressions similar to (A.21) whose explicit forms depend on $n \bmod 12$. Again, the leading terms are independent of this, so that we can write

$$\begin{aligned} \tilde{b}(n) = {} & \frac{n^{11}}{551809843200} + \frac{n^{10}}{3344302080} + \frac{29\, n^9}{1337720832} + \frac{5\, n^8}{5505024} \\ & + \frac{16949\, n^7}{696729600} + \mathcal{O}(n^6). \end{aligned} \tag{A.33}$$

We can now define $\tilde{p}(N, K)$ analogously to (A.24), (A.25) and evaluate

$$\begin{aligned} & \tilde{p}(N, K_1) - 4\tilde{p}(N-1/2, K_2) + 6\tilde{p}(N-1, K_3) - 4\tilde{p}(N-3/2, K_4) \\ & \quad + \tilde{p}(N-2, K_5), \end{aligned} \tag{A.34}$$

which leads to

$$\frac{N^{15/2}}{1751349600\sqrt{2}} + \mathcal{O}(N^{13/2}). \tag{A.35}$$

Since sums of terms of order n^6 give contributions of at most N^7, this also shows that it was sufficient to consider (A.33) only up to n^6. The coefficients of the truncated η^{-3} expansion grow as in (B.20), and the rest of the argument is then completely analogous to the case of R_2.

B. Rademacher expansions

The proofs in Appendix A require some asymptotic expansions for coefficients of some modular forms. We collect these here.

First, we apply the expansion to the modular vector

$$f_1 = \frac{1}{2}\frac{\vartheta_3 + \vartheta_4}{\eta^3} = q^{-1/8}\sum_{n=0}^{\infty} F_1(n) q^n, \tag{B.1}$$

$$f_2 = \frac{1}{2}\frac{\vartheta_3 - \vartheta_4}{\eta^3} = q^{3/8}\sum_{n=0}^{\infty} F_2(n) q^n, \tag{B.2}$$

$$f_3 = \frac{\vartheta_2}{\eta^3} = \sum_{j=0}^{\infty} F_3(n) q^n. \tag{B.3}$$

We have weight $w = -1$, the representation is manifest for T, and for S it is computed from

$$f_1(-1/\tau) = (-i\tau)^{-1}\frac{1}{2}(f_1 + f_2 + f_3)(\tau), \tag{B.4}$$

$$f_2(-1/\tau) = (-i\tau)^{-1}\frac{1}{2}(f_1 + f_2 - f_3)(\tau), \tag{B.5}$$

$$f_3(-1/\tau) = (-i\tau)^{-1}(f_1 - f_2)(\tau). \tag{B.6}$$

We now have convergent expansions

$$F_1(n) = \frac{\pi}{8}(n - 1/8)^{-1} I_2\left(4\pi\sqrt{\frac{1}{8}\left(n - \frac{1}{8}\right)}\right) + \mathcal{O}(e^{2\pi\sqrt{n/8}}), \tag{B.7}$$

$$F_2(n) = \frac{\pi}{8}(n + 3/8)^{-1} I_2\left(4\pi\sqrt{\frac{1}{8}\left(n + \frac{3}{8}\right)}\right) + \mathcal{O}(e^{2\pi\sqrt{n/8}}), \tag{B.8}$$

$$F_3(n) = \frac{\pi}{8}(n)^{-1} I_2\left(4\pi\sqrt{\frac{1}{8}n}\right) + \mathcal{O}\left(e^{2\pi\sqrt{n/8}}\right). \tag{B.9}$$

Now use

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \left(1 - \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + \cdots\right) \tag{B.10}$$

for $x \to +\infty$ to get

(B.11)
$$F_1(n) = (8n)^{-5/4} e^{4\pi\sqrt{n/8}} \left(1 - \frac{\pi^2+15}{8\sqrt{2}\pi}\frac{1}{n^{1/2}} + \frac{\pi^4+70\pi^2+105}{256\pi^2}\frac{1}{n} + \cdots\right)$$

(B.12)
$$F_2(n) = (8n)^{-5/4} e^{4\pi\sqrt{n/8}} \left(1 + \frac{3(\pi^2-5)}{8\sqrt{2}\pi}\frac{1}{n^{1/2}} + \frac{3(3\pi^4-70\pi^2+35)}{256\pi^2}\frac{1}{n} + \cdots\right).$$

We also need the asymptotic expansion of functions that are obtained from η^{-3} by dropping the first few factors in the product formula. Defining

$$\eta^{-3} = q^{-1/8} \sum_n p_3(n) q^n \tag{B.13}$$

(with $p_3(n) = 0$ for $n < 0$), we have the Rademacher formula

$$p_3(n) = 2\pi(8n-1)^{-5/4} I_{3/2}(\pi\sqrt{2(n-1/8)}) + \mathcal{O}(e^{\pi\sqrt{n/2}}). \tag{B.14}$$

Note that the Bessel function is elementary

$$I_{3/2}(x) = \frac{2}{\sqrt{2\pi x}} \left(\cosh x - \frac{\sinh x}{x}\right). \tag{B.15}$$

Define

$$(1-q)^3(1-q^2)^3\eta^{-3} = q^{-1/8} \sum_n \hat{p}_3(n) q^n , \tag{B.16}$$

which is a kind of sixth-order discrete derivative:

$$\begin{aligned} \hat{p}_3(n) &= p_3(n) - 3p_3(n-1) + 8p_3(n-3) - 6p_3(n-4) - 6p_3(n-5) \\ &\quad + 8p_3(n-6) - 3p_3(n-8) + p_3(n-9). \end{aligned} \tag{B.17}$$

Substituting the asymptotic expansion (B.14), one finds after some algebraic manipulations

$$\hat{p}_3(n) = \left(\frac{\pi^6}{8\sqrt{2}} n^{-9/2} + \mathcal{O}(n^{-5})\right) e^{\pi\sqrt{2n}}. \tag{B.18}$$

Similarly, the coefficients

$$(1 - q)^3(1 - q^2)^3(1 - q^3)^3(1 - q^4)^3\eta^{-3} = q^{-1/8}\sum_n \tilde{p}_3(n)q^n \tag{B.19}$$

have leading asymptotics

$$\tilde{p}_3(n) \sim \left(\frac{27\pi^{12}}{\sqrt{2}}n^{-15/2} + \mathcal{O}(n^{-8})\right)e^{\pi\sqrt{2n}}. \tag{B.20}$$

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